## Notes

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**Definition of a field:** Chapter 1, page 5. Examples:
real numbers $\mathbb{R}$; also complex numbers $\mathbb{C}$ and rational numbers $\mathbb{Q}$.

**Important points on Linear Algebra:**
Matrices: definition and the following operations
transpose, scalar multiplication, addition and multiplication of matrices;
zero matrix $\theta$ ($A+\theta=\theta+A=A$);
the identity matrix $I$ ($AI=A$ and $IB=B$);

**Properties:**
$A(BC)=(AB)C$; $AB$ need not be equal $BA$; $AB=I$ implies $BA=I$;

Typical exercise for this material:

**Exercise 1** *For* $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 11 & -1 \end{bmatrix}$ *and* $B=\begin{bmatrix} 0 & 8 \\ 4 & -2 \\ 5 & 1 \end{bmatrix}$ *find* $A^T$, $2A-3B$, $A^TB$, *and* $BA^T$.

**Vector spaces:** Definition 1, Chapter 3 page 2.
Examples:

- $K^{n\times m}$ – the family of all $n\times m$-matrices over the field $K$; e.g. $\mathbb{R}^{n\times m}$
- $\mathbb{R}^{n\times 1}$ – the family of all $n$-dimensional (column) matrices $[x_1\cdots x_n]^T$; often denotes as $\mathbb{R}^n$;
- $\mathbb{R}^2$, the classical plane vectors $[x\ y]^T$ often identified with $[x\ y]$ and written as $\langle x,y\rangle$; similarly $3D$-vectors $\mathbb{R}^3$;
- The family $\mathcal{F}(D,\mathbb{R})$ of all functions from a set $D\subset\mathbb{R}$ into $\mathbb{R}$; also the classes of: all polynomials; of all differentiable functions; of solutions of some differential equations; etc;

**Subspaces:** Definition 1, Chapter 3 page 10.

**Theorem 1** *If $V$ is a vector space (over the field $K$) and $W$ is non-empty subset of $V$, then $V$ is a subspace if, and only if, $v+w$ and $cv$ are in $W$ for every $c$ from $K$ and $x, \in W$.*

Examples:

- $W = \{\langle x, 3x\rangle \colon x \in \mathbb{R}\}$, a line in the plane $\mathbb{R}^2$ is a vector subspace of $\mathbb{R}^2$;
- polynomials forms a vector subspace of $\mathcal{F}(D, \mathbb{R})$; so are differentiable functions;

**System of linear equations $A\mathbf{x} = \mathbf{b}$:**

For a system $A\mathbf{x} = \mathbf{b}$ of $m$ equations with $n$ unknowns $x_1, \ldots, x_n$, $A$ is $m \times n$ coefficient matrix, $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{b} = [b_1, \ldots, b_n]^T$.

**Solutions of $A\mathbf{x} = \mathbf{b}$ via Gauss elimination:**

Use of Gauss elimination, that is, using augmented matrix approach. If the system is consistent (i.e., has at least one solution), the solution must be expressed in the vertical vector form:

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

From the text: Example # 1, Ch. 4, Pg. 8
From the text: Example # 2, Ch. 4, Pg. 19

**Class of January 22, 2013:**

Next class: Quiz # 1, material as in the page 1 of the Sample Test # 1, available at

http://www.math.wvu.edu/~kcies/teach/CurrentTeaching.html

Solve exercise 2 from the Sample Test # 1, via Gauss elimination.

**System of linear equations $A\mathbf{x} = \mathbf{b}$, revisited:**

For a system $A\mathbf{x} = \mathbf{b}$ of $m$ equations with $n$ unknowns $x_1, \ldots, x_n$, $A$ is $m \times n$ coefficient matrix, $\mathbf{x} = [x_1, \ldots, x_n]^T$, and $\mathbf{b} = [b_1, \ldots, b_n]^T$.

When $\mathbf{b} = \mathbf{0} = [b_1, \ldots, b_n]^T$, then the system $A\mathbf{x} = \mathbf{b}$ (i.e., $A\mathbf{x} = \mathbf{0}$) is a homogeneous system.

The solutions $\mathbf{x}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, that is, $V = \{\mathbf{x} \colon A\mathbf{x} = \mathbf{0}\}$, is a *vector space:*

$\mathbf{0} \in V$ and $\alpha\mathbf{x} + \beta\mathbf{y} \in V$ for every $\mathbf{x}, \mathbf{y} \in V$.

In other words, $V$ is a null space of the operator $A$: $\mathbf{x} \mapsto A\mathbf{x}$.

[A function $T$ from a vector space into another is a linear operator when

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}).$$

Its null space is the set of all vectors $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{0}$. Null space of any linear operator is also a vector space. ]

In particular $A\mathbf{x} = \mathbf{0}$ has either one, or infinitely many solutions.

If $\mathbf{x}_p$ is a solution for $A\mathbf{x} = \mathbf{b}$, then

$\mathbf{x}$ solution for $A\mathbf{x} = \mathbf{b}$ if, and only if, it is of the form $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution for $A\mathbf{x} = \mathbf{0}$.

**Inverse of a square, $n \times n$, matrix $A$** If there exists a matrix $B$ such that $BA = I$, then also $AB = I$ and $B$ is unique. It is denoted as $A^{-1}$ and referred to as the inverse of $A$. Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then the inverse of $A$ exists and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Note, $A^{-1} \neq \frac{1}{A}$. In fact, quotient $\frac{1}{A}$ has no sense at all!
$A$ is *singular* if $A^{-1}$ does not exists; otherwise, it is *non-singular.*

**Q.** What $A^{-1}$ is useful for?

**A.** Many uses. E.g.: $A\mathbf{x} = \mathbf{b}$ if, and only if, $\mathbf{x} = A^{-1}\mathbf{b}$.

Also, in deter mining when vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ are *linearly independent* (form a basis) — notions to be discussed.

**Q.** When does $A^{-1}$ exist (i.e., when $A$ is non-singular)?

**A.** E.g.: when the *determinant* of $A$, denoted $|A|$ or $\det A$, is $\neq 0$. Calculation of the determinants to be discussed, chapter 7.

**Q.** When $A$ is non-singular, how to find $A^{-1}$?

**A.** Gaussian elimination (again), to be explained.

**Calculation of the determinant:** Via arbitrary row (or column) expansion, definition, Example on page Ch. 6, Pg 4. Also, via Gaussian elimination, see Ch. 6, Pg 6.

**Solving $A$x = b via *Cramer Rule*: application of determinants.** Just state (Ch. 6, Pg 7), no exercises.

**Finding $A^{-1}$ via Gaussian elimination:** Chapter 9. To find $A^{-1}$: (1) write augmented matrix $[A; I]$; (2) Gaussian elimination to transform it to a matrix $[I; B]$; (3) declare that $A^{-1}$ equals $B$.

Go over Exercises 4, 5 from the sample test and Example 1, Ch. 7, Pg 5.

**Solving $A$x = b via *Cramer Rule*: application of determinants.** Just state (Ch. 6, Pg 7), no exercises.

**Linear independents of vectors and basis:** Just discuss these notions. No exercises.