MATH 251 NAME (print): ____________________

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**Solutions for SAMPLE TEST # 1**

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

**Ex. 1.** Evaluate

(a) $$3\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 11 & -1 \end{bmatrix} - 5\begin{bmatrix} 0 & 8 \\ 4 & -2 \\ 5 & 1 \end{bmatrix} =$$

**Sol:** $$\begin{bmatrix} 6 & 9 \\ 12 & 15 \\ 33 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -40 \\ -20 & 10 \\ -25 & -5 \end{bmatrix} = \begin{bmatrix} 6 & -31 \\ -8 & 25 \\ 8 & -8 \end{bmatrix}.$$

(b) $$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 3 \end{bmatrix} =$$

**Sol:** $$\begin{bmatrix} 1+0+2 & 1-2+3 \\ -1+0+6 & -1-6+9 \\ 0+0+4 & 0-1+6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 2 \\ 4 & 5 \end{bmatrix}.$$

(c) $$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} =$$

**Sol:** $$\begin{bmatrix} 1-14-3 \end{bmatrix} = \begin{bmatrix} -16 \end{bmatrix}.$$

(d) $$\begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} =$$

**Sol:** $$\begin{bmatrix} 1 & -2 & 3 \\ 7 & -14 & 21 \\ -1 & 2 & -3 \end{bmatrix}.$$

**Ex. 2.** Solve each of the following systems of linear equations by representing as augmented matrix and transforming it to the row reduced echelon form. If the system inconsistent, give a reason for it explain the meaning of this fact in terms of solutions. If it is consistent, express its solution in the vertical vector *form* as $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 5 \\ 11 \end{bmatrix}$.

(a) $\begin{cases} a + b - c = 0 \\ a - 4b + 2c = -1 \\ 2a - 3b + c = 1 \end{cases}$

**Sol:**

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -4 & 2 & -1 \\ 2 & -3 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -2R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & -1 \\ 0 & -5 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The last row leads to the equation $0a + 0b + 0c = 2$, which is never satisfied.

Answer: System is inconsistent, no solution.

(b) $\begin{cases} a + b - c = 0 \\ a - 4b + 2c = 1 \\ 2a - 3b + c = 1 \end{cases}$

**Sol:**

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -4 & 2 & -1 \\ 2 & -3 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -2R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 1 \\ 0 & -5 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \times -\frac{1}{5} \\ \\ \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -R_2 \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We get: $a - \frac{2}{5}c = \frac{1}{5}$, that is, $a = \frac{1}{5} + \frac{2}{5}c$; $b - \frac{3}{5}c = -\frac{1}{5}$, that is, $b = -\frac{1}{5} + \frac{3}{5}c$

Ans: Consistent, infinitely many solutions: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5} + \frac{2}{5}c \\ -\frac{1}{5} + \frac{3}{5}c \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{5} \\ 0 \end{bmatrix} + c \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$.

(c) $\begin{cases} a + b - c = 0 \\ a + 2c = 1 \\ 2a - 3b + c = 1 \end{cases}$ **Sol:**

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & -3 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -2R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -5 & 3 & 1 \end{bmatrix} \begin{matrix} +R_2 \\ \\ -5R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} \begin{matrix} \\ \times -1 \\ \times\frac{-1}{12} \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \begin{matrix} -2R_3 \\ +3R_3 \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}.$$

Answer: System is consistent, it has the unique solution: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$.

**Ex. 3.** Find the determinant of the matrix. Each time you expand the matrix, you **must** expand it over a row or column that has the largest number of zeros. If necessary, use the row (or column) reduction method to create additional zeros.

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

**Sol:** If we subtract from raw # 4 the raw # 2 and expand by the third column, we get

$$|A| = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 4 & 0 & 3 \end{vmatrix} = (-1)\cdot 1 \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$$

Next, subtracting from raw # 3 three times the raw # 2 and expanding again by the third column, we get

$$|A| = (-1)\begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -4 & -2 & 0 \end{vmatrix} = (-1)(-1)\begin{vmatrix} -1 & 2 \\ -4 & -2 \end{vmatrix} = 2-(-8) = 10.$$

**Ex. 4.** Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

**Sol:** We need to transform $[A; I]$ to $[I; B]$. Then $B = A^{-1}$.

$$[A; I] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ +R_1 \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} \\ \\ \times -\frac{1}{4} \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{matrix} -R_3 \\ -3R_3 \\ \\ \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Answer: $A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$

**Ex. 5.** Let $A$ be as below. Show that it is its own inverse, that is, that $A^{-1} = A$.

$$A = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**Sol:** Matrix $B$ is an inverse of $A$, $B = A^{-1}$, precisely when $AB = I$. Thus, for $A = A^{-1}$, we must have $AA = I$. Here is the checking:

$$AA = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1/4 + 3/4 + 0 & \sqrt{3}/4 - \sqrt{3}/4 + 0 & 0 + 0 + 0 \\ \sqrt{3}/4 - \sqrt{3}/4 + 0 & 3/4 + 1/4 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix} = I, \text{ as required.}$$

**Ex. 6.** Let $\mathbf{a} = \langle 0, 1, 2 \rangle$, $\mathbf{b} = \langle -1, 0, 7 \rangle$, and $\mathbf{c} = \langle 2, 3, -1 \rangle$. Evaluate: $2\mathbf{a} - \mathbf{b} + \mathbf{c}$, $|\mathbf{c}|$, and $(\mathbf{a} \cdot \mathbf{b})\ (\mathbf{b} \times \mathbf{c})$. (Do not confuse vectors with numbers. No partial credit for solutions with such errors.)

**Sol:**

$2\mathbf{a} - \mathbf{b} + \mathbf{c} = 2\langle 0, 1, 2 \rangle - \langle -1, 0, 7 \rangle + \langle 2, 3, -1 \rangle = \langle 0, 2, 4 \rangle + \langle 1, 0, -7 \rangle + \langle 2, 3, -1 \rangle = \langle 3, 5, -4 \rangle$

$|\mathbf{c}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

As $\mathbf{a} \cdot \mathbf{b} = \langle 0, 1, 2 \rangle \cdot \langle -1, 0, 7 \rangle = 0 + 0 + 14 = 14$ and

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 7 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 7 \\ 3 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 7 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} =$$

$\mathbf{i}(0 - 21) - \mathbf{j}(1 - 14) + \mathbf{k}(-3 - 0) = \langle -21, 13, -3 \rangle$, we have
$(\mathbf{a} \cdot \mathbf{b})\ (\mathbf{b} \times \mathbf{c}) = 14\ \langle -21, 13, -3 \rangle = \langle -14 \cdot 21, 14 \cdot 13, -3 \cdot 14 \rangle$.