MATH 261.007 NAME (print): ____________________

Instr. K. Ciesielski

Spring 2010

SAMPLE TEST # 2

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1. Find the general solution for each of the following differential equations:

(a) $y'' + 10y' + 25 = 0$

(b) $y'' + 10y' + 25y = 0$

(c) $y'' + 10y' + 29y = 0$

(d) $y'' + 10y' + 24y = 0$

(e) $2y'' + 3y' + y = t^2 + 3\sin t$

Ex. 2. Solve the initial value problem $y'' + y' - 2y = 2t$, $y(0) = 0$, $y'(0) = 1$.

Ex. 3. Find a particular solution of the equation $y'' + 3y = 3\sin 2t$.

Ex. 4. Given that $y_1(x) = e^x$ is a solution of the ODE $(x-1)y'' - xy' + y = 0$, $x > 1$, use the method of reduction of order to find a second independent solution of this equation.

Ex. 5. Use **the variation of parameters method** to find a particular solution of the equation $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$. (No credit for the solution found by another method.)

Ex. 6. A mass weighing 2 lb stretches a spring 6 in. The mass is pushed down additional 3 in and released with an upward velocity of 0.7 ft/sec. There is a resisting force equivalent to 8 lb when the velocity is 11 ft/sec. There is no external force. What is the ODE that gives the displacement at any time and what are the initial values? (Do not solve the ODE.)

Solutions to the SAMPLE TEST # 2

Ex. 1(a): $y'' + 10y' + 25 = 0$. This is first order linear ODE in terms of y'. That is, using $u = y'$, we get $u' + 10u = -25$.

Then $\mu = \exp\left(\int 10\ dt\right) = e^{10t}$. Also,
$u = \frac{1}{\mu}\int \mu(t)g(t)\ dt = e^{-10t}\int e^{10t}(-25)\ dt = e^{-10t}(\frac{1}{10}e^{10t}(-25) + C) = -2.5 + Ce^{-10t}$.
Thus, $y = \int u(t)\ dt = \int(-2.5 + Ce^{-10t})\ dt = -2.5t + C\frac{1}{-10}e^{-10t} + C_2$.
Replacing constant $C\frac{1}{-10}$ with a constant C_1 leads to
answer: $y(t) = -2.5t + C_1e^{-10t} + C_2$.

Ex. 1(b): $y'' + 10y' + 25y = 0$. This is second order linear homogeneous ODE with the characteristic equation $r^2 + 10r + 25 = 0$.

The equation is equivalent to $(r+5)^2 = 0$, so it has only one double root $r = -5$. Thus, a fundamental pair of solution of our ODE is: e^{-5t} and te^{-5t}, and its general solution
answer: $y(t) = C_1e^{-5t} + C_2te^{-5t}$.

Ex. 1(c): $y'' + 10y' + 29y = 0$. This is second order linear homogeneous ODE with the characteristic equation $r^2 + 10r + 29 = 0$. By quadratic formula its roots are

$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-10\pm\sqrt{100-4\cdot 29}}{2} = \frac{-10\pm\sqrt{-16}}{2} = \frac{-10\pm 4\mathbf{i}}{2} = -5 \pm 2\mathbf{i}$,
i.e., the equation has two complex roots, with real part $\lambda = -5$ and imaginary part $\mu = 2$.
Thus, a fundamental pair of solution of our ODE is: $e^{-5t}\cos(2t)$ and $e^{-5t}\sin(2t)$,
and its general solution
answer: $y(t) = C_1e^{-5t}\cos(2t) + C_2e^{-5t}\sin(2t)$.

Ex. 1(d): $y'' + 10y' + 24y = 0$. This is second order linear homogeneous ODE with the characteristic equation $r^2 + 10r + 24 = 0$.

The equation is equivalent to $(r+4)(r+6) = 0$,
so it has two real roots: $r = -4$ and $r = -6$. Thus,
a fundamental pair of solution of our ODE is: e^{-4t} and e^{-6t}, and its general solution
answer: $y(t) = C_1e^{-4t} + C_2e^{-6t}$.

Ex. 1(e): $2y'' + 3y' + y = t^2 + 3\sin t$. This is second order linear non-homogeneous ODE.

Its solution is of the form $y = y_0 + y_1 + y_2$, where
y_0 is a general solution of the homogeneous ODE $2y'' + 3y' + y = 0$,
y_1 is a particular solution of the non-homogeneous ODE $2y'' + 3y' + y = t^2$, and
y_2 is a particular solution of the non-homogeneous ODE $2y'' + 3y' + y = 3\sin t$.
Thus, this exercise splits into three separate problems, for finding y_0, y_1, and y_2.

Finding y_0: The ODE $2y'' + 3y' + y = 0$ has the characteristic equation $2r^2 + 3r + 1 = 0$,
that is, $(2r+1)(r+1) = 0$. So it has two real roots $r = -1$ and $r = -0.5$.
This gives a fundamental pair of solution for $2y'' + 3y' + y = 0$ as e^{-t} and $e^{-t/2}$ and
$y_0 = C_1e^{-t} + C_2e^{-t/2}$.

Finding y_1: The method of undetermined coefficients says that the ODE $2y'' + 3y' + y = t^2$
has solution of the form $y_1 = t^s(at^2 + bt + c)$. We can take $s = 0$, as $y_1 = at^2 + bt + c$
is not of the form of y_0, the solution of the corresponding homogeneous ODE.
Then $y_1' = 2at + b$ and $y_1'' = 2a$. Substituting this to $2y'' + 3y' + y = t^2$ gives

$2(2a)+3(2at+b)+(at^2+bt+c)1=t^2$, that is,
$at^2+(6a+b)t+(4a+3b+c)=1t^2+0t+0\cdot 1$.
The coefficients of these two polynomials must be equal, that is,
$a=1$, $6a+b=0$, and $4a+3b+c=0$.
This gives $a=1$, $b=-6a=-6$, and $c=-4a-3b=-4+18=14$ and leads to
$y_1=t^2-6t+14$.

Finding y_2: The method of undetermined coefficients says that the ODE $2y''+3y'+y=3\sin t$ has solution of the form $y_2=t^s(A\sin t+B\cos t)$. We can take $s=0$, as
$y_2=A\sin t+B\cos t$ is not of the form of y_0.
Then $y_2'=A\cos t-B\sin t$ and $y_2''=-A\sin t-B\cos t$.
Substituting this to $2y''+3y'+y=3\sin t$ gives
$2(-A\sin t-B\cos t)+3(A\cos t-B\sin t)+(A\sin t+B\cos t)1=3\sin t$, that is,
$(-2A-3B+A)\sin t+(-2B+3A+B)\cos t=3\sin t+0\cos t$.
The coefficients of sin and cos must be equal, that is,
$-A-3B=3$ and $3A-B=0$.
Hence, $B=3A$, $-A-3(3A)=3$, so $A=-0.3$ and $B=3A=-0.9$. So,
$y_2=-0.3\sin t-0.9\cos t$.

Therefore $y=y_0+y_1+y_2$ leads to the
final answer: $y(t)=C_1e^{-t}+C_2e^{-t/2}+t^2-6t+14-0.3\sin t-0.9\cos t$.

Ex. 2: $y''+y'-2y=2t$, $y(0)=0$, $y'(0)=1$.
This is initial value problem for second order linear non-homogeneous ODE.
Its solution is of the form $y=y_0+y_1$, where
y_0 is a general solution of the homogeneous ODE $y''+y'-2y=0$ and
y_1 is a particular solution of the non-homogeneous ODE $y''+y'-2y=2t$.
Thus, this exercise splits into three separate problems: finding y_0, finding y_1,
finding in $y=y_0+y_1$ the constants satisfying the initial condition.

Finding y_0: The ODE $y''+y'-2y=0$ has the characteristic equation $r^2+r-2=0$,
that is, $(r+2)(r-1)=0$. So it has two real roots $r=-2$ and $r=1$.
This gives a fundamental pair of solution for $2y''+3y'+y=0$ as e^{-2t} and e^t and
$y_0=C_1e^{-2t}+C_2e^t$.

Finding y_1: The method of undetermined coefficients says that the ODE $y''+y'-2y=2t$ has solution of the form $y_1=t^s(at+b)$. We can take $s=0$, as $y_1=at+b$
is not of the form of y_0, the solution of the corresponding homogeneous ODE.
Then $y_1'=a$ and $y_1''=0$. Substituting this to $y''+y'-2y=2t$ gives
$0+a-2(at+b)=2t$, that is, $-2at+(a-2b)=2t+0\cdot 1$.
The coefficients of these two polynomials must be equal, that is,
$-2a=2$ and $a-2b=0$. Hence $a=-1$ and $b=a/2=-1/2$,
giving $y_1=-t-0.5$. Thus, the general solution of ODE, $y=y_0+y_1$, is
$y=C_1e^{-2t}+C_2e^t-t-0.5$.

Solving initial value problem: Taking derivative of y we get $y'=-2C_1e^{-2t}+C_2e^t-1$.
$y(0)=0$ leads to $C_1e^0+C_2e^0-0-0.5=0$, that is, $C_1+C_2=0.5$.
$y'(0)=1$ leads to $-2C_1e^0+C_2e^0-1=1$, that is, $-2C_1+C_2=2$.
Subtracting first equation from the second leads to $-3C_1=1.5$, hence $C_1=-0.5$.
So, $C_2=0.5-C_1=1$ and we get
final solution: $y(t)=-0.5e^{-2t}+e^t-t-0.5$.

Ex. 3: Find a particular solution of the equation $y'' + 3y = 3\sin 2t$.

The method of undetermined coefficients says that the ODE has solution of the form $y = t^s(a\sin 2t + b\cos 2t)$.

The corresponding homogeneous ODE has the characteristic equation $r^2 + 3 = 0$, so it has two complex roots $\pm\sqrt{3}\mathbf{i}$ and fundamental solutions $\sin\sqrt{3}t$, $\cos\sqrt{3}t$.

These are not of the form of $y = a\sin 2t + b\cos 2t$, so we can take $s = 0$.

Then we calculate $y'' = -4a\sin 2t - 4b\cos 2t$.

Substitution of y and y'' to our ODE gives

$(-4a\sin 2t - 4b\cos 2t) + 3(a\sin 2t + b\cos 2t) = 3\sin 2t$, that is,

$-a\sin 2t - b\cos 2t = 3\sin 2t + 0\cos 2t$.

The coefficients of sin and cos must be equal, that is,

$-a = 3$ and $-b = 0$, so $a = -3$ and $b = 0$. This gives a final solution: $y = -3\sin 2t$.

Ex. 4: Find a second solution of $(x-1)y'' - xy' + y = 0$, if $y_1(x) = e^x$ is its solution.

The method of reduction of order, which we must use, says to use substitution $y = vy_1$, that is (in our case), $y = ve^x$. We need to find v. First, we calculate

$y' = v'e^x + ve^x = e^x(v' + v)$ and $y'' = (v''e^x + v'e^x) + (v'e^x + ve^x) = e^x(v'' + 2v' + v)$

Substituting these back to our ODE leads to

$(x-1)e^x(v'' + 2v' + v) - xe^x(v' + v) + ve^x = 0$, and after simplification,

$e^x[(x-1)(v'' + 2v' + v) - x(v' + v) + v] = 0$,

$e^x[(x-1)v'' + (2(x-1) - x)v' + ((x-1) - x + 1)v] = 0$,

$e^x[(x-1)v'' + (x-2)v'] = 0$. Dividing this by e^x and putting $u = v'$ we get ODE

$(x-1)u' + (x-2)u = 0$. By further algebra we get

$(x-1)\frac{du}{dx} = -(x-2)u$, so $\frac{du}{u} = -\frac{x-2}{x-1}dx$ and $\int \frac{du}{u} = -\int(1 - \frac{1}{x-1})dx$.

Hence $\ln|u| = -[x - \ln|x-1| + C]$. Thus

$|u| = \exp(-[x - \ln|x-1| + C]) = e^{-x}e^{\ln|x-1|}e^{-C} = e^{-x}|x-1|e^{-C}$. Since $x > 1$, we get

$|u| = e^{-x}(x-1)e^{-C}$ and $u = K(x-1)e^{-x}$, where $K = \pm e^{-C}$ is a constant.

From here $v = \int u\,dx = K\int(x-1)e^{-x}\,dx$. Then, by integration by parts,

$v = \int u\,dx = K[(x-1)(-e^{-x}) - \int 1(-e^{-x})\,dx] = K[e^{-x} - xe^{-x} - e^{-x} + c] = K[-xe^{-x} + c]$

Taking $c = 0$ and $K = -1$, we get $v = xe^{-x}$ and a second solution for our ODE:

$y = ve^x = xe^{-x}e^x = x$.

Answer: $y = x$.

Ex. 5: $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$; use the variation of parameters method.

The homogeneous part of our ODE has the characteristic equation

$r^2 + 4r + 4 =$, that is, $(r+2)^2 = 0$. There is only one, double root $r = -2$,

so $y'' + 4y' + 4y = 0$ has solutions $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$.

The method of the variation of parameters says that a particular solution is of the form

$y = u_1y_1 + u_2y_2$, where $u_1 = -\int \frac{y_2 g(t)}{W}\,dt$ and $u_2 = \int \frac{y_1 g(t)}{W}\,dt$.

Here $g(t) = t^{-2}e^{-2t}$ and W is Wronskian:

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = (e^{-2t})^2[(1-2t) - (-2t)] = e^{-4t} \neq 0$. So

$u_1 = -\int \frac{y_2 g(t)}{W}\,dt = -\int \frac{(te^{-2t})t^{-2}e^{-2t}}{e^{-4t}}\,dt = -\int t^{-1}\,dt = -\ln t$ and

$u_2 = \int \frac{y_1 g(t)}{W}\,dt = \int \frac{(e^{-2t})t^{-2}e^{-2t}}{e^{-4t}}\,dt = \int t^{-2}\,dt = -t^{-1}$. As $y = u_1y_1 + u_2y_2$, we get

Answer: $y = (-\ln t)e^{-2t} + (-t^{-1})te^{-2t}$, that is, $y = (-\ln t)e^{-2t} - e^{-2t}$, or just $y = (-\ln t)e^{-2t}$, as e^{-2t} is a solution to the homogeneous part.

Ex. 6. The ODE is of the form $mu''(t) + \gamma u'(t) + ku(t) = F(t)$.
Here there is no external force, that is, $F(t) = 0$.
We have the weight $w = 2lb$ and so mass $m = \frac{w}{g} = \frac{2lb}{32ft/sec^2} = \frac{1}{16}\frac{lb\ sec^2}{ft}$.
$\gamma = \frac{\text{resisting force}}{\text{velocity}} = \frac{8lb}{11ft/sec} = \frac{8}{11}\frac{lb\ sec}{ft}$
$k = \frac{\text{stretch weight}}{\text{displacement}} = \frac{2lb}{.05ft} = 4lb/ft$.
Clearly, initial position and velocity are $u(0) = 3in = 0.25ft$ and $u'(0) = -0.7ft/sec$. (Negative, as our positive direction is down.)
Answer: $\frac{1}{16}u'' + \frac{8}{11}u' + 4u = 0$, $u(0) = 0.25$, $u'(0) = -0.7$.