Notes

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Linear Algebra classes

Class # 1: January 11, 2018

Discussed syllabus.

Definition of a field: Chapter 1, page 5. Examples:
real numbers $\mathbb{R}$; also complex numbers $\mathbb{C}$ and rational numbers $\mathbb{Q}$.
Read briefly the rest of Chapter 1.

Chapter 2: Matrix Algebra
Matrices: definition and the following operations
transpose, scalar multiplication, addition, and multiplication of matrices. Basic properties.
Typical exercise for this material:

Exercise 1 *For* $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 11 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 8 \\ 4 & -2 \\ 5 & 1 \end{bmatrix}$ *find* A^T, $2A - 3B$, $A^T B$, *and* BA^T.

Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ but } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Definition of zero matrix θ. Properties: $A + \theta = \theta + A = A$.

Definition the identity matrix I ($AI = A$ and $IB = B$) and of inverse matrix A^{-1} of a square matrix A.

Properties of multiplication:
$A(BC) = (AB)C$; $AB = I$ implies $BA = I$; however, AB need not be equal BA:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ but } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Class # 2: January 16, 2018

Chapter 3, Vector spaces: State Definition 1, Chapter 3 page 2.
Examples:

- $K^{n\times m}$ – the family of all $n \times m$-matrices over the field K; e.g. $\mathbb{R}^{n\times m}$
- $\mathbb{R}^{n\times 1}$ – the family of all n-dimensional (column) matrices $[x_1 \cdots x_n]^T$; often denotes as $\mathbb{R}^n$;
- $\mathbb{R}^2$, the classical plane vectors $[x\ y]^T$ often identified with $[x\ y]$ and written as $\langle x, y\rangle$; similarly $3D$-vectors $\mathbb{R}^3$;
- The family $\mathcal{F}(D,\mathbb{R})$ of all functions from a set $D \subset \mathbb{R}$ into $\mathbb{R}$; also the classes of: all polynomials; of all differentiable functions; of solutions of some differential equations; etc;

Subspaces: Definition 1, Chapter 3 page 10.

Theorem 1 *If V is a vector space (over the field K) and W is non-empty subset of V, then W is a subspace if, and only if, $v + w$ and cv are in W for every c from K and $v, w \in W$.*

Examples:

- $W = \{\langle x, 3x\rangle : x \in \mathbb{R}\}$, a line in the plane $\mathbb{R}^2$ is a vector subspace of $\mathbb{R}^2$;
- polynomials forms a vector subspace of $\mathcal{F}(D,\mathbb{R})$; so are differentiable functions;

Chapter 4, System of linear equations $A\mathbf{x} = \mathbf{b}$:

For a system $A\mathbf{x} = \mathbf{b}$ of m equations with n unknowns $x_1, \ldots, x_n$, A is $m \times n$ coefficient matrix, $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{b} = [b_1, \ldots, b_n]^T$.

Solving $A\mathbf{x} = \mathbf{b}$ via Gauss elimination.
Went over Example # 1, Ch. 4, Pg. 8. (See also Example # 1 Pg. 19.)

$A\mathbf{x} = \mathbf{b}$ may have: no solutions, one solution, or infinitely many solutions.

Class # 3: January 18, 2018

Solutions of $A\mathbf{x} = \mathbf{b}$ via Gauss elimination:

Use of Gauss elimination, that is, using augmented matrix approach. If the system is consistent (i.e., has at least one solution), the solution must be expressed in the vertical vector form:

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

From the text: Example # 2, Ch. 4, Pg. 19
Solve exercise 2 from the Sample Test # 1, via Gauss elimination.
Solve exercise 1 from the Sample Test # 1, via Gauss elimination.
Solve problems fro Quiz #1 from the last semester.

Next class: Quiz # 1. Material as in Exercises 1 and 2 of the Sample Test # 1, just solved. The Sample Test # 1 is available at

http://www.math.wvu.edu/~kcies/teach/current/CurrentTeaching.html

Solve problems fro Quiz #1 from the last semester.

System of linear equations $A\mathbf{x} = \mathbf{b}$, revisited:

For a system $A\mathbf{x} = \mathbf{b}$ of m equations with n unknowns $x_1, \ldots, x_n$, A is $m \times n$ coefficient matrix, $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{b} = [b_1, \ldots, b_n]^T$.

When $\mathbf{b} = \mathbf{0} = [b_1, \ldots, b_n]^T$, then the system $A\mathbf{x} = \mathbf{b}$ (i.e., $A\mathbf{x} = \mathbf{0}$) is a homogeneous system.

The solutions $\mathbf{x}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, that is, $V = \{\mathbf{x}\colon A\mathbf{x} = \mathbf{0}\}$, is a *vector space:*

$\mathbf{0} \in V$ and $\alpha\mathbf{x} + \beta\mathbf{y} \in V$ for every $\mathbf{x}, \mathbf{y} \in V$.

In other words, V is a *null space* of the operator A: $\mathbf{x} \mapsto A\mathbf{x}$.

[A function T from a vector space into another is a linear operator when

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}).$$

Its *null space* is the set of all vectors $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{0}$. Null space of any linear operator is also a vector space.]

In particular $A\mathbf{x} = \mathbf{0}$ has either one, or infinitely many solutions.

If $\mathbf{x}_p$ is a solution for $A\mathbf{x} = \mathbf{b}$, then

$\mathbf{x}$ solution for $A\mathbf{x} = \mathbf{b}$ if, and only if, it is of the form $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution for $A\mathbf{x} = \mathbf{0}$.

Class # 4: January 23, 2018

Linear independence of vectors and basis

Inverse of a square, $n \times n$, matrix A If there exists a matrix B such that $BA = I$, then also $AB = I$ and B is unique. It is denoted as A^{-1} and referred to as the inverse of A. Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then the inverse of A exists and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Note, $A^{-1} \neq \frac{1}{A}$. In fact, the quotient $\frac{1}{A}$ has no sense at all!
A is *singular* if A^{-1} does not exist; otherwise, it is *non-singular*.

Q. What A^{-1} is useful for?

A. Many uses. E.g.: $A\mathbf{x} = \mathbf{b}$ if, and only if, $\mathbf{x} = A^{-1}\mathbf{b}$.

Also, in determining when vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ are *linearly independent* (form a basis) — notions briefly discussed.

Q. When does A^{-1} exist (i.e., when A is non-singular)?

A. E.g.: when the *determinant* of A, denoted $|A|$ or $\det A$, is $\neq 0$. Calculation of the determinants to be discussed, chapter 7.

Q. When A is non-singular, how to find A^{-1}?

A. Gaussian elimination (again), to be explained.

Finding A^{-1} via Gaussian elimination: Chapter 9. To find A^{-1}: (1) write augmented matrix $[A; I]$; (2) Gaussian elimination to transform it to a matrix $[I; B]$; (3) declare that A^{-1} equals B.

Go over Exercises 4, 5 from the sample test and Example 1, Ch. 9, Pg 5.

Class # 5: January 25, 2018

Calculation of the determinant: Via arbitrary row (or column) expansion (known as Laplace Expansion Method), definition (not in the "textbook"), Example on page Ch. 7, Pg 4. Take a look at Theorem Ch. 7, Pg 2, the properties of the determinant – leads to Gaussian elimination. Solve (the same problem) using Gaussian elimination, see Ch. 7, Pg 6.

Solving $A\mathrm{x} = \mathbf{b}$ via *Cramer Rule*: application of determinants. Just state (Ch. 6, Pg 7), no exercises.