Topology 2, Math 681, Spring 2018: Notes

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Class of Tuesday, January 9:

- Note that the last semester's abbreviated notes are still available on my web page, at:

  http://www.math.wvu.edu/~kcies/teach/Fall2017/Fall2017.html

Quick Review (Also definition of topological space and continuous maps)

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.
- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.
- A function $f: X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.
- If $\mathcal{B}$ a basis for Y, then $f: X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.
- *Product topology* $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ is generated by subbasis $\mathcal{S}_{prod} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$
- *If $f_\alpha: A \to X_\alpha$ and $f: A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*

  continuity of f implies the continuity of each f_α;

  continuity of all f_α's implies the continuity of $f: A \to \langle X, \mathcal{T}_{prod} \rangle$;
- A *metric space* is a pair $\langle X, d \rangle$, where d is a metric on X. $\mathcal{B}_d = \{B(x, \varepsilon): x \in X \ \& \ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).
- Subspace of a metric space is metric.

- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.
- A topological space $\langle X, \mathcal{T} \rangle$ is *first countable* provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x;
- *Let X be a first countable space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x.*

Class of January 11:

Quick Review, continuation.

- A topological space X is *connected* provided it **does not** exist a pair U, V of open, non-empty disjoint sets with $X = U \cup V$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.
- A closure of a connected space is connected.
- Finite product of connected spaces is connected: sketch proof.
- $\mathbb{R}^\omega$ with the product topology is connected: sketch proof.
- Continuous image of connected space is connected: sketch proof.
- $A \subset \mathbb{R}$ is connected if, and only if, A is convex (an interval).
- Intermediate Value Theorem.
- Definition of *path connectedness*.
- *Topologists sine curve*: it is connected but not path connected.
- Show that every continuous function $f \colon [0,1] \to [0,1]$ has a fixed point.

Written assignment for Tuesday, January 16: Exercise 4, page 162.

Sections 26 and part of 27, with mixed order: compactness

Definition 1 Let Y be a subset of a topological space X. A family $\mathcal{U}$ of subsets of X is a *covering* of Y provided $Y \subset \bigcup \mathcal{U}$. A covering $\mathcal{U}$ of Y is an *open covering* of Y provided every $U \in \mathcal{U}$ is open in X.

Definition 2 A topological space X is *compact* provided for every open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ that covers X (i.e., $\mathcal{U}_0 \subset \mathcal{U}$ is finite and $X = \bigcup \mathcal{U}_0$). Such a family $\mathcal{U}_0$ will be referred to as a (finite) *subcover* of $\mathcal{U}$.

Note: Although subcover $\mathcal{U}_0$ of $\mathcal{U}$ is defined in term of a union $\bigcup \mathcal{U}_0$, this union usually does not belong to $\mathcal{U}$!

Go over Examples 1 and 4: Neither $\mathbb{R}$ nor $(0, 1]$ are compact.

Go over Examples 2 and 3: Every finite space X is compact. So is $X = \{L\} \cup \{a_n : n = 1, 2, 3, \ldots\} \subset \mathbb{R}$, provided $\lim_n a_n = L$.

Lemma 1 (Lemma 26.1)

Class of January 16:

Collect homework. Be ready for a quiz next class.

Recall

- X is *compact* provided for every open cover $\mathcal{U}$ of X contains a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ that covers X.

New material

Theorem 2 (Theorem 26.2) *Closed subspace of compact space is compact.*

Theorem 3 (Theorem 26.3) *Every compact subspace of a Hausdorff space is closed.*

Go over Example 6. Proof of the theorem is based on:

Lemma 4 (Lemma 26.4) *Let X be Hausdorff. For every compact subspace Y of X and every $x \in X \setminus Y$ there exists disjoint open sets U and V in X such that $x \in U$ and $Y \subset V$.*

Theorem 5 (Theorem 27.1) *Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Corollary 6 (Corollary 27.3 for $\mathbb{R}$) *A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.*

Theorem 7 (Thm 26.5) *Continuous image of a compact space is compact.*

Theorem 8 (Theorem 26.6) *If $f: X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Corollary 9 (Thm 27.4: Extreme Value Theorem) *For every compact space X and continuous function $f: X \to \mathbb{R}$ there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$. In particular, this holds for $X = [a, b]$.*

Class of January 18:

Administer Quiz #1.
Recall

- X is *compact* provided for every open cover $\mathcal{U}$ of X contains a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ that covers X.
- Closed subspace of compact space is compact.
- Every compact subspace of a Hausdorff space is closed.
- For every compact subspace Y of a Hausdorff space X and every $x \in X \setminus Y$ there are disjoint open U and V such that $x \in U$ and $Y \subset V$.
- Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.
- A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.
- Continuous image of a compact space is compact.

Go over Exercises 1, 4, and 6 from section 26.

Written assignment for Tuesday, January 23: Exercise 5 p. 171.

New material

Example of a T_1 space X and compact $A \subset X$ such that $\mathrm{cl}(A)$ is not compact: $X = \mathbb{R}$, $\mathcal{T} = \{\emptyset\} \cup \{U \subset \mathbb{R} \colon \mathbb{N} \setminus U \text{ if finite}\}$, and $A = \mathbb{N}$.
$\mathrm{cl}(\mathbb{N}) = \mathbb{R}$ and $\mathcal{U} = \{N \cup \{x\} \colon x \in \mathbb{R}\}$ is its open cover with no finite subcover.

Theorem 10 (Thm 26.7) *Finite product of compact spaces is compact.*

Remark: Actually, arbitrary product of compact spaces is compact. This is Tychonoff Theorem. But its proof is more difficult.

Proof of Theorem 10 based on **very important**

Lemma 11 (Lem 26.8: The Tube Lemma) *Let Y be compact and $x \in X$. If an open set W of $X \times Y$ contains $\{x\} \times Y$, then there is an open set U in X such that $\{x\} \times Y \subset U \times Y \subset W$.*

Corollary 12 (Corollary 27.3 for $\mathbb{R}^n$) *A subspace X of $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded.*

Corollary 13 (Extreme Value Theorem for $\mathbb{R}^n$) *If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \le f(x) \le f(d)$ for every $x \in R$.*

Definition 3 A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.

Theorem 14 (Thm 26.9) *X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.*

Class of January 23:

Collect homework. Be ready for a quiz next class.

Recall

- Finite product of compact spaces is compact.
- **(Extreme Value Theorem for $\mathbb{R}^n$)** If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \le f(x) \le f(d)$ for every $x \in R$.
- A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.
- X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.

New material

Go over of Exercise 7 section 26.

Go over Exercise 4 section 27.

Written assignment for Tuesday, Jan. 30: Exercises 8 and 9 p. 171.

Definition 4 A function f from a metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$ is said to be *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_1, x_2 \in X$

$$d(x_0, x_1) < \delta \Longrightarrow \rho(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 15 (Thm 27.6, Uniform continuity theorem) *Let f be a continuous function from a compact metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$. Then f is uniformly continuous.*

Prove using the Lebesgue number lemma, where a *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as $\operatorname{diam}(D) = \sup\{d(x, y) \colon x, y \in D\}$.

Lemma 16 (Lem 27.5, the Lebesgue number lemma) *Let $\mathcal{A}$ be an open cover of a metric space $\langle X, d\rangle$. If X is compact, then there exists a $\delta > 0$, known as a* **Lebesgue number***, such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.*

Steps of the proof:

- For a metric space $\langle X, d\rangle$, $x \in X$, and non-empty $A \subset X$ we define the *distance from x to A* as $d(x, A) = \inf\{d(x, a) \colon a \in A\}$.
- Show that $X \ni x \mapsto d(x, A) \in \mathbb{R}$ is continuous.
- Prove the Lebesgue number lemma.
- Prove the uniform continuity theorem.

Class of January 25:

Administer Quiz #2.

Recall

- $f\colon \langle X, d\rangle \to \langle Y, \rho\rangle$ is *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(f(x_0), f(x_1)) < \varepsilon$ for every $x_1, x_2 \in X$ with $d(x_0, x_1) < \delta$.
- **Uniform continuity theorem:** Every continuous function from a compact metric space into a metric space is uniformly continuous.
- A *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as the number $\operatorname{diam}(D) = \sup\{d(x, y)\colon x, y \in D\}$.
- **Lebesgue number lemma:** For every open cover $\mathcal{A}$ of a metric space $\langle X, d\rangle$ there is a $\delta > 0$, *Lebesgue number*, such that for every $D \subset X$ of diameter $< \delta$, there is an $A \in \mathcal{A}$ with $D \subset A$.

New material

Definition 5 A point x in a topological space X is an *isolated point* provided $\{x\}$ is open in X.

Theorem 17 (Thm 27.7) *Let X be a compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Go over Exercise 5 page 178. This is Baire category theorem.

Students, try to solve Ex 11 page 171 (not homework).

Section 28: Limit Point Compactness

Definition 6 A space X is *limit point compact* provided every infinite subset of X has a limit point.

Theorem 18 (Thm 28.1) *If X is compact, then X is limit point compact, but not conversely.*

Go over Examples 1 and 2.

Definition 7 A space X is *sequentially compact* provided every sequence in X has a convergent subsequence.

Theorem 19 (Thm 28.2) *For a metrizable space X, the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Remark: "(2) implies (3)" requires only first countability of X.

Students, look at Exercise 7 pages 181–182 (not homework).

Class of January 30:

Collect homework. Be ready for Quiz next class.
Recall that:

- X is: *limit point compact* provided every infinite subset of X has a limit point; *sequentially compact* provided every sequence in X has a convergent subsequence.
- **Thm** For metrizable spaces, the three notions, *compactness*, *limit point compactness*, and *sequential compactness*, are equivalent.

New material

Mention Example 3: the space satisfies (1)–(3), but is not first countable, so not metrizable.
Go over Exercise 1 page 181.

Section 29: Local Compactness

Definition 8 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact.

Compact implies locally compact.
Go over Examples 1 and 2.
$\mathbb{Q}$ is not locally compact: Exercise 1 page 186.
State and prove (sketch only) Theorem 29.1.
Define *one-point compactification* of a locally compact space.
Go over Example 4.

Written assignment for Tuesday, February 6: Ex 7(a)&(b) p. 181-2.

Time permitting
State Theorem 29.2, Corollary 29.3, and Corollary 29.4.

To be covered in the remaining time

Countability axioms

- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.
- A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\mathrm{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Separation axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.
- (new) X is *completely regular* (or a $T_{3\frac{1}{2}}$ *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f[K] \subset \{1\}$.

Important related theorems

- *Urysohn Lemma*: Every T_4 space is a $T_{3\frac{1}{2}}$ space.
- *Tietze Extension Theorem*: If X is normal, $K \subset X$ is closed, and $f: K \to [0,1]$ is continuous, then f can be extended to a continuous $F: X \to [0,1]$.
- *Urysohn Metrization Theorem*: If X is regular and second countable, then it is metrizable.

The Tychonoff Theorem

- *Tychonoff Theorem*: Arbitrary product of compact spaces is compact.

Class of February 1:
Administer quiz. Collect homework.

Section 30: The Countability Axioms
The next definition and theorem were covered last semester.

Definition 9 A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Theorem 20 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

New material:

Definition 10 A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.

Go over Examples 1 and 2.
Go over Theorem 30.2.

Definition 11

- A subset A of a space X is *dense (in X)* provided $\mathrm{cl}(A) = X$.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\mathrm{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Go over Theorem 30.3.

Written assignment for Tuesday, February 6: Exercise 14, page 194. Hint: use the ideas form the proof, that the product of two compact spaces is compact.

Go over Examples 3 and 4. (Very important!)

Class of February 6:

Collect homework.

Recall that:

- A topological space X is: *second countable* provided X has a countable basis; *separable* provided X has a countable dense subset; *Lindelöf* provided every open cover of X has a countable subcover.
- Second countability implies: first countability, separability, and Lindelöf property. None of these implications can be reversed, as proved by $\mathbb{R}_\ell$.
- $\mathbb{R}_\ell$ is Lindelöf; main steps:
  - Note, that it is enough to consider only the covers $\{[a_\xi, b_\xi)\}_{\xi\in J}$.
  - $\mathcal{U} = \{(a_\xi, b_\xi)\}_{\xi\in J}$ is an open cover of $C = \bigcup_{\xi\in J}(a_\xi, b_\xi)$.
  - We can find countable $J_0 \subset J$ with $C = \bigcup_{\xi\in J_0}(a_\xi, b_\xi)$.
  - $\mathbb{R}\setminus C$ is countable. Find countable $J_1 \subset J$ with $\mathbb{R}\setminus C \subset \bigcup_{\xi\in J_1}[a_\xi, b_\xi)$.
  - $\mathcal{V}_0 = \{[a_\xi, b_\xi)\}_{\xi\in J_0\cup J_1} \subset \mathcal{V}$ is countable and covers $\mathbb{R} = C\cup(\mathbb{R}\setminus C)$.
- Product of Lindelöf spaces need not be Lindelöf, as proved by $\mathbb{R}_\ell$.

New material: (Last column of table not covered.)
Go over **Exercises 2, 4, and 5, page 194**; 16, page 195.
Suggested Exercises to examine by the students: 12 and 13, page 194.
Solve Exercise 10, page 194: *Show that a countable product of separable spaces is separable.*

Discuss the table

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	N, $[0,1)$; $[0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R}\setminus(0,1)$	N, $\mathbb{R}\setminus(0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
2nd count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex c
1st count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y, 9 p. 194	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex a

Class of February 8:

Collect homework. Next class I will give you the solutions of all remaining homework. (There exercises.) I will accept no rewrites after this.

Be ready for a quiz next class.

Go over Exercises 1 and 11 page 194. Discuss last column of the table.

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	N, $[0,1)$; $[0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R}\setminus(0,1)$	N, $\mathbb{R}\setminus(0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
2nd count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex c
1st count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex b,c
separable	N, $L\subset(\mathbb{R}_\ell)^2$	N, $L\subset(\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y, 9 p. 194	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex a

Here the space X_∞, in particular $\mathbb{R}_\infty$, is the one point compactification of a discrete space X, that is, $X_\infty = X\cup\{\infty\}$, where $\infty\notin X$, has the topology $\tau = \mathcal{P}(X)\cup\{X_\infty\setminus F\colon F \text{ is a finite subset of } X\}$.

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f\colon\langle X,\tau_d\rangle\to\langle X,\mathcal{T}\rangle$, given by $f(x)=x$, is continuous bijection.

(a) If $\mathcal{T}=\{\emptyset, X\}$ is anti-discrete topology and $X=\mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X=\mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X=\mathbb{N}$ and $\mathcal{T}$ be such that $\langle X,\mathcal{T}\rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (hence, it is not first countable).

Start new section:

Section 31: The Separation Axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

Go over Lemma 31.1.

Go over Exercises 1 and 2.

Go over Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular. (Same for Hausdorff spaces, as proved last semester.) Note that it is false for the normal spaces. (Point to where the subspace part of the proof for the regular spaces brakes for the normal spaces.)

Class of February 13:

Administer the quiz.
Collect homework. Hand the solutions of all remaining homework.

Recall

- X is *regular* (or a T_3 *space*) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$ (equivalently, for every open U and $x \in U$, there an exists open V with $x \in V \subset \text{cl}(V) \subset U$).
- X is *normal* (or a T_4 *space*) provided it is a T_1 space and: for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$ (equivalently, for every open U and closed $F \subset U$, there exists an open V with $F \subset V \subset \text{cl}(V) \subset U$).
- Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular.

Go over Example 1: $\mathbb{R}_K$ is Hausdorff but not regular.
Go over Exercise 4.
Go over Example 2: $\mathbb{R}_\ell$ is normal.
Go over Theorem 7.8 (set theoretical).
Use it, in Example 3, to show that $(\mathbb{R}_\ell)^2$ is not normal.

Note that the product of normal spaces need not be normal. Also, $(\mathbb{R}_\ell)^2$ is regular but not normal.

Latter we will prove that $(\mathbb{R}_\ell)^2$ is homeomorphic to a subspace of some normal spaces. So, a subspace of normal space need not be normal.

Thm 32.2: Every metrizable space is normal.
Thm 32.3: Every compact Hausdorff space is normal.

Written assignment for Tuesday, February 20: Exercise 5, page 199: *Let $f, g\colon X \to Y$ be continuous; assume that Y is Hausdorff. Show that the set $Z = \{x \in X\colon f(x) = g(x)\}$ is closed in X. (Do not assume on Y anything except being Hausdorff!)*

Class of February 15:

Recall

- $\mathbb{R}_K$ is Hausdorff but not regular.
- $\mathbb{R}_\ell$ is normal.
- $(\mathbb{R}_\ell)^2$ is not normal (but regular). So, product of normal spaces need not be normal.
- Thm 32.2: Every metrizable space is normal.
- Thm 32.3: Every compact Hausdorff space is normal.

Section 32: Normal spaces

Show that every regular Lindelöf space is normal. This is Ex 4 page 205. Proof the same as for Thm 32.1.

Corollary: the product of two Lindelöf spaces need not be Lindelöf, justified by $(\mathbb{R}_\ell)^2$.

Go (briefly) over Example 1.

Go over Exercises 1, 2, 3, 5.

Section 33: The Urysohn Lemma

State (proof next meeting):

Urysohn Lemma: *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Define *completely regular* (or $T_{3.5}$) spaces.

Prove Theorem 33.2.

Go over a part of Exercise 4, page 213:

(i) *If $f\colon X \to [0,1]$ is continuous, then $A = f^{-1}(0)$ is a G_δ-set (that is, A is an intersection of countably many open sets).*

Class of February 20:

Collect homework.

Recall

- Every regular Lindelöf space is normal.
- X is *completely regular* (or a $T_{3.5}$ *space*) when it is a T_1 space and for every closed $K \subset X$ and $x \in X \setminus K$ there is continuous $f\colon X \to [0,1]$ s.t. $f[K] \subseteq \{0\}$ and $f(x) = 1$.
- Thm 33.2: Subspace of a completely regular space is completely regular. Product of completely regular spaces is completely regular.

Section 33: The Urysohn Lemma

Prove:

Urysohn Lemma: *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Go over a part of Exercise 4, page 213:

(i) *If $f\colon X \to [0,1]$ is continuous, then $A = f^{-1}(0)$ is a G_δ-set*
(that is, A is an intersection of countably many open sets).

Written assignment for Tuesday, February 27: A more difficult direction of Exercise 4, page 213:

(a) Prove that if X is normal, then for every closed G_δ set $A \subset X$ there is a continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$.

Exercise 5, page 213 (a version of Urysohn Lemma): Let X be normal. *There exists a continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if, and only if, A and B are disjoint closed G_δ sets.*

PROOF. "$\Longrightarrow$" follows from (i).

"$\Longleftarrow$" By (a) there exists continuous functions $f_A, f_B\colon X \to [0,1]$ with $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$. Then $f(x) = \frac{f_A(x)}{f_A(x)+f_B(x)}$ is as needed. ∎

Suggestion: Look over Exercises 1 and 3, page 212; 7 and 8, page 213.

Go over the expanded table (raws regular, completely regular, normal):

	subspace	closed subspace	countable product	arbitrary product	continuous image
2nd countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex c
1st countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
compact	N, $[0,1); [0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R} \setminus (0,1)$	N, $\mathbb{R} \setminus (0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
regular	Y	Y	Y	Y	N, Ex a
completely reg	Y	Y	Y	Y	N, Ex a
normal	N, p. 203	Y, 1 p. 205	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	N, Ex a
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex a

Answers

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f\colon \langle X, \tau_d\rangle \to \langle X, \mathcal{T}\rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T}\rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Class of February 22:

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f: X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

New material:

Theorem 21 (Urysohn metrization theorem) *Every regular second countable space X is metrizable.*

PROOF.

1. Notice that every regular second countable space is normal (as it is regular Lindelöf), so we can use Urysohn Lemma.

2. Prove that there exists a countable family $\mathcal{F}$ of continuous functions $f: X \to [0,1]$ such that:

   (∗) For every open $U \subset X$ and $x \in U$ there is an $f \in \mathcal{F}$ such that $f(x) > 0$ and $f[X \setminus U] \subset \{0\}$.

   A family $\mathcal{F}$ of continuous functions $f: X \to \mathbb{R}$ satisfying (∗) is said to *separate points from closed sets* in X.

3. Prove that (Thm 34.2): *For any T_1 space X family $\{f_\alpha\}_{\alpha \in J}$ separating points from closed sets in X the mapping $F: X \to \mathbb{R}^J$, $F(x)(\alpha) = f_\alpha(x)$, is an imbedding.*

4. Notice that, by 1 and 2, our regular second countable space X can be imbedded into $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ is metrizable, so is X.

Notice that (second version of the proof of thm 34.1):

Every regular second countable space X can be imbedded into $\mathbb{R}^\omega$ considered with the uniform topology.

PROOF. By 2, there exists a family $\{f_n\}_{n=1}^\infty$ separating points from closed sets in X, with $f_n: X \to [0,1]$. Replacing f_n with f_n/n, if necessary, we can assume that $f_n: X \to [0, 1/n]$. Therefore, by 3, there is an imbedding F of X into $T = \prod_{n=1}^\infty [0, 1/n]$.

Then, the theorem follows from the fact that

The uniform topology on T coincides with the product topology.

State and prove Theorem 34.3.

Class of February 27:

Collect homework.

I plan to give you **Mid term test** in two parts, one on Tuesday, March 6, the other on Thursday, March 8. The first part will be only on the material covered in class (or handed to you). It will include few definition and statement of theorems. However, the main emphasis will be on *the proofs of different theorems* we proved in class and, possible, solutions of the exercises/problems solved in class/handed you. The second part will be on the original problems/exercises whose solutions were not discussed (at least, in details) in class.

Recall

- **Urysohn Lemma:** *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f: X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- **Urysohn metrization theorem**: *Every regular second countable space X is metrizable.*
- *X is completely regular iff X is homeomorphic to a subspace of $[0,1]^J$ for some J.*

New material:

State and prove **Tietze Extension Theorem**: I*f X is normal, $K \subset X$ is closed, and $f: K \to [0,1]$ is continuous, then f can be extended to a continuous $F: X \to [0,1]$.*

Notice, that if $X = \mathbb{R}$, then Tietze Extension Theorem is obvious: the linear interpolation of f is continuous.

Prove that Tietze Extension Theorem is true, when the interval $[0,1]$ is replaced with $\mathbb{R}$.

Go over Exercises 1 and 5(a), page 223.

Note that, for example, a circle does not have universal extension property (using Brower Fixed Point Theorem).

Go over Exercise 2, page 212.

Class of March 1:

Return homework with the solution.

I plan to give you **Mid term test** in two parts, one on Tuesday, March 6, the other on Thursday, March 8. The first part will be only on the material covered in class (or handed to you). It will include few definition and statement of theorems. However, the main emphasis will be on *the proofs of different theorems* we proved in class and, possible, solutions of the exercises/problems solved in class/handed you. The second part will be on the original problems/exercises whose solutions were not discussed (at least, in details) in class.

Review for the mid term test.

Consider going over Exercises 1, 2, and 3, page 218.

Class of March 6:

Mid term test, part 1.

Class of March 8:

Mid term test, part 2.

Class of March 20:

Returned tests. Discussed the results and some problems from the tests.

Next class students will present the following results:

JX Prove that every compact subset of a Hausdorff space is closed.

JQ Prove that every second countable space is separable.

JL Exercise 9 p. 171: Let A and B be subspaces of X and Y, respectively; let N be an open subset of $X \times Y$ containing $A \times B$. Show that if A and B are compact, then there exist open sets U and V in X and Y, respectively, such that $A \times B \subset U \times V \subset N$.

ZT **Exercise 8, page 213:** Let X be completely regular; let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f\colon X \to [0,1]$ such that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$.

Class of March 22:

Above students proved, as indicated. JL solved Ex 6 below, pp 24-27.

Other exercises to be solved, in a near future:

JQ 4/12: **Ex. 1. (TEE Fall 2012)** Consider the following subsets, $\vdash$ and $\models$, of $\mathbb{R}^2$, where $\mathbb{R}^2$ is endowed with the standard topology:

$$\vdash = (\{0\}\times[-2,2])\cup([0,2]\times\{0\}) \quad \& \quad \models = (\{0\}\times[-2,2])\cup([0,2]\times\{-1,1\}).$$

Prove, or disprove the following:

(i) There exists a continuous function from $\vdash$ onto $\models$.

(ii) There exists a continuous function from $\models$ onto $\vdash$.

Your argument must be precise, but no great details are necessary.

JQ 4/10: **Ex. 2. (TEE Fall 2012)** For the topologies τ and σ on $\mathbb{R}$ let symbol $C(\tau,\sigma)$ stand for the family of all continuous functions from $\langle\mathbb{R},\tau\rangle$ into $\langle\mathbb{R},\sigma\rangle$.

Let $\mathcal{T}_s$ be the standard topology on $\mathbb{R}$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on $\mathbb{R}$ such that $C(\mathcal{T}_1,\mathcal{T}_2)=C(\mathcal{T}_s,\mathcal{T}_s)$. Show that:

(i) $\mathcal{T}_2\subseteq\mathcal{T}_1$, that is, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(ii) $\mathcal{T}_2\neq\{\emptyset,\mathbb{R}\}$, that is, $\mathcal{T}_2$ is not trivial.

(iii) $\langle\mathbb{R},\mathcal{T}_1\rangle$ is connected.

(Notice that $C(\mathcal{T}_1,\mathcal{T}_2)=C(\mathcal{T}_s,\mathcal{T}_s)$ does not imply that either of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ must be equal to the standard topology $\mathcal{T}_s$.)

JQ 4/5: **Ex. 3. (TEE Fall 2015)** Let $\mathbb{Q}$ be the set of rational numbers considered with the standard topology. Prove that a topological space X is disconnected if, and only if, there exists a non-constant continuous function from X into $\mathbb{Q}$.

JQ 4/12: **Ex. 4. (TEE Fall 2015 plus extra)** Let X be a compact Hausdorff space and Y be Hausdorff. Assume that $f, g\colon X \to Y$ are the continuous functions.

(i) Show that the graph $G(f) = \{\langle x, f(x)\rangle\colon x \in X\}$ of f is compact (as a subspace of $X \times Y$).

(ii) Assume that $X \times Y$ is a metric space. Use part (i) to show that

(•) $\mathrm{dist}(G(f), G(g)) > 0$ if, and only if, $f(x) \neq g(x)$ for all $x \in X$.

Recall, that the distance $\mathrm{dist}(A, B)$ between the non-empty subsets A and B of a metric space $\langle Z, \rho\rangle$ is defined via formula $\mathrm{dist}(A, B) = \inf\{\rho(a, b)\colon a \in A \ \&\ b \in B\}$.

(iii) Show, by giving an example, that the characterization (•) is false if X is not compact.

****

JL 4/10: **Ex. 5. (TEE Spring 2015)** Let X be a compact topological space and let $f\colon X \to \mathbb{R}$ be an arbitrary, **not necessary continuous**, function. Assume that f is locally bounded, that is, that for every $x \in X$ there exists an open $U \ni x$ such that $f[U]$ is bounded in $\mathbb{R}$. Show that $f[X]$ is bounded in $\mathbb{R}$.

JL, 3/22: **Ex. 6. (TEE Fall 2014)** Let X be a regular space. Show that for every disjoint closed countable subsets A and B of X there exist disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$. Do not assume that X is normal!

JL 4/5: **Ex. 7. (TEE Fall 2013)** A topological space is a T_0-space provided for every distinct $x, y \in X$ there exists an open set U in X which contains precisely one of the points x and y. Show that X is a T_0-space if, and only if, $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$ for all distinct $x, y \in X$.

JL 4/10:: **Ex. 8. (TEE Fall 2013)** Let $\langle X, d\rangle$ be a metric space and let $A \subset X$ be such that it has no limit points in X, that is, such that $A' = \emptyset$. Show that there exists a family $\{U_a\}_{a\in A}$ of pairwise disjoint open sets such that $a \in U_a$ for every $a \in A$.

JL 4/12: **Ex. 9. (TEE Spring 2013)** Let X and Y be topological spaces and let $\pi\colon X \times Y \to X$ be a projection, that is, $\pi(x, y) = x$. Show that if Y is compact, then for every $S \subset X \times Y$ we have $\pi[\mathrm{cl}(S)] = \mathrm{cl}(\pi[S])$.

***

JX 4/12s: **Ex. 10.** Let X be connected and let $f, g\colon X \to [-1,1]$ be continuous functions. Assume that $f[X] = [-1,1]$. Show, that there exists an $x \in X$ such that $f(x) = g(x)$.

JX 4/10: **Ex. 11.** Prove or disprove the following assertion.

> For a closed subset X of $\mathbb{R}^3$, X is compact if, and only if, every continuous function $f\colon X \to \mathbb{R}$ is bounded.

Recall, that a function $f\colon X \to \mathbb{R}$ is bounded, when there exist numbers $m, M \in \mathbb{R}$ such that $m \le f(x) \le M$ for all $x \in X$.

JX: 3/27: **Ex. 12.** Let X be a compact topological space and $\langle Y, d\rangle$ be a metric space. Show that for every pair of continuous functions $f\colon X \to Y$ and $g\colon X \to Y$, the extended real number $B = \sup\{d(f(x_1), g(x_2))\colon x_1, x_2 \in X\} \in [0, \infty]$ is, in fact, a real number.

JX 4/12: **Ex. 13.** Let X be a topological space.

(a) Show that for every closed $F \subset X$ we have $\mathrm{int}(\mathrm{cl}(\mathrm{int}(F))) = \mathrm{int}(F)$.

(b) Give an example of an open $U \subset \mathbb{R}$, for which $\mathrm{int}(\mathrm{cl}(\mathrm{int}(U))) \neq \mathrm{int}(U)$. (This will show that part (a) need not hold for an open set F.)

JX 4/5: **Ex. 14.** Let X and Y be the topological spaces and let $A \subset X$.

Let $f\colon \mathrm{cl}(A) \to Y$ and $g\colon \mathrm{cl}(A) \to Y$ be continuous functions. Show that if $f \restriction A = g \restriction A$ and Y is Hausdorff, then $f = g$.

Ex. 15. Let $f\colon X \to Y$ and $g\colon X \to Z$ be continuous. Show that if A is a
JX 4/12: connected subset of X, then the set
$T = \{\langle f(a), g(a)\rangle\colon a \in A\}$ is a connected subset of $Y \times Z$.

Ex. 16. Let X be a topological space.

(a) Show that for every $A \subset X$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) = \mathrm{int}(\mathrm{cl}(A))$ (i.e., that the operation $\mathrm{int}(\mathrm{cl}(\cdot))$ is idempotent). You can use in your argument, without a proof, the fact that the operations int and cl are idempotent (i.e., that $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ and $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.)

ZT 4/12:

(b) Give an example of a subset A of $\mathbb{R}$, considered with the standard topology, for which $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \neq \mathrm{cl}(A)$.

Ex. 17. A family $\mathcal{A}$ of subsets of a topological space X is *locally discrete* when every $x \in X$ has a neighborhood that intersects at most one $A \in \mathcal{A}$. Show that if X is compact, then every locally discrete family in X is finite.

ZT 4/10:

Ex. 18. Assume that $f\colon X \to Y$ is a continuous function that is an open map, that is, such that $f[U]$ is open in Y for every open U in X.

ZT 4/12 by KC:

Show that, for such map, we have $\mathrm{cl}\left(f^{-1}(B)\right) = f^{-1}(\mathrm{cl}(B))$ for every $B \subset Y$.

Ex. 19. Prove or give a counterexample: If $\{X_\alpha\}_{\alpha \in J}$ is a family of connected metric spaces, then $Y = \Pi_{\alpha \in J} X_\alpha$ with the uniform topology is connected.

ZT 4/10:

Ex. 20. Let $\mathbb{I}$ be the set of irrational numbers and let $X \subset \mathbb{R}^2$ be such that $X \cup \mathbb{I}^2 = \mathbb{R}^2$.

ZT, 4/5 by KC:

Show, that X is connected.

Ex. 21. Let $\mathbb{R}$ be the set of real numbers.

(a) Define the following two topologies on $\mathbb{R}$: standard $\mathcal{T}_s$ and lower (i.e., Sorgenfrey) $\mathcal{T}_\ell$.

ZT 4/5:

(b) Show that if $f\colon \langle \mathbb{R}, \mathcal{T}_s \rangle \to \langle \mathbb{R}, \mathcal{T}_l \rangle$ is continuous if, and only if, f is constant.

Ex. 22. Let $\langle A_i \rangle_{i=1}^{\infty}$ be an arbitrary sequence of subsets of a topological space X. Show that

$$\mathrm{cl}\left(\bigcup_{i=1}^{\infty} A_i\right) = \left(\bigcup_{i=1}^{\infty} \mathrm{cl}(A_i)\right) \cup \bigcap_{k=1}^{\infty} \mathrm{cl}\left(\bigcup_{i=k}^{\infty} A_i\right).$$

ZT !!:

Class of March 27:

Students to present the following results:

JX Theorem 26.7 (compact finite product), including Lemma 26.8 (tube lemma).

JL Theorem 26.9 (fip), including the definition of fip.

ZT Theorems 24.1 (connectedness), for $[a, b]$ only.

Time permitting, solve some of the exercises from pages 24-27.

Class of March 29:

Students to present the following results:

JL Theorem 27.1 (compactness), for $[a, b]$ only.

JQ Theorem 23.6 (connectedness of finite product) using Star Lemma 23.3. Also Theorem 23.4 and indicate how it is used to prove connectedness of $\mathbb{R}^\omega$.

ZT Theorem 27.6 (uniform continuity), including Lemma 27.5 (on Lebesgue number).

JX Theorem 27.7.

Time permitting, solve some of the exercises from pages 24-27.

Class of April 3:

Students to present the following results:

ZT Use Lebesgue number lemma to prove Theorem 27.6 (uniform continuity). Prove that every compact space is limit point compact.

JQ Sketch a proof of the Urysohn Lemma

JX Sketch a proof of the Urysohn Metrization Theorem (assuming Urysohn Lemma)

JL Prove that a product of regular spaces is regular and that a subspace of regular spaces is regular. Indicate where the proofs of the similar results for normal spaces break.

Class of April 5:

Students will solve some of the exercises from pages 24-27.
Solved Exercises 3, 5, 7, 14, 20, 21.

Class of April 10:

Students will solve some of the exercises from pages 24-27.
Solved Exercises 2, 8, 11, 17, 19.

Class of April 12:

Students will solve some of the exercises from pages 24-27.
Solved Exercises 1,4,9,10,13,15,16,18.

Class of April 17:

Solve exercise 22 from page 27 and some exercises from Topology Entrance Exams of Spring and Fall 2017.

Possible also

- Prove that every *metric* limit point compact space is sequentially compact.

Review.

Classs of April 19, 24, 26:

Tychonoff Theorem

Go over incomplete proof of the Tychonoff Theorem, page 231.

Definition of a *filter* on a set X: *a non-empty family* $\mathcal{F} \subset \mathcal{P}(X)$ *(i.e., of subsets of* X*) such that:*

- *if* $A, B \in \mathcal{F}$*, then* $A \cap B \in \mathcal{F}$*;*
- *if* $A \in \mathcal{F}$ *and* $A \subset C \subset X$*, then* $C \in \mathcal{F}$*.*

Filter is *proper* when $\emptyset \notin \mathcal{F}$, that is, when $\mathcal{F} \neq \mathcal{P}(X)$. Note that the filter is proper if, and only if, it has the finite intersection property, fip. We will consider only proper filters.

Give an intuitive argument that any family having the finite intersection property can be extended to a maximal family having the finite intersection property. (Formal prove will be deduced from Zorn's Lemma.)

State Lemma 37.2 and the result that any family having the finite intersection property can be extended to a maximal family having the finite intersection property.

Use the above results to prove Tychonoff Theorem.