Solutions (without Ex. 2) to the SAMPLE TEST # 1

Ex. 1(a) $y' = \frac{e^x - e^{-x}}{3+4y}$, $y(0) = 1$.

Labeling This is *separable equation*, since $\frac{dy}{dx} = \frac{e^x - e^{-x}}{3+4y}$ is equivalent to $\int (3+4y)\, dy = \int (e^x - e^{-x})\, dx$.

Solution $3y + 2y^2 = e^x + e^{-x} + C$. This is a general implicit solution.

From the initial condition $y(0) = 1$ ($y = 0$ for $x = 0$) we have $3 \cdot 1 + 2 \cdot 1^2 = e^0 + e^0 + C$, that is, $5 = 1+1+C$, so $C = 3$. So, particular implicit solution is $3y+2y^2 = e^x+e^{-x}+3$.

Ex. 1(b) $\frac{y}{x} + 6x + (\ln x - 2)\frac{dy}{dx} = 0$, $x > 0$.

Labeling This is *exact equation*, since for $M(x,y) = \frac{y}{x} + 6x$ and $N(x,y) = \ln x - 2$ their partial derivatives $\frac{\partial M}{\partial y} = \frac{1}{x}$ and $\frac{\partial N}{\partial x} = \frac{1}{x}$ are equal.

Solution $\Psi(x,y) = \int M(x,y)\, dx = \int (\frac{y}{x} + 6x)\, dx = y \ln x + 3x^2 + K(y)$.

To find $K(y)$, note that $\frac{\partial \Psi}{\partial y} = N$, that is, $1 \ln x + 0 + K'(y) = (\ln x - 2)$.

Thus, $K'(y) = -2$ and $K(y) = -2y + c$. So, $\Psi(x,y) = y \ln x + 3x^2 - 2y + c$.

The general solution is of the implicit form $\Psi(x,y) = C$, which, in our case is $y \ln x + 3x^2 - 2y = C$.

(In this particular case, the explicit form of y can also be found.)

Ex. 1(c) $ty' - y = t^2 e^{-t}$, $t > 0$.

Labeling This is *linear equation* $y' + p(t)y = g(t)$, since it is equivalent to $y' - \frac{1}{t}y = te^{-t}$, with $p(t) = -\frac{1}{t}$ and $g(t) = te^{-t}$.

Solution $\mu(t) = \exp(\int p(t)\, dt) = \exp(\int -\frac{1}{t}\, dx) = e^{-\ln t} = t^{-1}$.

$y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t)\, dt = t\left(\int t^{-1} t e^{-t}\, dt\right) = t\left(\int e^{-t}\, dt\right) = t\left(-e^{-t} + C\right)$.

Thus $y(t) = -te^{-t} + Ct$ constitutes the general solution.

Ex. 1(d) $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$

Labeling This is a *homogeneous equation* $y' = h(y/x)$, since it is equivalent to the equation $y'(x) = \frac{(x^2+3y^2)/x^2}{2xy/x^2} = \frac{1+3(y/x)^2}{2(y/x)} = h(y/x)$ with $h(z) = \frac{1+3z^2}{2z}$. We make substitution $v = y/x$, leading to $y = xv$ and $y' = v + xv'$.

Solution Using substitution with $y' = \frac{1+3(y/x)^2}{2(y/x)}$ we get $v + xv' = \frac{1+3v^2}{2v}$, that is, xv' is equal to $\frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} = \frac{1+v^2}{2v}$.

Thus, $x\frac{dv}{dx} = \frac{1+v^2}{2v}$. This is a separable equation, equivalent to $\int \frac{2v}{1+v^2}\, dv = \int \frac{1}{x}\, dx$. In both integrals, numerator is the derivative of denominator, so

$\ln(1+v^2) = \ln|x| + C$. Thus, $\exp(\ln(1+v^2)) = \exp(\ln|x| + C)$, that is,

$1 + v^2 = e^C|x|$. Putting $K = \pm e^C$ we get $1 + v^2 = Kx$. Since $v = y/x$, the final general implicit solution is

$1 + (y/x)^2 = Kx$. (In this case, it can be also solved for y.)

Ex. 1(e) $x^2y' = y^3 - 2xy$, $x > 0$.

Labeling This is a *Bernouli equation* $y' + p(x)y = q(x)y^n$, since it is equivalent to the equation $y' + \frac{2}{x}y = \frac{1}{x^2}y^3$ with $p(x) = \frac{2}{x}$, $q(x) = \frac{1}{x^2}$, and $n = 3$. We make substitution $v = y^{1-3}$, leading to $y = v^{-1/2}$ and $y' = -\frac{1}{2}v^{-3/2}v'$.

Solution Using substitution with $y' + \frac{2}{x}y = \frac{1}{x^2}y^3$ we get $-\frac{1}{2}v^{-3/2}v' + \frac{2}{x}v^{-1/2} = \frac{1}{x^2}(v^{-1/2})^3$. So, $-\frac{1}{2v^{3/2}}v' + \frac{2}{x}\frac{1}{v^{1/2}} = \frac{1}{x^2}\frac{1}{v^{3/2}}$. Multiplication by $-2v^{3/2}$ results in linear ODE:

$v' - \frac{4}{x}v = -\frac{2}{x^2}$.

Here $\mu(x) = \exp(\int -\frac{4}{x}\ dx) = \exp(-4\ln|x|) = x^{-4}$. Therefore,

$v(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\ dx = x^4\left(\int x^{-4}(-\frac{2}{x^2})\ dx\right) = x^4\left(\int -2x^{-6}\ dx\right) = x^4(\frac{2}{5}x^{-5} + C)$.

Using again $y = v^{-1/2}$, we get

$y = (x^4(\frac{2}{5}x^{-5} + C))^{-1/2}$.

(This can be farther simplified to

$y = (x^4(\frac{2}{5x^5} + C))^{-1/2} = (\frac{x^4}{5x^5}(2 + 5Cx^5))^{-1/2} = (\frac{5x}{2+5Cx^5})^{1/2} = \sqrt{\frac{5x}{2+cx^5}}$.)

Ex. 1(f) $\frac{dy}{dx} + y = \frac{1}{1+e^x}$.

Labeling This is *linear equation* $y' + p(x)y = g(x)$ with $p(x) = 1$ and $g(x) = \frac{1}{1+e^x}$.

Solution $\mu(x) = \exp(\int p(x)\ dx) = \exp(\int 1\ dx) = e^x$.

$y(x) = e^{-x}\int \mu(x)g(x)\ dx = e^{-x}\int e^x\frac{1}{1+e^x}\ dx = e^{-x}\int \frac{e^x}{1+e^x}\ dx$.

Since in $\frac{e^x}{1+e^x}$ numerator is the derivative of denominator, we have

$\int \frac{e^x}{1+e^x}\ dx = \ln(1+e^x) + C$. Thus,

$y(x) = e^{-x}(\ln(1+e^x) + C) = \frac{\ln(1+e^x)+C}{e^x}$ constitutes the general solution of our ODE.

Ex. 2. Solution in class.

Ex. 3. This is linear ODE $y' + \frac{\cos x}{x^2-x-6}y = \frac{e^x}{x^2-x-6}$ with $p(x) = \frac{\cos x}{(x-3)(x+2)}$ and $g(x) = \frac{e^x}{(x-3)(x+2)}$. Functions p and g are undefined at -2 and 3, and are continuos on the remaining intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. In the initial condition $y(2) = 0$ we have fixed value of y for $x = 2$. Since number 2 is in the interval $(-2, 3)$, this interval constitutes the answer.

Ex. 4. We have $y(0) = -2$ and $y' = f(t, y)$ with $f(t, y) = \frac{4-ty}{1+y^2}$. Hence $y(0.1) \approx -1.92$, as
$$y(0.1) = y(0+h) \approx y(0) + hy'(0) = y(0) + hf(0, -2) = -2 + 0.1\frac{4-0}{1+(-2)^2} = -2 + 0.08.$$

Ex. 5. Let $S(t)$ denotes the amount of salt in the tank, in pounds, after t minutes.

Initially we have $S(0) = 100$. Rate *in* is $r_{in} = 1lb/gal \cdot 3gal/min = 3$ (in lb/min). Rate *out* is $r_{out} = 2gal/min \cdot S(t)/$"current tank solution holding" $= 2S(t)/(200 + (3-2)t) = \frac{2S}{200+t}$ (in lb/min). Since $S' = r_{in} - r_{out}$, our ODE is $S' = 3 - \frac{2S}{200+t}$.

ANSWER: $S' = 3 - \frac{2S}{200+t}$, $S(0) = 100$.