Foundations of Geometry, Math 535, Spring 2015: Notes

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Class of January 12:

The goal of the course is to build a solid understanding of Euclidean geometry (on the plane), as axiom-based theory. This includes not only the formal derivation of different consequences of the axioms (i.e., high school covered material), but also discussion of their independence, via describing different models that satisfies some of the axioms, but not others.

- Take a look at page 123, five *Euclid's postulates.*
- Read preliminary Chapters 1 (equivalence relations), 2 (mappings), and 3 (real numbers).

Written assignment for Wednesday, January 14: Exercise 3.2, p. 31:

Ex. 1. *Recall that a binary relation* $\circ$ *on a set* S *is:*

commutative *provided* $s \circ t = t \circ s$ *for every* $s, t \in S$*; and*

associative *provided* $(s \circ t) \circ u = s \circ (t \circ u)$ *for every* $s, t, u \in S$.

Define the following relations $\star$ *and* $\#$ *on the set* $\mathbb{R}$ *of real numbers:*

- $a \star b = a^2 + b^2$ *for ever* $a, b \in \mathbb{R}$ *and*
- $a \# b = |a| b$ *for ever* $a, b \in \mathbb{R}$.

Show that

1. $\star$ *is commutative but not associative, and*
2. $\#$ *is associative but not commutative.*

Read Section 4.1 (Axiom systems).

Mention the notions of *consistency* and *independence* of the axiom system.

Section 4.2, Incidence planes

Definition 1 An *incidence plane* is any triple $\langle \mathcal{P}, \mathcal{L}, \mathcal{F} \rangle$, where sets $\mathcal{P}$ and $\mathcal{L}$ are disjoint and $\mathcal{F}$ is a binary relation from $\mathcal{P}$ to $\mathcal{L}$. The elements of $\mathcal{P}$ are typically referred to as *points*, while the elements of $\mathcal{L}$ are typically referred to as *lines*. The relation $\langle P, \ell \rangle \in \mathcal{F}$ is usually (but not always) interpreted as "P is contained in ℓ."

Go over example in Figure 4.1.

Go over example *Real Cartesian Incidence Plane*.

***Affine plane* axiom system. (Explain *parallel*.)**

$A1$ For every distinct $P, Q \in \mathcal{P}$ there exists unique $\ell \in \mathcal{L}$ containing P and Q.

$A2$ If $P \in \mathcal{P}$ is not on $\ell \in \mathcal{L}$, then there exists unique line containing P parallel to ℓ.

$A3$ There exists four points such that no (distinct) three are on any line.

Go over two examples of (i.e., *models* for) the affine plane: standard Euclidean plane and the 4 point example, $\mathcal{M}_0$ from Figure 4.2 (edges of tetrahedron).

Note that in the example from Figure 4.2 notion of *perpendicular* is meaningless. Mention that *collineation* is an isomorphism between the incidence planes.

Note that the affine plane axiom system is not *categorical*, meaning that it has two models that are not collinear.

Theorem 1 *The affine plane axioms $A1$-$A3$ are independent.*

Proof.

Model for $\neg A1$, $A2$, and $A3$: $\mathcal{L}$ – all planes in $\mathbb{R}^3$ perpendicular to one of the coordinate axis; $\mathcal{P}$ – all points in $\mathbb{R}^3$; incidence – standard.

Model for $A1$, $A2$, and $\neg A3$: $\mathcal{P}$ –arbitrary set; $\mathcal{L} = \{\mathcal{P}\}$; incidence – standard.

Model for $A1$, $\neg A2$, and $A3$: 7 point model $\mathcal{M}_1 = \langle \mathcal{P}_1, \mathcal{L}_1, \mathcal{F}_1 \rangle$ from Figure 4.3. ∎

Class of January 14:

***Projective plane* axiom system.**

$A1$ For every distinct $P, Q \in \mathcal{P}$ there exists unique $\ell \in \mathcal{L}$ containing P and Q.

$A2'$ For every distinct lines ℓ and m there exists unique point on ℓ and on m.

$A3$ There exists four points such that no (distinct) three are on any line.

Theorem 2 *The projective plane axioms $A1$, $A2'$, and $A3$ are consistent.*

Proof. Provided by 7 points model $\mathcal{M}_1$ for $A1$, $\neg A2$, and $A3$. ∎

Theorem 3 *The projective plane axioms $A1$, $A2'$, and $A3$ are independent.*

Proof.
Model for $A1$, $\neg A2'$, and $A3$: 4 point model $\mathcal{M}_0$ for $A1$, $A2$, and $A3$.

Model for $A1$, $A2'$, and $\neg A3$: $\mathcal{P}$ –arbitrary set; $\mathcal{L} = \{\mathcal{P}\}$; incidence – standard.

Model for $\neg A1$, $A2'$, and $A3$: $\mathcal{P}$ –arbitrary 4 point set; $\mathcal{L}$ a single line containing no points; incidence – standard. ∎

Theorem 4 *The projective plane axiom system is not categorical (i.e., has different models).*

Proof. Two truly different models:

- 7 points model $\mathcal{M}_1$.
- An infinite model $\mathcal{M}_2 = \langle \mathcal{P}_2, \mathcal{L}_2, \mathcal{F}_2 \rangle$: Fix a point $O \in \mathbb{R}^3$.

  Let $\mathcal{P}_2$ be the set of all lines in $\mathbb{R}^3$ through O.

  Let $\mathcal{L}_2$ be the set of all planes in $\mathbb{R}^3$ through O.

  $\mathcal{F}_2$: $P \in \mathcal{P}_2$ is $\mathcal{F}_2$-incident to $\ell \in \mathcal{L}_2$ when P is contained (in R^3) in ℓ.

Note that $\mathcal{M}_2$ satisfies $A1$, $A2'$, and $A3$. ∎

See the interpretation of $\mathcal{M}_2$ as in Figure 4.4.

This means that $\mathcal{M}_2$ contains a copy of the standard Euclidean plane model for the affine plane.

***Hyperbolic plane* axiom system.**

- $A1$ For every distinct $P, Q \in \mathcal{P}$ there is unique $\ell \in \mathcal{L}$ containing P and Q.
- $A2''$ If $P \in \mathcal{P}$ is not on $\ell \in \mathcal{L}$, then there exist two distinct lines containing P, each parallel to ℓ.
- $A3'$ There exists four points such that no (distinct) three are on any line. Each line contains a point.

Theorem 5 *The hyperbolic plane axioms $A1$, $A2''$, and $A3$ are consistent.*

PROOF. Given by model Q_1, *Quadrant Incidence Plane*, from Fig 4.5(a). ∎

Theorem 6 *The hyperbolic plane axiom system is not categorical (i.e., has different models).*

PROOF. Two truly different models: infinite Q_1 and finite (10 points, 25 lines) described in the text. (No proof, for the finite model). ∎

Theorem 7 *The hyperbolic plane axioms $A1$, $A2''$, and $A3$ are independent.*

PROOF.
Model for $A1$, $\neg A2''$, and $A3$: 4 point model $\mathcal{M}_0$ for $A1$, $A2$, and $A3$. Also, model Q_2, *Halfplane Incidence Plane*, from Figure 4.5(b).

Model for $A1$, $A2''$, and $\neg A3$: $\mathcal{P}$ –arbitrary set; $\mathcal{L} = \{\mathcal{P}\}$; incidence – standard.

Model for $\neg A1$, $A2''$, and $A3$: $\mathcal{P}$ –arbitrary 4 point set; $\mathcal{L}$ 8 lines, each containing precisely one point, each point contained in two distinct lines; incidence – standard. ∎

Theorem 8 *There exists an incidence plane satisfying $A1$ and $A3$ which satisfies neither of the axioms: $A2$, $A2'$, and $A2''$.*

PROOF. Model Q_3, *Missing-Quadrant Incidence Plane*, from Figure 4.5(c). ∎

Written assignment for Wednesday, January 21: Solve Ex 4.1, p. 45: *Show that parallelism is an equivalence relation on the set of all lines of an affine plane, but parallelism is not an equivalence relation on the set of all lines of a hyperbolic plane.*

Class of January 21: Chapter 6

We start describing the Axioms of Euclidean plane and derive their consequences. Beside the axioms, we assume basics consisting of: *logic, set theory, real numbers.*

The axioms will describe the properties of four classes of objects: $\mathcal{P}$ – referred to as *points*; $\mathcal{L}$ – referred to as *lines*; d – referred to as *distance function* (between points); m – referred to as the *measure of the angle*.

Axiom 1: *Incidence Axiom*

(a) $\mathcal{P}$ and $\mathcal{L}$ are sets; any $\ell \in \mathcal{L}$ is a subset of $\mathcal{P}$.

(b) If $P, Q \in \mathcal{P}$ are distinct, then there is unique $\ell \in \mathcal{L}$ containing both $\mathcal{P}$ and Q.

(c) There exists three elements of $\mathcal{P}$ not all in any element of $\mathcal{L}$.

The axiom does not preclude the situation, that a line contains no points!

Definition 2 (See Definition 6.1)

- If $\ell, m \in \mathcal{L}$ are distinct, and $P \in \mathcal{P}$ is in ℓ and in m, then ℓ *and* m *intersect at* P.
  (By (b), the intersection cannot contain more than one point!)
- $\ell \in \mathcal{L}$ is *parallel to* $m \in \mathcal{L}$, denoted $\ell \parallel m$, if either $\ell = m$ or ℓ and m do not intersect.
- A set of points is *collinear* if it is contained in some line $\ell \in \mathcal{L}$.
- The unique line from (b) is denoted $\overleftrightarrow{PQ}$.

The following follows from (b).

Theorem 9 *(Thm 6.2) If distinct $R, S \in \mathcal{P}$ are on $\overleftrightarrow{PQ}$, then $\overleftrightarrow{RS}$=$\overleftrightarrow{PQ}$.*

The following follows from the definition of parallel.

Theorem 10 *(Thm 6.3) If $\ell, m \in \mathcal{L}$, then $\ell \parallel \ell$ and $\ell \parallel m$ implies $m \parallel \ell$.*

Note: at this point, we do not know if parallelism is transitive!
Go over Theorem 6.4.
Note that $\mathcal{P} = \{A, B, C\}$ and $\mathcal{L} = \{\{A, B\}, \{B, C\}, \{A, C\}, \emptyset\}$ form a "trivial" model for Axiom 1.

Axiom 2: *Ruler Postulate*

There exists a mapping $d\colon \mathcal{P}\times\mathcal{P}\to\mathbb{R}$, *with* $d(P,Q)$ *denoted* PQ, *having the following property.*

For every $\ell\in\mathcal{L}$ *there exists a bijection* $f:\ell\to\mathbb{R}$ *such that*

$$PQ = |f(P)-f(Q)| \text{ for every } P \text{ and } Q \text{ on } \ell.$$

If $f:\ell\to\mathbb{R}$ is as in Axiom 2, then f is called *coordinate system* for ℓ and $f(P)$ is a *coordinate* of P on ℓ (with respect to f).

The axiom insures that each line contains many points.

Go over Theorem 6.6, with proof.

Written assignment for Monday, January 26:

Ex 1: (Ex. 6.6) *Show that Axioms 1 and 2 do not ensure that* $PQ+QR\geq PR$ *by considering a model of the standard Euclidean plane with a function* d *that insures Axiom 2, but for which the inequality fails for some points* P, Q, *and* R.

Ex 2: *Consider the Rational Cartesian Incidence Plane model* M_2 *defined with* $\mathcal{P}=\mathbb{Q}^2$ *and lines* $\mathcal{L}$ *of the form* $\ell=\{\langle x,y\rangle\in\mathcal{P}\colon ax+by+c=0\}$ *for some* $a,b,c\in\mathbb{Q}$ *with* $a^2+b^2>0$.

- *(a) (Ex. 6.8) Show that there is no map* d *satisfying Axiom 2 for* M_2.
- *(b) Show that if Axiom 2' is obtained by replacing in Axiom 2 the phrase "bijection* $f:\ell\to\mathbb{R}$*" with the phrase "bijection* $f:\ell\to\mathbb{Q}$*," then there exists a function* d *insuring that Axiom 2' is satisfied. (Can such* d *be the standard Euclidean distance?)*

Go over Theorem 6.7.

Go over Theorem 6.8.

Go over Exercise 6.3: Axioms 1 and 2 are independent:
– Euclidean line satisfies A2, but not A1.
– Our finite models for Affine Plane satisfy A1, but not A2.

Class of January 26: Chapter 7 – Betweenness

Definition 3 A point B is *between* points A and C, denoted $A - B - C$, provided: (1) there are all distinct; (2) they are collinear; (3) $AB+BC = AC$.

Remark: The definition depends, explicitly on the distance-like function d from Axiom 2, as AC is defined as $d(A,C)$. However, there is no (explicit) dependence of the coordinate function f of $\ell = \overleftrightarrow{AC}$.

Theorem 11 *(Thm 7.2) $A - B - C$ implies $C - B - A$.*

Theorem 12 *(Thm 7.4) If $A-B-C$, then neither $A-C-B$ nor $C-A-B$.*

Theorem 13 *(Thms 7.3 & 7.5) $A - B - C$ if, and only if, A, B, and C are collinear and $f(B)$ is between $f(A)$ and $f(C)$ for any coordinate system f of $\overleftrightarrow{AB}$.*

Theorem 14 *(Thm 7.6, known as* **Cantor-Dedekind Axiom***) For any line ℓ there is an* **order-preserving** *bijection $f : \ell \to \mathbb{R}$.*

Theorem 15 *(Thm 7.7) If P is in $\overleftrightarrow{AB}$, then exactly one the following holds: $P - A - B$, $P = A$, $A - P - B$, $P = B$, $A - B - P$.*

Theorem 16 *(Thm 7.8) If $A, C \in \mathcal{P}$ are distinct, then there exist $B, D \in \mathcal{P}$ with $A - B - C$ and $A - C - D$.*

Theorem 17 *(Thm 7.9) If $A - B - C$ and $A - B - D$, then $C = D$, or $B - C - D$* or *$B - D - C$.*

Definition 4 $A-B-C-D$ if, and only if, $A-B-C$, $A-B-D$, $A-C-D$, and $B - C - D$.

Theorem 18 *(Thm 7.11) If $A-B-C$ and $B-C-D$, then $A-B-D-C$.*

Theorem 19 *(Thm 7.12) Any four distinct collinear points can be named A, B, C, and D such that $A - B - D - C$.*

Class of January 28

Review new remark on the definition of $A - B - C$:

Remark: The definition of $A-B-C$ depends, explicitly on the distance-like function d from Axiom 2, as AC is defined as $d(A, C)$. However, there is no (explicit) dependence of the coordinate function f of $\ell = \overleftrightarrow{AC}$.

7.2: Taxicab geometry: review and new results.

Why assumption "(2) points are collinear" in the definition of $A-B-C$? Consider weaker definition:

Definition 5 A point B is $*$-*between* points A and C, denoted $A * B * C$, provided: (1) there are all distinct, and (3) $AB+BC = AC$. (No requirement that (2): the points are collinear!)

Theorem 20 *The "$*$-between" notion is essentially weaker than the standard notion of "between."*

PROOF. Justified by the taxicab distance, details below.

Definition 6

- Euclidean distance function d from Axiom 2:
  $d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where $P = (x_1, y_1)$, $Q = (x_2, y_2)$.
- *Taxicab distance* function t:
  $t(P, Q) = |x_2 - x_2| + |y_2 - y_1|$, where $P = (x_1, y_1)$, $Q = (x_2, y_2)$.

Theorem 21 *Both distances, d and t, satisfy Axiom 2.*

PROOF. **The case of Euclidean distance d:** For a non-vertical line ℓ with equation $y = mx + b$, define the coordinate function $f\colon \ell \to \mathbb{R}$ via

$$f(P) = \sqrt{1 + m^2}\ x \text{ for every } P = (x, mx + b) \in \ell.$$

Clearly it is a bijection and we have desired Axiom 2 property for every $P = (x_1, mx_1 + b)$ and $Q = (x_2, mx_2 + b)$ on ℓ:

$$\begin{aligned} PQ = d(P, Q) &= \sqrt{(x_1 - x_2)^2 + ((mx_1 + b) - (mx_2 + b))^2} \\ &= \sqrt{(x_1 - x_2)^2 + (m(x_1 - x_2))^2} = \sqrt{1 + m^2}\ |x_1 - x_2| \\ &= |\sqrt{1 + m^2}\ x_1 - \sqrt{1 + m^2}\ x_2| = |f(P) - f(Q)|. \end{aligned}$$

For a vertical line ℓ with equation $x = c$, define the coordinate function $f\colon \ell \to \mathbb{R}$ via $f(P) = y$ for every $P = (c, y) \in \ell$. It is obviously a bijection with the desired property.

The case of the taxicab distance t: For a vertical line the same function, $f(P) = y$ for $P = (c, y)$, is also a coordinate system.

For a non-vertical line ℓ with equation $y = mx + b$, define the coordinate function $f\colon \ell \to \mathbb{R}$ via

$$f(P) = (1 + |m|)x \text{ for every } P = (x, mx + b) \in \ell.$$

Clearly it is a bijection and we have desired Axiom 2 property for every $P = (x_1, mx_1 + b)$ and $Q = (x_2, mx_2 + b)$ on ℓ:

$$\begin{aligned} PQ = d(P,Q) &= |x_1 - x_2| + |(mx_1 + b) - (mx_2 + b)| \\ &= |x_1 - x_2| + |m||x_1 - x_2| = (1 + |m|)|x_1 - x_2| \\ &= |(1 + |m|)x_1 - (1 + |m|)x_2| = |f(P) - f(Q)|. \end{aligned} \quad \blacksquare$$

For $A = (0, 0)$ and $C = (2, 4)$ draw all points B with $A * B * C$. Notice, that there is B with $A * B * C$ but not $A - B - C$.

More specifically, finish the proof of theorem 20 by noticing that for the Axiom 2 satisfied by the taxicab distance t and points $A = (0, 0)$, $B = (2, 2)$ and $C = (2, 4)$, we have

- $A * B * C$, as $AB + BC = 4 + 2 = 6 = AC$, but
- $A - B - C$ fails, as points A, B, and C are no co-linear. $\blacksquare$

Written assignment for Monday, February 2: Solve Ex 7.11, p. 82: *On the Euclidean plane, with points $\mathcal{P} = \mathbb{R}^2$, define function:*

$$t(P,Q) = \max\{|x_2 - x_1|, |y_2 - y_1|\}, \text{ where } P = (x_1, y_1),\ Q = (x_2, y_2).$$

Show that the function r satisfies Axiom 2 (as d).

Go over the solutions of the three exercises, assigned on January 14 .

What follows, not covered, worth to look over.

Note that taxicab distance satisfies the triangle inequality.

Describe taxicab "circle:" $B(A, \varepsilon) = \{B \in \mathcal{P} \colon t(A, B) < \varepsilon\}$.

Describe, for the taxicab distance, the set $L(A, B) = \{C \in \mathcal{P} \colon AP = BP\}$ of all points P equidistant from A and B, for:

(1) $A = (0, 0)$ and $B = (0, 2)$ (see Fig. 7.8a),
(2) $A = (0, 0)$ and $B = (4, 2)$ (see Fig. 7.8b), and
(3) $A = (0, 0)$ and $B = (2, 2)$ (see Fig. 7.8c).

Class of February 2

Go over the solutions of the two exercises, assigned on January 21.

Section 8.1: Segments and Rays

Read pages 84 and 85 on terminology changes and on how rigorous proofs should/should not be.

Definition 7 For distinct points A and B, we define

- *segment (with endpoints A & B)* as $\overline{AB} = \{A, B\} \cup \{P \in \mathcal{P} \colon A{-}P{-}B\}$;
- *ray (with vertex A)* as $\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} \colon \text{ not } P - A - B\}$.

Theorem 22 *(Thm 8.2) For every distinct points A and B:*

1. $\overline{AB} = \overline{BA}$
2. $\overrightarrow{AB} \neq \overrightarrow{BA}$
3. $\overrightarrow{AB} = \overline{AB} \cup \{P \in \mathcal{P} \colon A - B - P\}$
4. $\overline{AB} \subsetneq \overrightarrow{AB} \subsetneq \overleftrightarrow{AB}$

Theorem 23 *(Thm 8.3)* $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$ *and* $\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ *for every distinct points A and B.*

Theorem 24 *(Thm 8.4) For every distinct points A and B:* $\overline{AB} = \overline{CD}$ *iff* $\{A, B\} = \{C, D\}$.

Definition 8

- The *length* of $\overline{AB}$ is defined as AB.
- $\overline{AB}$ is *congruent* to $\overline{CD}$, denoted $\overline{AB} \simeq \overline{CD}$, provided $AB = CD$. That is, $\overline{AB} \simeq \overline{CD}$, provided the segments have the same length.

Theorem 25 *(Thm 8.6) The congruence $\simeq$ is an equivalence relation on the family of all segments.*

Theorem 26 *(Thm 8.7) For every ray $\overrightarrow{VA}$ there exists a unique coordinate system f for $\overleftrightarrow{VA}$ such that $f(V) = 0$ and $\overrightarrow{VA} = \{P \in \overleftrightarrow{VA} : f(P) \geq 0\}$.*

Theorem 27 *(Thm 8.8, on segment construction) For every segment $\overline{AB}$ and ray $\overrightarrow{VC}$ there exists a unique point D in $\overrightarrow{VA}$ with $\overline{AB} \simeq \overline{VD}$.*

Notice that the above theorem is non-trivial, in a sense that it fails in the Rational Cartesian Incidence Plane model M_2 we discussed in Ex 2(b) in a homework assigned on January 21.

Written assignment for Monday, February 9: Solve Ex 8.2 p. 92: *Prove Thm 8.10, on segment subtraction.*

Class of February 4

Recall that:

- *segment* $\overline{AB} = \{A, B\} \cup \{P \in \mathcal{P} \colon A - P - B\}$;
- *ray* $\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} \colon \text{ not } P - A - B\}$;
- $\overline{AB} = \overline{BA}$; $\overrightarrow{AB} \neq \overrightarrow{BA}$; $\overrightarrow{AB} = \overline{AB} \cup \{P \in \mathcal{P} \colon A - B - P\}$;
- $\overline{AB} = \overline{CD}$ iff $\{A, B\} = \{C, D\}$
- $\overline{AB} \simeq \overline{CD}$, provided $AB = CD$; $\simeq$ is an equivalence relation;
- Thm: For every $\overrightarrow{VA}$ there exists a unique coordinate system f for $\overleftrightarrow{VA}$ such that $f(V) = 0$ and $\overrightarrow{VA} = \{P \in \overleftrightarrow{VA} \colon f(P) \geq 0\}$.
- Thm: For every $\overline{AB}$ and $\overrightarrow{VC}$ there is unique $D \in \overrightarrow{VA}$ with $\overline{AB} \simeq \overline{VD}$.

Review both theorems above, with proofs, and the remark related to the Rational Cartesian Incidence Plane model M_2.

New material:

Theorem 28 *(Thm 8.9, on segment addition) If $A - B - C$, $D - E - F$, $\overline{AB} \simeq \overline{DE}$, and $\overline{BC} \simeq \overline{EF}$, then $\overline{AC} \simeq \overline{DF}$.*

Theorem 29 *(Thm 8.10, on segment subtraction) If $A - B - C$, $D - E - F$, $\overline{AB} \simeq \overline{DE}$, and $\overline{AC} \simeq \overline{DF}$, then $\overline{BC} \simeq \overline{EF}$.*

Theorem 30 *(Thm 8.11) B is on $\overrightarrow{VA}$ and $B \neq V$ iff $\overrightarrow{VB} = \overrightarrow{VA}$.*

Theorem 31 *(Thm 8.12) If $\overrightarrow{VA} = \overrightarrow{WB}$, then $V = W$. In particular, every ray has unique vertex.*

Notice, how much work was required to prove that the axioms imply uniqueness of vertex in any ray!

Written assignment for Monday, February 9: Solve Ex 8.9 p. 93:

In the Taxicab Geometry find a counterexample to the statement that $\overline{AB} = \{P \in \mathcal{P} \colon AP + PB = AB\}$.

Section 8.2: Convex sets

Definition 9 For distinct points A and B,

- a *mid point of* $\overline{AB}$ is an M with $A - M - B$ and $AM = MB$;
- $M \in \mathcal{P}$ is a *mid point of* A *and* B if either $A = M = B$ or M is a mid point of $\overline{AB}$;
- $\overrightarrow{VA}$ is an *opposite ray* of $\overrightarrow{VB}$ provided $A - V - B$.

Theorem 32 *(Thm 8.15) Every segment has a unique midpoint. For every $A, B \in \mathcal{P}$ there exist unique $M, N \in \mathcal{P}$ such that M is a mid point of A and B, while B is a mid point of A and N.*

Theorem 33 *(Thm 8.16) Every ray has an opposite ray. If $\overrightarrow{VA}$ is an opposite ray of $\overrightarrow{VB}$, then $\overrightarrow{VB}$ is an opposite ray of $\overrightarrow{VA}$.*

Theorem 34 *(Thm 8.17) If $P, Q \in \overleftrightarrow{AB}$, $AP = AQ$, and $BP = BQ$, then $P = Q$.*

Definition 10

- The *interior of* $\overline{AB}$, denoted $\text{int}(\overline{AB})$, is $\overline{AB} \setminus \{A, B\}$;
- The *interior of* $\overrightarrow{VA}$, denoted $\text{int}(\overrightarrow{VA})$, is $\overrightarrow{VA} \setminus \{V\}$;
- A *half line* is the interior of a ray;
- A set $T \subset \mathcal{P}$ is *convex* provided $\overline{AB} \subset T$ for every distinct $A, B \in T$.

See Fig 8.5 for examples of sets that are not convex in the Euclidean plane.

Theorem 35 *(Thm 8.19) The intersection of two or more convex sets is convex.*

Theorem 36 *(Thm 8.20) If $A, B \in \mathcal{P}$ are distinct, that each of the following sets is convex: $\emptyset$, $\{A\}$, $\overline{AB}$, $\text{int}(\overline{AB})$, $\overrightarrow{AB}$, $\text{int}(\overrightarrow{AB})$, $\overleftrightarrow{AB}$, $\mathcal{P}$.*

Theorem 37 *(Thm 8.21) For every $V \in \ell \in \mathcal{L}$ there exists convex sets H_1 and H_2 such that:*

(i) $\ell \setminus \{V\} = H_1 \cup H_2$,

(ii) *if $P \in H_1$, $Q \in H_2$, and $P \neq Q$, then $\overline{PQ} \cap \{V\} \neq \emptyset$ (i.e., $V \in \overline{PQ}$).*

Note, that there are models with points $\mathcal{P} = \mathbb{R}^2$ (i.e., $M10$ from the text) in which convex sets are not convex in the standard Euclidean plane model.

Class of February 9

Recall that:

- The *interiors:* $\text{int}(\overline{AB}) = \overline{AB} \setminus \{A, B\}$, $\text{int}(\overrightarrow{VA}) = \overrightarrow{VA} \setminus \{V\}$.
- A set $T \subset \mathcal{P}$ is *convex* provided $\overline{AB} \subset T$ for every distinct $A, B \in T$.

Describe Moulton model (i.e., $M10$ from the text, pages 10–11, with points $\mathcal{P} = \mathbb{R}^2$) in which convex sets are not convex in the standard Euclidean plane model, see Figure 8.6.

Chapter 9: Angles and Triangles

Definition 11 If A, V, and B are three distinct noncollinear points, then an *angle* $\angle AVB$ (having *vertex* V and *sides* $\overrightarrow{VA}$ and $\overrightarrow{VB}$) is defined as $\angle AVB = \overrightarrow{VA} \cup \overrightarrow{VB}$.

Notice that 180° *straight angle* is not angle according to this definition. Similarly 0° angle.

Theorem 38 *(Thm 9.2) If A, B, and C are three distinct noncollinear points, then $\angle ABC = \angle CBA \neq \angle ACB$.*

Theorem 39 *(Thm 9.3) Given $\angle AVB$, if $C \in \text{int}(\overrightarrow{VA})$ and $D \in \text{int}(\overrightarrow{VB})$, then $\angle AVB = \angle CVD$.*

Theorem 40 *(Thm 9.4) If $\angle AVB = \angle CVD$, then either $\overrightarrow{VA} = \overrightarrow{VC}$ or $\overrightarrow{VA} = \overrightarrow{VD}$.*

Theorem 41 *(Thm 9.5) If $\angle AVB = \angle AWB$, then $V = W$.*

This uniqueness of a vertex in the angle is not that obvious in the Moulton model $M10$, see Figure 9.2

Theorem 42 *(Thm 9.6) If $\angle AVB = \angle CWD$, then $V = W$ and either $\overrightarrow{VA} = \overrightarrow{VC}$ or $\overrightarrow{VA} = \overrightarrow{VD}$.*

Class of February 11

Note: The solutions of the homework that was due February 9 will be handed on Monday, February 16. No corrections afterword.

Note: I *will* administer a 20-30 minutes quiz on Monday, February 16!

Few words what problem 8.9, page 93, is about.

Recall that:

- *Angle:* $\angle AVB = \overrightarrow{VA} \cup \overrightarrow{VB}$.
- If $\angle AVB = \angle CWD$, then $V = W$ and either $\overrightarrow{VA} = \overrightarrow{VC}$ or $\overrightarrow{VA} = \overrightarrow{VD}$.

New material

Definition 12 Given $\angle AVB$, $A - V - A'$ and $B - V - B'$,

- $\angle AVB$ and $\angle A'VB'$ are *vertical angles*;
- $\angle AVB$ and $\angle A'VB$ are *linear pair* of angles.

Theorem 43 *(Thm 9.8) Given $\angle AVB$, if $\overrightarrow{VA'}$ is opposite to ray $\overrightarrow{VA}$ and $\overrightarrow{VB'}$ is opposite to ray $\overrightarrow{VB}$, then*

1. *$\angle AVB$ and $\angle A'VB'$ are vertical angles;*
2. *$\angle AVB'$ and $\angle A'VB$ are vertical angles;*
3. *$\angle AVB$ and $\angle BVA'$ are a linear pair;*
4. *$\angle AVB$ and $\angle AVB'$ are a linear pair;*
5. *$\angle BVA'$ and $\angle A'VB'$ are a linear pair;*
6. *$\angle A'VB'$ and $\angle B'VA$ are a linear pair.*

Definition 13 If A, B, and C are three distinct noncollinear points, then a *triangle* $\triangle ABC$ (with *vertices* A, B, and C and with *sides* $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$) is defined as $\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$.

The angles $\angle BAC$, $\angle ABC$, and $\angle ACB$ are *angles of* $\triangle ABC$ and are often simply denoted as $\angle A$, $\angle B$, and $\angle C$, when no confusion is likely.

Theorem 44 *(Thm 9.10) If A, B, and C are three distinct noncollinear points, then $\triangle ABC = \triangle CAB = \triangle ACB$ and $\overline{AB} = \triangle ABC \cap \overleftrightarrow{AB}$.*

Theorem 45 *(Thm 9.11) If $\triangle ABC = \triangle DEF$, then $\{A, B, C\} = \{D, E, F\}$.*

Intuitive notion of how triangle looks breaks in the Moulton model $M10$, see Figure 9.6.

Definition 14

- Point P is in the *ray-interior* of $\angle AVB$ if there exist $C \in \text{int}(\overrightarrow{VA})$, $D \in \text{int}(\overrightarrow{VB})$, and $E \in \text{int}(\overline{CD})$ such that $P \in \text{int}(\overrightarrow{VE})$. In such case, $\overrightarrow{VB}$ is an *interior ray* of $\angle AVB$.
- Point P is in the *inside* or *segment-interior* of $\angle AVB$ if there exist $C \in \text{int}(\overrightarrow{VA})$ and $D \in \text{int}(\overrightarrow{VB})$ such that $P \in \text{int}(\overline{CD})$.

Theorem 46 *(Thm 9.14) The inside of $\angle AVB$ is contained in the ray-interior $\angle AVB$.*

But, in general, not other way around! This happens in the *weird plane* model to be discussed next class.

Why difficulties? Why two different definitions?

Definition 15 *Pasch's Postulate*: If a line intersects a side of a triangle not at the vertex, then it intersects another side of the triangle.

Theorem 47 *Pasch's Postulate does not follow from our two axioms!*

Proof. This happens in the *missing strip* model $(M8, e)$, (see page 56): see Figure 9.11.

We will discuss missing strip model $(M8, e)$ in some details. ∎

Written assignment for Monday, February 16: Solve Ex 9.9 p. 109.

Give a specific distance formula e for the missing strip *model $(M8, e)$. Show that the provided function e satisfies Axiom 2 (Ruler Postulate).*

Definition 16 *Plane-Separation Postulate, PSP*: For any line ℓ there exists two convex sets H_1 and H_2 such that $\mathcal{P} \setminus \ell \subset H_1 \cup H_2$ and for every $P \in H_1$ and $Q \in H_2$, if $P \neq Q$, then $\overline{PQ} \cap \ell \neq \emptyset$.

Fact: PSP also fails in the missing strip model (as we will later prove the two axioms imply that PSP and Pasch's Postulate are equivalent).

Class of February 16

Start with 20 minutes quiz. Then recall:

- Point P is in: the *inside* of $\angle AVB$ if there exist $C \in \operatorname{int}(\overrightarrow{VA})$ and $D \in \operatorname{int}(\overrightarrow{VB})$ such that $P \in \operatorname{int}(\overline{CD})$;
  the *ray-interior* of $\angle AVB$ if there exist $C \in \operatorname{int}(\overrightarrow{VA})$, $D \in \operatorname{int}(\overrightarrow{VB})$, and $E \in \operatorname{int}(\overline{CD})$ such that $P \in \operatorname{int}(\overrightarrow{VE})$;
  P is the inside of $\angle AVB$ implies that P is in the ray-interior of $\angle AVB$.
- *Pasch's Postulate*: If a line intersects a side of a triangle not at the vertex, then it intersects another side of the triangle.
  Pasch's Postulate does not follow from our two axioms. This is justified by the *missing strip* model $(M8, e)$.

New material: several new models.

The *Space Incidence Plane* model $(M4, d)$: $\mathcal{P} = \mathbb{R}^3$ (!!!), lines and distance d as in Euclidean 3D space.

Theorem 48 *In the Space Incidence Plane model $M4$:*

- *the ray-interior of an angle is the same as the inside of this angle;*
- *Pasch's Postulate fails;*
- *the following "strong" version of Crossbar (defined below) holds.*

Definition 17 *Crossbar:* If P is a point in the ray-interior of $\angle AVB$, then $\overrightarrow{VP}$ intersects $\overline{AB}$.

Theorem 49 *Crossbar fails in the missing strip model $(M8, e)$.*

PROOF. See Figure 9.11: G is in the inside (so, also ray-interior) of $\angle AVB$, while $\overrightarrow{VG}$ does not intersect $\overline{AD}$. ∎

Remark *In $(M8, e)$, the ray-interior of an angle is the same as the inside of this angle.*

Theorem 50 *Let M be a model that satisfies Axiom 1, Incidence Axiom. If for every line ℓ there exist a bijection $f_\ell \colon \ell \to \mathbb{R}$, then M satisfies also Axiom 2, Ruler Postulate.*

PROOF. For points $P, Q \in \mathcal{P}$ we put: $d(P,Q) = 0$ provided $P = Q$, and $d(P,Q) = \left| f_{\overleftrightarrow{PQ}}(P) - f_{\overleftrightarrow{PQ}}(Q) \right|$ when $P \neq Q$. ∎

Cayley-Klein Incidence Plane model $M13$: $\mathcal{P} = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$, lines – $\ell \cap \mathcal{P} \neq \emptyset$, where ℓ standard Euclidean line in the plane.

$M13$ satisfies Axioms 1 and 2: Ruler Postulate follows from Theorem 50.

Theorem 51 *In the Cayley-Klein Incidence Plane model $M13$:*

(i) the ray-interior of an angle is **not** *the same as the inside of the angle;*

(ii) Pasch's Postulate holds;

(iii) Crossbar ("strong" version) holds.

PROOF. For (i), see Figure 9.13 and point E. (ii) Pasch holds, since the only triangles in $M13$ are the "true" Euclidean triangles (no details); (iii) no detailed proof. ∎

Summary: Let $I = RI$ stand for " the ray-interior of an angle is the same as the inside of this angle." Here at the models for the two axioms that describe the relations between Pasch, Crossbar, and $I = RI$:

- Pasch & Crossbar & $I = RI$ hold in the Euclidean plane model $M1$
- ¬ Pasch & Crossbar & $I = RI$ hold in the Space Incidence model $M4$
- ¬ Pasch & ¬Crossbar & $I = RI$ hold in the missing strip model $M8$
- Pasch & Crossbar & $\neg I = RI$ hold in the Cayley-Klein model $M13$

Next: *Weird Plane* model $(M1, d')$: standard $M1$ points and lines, but weird distance d'. Let $h, k \colon \mathbb{R} \to \mathbb{R}$ be defined as: $h(x) = x+2$ and $k(x) = -x$ when x is and integer an as $h(x) = k(x) = x$ otherwise.

For the point $P = (x,y)$ on a line ℓ we define

$$f_\ell(P) = \begin{cases} h(x) & \text{when } \ell \text{ is horizontal,} \\ k(y) & \text{when } \ell \text{ is vertical,} \\ \sqrt{1+m^2}\,x & \text{otherwise, for } \ell \text{ with } y = mx + b \end{cases}$$

($f_\ell(P) = \sqrt{1+m^2}\,x$ is the standard Euclidean plane coordinate system.) This, by Theorem 50, leads to a distance function, d'.

Class of February 18

Start with 20 minutes quiz.

Weird plane, $(M1, d')$**:** Standard points $\mathcal{P} = \mathbb{R}^2$ and lines. Let $h, k\colon \mathbb{R} \to \mathbb{R}$ be defined as: $h(x) = x + 2$ and $k(x) = -x$ when x is an integer and as $h(x) = k(x) = x$ otherwise. If d is the standard Euclidean distance, put

$$d'((x_1, y_1), (x_2, y_2)) = \begin{cases} |h(x_2) - h(x_1)| & \text{when } y_1 = y_2, \\ |k(y_2) - k(y_1)| & \text{when } x_1 = x_2, \\ d((x_1, y_1), (x_2, y_2)) & \text{otherwise.} \end{cases}$$

It satisfies Axiom 2, where coordinate system for line ℓ is given as

$$f_\ell((x, y)) = \begin{cases} h(x) & \text{when } \ell \text{ is horizontal,} \\ k(y) & \text{when } \ell \text{ is vertical,} \\ \sqrt{1 + m^2}\, x & \text{otherwise, for } \ell \text{ with } y = mx + b. \end{cases}$$

Discuss some segments and rays for the weird plane depicted in Figure 9.14.

Discuss an angle for the weird plane depicted in *corrected* Figure 9.15.

Discuss the two triangles for the weird plane depicted in Figure 9.16.

Written assignment for Monday, February 23: Solve Ex 9.7 p. 109.

Show, that Pasch's Postulate does not hold in the weird plane, $(M1, d')$*. Hint: Either of the triangles depicted in Figure 9.16 could be used in a counterexample. But you need to give specific coordinates of the vertices of the used triangle and indicate for which line the postulate fails.*

Section 12.1: Axiom 3, PSP

- *Pasch's Postulate, PASCH*: If a line intersects a triangle not at the vertex, then it intersects two sides of the triangle.
- *Plane-Separation Postulate, PSP*: For any line ℓ there exists two convex sets H_1 and H_2 such that $\mathcal{P} \setminus \ell = H_1 \cup H_2$ and $\overline{PQ} \cap \ell \neq \emptyset$ for every $P \in H_1$ and $Q \in H_2$ such that $P \neq Q$.

Theorem 52 *(Thm 12.1) Assuming Axioms 1 and 2, PASCH implies PSP.*

We will skip a proof of this theorem. (The reverse implication is also true, to be proved.)

Axiom 3: *PSP holds.*

Sets H_1 and H_2 from PSP are called *halfplanes* of line ℓ, while ℓ is an *edge* of each halfplane.

From now on, all theorem will be proved assuming Axioms 1–3.

Class of February 23

Recall

- *Pasch's Postulate, PASCH*: If a line intersects a triangle not at the vertex, then it intersects two sides of the triangle.
- *Plane-Separation Postulate, PSP*: For any line ℓ there exists two convex sets H_1 and H_2 such that $\mathcal{P} \setminus \ell = H_1 \cup H_2$ and $\overline{PQ} \cap \ell \neq \emptyset$ for every $P \in H_1$ and $Q \in H_2$ such that $P \neq Q$.
- **Axiom 3:** *PSP holds.*

From now on, all theorem will be proved assuming Axioms 1–3.

New material:

Theorem 53 *(Thm 12.4) If points A and B are off line ℓ and $\overline{AB} \cap \ell \neq \emptyset$, then A and B are not on the same halfplane of ℓ.*

Theorem 54 *(Thm 12.5) If H_1 and H_2 are a pair of halfplanes of line ℓ, then $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$, but $H_1 \cap H_2 = \emptyset$.*

Theorem 55 *(Thm 12.6) If H_1 and H_2 are a pair of halfplanes of line ℓ, then $\mathcal{P}$ is a union of three mutually disjoint sets: H_1, H_2, and ℓ.*

Theorem 56 *(Thm 12.7) Up to an order, the halfplanes of a line ℓ are unique. If $A \in \mathcal{P} \setminus \ell$, then the halfplanes are: $\{P \in \mathcal{P} \setminus \ell \colon \overline{AP} \cap \ell \neq \emptyset\}$ and $\{A\} \cup \{Q \in \mathcal{P} \setminus \ell \colon \overline{AQ} \cap \ell = \emptyset\}$.*

Theorem 57 *(Thm 12.8) No two lines have the same halfplanes. The edge of a halfplane is unique.*

Definition 18 A halfplane of $\overleftrightarrow{AB}$ is a *side* of $\overleftrightarrow{AB}$. Each of the two halfplanes of a line is *opposite* of the other.

Theorem 58 *(Thm 12.10) Let A and B be points on opposite sides of line ℓ. Then*

- *A and B are not on the same side of ℓ;*
- *if B and C are points on opposite sides of ℓ, then A and C are on the same side of ℓ;*
- *if B and D are points on same side of ℓ, then A and D are on the opposite sides of ℓ.*

Theorem 59 *(Thm 12.11) If line ℓ contains no vertex of $\triangle ABC$, then ℓ cannot intersects all three sides of $\triangle ABC$.*

Theorem 60 *(Thm 12.12) The three axioms imply PASCH.*

Written assignment for Wednesday, February 25: Solve Ex 12.1 p. 140.

Prove PASCH.

Class of February 25

Mid term test will be in class, February 9, 2015. I can extend the test time for up to two hours.

Recall

- If H_1 and H_2 are a pair of halfplanes of line ℓ, then $\mathcal{P}$ is a union of three mutually disjoint sets: H_1, H_2, and ℓ.
- The edge of a halfplane is unique.
- The three axioms imply PASCH.

New material

Theorem 61 *(Thm 12.13)* **Peano's Postulate:** *If $\triangle ABC$, $B-C-D$, and $A-E-C$, then there is an $F \in \overleftrightarrow{DE}$ such that $A-F-B$. (See Figure 12.3.)*

Theorem 62 *(Thm 12.14) If $\triangle ABC$, $B-C-D$, and $A-F-B$, then there is an $E \in \overleftrightarrow{DF}$ such that $A-E-C$ and $D-E-F$. (See Figure 12.4.)*

Theorem 63 *(Thm 12.15) If $\triangle ABC$, then every point lies on a line that intersects the triangle at two points.*

Theorem 64 *(Thm 12.16)* **Sylvester's Theorem:** *If n points are not all collinear, then there exists a line containing exactly two of the points.*

Interesting, but we will not prove this theorem.

Section 13.1: More incidence theorems

Theorem 13.2 is, essentially, a restatement of theorem 12.10.

Theorem 65 *(Thm 13.3) If a non-empty convex set S does not intersect line ℓ, then it is on one side of ℓ.*

Theorem 66 *(Cor 13.4) Theorem 65 applies when S is a line, ray, or segment. If line ℓ intersects $\overleftrightarrow{AC}$ at the single point V such that $A-V-C$, then $\mathrm{int}(\overrightarrow{VA})$ and $\mathrm{int}(\overline{VA})$ are on the same side of ℓ, but $\mathrm{int}(\overrightarrow{VA})$ and $\mathrm{int}(\overrightarrow{VC})$ are on the opposite sides of ℓ.*

Definition 19 The *interior* of $\angle AVB$, denoted $\text{int}(\angle AVB)$, is the intersection of the side of $\overleftrightarrow{VA}$ that contains B and the side of $\overleftrightarrow{VB}$ that contains A.

Theorem 67 *(Thm 13.6) Point P is in $\text{int}(\angle AVB)$ if, and only if, A and P are on the same side of $\overleftrightarrow{VB}$ and B and P are on the same side of $\overleftrightarrow{VA}$.*
Given $\angle AVB$, if $A - P - B$, then P is in $\text{int}(\angle AVB)$.
Given $\triangle ABC$, then $\text{int}(\overline{AB})$ is in $\text{int}(\angle ACB)$.

Written assignment for Monday, March 2: Solve Ex 13.1 p. 152, that is, prove Theorem 75 (Thm 13.14 from the text):
If $\triangle ABC$, $B-C-D$, $A-E-C$, and $B-E-F$, then F is in $\text{int}(\angle ACD)$.

Class of March 2

Recall

- The *interior* of $\angle AVB$, denoted $\operatorname{int}(\angle AVB)$, is the intersection of the side of $\overleftrightarrow{VA}$ that contains B and the side of $\overleftrightarrow{VB}$ that contains A.

New material (we repeat the proofs of theorems 68-70)

Theorem 68 *(Thm 13.7) If P is in $\operatorname{int}(\angle AVB)$ and $\overrightarrow{AP}$ intersects $\overrightarrow{VB}$ at D, then $A - P - D$.*

Theorem 69 *(Thm 13.8,* **crossbar***) If P is in $\operatorname{int}(\angle AVB)$, then $\overrightarrow{VP}$ intersects $\operatorname{int}(\overline{AB})$.*

Proof. See Figure 13.2. (1) P and C are on the opposite sides of $\overleftrightarrow{VA}$ as V and B are on the same side, while C and B on the opposite sides. (2) $\operatorname{int}(\overline{AC})$ is disjoint with $\operatorname{int}(\overrightarrow{VP})$, since each is convex and they are on the opposite sides of $\overleftrightarrow{VA}$. (3) $\operatorname{int}(\overline{AC})$ is disjoint with $\overleftrightarrow{VP}$, since the interior of the ray opposite to $\overrightarrow{VP}$ does not intersect $\operatorname{int}(\overline{AC})$, each being convex and on the opposite sides of $\overleftrightarrow{VA}$. (4) $\overleftrightarrow{VP}$ intersects $\operatorname{int}(\overline{AB})$, at some point Q, by PASCH applied to $\triangle ABC$. (5) Q is on $\operatorname{int}(\overrightarrow{VP})$, since otherwise Q and P would be on the opposite sides of $\overleftrightarrow{VB}$, which is impossible, as P and $\operatorname{int}(\overline{AB}) \ni Q$ are on the same side of $\overleftrightarrow{VB}$. ∎

Theorem 70 *(Thm 13.9) Given $\operatorname{int}(\angle AVB)$, if $\overrightarrow{VP}$ intersects $\operatorname{int}(\overline{AB})$, then P is in $\operatorname{int}(\angle AVB)$.*

Theorem 71 *(Thm 13.10) If points B and P are on the same side of $\overleftrightarrow{VA}$, then P is in $\operatorname{int}(\angle AVB)$ if, and only if, points A and B are on the opposite sides of $\overleftrightarrow{VP}$.*

If $A - V - C$, then P is in $\operatorname{int}(\angle AVB)$ if, and only if, B is in $\operatorname{int}(\angle CVP)$.

If points B and P are on the same side of $\overleftrightarrow{VA}$, then either $\overrightarrow{AB} = \overrightarrow{AP}$, point P is in $\operatorname{int}(\angle AVB)$, or B is in $\operatorname{int}(\angle AVP)$.

Theorem 72 *(Thm 13.11) If* $\angle AVB = \angle CVD$ *and* $\overrightarrow{VE}$ *intersects* $\text{int}(\overline{CD})$, *then* $\overrightarrow{VE}$ *intersects* $\text{int}(\overline{AB})$.

Theorem 73 *(Thm 13.12) Given* $\angle AVB$, *the following are equivalent*

(a) P *is in* $\text{int}(\angle AVB)$.

(b) $\overrightarrow{VP}$ *intersects* $\text{int}(\overline{AB})$.

(c) Point P *is in the ray-interior of* $\angle AVB$.

(d) $\overrightarrow{VP}$ *is an interior ray of* $\angle AVB$.

Theorem 74 *(Cor 13.13) The ray-interior of* $\angle AVB$ *is the interior of* $\angle AVB$.

Theorem 75 *(Thm 13.14) If* $\triangle ABC$, $B-C-D$, $A-E-C$, *and* $B-E-F$, *then* F *is in* $\text{int}(\angle ACD)$.

Definition 20 For a point P off $\ell \in \mathcal{L}$ let $H_P(\ell)$ be a side of ℓ containg P. Them the *interior* of $\triangle ABC$ is defined as

$$\text{int}(\triangle ABC) = H_A(\overleftrightarrow{BC}) \cap H_B(\overleftrightarrow{AC}) \cap H_C(\overleftrightarrow{AB}).$$

Theorem 76 *(Thm 13.16)* $\text{int}(\angle ABC)$ *and* $\text{int}(\triangle ABC)$ *are convex sets.*

Theorem 77 *(Thm 13.17) Given* $\triangle ABC$,

$$\text{int}(\triangle ABC) = \text{int}(\angle A) \cap \text{int}(\angle B) = \text{int}(\angle A) \cap \text{int}(\angle C) = \text{int}(\angle B) \cap \text{int}(\angle C)$$

Class of March 4

As a review for a test, go over some of the past homework exercises.

Recall

- For a point P off $\ell \in \mathcal{L}$ let $H_P(\ell)$ be a side of ℓ containg P. Then the *interior* of $\triangle ABC$ is defined as

$$\mathrm{int}(\triangle ABC) = H_A(\overleftrightarrow{BC}) \cap H_B(\overleftrightarrow{AC}) \cap H_C(\overleftrightarrow{AB}).$$

.

Theorem 78 *(Thm 13.18,* **Line-Triangle Theorem***) If a line intersects the interior of a triangle, then the line intersects the triangle exactly twice.*

Proof. See Figure 13.4.

Definition 21 *Quadrilateral* is defined as $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ provided no three of the points A, B, C, and D are collinear and such that no two of $\mathrm{int}(\overline{AB})$, $\mathrm{int}(\overline{BC})$, $\mathrm{int}(\overline{CD})$, and $\mathrm{int}(\overline{DA})$ intersect each other. For other related and natural definitions associated with $\square ABCD$ see the text.

Theorem 79 *(Thm 13.20) Given $\square ABCD$ we have $\square ABCD = \square DCBA$ and $\square ABCD = \square BCDA \square CDAB = \square DABC$. However, if $\square ABCD$ and $\square ABDC$ exist, then they are not equal.*

Class of March 9

Mid term test.

Class of March 11

Return the in-class part of the mid term test.

Collect take home part of the mid term test.

Discuss the solutions of all mid term test, except the bonus exercise, # 7.

Note in the solution of Exercise 5, points D and E need to be switched.

Class of March 16

Collect the bonus exercise, # 7, from the mid term test.

Return the take home part of the mid term test.

Shortly discuss the results of the mid term test.

Recall

- *Quadrilateral* is defined as $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ provided no three of the points A, B, C, and D are collinear and such that no two of $\text{int}(\overline{AB})$, $\text{int}(\overline{BC})$, $\text{int}(\overline{CD})$, and $\text{int}(\overline{DA})$ intersect each other.
- (Theorem 13.20) Given $\square ABCD$ we have $\square ABCD = \square DCBA$ and $\square ABCD = \square BCDA \square CDAB = \square DABC$. However, if $\square ABCD$ and $\square ABDC$ exist, then they are not equal.

Theorem 80 *(Thm 13.21) The four vertices, the four sides, the two diagonals, and the four angles of a quadrilateral are unique.*

Definition 22 A *convex quadrilateral* is a quadrilateral with the property that each side of the quadrilateral is on a halfplane of the opposite side of the quadrilateral.

Theorem 81 *(Thm 13.23) A quadrilateral is a convex quadrilateral if, and only if, the vertex of each angle of the quadrilateral is in the interior of its opposite angle.*

Theorem 82 *(Thm 13.24) The diagonals of a convex quadrilateral intersect each other. Conversely, if the diagonals of a quadrilateral intersect each other, then the quadrilateral is a convex quadrilateral.*

Section 14.1 – Axiom 4: The Protractor Postulate

Axiom 4, *The Protractor Postulate: There exists a mapping m from the set of all angles into $R = \{x \in \mathbb{R} \colon 0 < x < \pi\}$ such that*

(a) if $\overrightarrow{VA}$ is a ray on the edge of halfplane H, then for every $r \in R$ there exists exactly one ray $\overrightarrow{VP}$ with P in H such that $m\angle AVP = r$;

(b) If B is a point in the interior of $\angle AVC$, then $m\angle AVB + m\angle BVC = m\angle AVC$.

Definition 23 The mapping m from the Protractor Postulate is called *angle measure function*. Angles are *congruent* if they have the same measure. See also book on how degrees are related to radians.

Class of March 18

Recall

Axiom 4, *The Protractor Postulate: There exists a mapping m from the set of all angles into $R = \{x \in \mathbb{R} \colon 0 < x < \pi\}$ such that*

(a) *if $\overrightarrow{VA}$ is a ray on the edge of halfplane H, then for every $r \in R$ there exists exactly one ray $\overrightarrow{VP}$ with P in H such that $m\angle AVP = r$;*

(b) *If B is a point in the interior of $\angle AVC$, then $m\angle AVB + m\angle BVC = m\angle AVC$.*

- The mapping m from the Protractor Postulate is called *angle measure function*. Angles are *congruent*, denoted by $\angle AVB \simeq \angle CWD$, if they have the same measure.

New material

Theorem 83 *(Thm 14.2) Congruence of angles is an equivalence relation on the set of all angles.*

Theorem 84 *(Thm 14.3: Angle-Construction Theorem) Given $\angle AVB$, $\overrightarrow{WC}$, and half plane H of $\overleftrightarrow{WC}$, there exists exactly one ray $\overrightarrow{WD}$ such that D is in H and $m\angle AVB = m\angle CWD$.*

Corollary 85 *(Cor 14.4: Angle-Segment-Construction Theorem) If $0 < a < \pi$, $r > 0$ and H is a half plane of $\overleftrightarrow{AB}$, then there exists a unique point C in H such that $m\angle ABC = a$ and $BC = r$.*

Theorem 86 *(Theorem 14.5: Angle-Addition Theorem) Suppose that $C \in \text{int}(\angle AVB)$, $C' \in \text{int}(\angle A'V'B')$, and $\angle AVC$ is congruent to $\angle A'V'C'$. Then $\angle AVB$ is congruent to $\angle A'V'B'$ if, and only if, $\angle CVB$ is congruent to $\angle C'V'B'$.*

Written assignment for Monday, March 30. Solve Exercise 14.1, p. 169: Prove Angle-Addition Theorem.

Theorem 87 *(Thm 14.6) If B and C are two points on the same side of $\overleftrightarrow{VA}$ and $m\angle AVB < m\angle AVC$, then $B \in \operatorname{int}(\angle AVC)$.*

Definition 24 If $m\angle AVB = m\angle BVC$ and $B \in \operatorname{int}(\angle AVC)$, then $\overrightarrow{VB}$ is an *angle bisector* of $\angle AVC$. If $m\angle AVB + m\angle CWD = \pi$, then angles $\angle AVB$ and $\angle CWD$ are *supplementary*; if $m\angle AVB + m\angle CWD = \pi/2$, then angles $\angle AVB$ and $\angle CWD$ are *complementary*.

Theorem 88 *(Thm 14.8) Every angle has a unique angle bisector.*

Theorem 89 *(Thm 14.9: Euclid's Proposition I.13) If two angles are linear pair, then the two angles are supplementary.*

Theorem 90 *(Thm 14.10: Euclid's Proposition I.14) If points A and C are on the opposite sides of $\overleftrightarrow{VB}$, then $m\angle AVB + m\angle BVC = \pi$ implies $A - V - C$.*

Written assignment for Monday, March 30. Solve Exercise 14.2, p. 169: Prove Euclid's Proposition I.14.

Written assignment for Monday, March 30, *optional, for students that need to improve their grade*: Solve Exercise 14.3, p. 169.

Class of March 30

Recall

(a) If $m\angle AVB = m\angle BVC$ and $B \in \text{int}(\angle AVC)$, then $\overrightarrow{VB}$ is an *angle bisector* of $\angle AVC$. If $m\angle AVB + m\angle CWD = \pi$, then angles $\angle AVB$ and $\angle CWD$ are *supplementary*; if $m\angle AVB + m\angle CWD = \pi/2$, then angles $\angle AVB$ and $\angle CWD$ are *complementary*.

(b) (Thm 14.9: Euclid's Proposition I.13) If two angles are linear pair, then the two angles are supplementary.

New material

Theorem 91 *(Thm 14.11: Euclid's Proposition I.15) Vertical angles are congruent. Also, if $A - V - C$ and points B and D are on the opposite sides of $\overleftrightarrow{AC}$, then the fact that $\angle AVB$ is congruent to $\angle CVD$ implies that $B - V - D$.*

Definition 25 If $m\angle AVB = \pi/2$, then angle $\angle AVB$ is *right*.
If $m\angle AVB < \pi/2$, then angle $\angle AVB$ is *acute*.
If $m\angle AVB > \pi/2$, then angle $\angle AVB$ is *obtuse*.

Theorem 92 *(Thm 14.13) If two congruent angles are a linear pair, then each of the angles is a right angle.*

Theorem 93 *(Thm 14.14: Four-Angle Theorem) If $A-V-A'$, $B-V-B'$, and $\angle AVB$ is a right angle, then each of the angles $\angle AVB'$, $\angle A'VB$, and $\angle A'VB'$ is a right angle.*

Theorem 94 *(Thm 14.15) If $m\angle AVB + m\angle BVC = m\angle AVC$, then $B \in \text{int}(\angle AVC)$.*

PROOF. This follows from Theorem 14.6, if B and C are on the same side of $\overleftrightarrow{AV}$. So, assume that B and C are on the opposite sides of $\overleftrightarrow{AV}$.

Let $A - V - A'$.

If A and C are on the same side of $\overleftrightarrow{VB}$, then, by Theorem 13.10, $A \in \text{int}(\angle BVC)$ and, by the Protractor Postulate, $m\angle BVA + m\angle AVC = m\angle BVC$, so that $m\angle AVC > m\angle BVC$ contradicting $m\angle AVB + m\angle BVC = m\angle AVC$.

Thus, A and C must be on the opposite sides of $\overleftrightarrow{VB}$ and so, A' and C are on the same side of $\overleftrightarrow{VB}$. So, by Theorem 13.10, $A' \in \text{int}(\angle BVC)$. Thus, by the Protractor Postulate, $m\angle BVA' + m\angle A'VC = m\angle BVC$. Moreover, by Euclid's Proposition I.13, $\angle AVB + \angle BVA' = \pi$, so that we have $m\angle AVC = m\angle AVB + m\angle BVC = m\angle AVB + m\angle BVA' + m\angle A'VC = \pi + m\angle A'VC > \pi$, a contradiction. ∎

Definition 26 The lines ℓ and m are *perpendicular*, denoted as $\ell \perp m$, provided $\ell \cup m$ contains a right angles. We will also write $a \perp b$ provided $\ell \perp m$ and a (b, respectively) is a segment, ray, or line on ℓ (on m, respectively).

Theorem 95 *(Thm 14.17) If each of a and b is a segment, ray, or line, then $a \perp b$ implies $b \perp a$.*

Theorem 96 *(Thm 14.18) If P is on line ℓ, then there is a unique line through P that is perpendicular to ℓ.*

Proof. Existence is easy. Uniqueness requires some work. ■

Models for Axioms 1-4

- Standard Euclidean model $(M1, d, m)$, where m is the *usual* angle measure function.
- Taxicab Geometry model $(M1, t, m)$, with m as above.
- The Cayley-Klein model $(M13, d, m)$, with m as above. (Recall that $\mathcal{P} = \{\langle x, y\rangle \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$.)
- The Moulton "bent lines" model $(M10, s, m_2)$, where, in case of when the angle measure is measured at the bending point, we use the usual angle measure function m to the *left* rays. (Requires some work to check, that Axiom 4 is satisfied.)

Theorem 97 *Axioms 1-4 apply neither of the following:*

- *Side-Angle-Side, SAS, Theorem (for the triangles);*
- *that the sum of the angles in a triangle is π;*
- *uniqueness of a perpendicular to a line ℓ through a point P off ℓ;*
- *existence of a perpendicular to a line ℓ through a point P off ℓ.*

Proof. All of this holds in the Moulton "bent lines" model $(M10, s, m_2)$, see Figure 14.7. (No detailed proof.) ■

The theorem shows need for yet another axiom, which will be SAS.

Theorem 98 *Axioms 1-3 imply Axiom 4!*

Proof. See Section 14.2. We will not go over this proof. ■

Class of April 1: MIRROR, SAS, and congruence of triangles

Definition 27 (Definition 16.1) **MIRROR:** For every line m there exists a collineation (i.e., a bijection from $\mathcal{P}$ onto $\mathcal{P}$) that: (1) preserves distance and angle measure, (2) fixes m pointwise, and (3) interchanges the half-planes of m.

Definition 28 (Definition 16.2) **Side-Angle-Side, SAS:** Given $\triangle ABC$ and $\triangle A'B'C'$, if $\overline{AB} \simeq \overline{A'B'}$, $\angle A \simeq \angle A'$, and $\overline{AC} \simeq \overline{A'C'}$, then $\angle B \simeq \angle B'$, $\overline{BC} \simeq \overline{B'C'}$, and $\angle C \simeq \angle C'$.

Similarly, we define propositions: ASA, SAA, ASS, SSS, and AAA.

Theorem 99 *(Thm 16.3) Under Axioms 1-4, MIRROR implies SAS.*

PROOF. We will not go over it, since we will use SAS as Axiom 5. We will also prove soon that, under Axioms 1-4, SAS implies MIRROR. ∎

Theorem 100 *(Thm 16.5) Under Axioms 1-4, ASA implies SAS.*

Written assignment for Monday, April 6. Solve Exercise 16.3, p. 189: Prove that ASS is false in any model of Axioms 1-4.

Definition 29 (Version of definition 17.2) Suppose $\triangle ABC$ and $\triangle DEF$ exist. Then $\triangle ABC$ is **congruent** to $\triangle DEF$ provided there are points A', B', C' such that $\triangle A'B'C' = \triangle DEF$ and $\overline{AB} \simeq \overline{A'B'}$, $\overline{AC} \simeq \overline{A'C'}$, $\overline{BC} \simeq \overline{B'C'}$, $\angle A \simeq \angle A'$, $\angle B \simeq \angle B'$, and $\angle C \simeq \angle C'$.

Note, that definition 17.1 from the text, of $\triangle ABC \simeq \triangle DEF$, is **incorrect** in a sense that we may have $\triangle ABC \simeq \triangle DEF$ and not $\triangle CAB \simeq \triangle DEF$, in spite the fact that $\triangle CAB = \triangle ABC$! This definition should be used for the ordered triangles, $\vec{\triangle} ABC$, instead. The notion from definition 17.1 (of $\triangle ABC \simeq \triangle DEF$, that should have been $\vec{\triangle} ABC \simeq \vec{\triangle} DEF$) is used in the text in some proofs.

Axiom 5, *SAS:* Given $\triangle ABC$ and $\triangle A'B'C'$, if $\overline{AB} \simeq \overline{A'B'}$, $\angle A \simeq \angle A'$, and $\overline{AC} \simeq \overline{A'C'}$, then $\vec{\triangle} ABC \simeq \vec{\triangle} DEF$.

Theorem 101 *(Thm 17.5)* **Pons Asinorum:** *Given $\triangle ABC$, if $\overline{AB} \simeq \overline{AC}$, then $\angle B \simeq \angle C$.*

Theorem 102 *(Thm 17.6) Given $\triangle ABC$, if $A - C - D$, then $m\angle BCD > m\angle B$.*

Definition 30 (Def 17.7) If $\triangle ABC$ and $A - C - D$, then $\angle BCD$ is an *exterior angle* of $\triangle ABC$ with *remote interior angles* $\angle A$ and $\angle B$.

Read the rest of Def 17.7 for the notions of: *opposite side and angle*, *isosceles triangle*, *base angles*, *scalene triangle*, *equilateral triangle*, and *equiangular triangle*.

Read Def 17.8 for the notions of *larger* and *smaller* angles.

Written assignment for Monday, April 6. Solve Exercise 17.5, p. 201:

Prove Euclid's Proposition I.17: Given $\triangle ABC$, $m\angle A + m\angle B < \pi$.

Class of April 6:

Recall:

- **Axiom 5:** *SAS.*
- **Pons Asinorum:** Given $\triangle ABC$, if $\overline{AB} \simeq \overline{AC}$, then $\angle B \simeq \angle C$.
- Thm 17.6: Given $\triangle ABC$, if $A - C - D$, then $m\angle BCD > m\angle B$.

New material:

Theorem 103 *(Thm 17.9: Euclid's Proposition I.16) An exterior angle of a triangle is larger than either of its remote interior angles.*

Theorem 104 *(Thm 17.10) The base angles of an isosceles triangle are acute.*

Theorem 105 *(Thm 17.12) ASA*

Theorem 106 *(Thm 17.13) SAA*

Theorem 107 *(Thm 17.14) SSS*

Proof. Use Philo's argument.

Theorem 108 *(Thm 18.1) Given point P and line ℓ, there exists a unique line through P that is perpendicular to ℓ.*

Proof. Similar to Philo's argument.

Theorem 109 *(Thm 18.2) If two lines are perpendicular to the same line, then the two lines are parallel.*

Theorem 110 *(Thm 18.3) Given a point P off line ℓ, there exists a line through P that is parallel to ℓ.*

This is **not** famous parallel axiom. No uniqueness!

Define *perpendicular bisector*, Def 18.4.

Theorem 111 *(Thm 18.5) The locus of all points equidistant from distinct points P and Q is the perpendicular bisector of $\overline{PQ}$.*

Theorem 112 *(Thm 18.6) If P is off line ℓ, there exists a unique point Q such that ℓ is a perpendicular bisector of $\overline{PQ}$.*

Theorem 113 *(Thm 18.7) Given $\triangle ABC$, if $\angle B \simeq \angle C$, then $\overline{AB} \simeq \overline{AC}$.*

Theorem 114 *(Thm 18.8) A triangle is equilateral if, and only if, it is equiangular.*

Class of April 8:

Theorem 115 *(Thm 18.10) If two sides of a triangle are not congruent, then the angle opposite the longer side is larger than the angle opposite the shorter side.*

Theorem 116 *(Thm 18.11) If two angles of a triangle are not congruent, then the side opposite the larger angle is longer than the side opposite the smaller angle.*

Theorem 117 *(Thm 18.12) In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.*

Theorem 118 *(Thm 18.13,* **triangle inequality***) For any points A, B, C:*

- $AB + BC \geq AC$;
- $AB + BC = AC$ *if, and only if, either* $A = B = C$ *or* $B \in \overline{AC}$;
- $\triangle ABC$ *iff* $AB + BC > AC$, $AC + CB > AB$, *and* $BA + AC > BC$.

Theorem 119 *(Thm 18.16: Euclid's Proposition I.21) If $D \in \mathrm{int}(\triangle ABC)$, then $BD + BC < BA + AC$ and $\angle BDC > \angle BAC$.*

Theorem 120 *(Thm 18.17) If $\triangle ABC$, $A - D - B$, and $BC \geq AC$, then $CD < BC$.*

Theorem 121 *(Thm 18.18: Euclid's Proposition I.24) Suppose $\triangle ABC$ and $\triangle DEF$ are such that $\overline{AB} \simeq \overline{DE}$, $\overline{AC} \simeq \overline{DF}$, and $\angle A > \angle D$, then $BC > EF$.*

Proof. Read from the text.

Theorem 122 *(Thm 18.19: Euclid's Proposition I.25) Suppose $\triangle ABC$ and $\triangle DEF$ are such that $\overline{AB} \simeq \overline{DE}$, $\overline{AC} \simeq \overline{DF}$, and $BC > EF$, then $\angle A > \angle D$.*

Theorem 123 *(Thm 18.20) A triangle has at most one right angle. If a triangle has a right or an obtuse angle, than the other angles are acute angles.*

Theorem 124 *(Thm 18.22) A hypotenuse of a right triangle is longer than either leg. The shortest segment joining a point to a line is the perpendicular segment.*

Theorem 125 *(Thm 18.23) Let F be the foot of the perpendicular from A to $\overleftrightarrow{BC}$. If $\overline{BC}$ is the longest side of $\triangle ABC$, then $B - F - C$.*

Read briefly Chapter 19 on reflections.

You will not be responsible for the proofs. But in what follows I may use, without proofs, some of the results presented there.

Chapter 20, Circles: Read Def 20.1 on standard terminology on circles.

Theorem 126 *(Thm 20.2, part) Three points on a circle determine the circle. The center and the radius of a circle are unique.*

Read the remaining parts of the theorem.

Class of April 13:

Recall:

- (Thm 20.2, part) Three points on a circle determine the circle. The center and the radius of a circle are unique.

Theorem 127 *(Thm 20.9,* **Compass-Construction Axiom***) If $\overrightarrow{PQ} \perp \overline{CP}$ and $CP < r$, then there exists a unique point T on $\overrightarrow{PQ}$ such that $CT = r$.*

PROOF. This is a deep theorem, with proof based on the Intermediate Value Theorem!

Remark: Note that Compass-Construction Axiom fails in the Rational Cartesian Plane model $M2$.

Theorem 128 *(Thm 20.13) If $\triangle ABC$ has a right angle at C, $\triangle A'B'C'$ has a right angle at C', $AB = A'B'$, and $AC > A'C'$, then $BC < B'C'$.*

Theorem 129 *(Thm 20.14) If chords $\overline{PR}$ and $\overline{QS}$ of a circle with center C and radius r are perpendicular to a diameter of the circle at X and Y, respectively, such that $C - X - Y$, then $r > PX > QY$.*

Theorem 130 *(Thm 20.15,* **Triangle Theorem***) For positive real numbers a, b, and c*

- *a, b, and c are the lengths of the sides of some triangle if, and only if, each of the numbers is less than the sum of the other two.*

PROOF. Once again, the proof is based on the Intermediate Value Theorem!

Theorem 131 *(Thm 20.16,* **Two-Circle Theorem***) If $\mathcal{C}_A$ is a circle with center A and radius a, $\mathcal{C}_B$ is a circle with center b and radius b, $AB = c$, and each of a, b, and c is less than the sum of the other two, then $\mathcal{C}_A$ and $\mathcal{C}_B$ intersect in exactly two points, one each on each side of $\overleftrightarrow{AB}$.*

Theorem 132 *(Thm 21.1: Euclid's Proposition I.1) Given $\overline{AB}$, there exists an equilateral triangle with side $\overline{AB}$.*

PROOF. Follows from Two-Circle Theorem. ∎

Class of April 15:

Recall:

- **Compass-Construction Axiom:** If $\overrightarrow{PQ} \perp \overline{CP}$ and $CP < r$, then there exists a unique point T on $\overrightarrow{PQ}$ such that $CT = r$.
- **Triangle Theorem:** For positive real numbers a, b, and c: *a, b, and c are the lengths of the sides of some triangle if, and only if, each of the numbers is less than the sum of the other two.*
- **Euclid's Proposition I.1:** Given $\overline{AB}$, there exists an equilateral triangle with side $\overline{AB}$.

New material:

Theorem 133 *(Thm 21.2: Euclid's Proposition I.2) Given $\overline{BC}$ and a point A, there exists a point L such that $\overline{AL} \simeq \overline{BC}$.*

Definition 31 (Def 21.3) Define the notions of: *cut by transversal*, *interior angle*, *alternate interior angles*, and *corresponding angles* see Figure 21.2.

Theorem 134 *(Thm 21.4) If two lines are cut by transversal, then the following are equivalent:*

(a) The angles in a pair of alternate interior angles are congruent.

(b) The angles in a pair of corresponding angles are congruent.

(c) Each of the four pairs of corresponding angles is a pair of congruent angles.

(d) Each of the two pairs of alternate interior angles is a pair of congruent angles.

(e) The interior angles intersecting the same side of transversal are supplementary.

Theorem 135 *(Thm 21.5) If two lines are cut by transversal such that a pair of alternate interior angles are congruent, then the two lines have a common perpendicular.*

Theorem 136 *(Cor 21.6: Euclid's Proposition I.27) If two lines are cut by transversal such that a pair of alternate interior angles are congruent, then the two lines are parallel.*

Theorem 137 *(Cor 21.7: Euclid's Proposition I.28) If two lines are cut by transversal such that a pair of corresponding angles are congruent or such that the interior angles intersecting the same side of transversal are supplementary, then the two lines are parallel.*

Note that none of the last two theorems claim that the implication can be reversed. The reversibility is actually equivalent to:

Euclid's Parallel Postulate: *If A and D are on the same side of $\overleftrightarrow{BC}$ and $m\angle ABC + m\angle BCD < \pi$, then $\overrightarrow{BA}$ and $\overrightarrow{CD}$ intersect.*

In *Elements* Euclid use the postulate to prove

Theorem 138 *(Pege 242: Euclid's Proposition I.29) If A and D are on the same side of $\overleftrightarrow{BC}$ and $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$, then $m\angle ABC + m\angle BCD = \pi$.*

Euclid's Parallel Postulate is equivalent to our Axiom 6:

Playfair's Parallel Postulate: *If point P is off line ℓ, then there exists a* **unique** *line through P that is parallel to ℓ.*

Notice that the statement of Playfair's Parallel Postulate is almost identical to Theorem 18.3. The only difference is the **uniqueness** requirement!

Theorem 139 *Under Axioms 1-5, the following are equivalent:*

(i) Euclid's Parallel Postulate,

(ii) Euclid's Proposition I.29,

(iii) Playfair's Parallel Postulate.

Proof. Prove (see pages 242 and 243) that: (i) implies (ii), (ii) implies (iii), and (iii) implies (i). ■

Either of the Parallel Postulates independent of Axioms 1-5. The model satisfying Axioms 1-5 and the negation of Playfair's Parallel Postulate is the Cayley-Klein Incidence Plane model $M13$.

Class of April 20:

Recall that the following are equivalent under Axioms 1-5.

- **Euclid's Parallel Postulate:** *If A and D are on the same side of $\overleftrightarrow{BC}$ and $m\angle ABC + m\angle BCD < \pi$, then $\overrightarrow{BA}$ and $\overrightarrow{CD}$ intersect.*
- **Euclid's Proposition I.29:** *If A and D are on the same side of $\overleftrightarrow{BC}$ and $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$, then $m\angle ABC + m\angle BCD = \pi$.*
- **Playfair's Parallel Postulate:** *If point P is off line ℓ, then there exists a* **unique** *line through P that is parallel to ℓ.*

New material:

Recall/complete the proof that Playfair's Parallel Postulate implies Euclid's Parallel Postulate.

Theorem 140 *Euclid's Parallel Postulate is independent of Axioms 1-5.*

Sketch of proof. Clearly Axioms 1-5 and Euclid's Parallel Postulate are satisfied in the Euclidean's (Cartesian) Incidence Plane model $M1$.

We will show, following section 23.2, that Axioms 1-5 and the **negation** of Playfair's Parallel Postulate are satisfied in the Cayley-Klein Incidence Plane model $M13$: $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$, lines – $\ell \cap \mathcal{P} \neq \emptyset$, where ℓ standard Euclidean line in the plane.

Axiom 1 and the negation of Playfair's Parallel Postulate are obvious in $M13$.

Axiom 2, Ruler Postulate: It is satisfied with the standard Euclidean distance d. To see this, choose a line ℓ in $\mathbb{R}^2$ intersecting $S^1 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ at two distinct points S and T. Then, for $f_\ell \colon \ell \cap \mathcal{P} \to \mathbb{R}$,

$$f_\ell(P) = \frac{1}{2} \ln \frac{d(P, T)}{d(P, S)}$$

we have desired: $PQ = |f_\ell(P) - f_\ell(Q)|$ for every P and Q on $\ell \cap \mathcal{P}$.

Define $h \colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ for $M13$ as $h(P, P) = 0$ and, for $P \neq Q$ and $\ell = \overleftrightarrow{PQ}$,

$$h(P, Q) = |f_\ell(Q) - f_\ell(P)| = \frac{1}{2} \ln \frac{d(P, S)d(Q, T)}{d(P, T)d(Q, S)}.$$

Axiom 3, PSP: To see this notice that if $A-B-C$ in $M13$, then $A-B-C$ in $M1$. Now if ℓ is a line in $M13$ with equation $Ax+By+C=0$, then H_1 and H_2 are given, respectively, by the inequalities $Ax+By+C>0$ and $Ax+By+C<0$.

Sets H_1 and H_2 are convex (in $M1$, so in $M13$), as intersection convex sets: half-planes and a disk. The fact that

$$\overline{PQ}\cap\ell\neq\emptyset \textit{ for every } P\in H_1 \textit{ and } Q\in H_2 \textit{ such that } P\neq Q$$

follows from the same fact in $M1$.

Axiom 4, the Protractor Postulate follows from Axioms 1-3, so we need not prove it. However, it also follow from our argument below.

Axiom 5, SAS: Here we will sketch the proof that MIRROR holds in $M13$. As, under Axioms 1-3, MIRROR implies SAS, this will finish the argument.

For a line ℓ in $M13$ (with equation $Ax+By+C=0$) we will define a mapping (reflexion) $\sigma_\ell\colon \mathcal{P}\to\mathcal{P}$ satisfying MIRROR for ℓ.

The formula is given on page 282. Geometrically, if ℓ passes through $(0,0)$, then σ_ℓ is the standard reflection with respect to ℓ.

If ℓ does not pass through $(0,0)$ and ℓ (with equation $Ax+By+C=0$) intersects S^1 at S and T in $\mathbb{R}^2$, then the lines m_S and m_T tangent to S^1 and containing, respectively, S and T are not parallel. Let R be a point of intersection of m_S and m_T. It can be checked that $R=(-A/C,-B/C)$. (Compare Figure 23.2.)

If $P\in\mathcal{P}$, let F, the foot of P, be the point of intersection, in $M1$, of ℓ with $\overleftrightarrow{PR}$. If $P=F$ we define $\sigma_\ell(P)=P$. Otherwise, $P'=\sigma_\ell(P)$ is the only point on $\overleftrightarrow{PR}\cap\mathcal{P}$ that is on the opposite side of ℓ than P and such that $h(P',F)=h(P,F)$. The algebraic formula for σ_ℓ can be found on page 282. It needs to be proved that σ_ℓ:

- is a collineation (i.e., a bijection from $\mathcal{P}$ onto $\mathcal{P}$) — clear;
- fixes ℓ pointwise — obvious;
- interchanges the half-planes of ℓ — clear;
- preserves distance h — obvious for points on lines through R; other cases require some algebra, see pages 282-283;
- preserves angle measure — see pages 283-284.

Class of April 22:

Note: Euclid's postulates and axioms are not strong enough to conclude all his Propositions! (See e.g. page 124.) The axioms we presented are!

Some results on Saccheri quadrilaterals with little or no proofs.

Definition 32 (Def 21.9) $\square ABCD$ is a *Saccheri quadrilateral* provided $m(\angle A) = m(\angle D) = \pi/2$ and $AB = CD$. Read the rest of the definition.

Theorem 141 *(Thm 21.10) If $\square ABCD$ is a Saccheri quadrilateral then: $\square ABCD$ is a quadrilateral convex; $m(\angle B) = m(\angle C)$; and $\overline{AC} \simeq \overline{BD}$.*
The opposite sides of a rectangle are congruent.

PROOF. Convexity: see Thm 21.8. See also Fig 21.9.
$\overline{AC} \simeq \overline{BD}$ and $m(\angle ABD) = m(\angle DCA)$ follows from SAS.
Then, by SSS, $m(\angle CBC) = m(\angle ACB)$, so $m(\angle B) = m(\angle C)$. ■

Theorem 142 *(Thm 21.11) If $\square ABCD$ is a Saccheri quadrilateral then a line through mid points of the bases is perpendicular to each base.*

Theorem 143 *(Cor 21.12) If $\square ABCD$ is a Saccheri quadrilateral, then its bases are parallel.*

Definition 33 (Def 22.2)

Hypothesis of the Acute Angle: There exists a Saccheri quadrilateral with acute upper basis angles.

Hypothesis of the Right Angle: There exists a Saccheri quadrilateral with right upper basis angles.

Hypothesis of the Obtuse Angle: There exists a Saccheri quadrilateral with obtuse upper basis angles.

Theorem 144 *(Cor 22.10, Saccheri's Propositions V, VI, and VII)*

- *Hypothesis of the Acute Angle implies that every Saccheri quadrilateral has acute upper basis angles.*
- *Hypothesis of the Right Angle implies that every Saccheri quadrilateral has right upper basis angles.*
- *Hypothesis of the Obtuse Angle implies that every Saccheri quadrilateral has obtuse upper basis angles.* ■

Theorem 145 *(Thm 22.12, Saccheri's Propositions IX) Let $\triangle ABC$ have a right angle at C.*

(i) Hypothesis of the Acute Angle implies that $m\angle A + m\angle B < \pi/2$.

(ii) Hypothesis of the Right Angle implies that $m\angle A + m\angle B = \pi/2$.

(iii) Hypothesis of the Obtuse Angle implies that $m\angle A + m\angle B > \pi/2$.

Theorem 146 *(Cor 22.13, Saccheri's Propositions IX) If $\triangle ABC$, then*

(i) Hypothesis of the Acute Angle implies that $m\angle A + m\angle B + m\angle C < \pi$.

(ii) Hypothesis of the Right Angle implies that $m\angle A + m\angle B + m\angle C = \pi$.

(iii) Hypothesis of the Obtuse Angle implies that $m\angle A + m\angle B + m\angle C > \pi$.

Proof. Follows easy from the above theorem. ∎

Theorem 147 *(i) Hypothesis of the Acute Angle is consistent with Axioms 1-5 of Euclid's Absolute Geometry.*

(ii) Hypothesis of the Right Angle is consistent with Axioms 1-5 of Euclid's Absolute Geometry.

(iii) Hypothesis of the Obtuse Angle **contradicts** *Axioms 1-5 of Euclid's Absolute Geometry.*

Proof. (i) is justified by the Cayley-Klein Incidence Plane model $M13$.
(ii) is justified by the Cartasian Incidence Plane model $M1$.
(iii) is proved as Thm 22.18. (See also Cor. 22.19.) ∎

The remaining two classes we will be used for review.