**Real Analysis 2, Math 651, Spring 2005**

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# 1/12/05: sec 3.1 and my article: *How good is the Lebesgue measure?*, Math. Intelligencer 11(2) (1989), 54-58.

$\mathbb{N} = \{1, 2, 3, \ldots\}$.

*Measure* (on $\mathbb{R}^n$) is a function $m$ from a family $\mathcal{M}$ of subsets of $\mathbb{R}^n$ into $[0, \infty]$. Desired properties for a measure:

(i) $\mathcal{M}$ is a $\sigma$-algebra on $\mathbb{R}^n$ with $[0, 1] \in \mathcal{M}$.

(i$^+$) $\mathcal{M} = \mathcal{P}(\mathbb{R}^n)$.

(ii) $m([0, 1]) = \ell([0, 1]) = 1$.

(iii) (measure $m$ is countably additive) If $\langle E_n \in \mathcal{M} \colon n \in \mathbb{N}\rangle$, then we have $m\left(\bigcup_n E_n\right) = \sum_n m(E_n)$.

(iii$^-$) ($m$ is finitely additive) If $E, F \in \mathcal{M}$, then $m(E \cup F) = m(E) + m(F)$.

(iv) ($m$ is translation invariant) If $E \in \mathcal{M}$ and $x \in \mathbb{R}^n$, then $x + E \in \mathcal{M}$ and $m(x + E) = m(E)$.

(iv$^+$) ($m$ is isometrically invariant) If $E \in \mathcal{M}$ and and $i$ is an isometry of $\mathbb{R}^n$, then $i[E] \in \mathcal{M}$ and $m(i[E]) = m(E)$.

Facts discussed:

- There is no measure satisfying (i$^+$)-(ii)-(iii)-(iv). (To be shown later.)
- Lebesgue measure we will construct will satisfy (i)-(ii)-(iii)-(iv)$^+$. (We will construct it only for $n = 1$, but it works also for any $n$.)
- There are many measures satisfying (i$^+$)-(iii)-(iv$^+$). E.g. $m \equiv 0$, $m \equiv \infty$, $m$ being counting measure.
- There are measures satisfying (i$^+$)-(ii)-(iii$^-$)-(iv). E.g. $m \equiv 0$, $m \equiv \infty$, $m$ being counting measure.

- There are measures satisfying ($i^+$)-(ii)-($iii^-$)-($iv^+$) for $n \leq 2$, but there are no such measures for $n \geq 3$ (by Banach-Tarski Paradox.)
- The existence of measures satisfying ($i^+$)-(ii)-(iii) cannot be (easily) decided with the framework of usual axioms of set theory.

# 1/13/05: sec 3.2.

Homework:

**Ex. 1.** Prove that for any measure $m\colon \mathcal{M} \to [0, \infty]$ satisfying (i)-(ii)-(iii)-(iv) we have also

($ii^+$) every interval $J$ belongs to $\mathcal{M}$ and $m(J)$ is equal to its length $\ell(J)$.

**Hint** Prove the statement showing the following steps.

1. Every interval $J$ belongs to $\mathcal{M}$.
2. $m(\{x\}) = 0$ for every $x \in \mathbb{R}$.
3. $m([0, 2^{-n})) = 2^{-n}$ for every $n = 0, 1, 2, \ldots$.
4. $m([a, b)) = \ell([a, b))$ for every $a, b \in \mathbb{R}$, $a < b$.
5. $m(J) = \ell(J)$ for every interval $J$.

We proved defined *outer measure* $m^*$ and proved Proposition 1, page 56 that $m^*(J) = \ell(J)$ for every interval $J$.

# 1/19/05: sec 3.2 and 3.3

**Proposition 2, page 57:** Outer measure $m^*$ is subattitive: for every sets $A_n \subset \mathbb{R}$ we have $m^*\left(\bigcup_n A_n\right) = \sum_n m(^*A_n)$.

**Corollary 3, page 58:** If $A \subset \mathbb{R}$ is countable, then $m^*(A) = 0$.

**Corollary 4, page 58:** The interval $[0, 1]$ is not countable.

Proved Proposition 5. Asked to try Exercise 8 for the next day,

A set $E \subset \mathbb{R}$ is *measurable* ($E \in \mathcal{M}$) provided for every $A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

or, equivalently,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E}).$$

A *(Lebesgue) measure* $m$ is equal to $m^*$ restricted to $\mathcal{M}$.

We need to show that $\mathcal{M}$ is a $\sigma$-algebra containing all intervals. This will be done in the following steps.

(1) If $E \in \mathcal{M}$, then $\tilde{E} \in \mathcal{M}$, as the definition is symmetric in $E$ and $\tilde{E}$.

(2) $\emptyset, \mathbb{R} \in \mathcal{M}$.

(3) (Lemma 6) If $m^*(E) = 0$, then $E \in \mathcal{M}$.

## 1/20/05: sec 3.3

(4) (Lemma 7) If $E_1, E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$.

(5) (Corollary 8) $\mathcal{M}$ is an algebra on $\mathbb{R}$.

(6) (Lemma 9) $m$ is finitely additive. Moreover, if $E_1, \ldots, E_n \in \mathcal{M}$ are pairwise disjoint and $A \subset \mathbb{R}$, then $m^*\left(A \cap \left[\bigcup_{i=1}^n E_i\right]\right) = \sum_{i=1}^n m^*\left(A \cap E_i\right)$.

(7) (Theorem 10) $\mathcal{M}$ is a $\sigma$-algebra on $\mathbb{R}$ containing all sets of outer measure 0.

(8) (Lemma 11) $(a, \infty) \in \mathcal{M}$.

(9) (Theorem 12) $\mathcal{M}$ contains all Borel sets.

(10) (Exercise 9) $\mathcal{M}$ is translation invariant.

Proposition 13: $m$ is countably additive.

**Homework**. Exercises 7 & 8 page 58 and exercise 9, page 64.

**Solution to Exercise 1:** Prove that for any measure $m\colon \mathcal{M} \to [0,\infty]$ satisfying (i)-(ii)-(iii)-(iv) we have also *"every interval $J$ belongs to $\mathcal{M}$ and $m(J)$ is equal to its length $\ell(J)$."*

Proof. Step 1: *Every interval $J$ belongs to $\mathcal{M}$.* By (iv), for every $x \in \mathbb{R}$ we have $[x, x+1] = x + [0,1] \in \mathcal{M}$ and so, by the property (iii), the sets $[x,\infty) = \bigcup_{n=0}^{\infty}[x+n, x+n+1]$ and $(-\infty, x] = \bigcup_{n=1}^{\infty}[x-n, x-n+1]$ belong to $\mathcal{M}$. Thus, by (i), $\mathbb{R} = (-\infty,\infty) = (-\infty, 0] \cup [0,\infty) \in \mathcal{M}$ and so, for every $x \in \mathbb{R}$, the intervals $(-\infty, x) = \mathbb{R} \setminus [x,\infty)$ and $(x,\infty) = \mathbb{R} \setminus (-\infty, x]$ are in $\mathcal{M}$. We proved that every unbounded interval is in $\mathcal{M}$. Since every bounded interval is an intersection of two of such intervals, the claim of Step 1 follows.

Step 2: $m(\{x\}) = 0$ *for every* $x \in \mathbb{R}$. Let $a = m(\{0\}) \in [0,\infty]$. By (iv) for every $x \in \mathbb{R}$ we have $m(\{x\}) = m(x + \{0\}) = m(\{0\}) = a$. Since $\bigcup_{n=1}^{\infty}\{2^{-n}\} \subset [0,1]$, by (ii) and monotonicity of $m$ we get

$$\textstyle\sum_{n=1}^{\infty} a = \sum_{n=1}^{\infty} m(\{2^{-n}\}) = m\left(\bigcup_{n=1}^{\infty}\{2^{-n}\}\right) \le m([0,1]) = 1.$$

Thus, $a = 0$ and the claim of Step 2 follows.

Step 3: $m([0,2^{-n})) = 2^{-n}$ *for every* $n = 0, 1, 2, \ldots$. For $n = 0$ this is true, since, by (iii), $m([0,1)) = m([0,1)) + 0 = m([0,1)) + m(\{1\}) = m[0,1] = 1$. So, assume that for some $n$ we have $m([0,2^{-n})) = 2^{-n}$. We will prove this for $n+1$. Indeed, by (iv), additivity and the inductive assumption, we have

$$\begin{aligned}
2\, m\left(\left[0, 2^{-(n+1)}\right)\right) &= m\left(\left[0, 2^{-(n+1)}\right)\right) + m\left(2^{-(n+1)} + \left[0, 2^{-(n+1)}\right)\right)\\
&= m\left(\left[0, 2^{-(n+1)}\right)\right) + m\left(\left[2^{-(n+1)}, 2^{-n}\right)\right)\\
&= m\left(\left[0, 2^{-(n+1)}\right) \cup \left[2^{-(n+1)}, 2^{-n}\right)\right)\\
&= m\left(\left[0, 2^{-n}\right)\right) = 2^{-n}.
\end{aligned}$$

Thus, $m\left(\left[0, 2^{-(n+1)}\right)\right) = 2^{-(n+1)}$ and, by induction, we conclude that the claim of Step 3 holds true.

Step 4: $m([0,x)) = x$ *for every* $x \in [0,1]$. Let $.i_1 i_2 \ldots$ be a binary representation of $x$, that is, $i_n \in \{0,1\}$ for every $n$ and $x = \sum_{n=1}^{\infty} i_n/2^n$. Put $x_0 = 0$ and for every $k = 1, 2, \ldots$ put $x_k = \bigcup_{n=1}^{k} i_n/2^n$ and $I_k = [x_{k-1}, x_k)$. Notice that the intervals $I_k$ are pairwise disjoint and that $[0,x) = \bigcup_{k=1}^{\infty} I_k$. Note also that $m(I_k) = m(x_{k-1} + [0, i_k/2^k)) = m([0, i_k/2^k)) = i_k/2^k$, where for $i_k = 1$ this follows from Step 3, and for $i_k = 0$ it is obvious, since then $I_k = \emptyset$. Thus, by (iii), we have $m([0,x)) = m\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} m(I_k) = \sum_{k=1}^{\infty} i_k/2^k = x$, concluding the argument for Step 4.

Step 5: $m([a,b)) = b - a = \ell([a,b))$ *for every* $a, b \in \mathbb{R}$, $a < b$. First note that $m([0,x)) = x$ for every $x > 0$. If $x < 1$ this follows from Step 4. So, assume that $x \geq 1$ and let $n \in \{1, 2, 3, \ldots\}$ be maximal such that $n \leq x$. Then $x - n \in [0,1)$ and $[0,x) = [n,x) \cup \bigcup_{k=1}^{n}[k-1,k)$, and all the intervals in this representations and disjoint. So, by Step 4,

$$\begin{aligned} m([0,x)) &= m\left([n,x) \cup \bigcup_{k=1}^{n}[k-1,k)\right) \\ &= m([n,x)) + \sum_{k=1}^{n} m([k-1,k)) \\ &= m(n + [0, x-n)) + \sum_{k=1}^{n} m(k-1+[0,1)) \\ &= (x-n) + n \cdot 1 = x. \end{aligned}$$

The general case follows, as $m([a,b)) = m(a+[0,b-a)) = m([0,b-a)) = b-a$.

Step 6: $m(J) = \ell(J)$ *for every interval* $J$. Let $a < b$ be real numbers. Then $m((a,b)) = 0 + m((a,b)) = m(\{a\}) + m((a,b)) = m([a,b)) = b - a$ and $m([a,b]) = m([a,b)) + m(\{b\}) = m([a,b)) = b - a$. Since for every bounded interval $J$ there are $a < b$ such that $(a,b) \subseteq J \subseteq [a,b]$ for such $J$ we have $b - a = m((a,b)) \leq m(J) \leq m([a,b]) = b - a$ and so $m(J) = b - a = \ell(J)$. Thus, $m(J) = \ell(J)$ holds for every bounded interval $J$. To prove that it also holds for any unbounded interval, notice that for every $a \in \mathbb{R}$ we have $m((a,\infty)) \geq m\left(\bigcup_{n=1}^{\infty}[a+n, a+n+1)\right) = \sum_{n=1}^{\infty} m\left(a+n+[0,1)\right) = \infty$ and $m((-\infty,a)) \geq m\left(\bigcup_{n=1}^{\infty}[a-n, a-n+1)\right) = \sum_{n=1}^{\infty} m\left(a-n+[0,1)\right) = \infty$. Since any unbounded interval $J$ contains an interval of the form $(a,\infty)$ or $(-\infty,a)$, we conclude that for such $J$ we have $m(J) \geq \infty$. Therefore, we have $m(J) = \infty = \ell(J)$.

**Exercise 8 page 58:** *Prove that if* $m^*(A) = 0$, *then* $m^*(A \cup B) = m^*(B)$.

Proof. (Too complicated.) Clearly $m^*(B) \leq m^*(A \cup B)$. Fix an $\varepsilon > 0$. Then there exist the families $\{I_{2n}\}$ and $\{I_{2n+1}\}$ of open intervals such that $A \subset \bigcup_n I_{2n}$ and $B \subset \bigcup_n I_{2n+1}$ with the property $\sum_n \ell(I_{2n}) \leq m^*(A) + \varepsilon = \varepsilon$ and $\sum_n \ell(I_{2n+1}) \leq m^*(B) + \varepsilon$. Then $A \cup B \subset \bigcup_n I_n$ and

$$m^*(A \cup B) \leq \textstyle\sum_n \ell(I_n) = \sum_n \ell(I_{2n}) + \sum_n \ell(I_{2n+1}) \leq \varepsilon + m^*(B) + \varepsilon.$$

So, $m^*(B) \leq m^*(A \cup B) \leq m^*(B)$, proving the result.

# 1/26/05: sec 3.3

**Proposition A** *Let $m$ be an arbitrary measure satisfying (i) and (iii). Let $E_1 \subseteq E_2 \subseteq \cdots$ be measurable. Then $m\left(\bigcup_n E_n\right) = \lim_{n\to\infty} m(E_n)$.*

PROOF. Let $E_0 = \emptyset$ and $F_k = E_k \setminus E_{k-1}$ for every $k = 1, 2, 3, \ldots$. Then $m\left(\bigcup_n E_n\right) = m\left(\bigcup_n F_n\right) = \sum_{n=1}^{\infty} m\left(F_n\right) = \lim_{k\to\infty} \sum_{n=1}^{k} m\left(F_n\right)$. Therefore, $m\left(\bigcup_n E_n\right) = \lim_{k\to\infty} \sum_{n=1}^{k} m\left(F_n\right) = \lim_{k\to\infty} m\left(\bigcup_{n=1}^{k} F_n\right) = \lim_{k\to\infty} m\left(E_k\right)$ finishing the proof. ∎

**Proposition B (version of Proposition 14 page 62)** *Let $m$ be an arbitrary measure satisfying (i) and (iii). Let $F_1 \supseteq F_2 \supseteq \cdots$ be measurable. If $m(F_1) < \infty$, then $m\left(\bigcap_n F_n\right) = \lim_{n\to\infty} m(F_n)$.*

PROOF. Note that if $A \subseteq B$ are measurable and $m(B) < \infty$ then

$$m(B \setminus A) = m(B) - m(A),$$

since $m(B) = m(A \cup (B \setminus A)) = m(A) + m(B \setminus A)$.

Let $E_n = F_1 \setminus F_n$. Then $E_n$'s satisfy Proposition A. Thus

$$\begin{aligned}
m(F_1) - m\left(\bigcap_n F_n\right) &= m\left(F_1 \setminus \bigcap_n F_n\right) \\
&= m\left(\bigcup_n E_n\right) \\
&= \lim_{n\to\infty} m(E_n) \\
&= \lim_{n\to\infty} m(F_1 \setminus F_n) \\
&= \lim_{n\to\infty} [m(F_1) - m(F_n)] \\
&= m(F_1) - \lim_{n\to\infty} m(F_n).
\end{aligned}$$

So, $m\left(\bigcap_n F_n\right) = \lim_{n\to\infty} m(F_n)$. ∎

**Ex. 11:** If $E_1, E_2 \in \mathcal{M}$, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

**Homework**. Exercise 11, page 64.

Proposition 15, with the proof: (i)$\Longrightarrow$ (ii)$\Longrightarrow$(iv)$\Longrightarrow$(i).

# 1/27/05: sec 3.3

Proof of Proposition 15: (ii)$\Leftrightarrow$(iii), (iv)$\Leftrightarrow$(v), and (i)$\Leftrightarrow$(vi).

Proof of (i)$\Longrightarrow$(vi): Fix $\varepsilon > 0$. Let $\{I_n : n \in \mathbb{N}\}$ be a family of open intervals such that $E \subset \bigcup_n I_n$ and $\sum_n \ell(I_n) < m^*(E) + \varepsilon/2$. Let $V = \bigcup_n I_n$. Then $m(V) = m\left(\bigcup_n I_n\right) \leq \sum_n m(I_n) = \sum_n \ell(I_n) < m(E) + \varepsilon/2$. As $m^*(E) < \infty$, we conclude that $m(V \setminus E) = m(V) - m(E) < \varepsilon/2$.

By Proposition A we have $\lim_{N\to\infty} m\left(\bigcup_{n=1}^N I_n\right) = m\left(\bigcup_{n=1}^\infty I_n\right) = m(V)$. Thus, there is an $N$ for which $m(V) - m\left(\bigcup_{n=1}^N I_n\right) < \varepsilon/2$. Let $U = \bigcup_{n=1}^N I_n$. Then $U$ is a finite union of open intervals. We will show that it satisfies (vi). For this first note that $\mu(V) < \infty$ implies $m(V \setminus U) = m(V) - m(U) < \varepsilon/2$. Thus, since $V$ contains $U$ and $E$,

$$m^*(U \triangle E) \leq m^*(U \setminus E) + m^*(E \setminus U) \leq m^*(V \setminus E) + m^*(V \setminus U) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Proof of (vi)$\Longrightarrow$(i): In fact, it is enough to show that (vi) implies (ii). So, fix an $\varepsilon > 0$. By (vi) there exists an open set $U$ (which is a finite union of intervals) such that $m^*(U \triangle E) < \varepsilon/4$. Then, $m^*(E \setminus U) \leq m(U \triangle E) < \varepsilon/4$, since $E \setminus U \subset U \triangle E$. Thus, there exists an open set $W \supset E \setminus U$ such that $m^*(W) < m^*(E \setminus U) + \varepsilon/4 < e/4 + \varepsilon/4 = \varepsilon/2$. Let $V = U \cup W$. Then $V$ is open and contains $E$. Moreover,

$$\begin{aligned} m^*(V \setminus E) &= m^*((U \setminus E) \cup (W \setminus E)) \\ &\leq m^*(U \setminus E) + m^*(W \setminus E) \\ &\leq m^*(U \triangle E) + m^*(W) \\ &< \varepsilon/4 + \varepsilon/2 < \varepsilon. \end{aligned}$$

So, $V$ satisfies (ii). ∎

Solved Ex 14(a): Show that Cantor ternary set has measure zero.

**Homework**. Exercise 14(b), page 64.

## 2/2/05: sec 3.4 and 3.5

Construction of a non-measurable (Vitali) set as in *How good is the Lebesgue measure?* page 55
(see http://Jacobi.math.wvu.edu/ kcies/Other/ElectronicReprints/12.pdf):

For $x \in \mathbb{R}$ let $E_x = \{y \in \mathbb{R}\colon y - x \in \mathbb{Q}\} = x + \mathbb{Q}$. Notice that $E_x$ is an equivalence class of $x$ with respect to the equivalence relation $\sim$ on $\mathbb{R}$ defined as $x \sim y$ if and only if $y - x \in \mathbb{Q}$. Thus $\mathcal{E} = \{E_x\colon x \in \mathbb{R}\}$ is a family of non-empty pairwise disjoint sets, and so is $\mathcal{E}_0 = \{E_x \cap [0,1]\colon x \in \mathbb{R}\}$. So, by the Axiom of Choice, there exists a set $V$ such that $V \cap E$ contains precisely one element for every $E \in \mathcal{E}_0$.

Step 1 (Exercises 15 page 66): If $E \subset V$ is measurable then $m(E) = 0$.

Indeed, as $\sum_{q\in\mathbb{Q}\cap[0,1]} m(q+E) = m\left(\bigcup_{q\in\mathbb{Q}\cap[0,1]} q+E\right) \le m([0,2]) = 2$ and $m(q+E) = m(E)$, we must have $m(E) = 0$ as set $\mathbb{Q} \cap [0,1]$ is infinite.

Step 2: $V$ cannot be measurable.

By way of contradiction assume $V \in \mathcal{M}$. Then, by Step 1, $m(V) = 0$ and we would have

$$m(\mathbb{R}) = m\left(\bigcup_{q\in\mathbb{Q}} q+V\right) = \sum_{q\in\mathbb{Q}} m(q+V) = 0,$$

a contradiction.

Solve Exercise 16 page 66, by noting that if $m^*(A) > 0$, then one of the sets $A \cap (q+V)$, $q \in \mathbb{Q}$, must be non-measurable.

**Homework (bonus)**. Exercise 17, page 66.

Proposition 18, page 66.

Note that (i) is equivalent to: $f^{-1}(U) \in \mathcal{M}$ for every open set $U \subset [-\infty,\infty]$.

Let $E \in \mathcal{M}$ and $f\colon E \to [-\infty,\infty]$. We say that $f$ is *measurable* if one of the above conditions hold.

Solve Exercise 19, page 70.

## 2/3/05: sec 3.5

Exercise 24, page 71: If $f\colon E \to [-\infty, \infty]$ is measurable, then $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \subset [-\infty, \infty]$.

Let $\mathcal{F} = \{S \subset [-\infty, \infty]\colon f^{-1}(B) \in \mathcal{M}\}$. Then $\mathcal{F}$ contains all open sets and is a $\sigma$-algebra. So, it contains all Borel sets.

Proposition 19, page 67.

Theorem 20, page 68.

Discuss Exercises 25 and 28, page 71.

Define *almost everywhere* (abbreviated as a.e.)

Proposition 21, page 69.

Proposition 22, page 69.

**Homework (project)**. Exercise 23, page 71: prove Prop. 22, page 69.

Define *simple function*.

**Homework**. Exercise 20, page 70.

# 2/9/05: sec 3.6 and 4.2

## Three Littlewood's Principles

### Every measurable $E \subset \mathbb{R}$ of finite measure is nearly a finite union of intervals

Proposition 15(vi): *For every $\varepsilon > 0$ there is an $U \subset [a, b]$ which is a finite union of open intervals such that $m(U \triangle E) < \varepsilon$.*

Also, other parts of Proposition 15. E.g., For every $\varepsilon > 0$ there is a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$

### Every measurable function $f\colon [a, b] \to [-\infty, \infty]$ is nearly continuous

Proposition 22 (part): *For every measurable function $f\colon [a, b] \to \mathbb{R}$ and $\varepsilon > 0$ there is a continuous function $h\colon [a, b] \to \mathbb{R}$ such that*

$$m(\{x \in [a, b]\colon |f(x) - h(x)| \geq \varepsilon\}) < \varepsilon.$$

**Lusin's Theorem** (Exercise 31, page 74): *For every measurable function $f\colon [a, b] \to \mathbb{R}$ and $\varepsilon > 0$ there is a continuous function $h\colon [a, b] \to \mathbb{R}$ such that*

$$m(\{x \in [a, b]\colon f(x) \neq h(x)\}) < \varepsilon.$$

### Every convergent sequence $f_n\colon [a, b] \to [-\infty, \infty]$ of measurable functions is nearly uniformly convergent.

**Egoroff's Theorem** (Exercise 30, page 73): *Let $E$ be measurable of finite measure and assume that the sequence $f_n\colon E \to \mathbb{R}$ of measurable functions converges to $f\colon E \to \mathbb{R}$ a.e. on $E$. Then for every $\eta > 0$ there is an $A \subset E$ with $m(A) < \eta$ such that $f_n$ converges uniformly to $f$ on $E \setminus A$.*

## Proofs

Proposition 15 was proved.

Proposition 24, page 73 (Weak version of Egoroff's Theorem): *Let $E$ be measurable set of finite measure and assume that the sequence $f_n\colon E \to \mathbb{R}$ of measurable functions converges to the function $f\colon E \to \mathbb{R}$ a.e. on $E$. Then for every $\varepsilon, \delta > 0$ there is an $A \subset E$ with $m(A) < \delta$ and an $N$ such that $|f_n(x) - f(x)| < \varepsilon$ for every $x \in E \setminus A$ and $n \geq N$.*

Proof: As in the text, for Proposition 23.

Proof of Egoroff's Theorem: For every $n \in \mathbb{N}$ use Prop. 24 with $\varepsilon = 1/n$ and $\delta_n = 2^{-n}\eta$ to find an appropriate $A_n \subset E$. Then $A = \bigcup_n A_n$ works.

Proof of Proposition 22: left as an exercise.

Proof of Lusin's Theorem: For every $n \in \mathbb{N}$ use Prop. 22 with $\varepsilon = 2^{-n-1}\delta$ to find an $A_n \subset [a,b]$ such that $m(A_n) < 2^{-n-1}\delta$ and a continuous function $h_n\colon [a,b] \to \mathbb{R}$ such that $|f(x) - h_n(x)| < 2^{-n-1}\delta$ for every $x \in [a,b] \setminus A_n$. Let $A = \bigcup_n A_n$. Then $m(A) < \delta/2$ and $\langle h_n \rangle$ converges to $f$ uniformly on $[a,b] \setminus A$. In particular, $f$ is continuous on $[a,b] \setminus A$. (Uniform limit of continuous functions is continuous.) By Proposition 15(iii) there is a closed set $F \subset [a,b] \setminus A$ such that $m(([a,b] \setminus A) \setminus F) < \delta/2$. Let $B = [a,b] \setminus F$. Then $m(B) \leq m(A) + m(([a,b] \setminus A) \setminus F) < \delta$ and $f \restriction F$ is continuous. Let $h\colon [a,b] \to \mathbb{R}$ be a continuous extension of $f \restriction F$. Then $h$ is as desired, as $\{x \in [a,b]\colon f(x) \neq h(x)\} \subset [a,b] \setminus F = B$.

# 2/10/05: sec 4.1 and 4.2

**Remark.** Lusin's Theorem is true for every measurable function $f\colon E \to \mathbb{R}$. First notice that this is true for $E = \mathbb{R}$ by applying our version of Lusin's Theorem to $f \restriction [n, n+1]$ for every $n \in \mathbb{Z}$. Then glue the pieces together. For arbitrary $E$ extend $f$ to $\mathbb{R}$ by assigning value 0 on $\tilde{E}$ and then apply version with $E = \mathbb{R}$.

**Remark.** In Lusin's Theorem the set $\{x \in [a,b] \colon f(x) \neq h(x)\}$ cannot be required to have measure 0. For example, this is the case for $f \colon [0,2] \to \{0,1\}$ defined as $f(x) = 1$ if and only if $x \in [0,1]$. (That is, $f$ is the characteristic function of $[0,1]$.)

**Remark.** In Egoroff's Theorem the assumption that $E$ has finite measure is essential, even if all functions are continuous. For example, take $f_n \colon \mathbb{R} \to \mathbb{R}$, defined ad $f(x) = x/n$.

For $E \subset \mathbb{R}$ a *characteristic function* $\chi_E$ of $E$ is defined as $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \in \mathbb{R} \setminus E$. A function $\psi \colon \mathbb{R} \to \mathbb{R}$ is a *simple function* provided $\psi = \sum_{i=1}^n a_i \chi_{E_i}$ for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$, and $E_1, \ldots, E_n \in \mathcal{M}$. Moreover, if each $E_i$ is an interval, then $\psi$ is a *step function*.

**Fact.** *$\psi \colon \mathbb{R} \to \mathbb{R}$ is a simple function if and only if $\psi$ is measurable and $\psi[\mathbb{R}]$ is finite.*

For a simple function $\psi \colon \mathbb{R} \to \mathbb{R}$ let $\{a_1, \ldots, a_n\} = \psi[\mathbb{R}] \setminus \{0\}$ and for every $i \in \{1, \ldots, n\}$ let $A_i = \psi^{-1}(\{a_i\})$. Then $\psi = \sum_{i=1}^n a_i \chi_{A_i}$ is the *canonical representation of $\psi$.*

For a simple function $\psi \colon \mathbb{R} \to \mathbb{R}$ with $m\left(\psi^{-1}(\mathbb{R} \setminus \{0\})\right) < \infty$ we define

$$\int \psi(x)\, dx = \sum_{i=1}^n a_i m(A_i),$$

where $\psi = \sum_{i=1}^n a_i \chi_{A_i}$ is the canonical representation of $\psi$. Also, for $E \in \mathcal{M}$,

$$\int_E \psi(x)\, dx = \int (\chi_E \cdot \psi)(x)\, dx.$$

**We will assume that $0 \cdot \infty = 0$! Then $0$ can be between $a_i$'s.**

**Fundamental Fact.** *If $\psi \colon \mathbb{R} \to \mathbb{R}$ is a simple function with the property that $m\left(\psi^{-1}(\mathbb{R} \setminus \{0\})\right) < \infty$, then $\int \psi(x)\, dx = \sum_{i=1}^k b_i m(E_i)$ for every representation $\psi = \sum_{i=1}^k b_i \chi_{E_i}$ of $\psi$.*

Proof: Prove Lemma 1 and Proposition 2 from page 78.

# 2/16/05: sec 4.1 and 4.2

For a function $f\colon E \to [-\infty, \infty]$ we define support $\mathrm{supp}(f)$ of $f$ as a set $\{x \in E\colon f(x) \neq 0\}$. Recall that for a simple function $\psi = \sum_{i=1}^{n} a_i \chi_{A_i}$ with $m(\mathrm{supp}(f)) < \infty$ and $E \in \mathcal{M}$ we have:

$$\int \psi(x)\, dx = \sum_{i=1}^{n} a_i m(A_i) \qquad \& \qquad \int_E \psi(x)\, dx = \int (\chi_E \cdot \psi)(x)\, dx.$$

**Simple Fact (motivational).** If $f\colon [a,b] \to \mathbb{R}$ is bounded, then $f$ is Riemann integrable if and only if

$$\inf\left\{\int_{[a,b]} \psi\colon \psi \geq f \text{ is step function}\right\} = \sup\left\{\int_{[a,b]} \varphi\colon \varphi \leq f \text{ is step function}\right\}.$$

If the equation holds, then the Riemann itegral of $f$ is equal to this number.

**Definition.** If $E \in \mathcal{M}$ is of finite measure and $f\colon E \to \mathbb{R}$ is bounded measurable, then the *(Lebesgue) integral of $f$ over $E$* is defined as

$$\begin{aligned} \int_E f(x)\, dx &= \inf\left\{\int_E \psi\colon \psi \geq f \text{ is simple function}\right\} \\ &= \sup\left\{\int_E \varphi\colon \varphi \leq f \text{ is simple function}\right\}. \end{aligned}$$

**IMPORTANT REMARK.** If $f$ is simple, then we have two different formulas for $\int_E f = \int_E f(x)\, dx$. However, they coincide!

**Proposition 3, page 79.** If $E \in \mathcal{M}$ is of finite measure and $f\colon E \to \mathbb{R}$ is bounded, then $f$ is measurable if and only if

$$\inf\left\{\int_E \psi\colon \psi \geq f \text{ is simple function}\right\} = \sup\left\{\int_E \varphi\colon \varphi \leq f \text{ is simple function}\right\}.$$

Go over the proof of Proposition 3. Notice that, at the end of the proof, $\psi^* - \varphi^* \geq \frac{1}{\nu}\chi_{\Delta_\nu}$, so $\frac{1}{n} > \int(\psi_n - \varphi_n) \geq \int(\psi^* - \varphi^*) \geq \int \frac{1}{\nu}\chi_{\Delta_\nu} = \frac{1}{\nu} m(\Delta_\nu)$. Thus, $m(\Delta_\nu) < \frac{\nu}{n}$ for every $n$, that is, $m(\Delta_\nu) = 0$.

**Proposition 4, page 81.** *If $f\colon E \to \mathbb{R}$ is bounded and Riemann integrable, then $f$ is measurable and the Riemann and Lebesgue integrals of $f$ are equal.*

# 2/17/05: sec 4.2 and 4.3

**Proposition 5, page 82.** *Lebesgue integral is linear etc.*

**Bounded Convergence Theorem (Proposition 6, page 83.)** *Let $E \in \mathcal{M}$ be of finite measure and let $\langle f_n \rangle$ be a sequence of measurable functions from $E$ into $\mathbb{R}$ which are pointwise convergent to $f\colon E \to \mathbb{R}$. If there exists a number $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for every $n \in \mathbb{N}$ and $x \in E$, then $\int_E f = \lim_n \int_E f_n$.*

**Proposition 7, page 85.** *A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if the set $\{x \in [a,b]\colon f$ is discontinuous at $x\}$ has measure zero.*

For measurable $f\colon E \to [0,\infty]$ we define $\int_E f(x)\, dx$ as

$$\sup \left\{ \int_E h\colon h \leq f \text{ is bounded, measurable, and } m(h^{-1}(\mathbb{R} \setminus \{0\}) < \infty \right\}.$$

**Remark.** If $f$ is bounded and measurable with $m(f^{-1}(\mathbb{R} \setminus \{0\})$, then we have two different formulas for $\int_E f = \int_E f(x)\, dx$. However, they coincide.

**Homework**. Exercise 4, page 89.

**Proposition 8, page 85.** *The integral of non negative function is linear.*

# 2/23/05: sec 4.3

**Fatou's Lemma (Theorem 9, page 86.** *If $f_n\colon E \to [0,\infty]$ are measurable and converge a.e. to $f$, then*

$$\int_E f \leq \liminf_n \int_E f_n.$$

**Monotone Convergence Theorem (Theorem 10, page 87).** *If the functions $f_n\colon E \to [0,\infty]$ are measurable and $\langle f_n \rangle$ is an increasing sequence (in sense of $\leq$) converging to $f$, then $\int_E f = \lim_n \int_E f_n$.*

**Exercise 7(a), page 89.** The inequality in Fatou's Lemma can be strict.

**Exercise 7(b), page 89.** The Monotone Convergence Theorem does not hold for decreasing sequences.

Go over Exercise 6, page 89.

Go over Corollary 11 and Proposition 12, page 87.

A measurable function $f\colon E \to [0,\infty]$ is *integrable* provided $\int_E f < \infty$.

Go over Propositions 13 and 14, page 88.

# 2/24/05: sec 4.4

If $f: E \to [-\infty, \infty]$, then we define $f^+, f^-: E \to [0, \infty]$ by

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{-f(x), 0\}.$$

Note that if $f$ is measurable, then so are $f^+$ and $f^-$. Also,

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

A measurable function $f: E \to [-\infty, \infty]$ is *integrable (over $E$)* provided so are $f^+$ and $f^-$. In this case we define

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Go over Exercise 10(a), page 93.

**IMPORTANT REMARK.** If $m(E) < \infty$ and $f$ is bounded, then we have two different formulas for $\int_E f = \int_E f(x)\,dx$. However, they coincide!

**Proposition 15, page 90.** *The integral of integrable function is linear.*

**Lebesgue (Dominated) Convergence Theorem (Theorem 16, p. 91).** *If $g: E \to [0, \infty]$ is integrable, the functions $f_n: E \to [-\infty, \infty]$ are measurable and such that $|f_n| \le g$ for every $n = 1, 2, 3, \ldots$, and functions $f_n$ converge to $f: E \to [-\infty, \infty]$ a.e., then $\int_E f = \lim_n \int_E f_n$.*

**Theorem 17, p. 92.** *Assume that $g_n: E \to [0, \infty]$ is a sequence of integrable functions which converge a.e. to an integrable $g: E \to [0, \infty]$. If functions $f_n: E \to [-\infty, \infty]$ are measurable, such that $|f_n| \le g_n$ for every $n = 1, 2, 3, \ldots$, and functions $f_n$ converge to $f: E \to [-\infty, \infty]$ a.e., then*

$$\int_E g = \lim_n \int_E g_n \textit{ implies } \int_E f = \lim_n \int_E f_n.$$

**Homework**. Exercise 14, page 93.

**Solution to Exercise 23, page 71.**

**Lemma a.** *Let $f: E \to [-\infty, \infty]$ be measurable such that $m(E) < \infty$ and $A = \{x \in E: f(x) \notin \mathbb{R}\}$ has measure zero. Then for every $\varepsilon > 0$ there is an $M > 0$ such that the set $A_M = \{x \in E: f(x) \notin (-M, M)\}$ has measure less than $\varepsilon/3$.*

PROOF. Notice that $E \supset A_1 \supset A_2 \supset A_3 \cdots$ and that $\bigcap_{m=1}^{\infty} A_m = A$. Since $\min_m m(A_m) = m\left(\bigcap_{m=1}^{\infty} A_m\right) = m(A) = 0$, there exists an $M$ with $m(A_M) < \varepsilon/3$. ∎

**Lemma b.** *Let $f: E \to [-\infty, \infty]$ be measurable such that $m(E) < \infty$. For every $\varepsilon > 0$ and $M > 0$ there exists a simple function $\psi: E \to [-M, M]$ such that $|f(x) - \psi(x)| < \varepsilon$ for every $x$ in the set $B = \{x \in E: |f(x)| < M\}$.*

PROOF. Let $n$ be such that $2M/n < \varepsilon$ and let $\delta = 2M/n$. For $i = 0, 1, \ldots, n$ let $a_i = -M + i\delta$ and note that $-M = a_0 < \cdots < a_n = M$. For $i = 1, \ldots, n$ let $E_i = f^{-1}((a_{i-1}, a_i])$ and notice that it is measurable as a preimage of a Borel set. In addition, the sets $E_i$ are pairwise disjoint and we also have $B = f^{-1}((-M, M)) \subset f^{-1}((-M, M]) = \bigcup_{i=1}^{n} f^{-1}((a_{i-1}, a_i]) = \bigcup_{i=1}^{n} E_i$.

Define $\psi = \bigcup_{i=1}^{n} a_i \chi_{E_i}$. Then $\psi$ is simple and and every $x \in B$ belongs to some $E_i$, for which we have $a_i - \delta = a_{i-1} < f(x) \le a_i = \psi(x)$. In particular, $|f(x) - \psi(x)| < \delta < \varepsilon$. ∎

**Notation.** For $f, g: E \to [-\infty, \infty]$ we put $[f \neq g] = \{x \in E: f(x) \neq g(x)\}$.

**Lemma c.** *For every simple function $\psi = \sum_{i=1}^{n} a_i \chi_{E_i}$ from $[a, b]$ into $[m, M]$ there is a step function $g: [a, b] \to [m, M]$ such that $m([\psi \neq g]) < \varepsilon/3$.*

PROOF. Let $\delta = \varepsilon/3n$. For every $i = 1, \ldots, n$ there is a set $U_i$ which is a union of open disjoint intervals such that $m(U_i \triangle E_i) < \delta$. Then $a_i \chi_{U_i}$ is a step function and $[a_i \chi_{U_i} \neq a_i \chi_{E_i}] = U_i \triangle E_i$ has measure less then $\delta$. Let $g = \sum_{i=1}^{n} a_i \chi_{U_i}$. Then $g$ is a step function, as a finite union of step functions, and we have $[\psi \neq g] \subset \bigcup_{i=1}^{n} [a_i \chi_{U_i} \neq a_i \chi_{E_i}]$. Therefore

$$m([\psi \neq g]) \le \sum_{i=1}^{n} m([a_i \chi_{U_i} \neq a_i \chi_{E_i}]) < \sum_{i=1}^{n} \delta = \varepsilon/3. \qquad \blacksquare$$

**Lemma d.** *For every step function* $g\colon [a,b] \to [m,M]$ *there is a continuous function* $h\colon [a,b] \to [m,M]$ *such that* $m([g \neq h]) < \varepsilon/3$.

PROOF. Let $g = \sum_{i=1}^{n} a_i \chi_{J_i}$, where $J_i$'s are pairwise disjoint intervals. Let $\delta = \varepsilon/3n$. For every $i = 1, \ldots, n$ let $I_i$ be a closed interval (possible empty) which is contained in the interior of $J_i$ and such that $m(J_i \setminus I_i) < \delta$ and let $h_i$ be a continuous hat-like function that agrees with $a_i \chi_{J_i}$ on the complement of $J_i \setminus I_i$. Let $h = \sum_{i=1}^{n} h_i$. Then $h$ is continuous and we have

$$[g \neq h] \subset \bigcup_{i=1}^{n} [a_i \chi_{J_i} \neq h_i] = \bigcup_{i=1}^{n} (J_i \setminus I_i).$$

So, $m([g \neq h]) \leq \sum_{i=1}^{n} m(J_i \setminus I_i) < \sum_{i=1}^{n} \delta = \varepsilon/3$. ■

PROOF OF PROPOSITION 22. Let $M$ be as in Lemma a, $\psi$ be as in Lemma b, $g$ be as in Lemma c, and $h$ be as in Lemma d. Then

$$\begin{aligned} \{x \in E\colon |f(x) - h(x)| \geq \varepsilon\} &\subset \{x \in E\colon |f(x) - \psi(x)| \geq \varepsilon\} \cup [\psi \neq h] \\ &\subset \tilde{B} \cup [\psi \neq h] \\ &\subset A_M \cup [\psi \neq g] \cup [g \neq h], \end{aligned}$$

so $m(\{x \in E\colon |f(x) - h(x)| \geq \varepsilon\}) \leq m(A_M) + m([\psi \neq g]) + m([g \neq h]) < \varepsilon$. ■

## 3/2/05: sec 4.5

Let $E \in \mathcal{M}$ and $f, f_n\colon E \to [-\infty, \infty]$ be measurable.

**uniform convergence** $f_n \rightrightarrows f$ provided

$$\forall \varepsilon > 0 \;\; \exists N \;\; \forall x \in E \;\; \forall n \geq N \;\; |f_n(x) - f(x)| < \varepsilon.$$

**pointwise convergence** $f_n \to f$ provided

$$\forall x \in E \;\; \forall \varepsilon > 0 \;\; \exists N \;\; \forall n \geq N \;\; |f_n(x) - f(x)| < \varepsilon.$$

**almost everywhere convergence** $f_n \to f$ a.e., provided there exists an $A \subset E$ such that $m(A) = 0$ and

$$\forall \varepsilon > 0 \;\; \exists N \;\; \forall x \in E \setminus A \;\; \forall n \geq N \;\; |f_n(x) - f(x)| < \varepsilon.$$

**convergence in measure** $f_n \to f$ in measure, provided

$$\forall \varepsilon > 0 \ \ \exists N \ \ \forall n \geq N \ \ m(\{x \in E \colon |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon.$$

**Theorem**

(a) $f_n \rightrightarrows f \quad \Longrightarrow \quad f_n \to f \quad \Longrightarrow \quad f_n \to f$ a.e.

(b) If $m(E) < \infty$, then $\quad f_n \to f$ a.e. $\quad \Longrightarrow \quad f_n \to f$ in measure.

(c) If $m(E) < \infty$, then $\quad f_n \rightrightarrows f \quad \Longrightarrow \quad \int_E |f_n - f| \to 0$.

(d) $\int_E |f_n - f| \to 0 \quad \Longrightarrow \quad f_n \to f$ in measure.

Moreover, none of these implications can be reversed.

PROOF. (a) is obvious. (b) follows from Proposition 3.23. (c) is very easy. Show (d) by contradiction.

The fact that convergence in measure does not imply any of the other convergences is justified by the following example.

Let $\{f_n \colon [0,1] \to [0,1]\}$ be a enumeration of

$$\{2^n \chi_{[k \cdot 2^{-n}, (k+1) \cdot 2^{-n}]} \colon n = 1, 2, 3, \ldots \ \& \ k = 0, 1, \ldots, 2^n - 1\}.$$

Then $f_n$ converges to $f \equiv 0$ in measure, but it does not converge pointwise and $\int |f_n - f| \to 1 \neq 0$.

**Proposition 20, page 96** Fatou Lemma, Monotone Convergence Theorem, and Dominated Convergence Theorem remain valid if *convergence a.e.* is replaced by the *convergence in measure.*

PROOF. Check this at home.

**Proposition 18, page 95** If $f_n \to f$ in measure, then there exists a subsequence $\langle f_{n_k} \rangle_k$ which converge to $f$ a.e.

PROOF. Go over it.

**Corollary 19, page 96** Let $f_n, f \colon E \to \mathbb{R}$ be measurable and $m(E) < \infty$. Then $f_n \to f$ in measure if and only if every subsequence $\langle f_{n_k} \rangle_k$ of $\langle f_n \rangle_n$ contain a farther subsequence $\langle f_{n_{k_i}} \rangle_i$ which converges a.e. to $f$.

PROOF. **Bonus Homework Exercise.** (Exercise 23, page 96.) Will accept it till the end of March.

**Solution to Exercise 14, page 93.**

**Part (a).** *Under the hypothesis of Theorem 17,* $\int |f - f_n| \to 0$.

Proof. Since $|f_n| \leq g_n$ a.e., then $|f| = \lim_n |f_n| \leq \lim_n g_n = g$ a.e. Thus, by Exercise 10, $\int |f| \leq \int |g| < \infty$, so $|f|$ is integrable. Then functions $G_n = g_n + |f|$ and $G = g + |f|$ are integrable as well and

$$\int G = \int g + \int |f| = \lim_n \int g_n + \int |f| = \lim_n \int G_n.$$

Also, if $F_n = |f_n - f|$, then $F_n \leq |f_n| + |f| \leq g_n + |f| = G_n$. Since $F_n \to 0$ a.e., functions $G_n$, $G$, $F_n$, and $F \equiv 0$ satisfy the assumptions of Theorem 17. Thus, $\lim_n \int F_n = \int F = 0$.

**Part (b).** *Let* $f_n, f\colon E \to [-\infty, \infty]$ *be integrable such that* $f_n \to f$ *a.e. Then* $\int |f - f_n| \to 0$ *if and only if* $\int |f_n| \to \int |f|$.

Proof. "$\Longrightarrow$" By a version of triangle inequality $|\,|f| - |f_n|\,| \leq |f - f_n|$ we have

$$0 \leq \left| \int (|f| - |f_n|) \right| \leq \int |\,|f| - |f_n|\,| \leq \int |f - f_n|,$$

where the second inequality follows from Exercise 10(a). Therefore, by squeeze theorem, $\left|\int |f| - \int |f_n|\right| = \left|\int (|f| - |f_n|)\right| \to 0$. So, $\int |f_n| \to \int |f|$.

"$\Longleftarrow$" If $G_n = |f_n|$ and $G = |f|$, then functions $G_n$, $G$, $f_n$, and $f$ satisfy the assumptions of Theorem 17, and so $\int |f - f_n| \to 0$ follows from part (a).

**Remark.** *The conclusion of Theorem 17,* $\int (f - f_n) \to 0$ *does not imply* $\int |f - f_n| \to 0$.

Proof. Indeed, let $f_n(x) = \frac{x}{n} \cdot \chi_{[-n,n]}$. Then $f_n$ are integrable and they converges to $f \equiv 0$. Also, $\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f = 0$. However, $\int_{\mathbb{R}} |f_n - f| = 2 \int_0^n \frac{x}{n} = n$ does not converge to 0. ∎

# 3/3/05: sec. 5.1–5.3

Main next goal: to prove that for any integrable $f\colon [a,b] \to \mathbb{R}$

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x) \text{ for almost all } x.$$

Main notion needed (section 5.2): $f$ is of *bounded variation* ($f \in BV$) provided its *total variation*

$$T = \sup\left\{ t = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \colon a = x_0 < x_1 < \cdots < x_k = b \right\}$$

is finite.

**Fact:** Every monotone $f$ is of bounded variation.

PROOF. Clearly every $t$ is equal to $|f(b) - f(a)|$. ∎

**Theorem A** (Theorem 5, page 103): $f \in BV$ if and only if $f = g - h$ for some increasing $g$ and $h$.

**Theorem B** (Theorem 3, page 100): If $f\colon [a,b] \to \mathbb{R}$ is increasing, then $f$ is differentiable almost everywhere, $f'$ is measurable, and

$$\int_a^b f'(x)\,dx \le f(b) - f(a).$$

Theorems A and B will be proved later. Now, we will use them to show the main theorem of this part.

**Section 5.3:** Fix integrable $f\colon [a,b] \to \mathbb{R}$ and put $F(x) = \int_a^x f(t)\,dt$.

**Lemma A** (Lemma 7, page 105): $F$ is of bounded variation.

**Lemma B** (Lemma 8, page 105): If $F \equiv 0$, then $f(x) = 0$ a.e.

**Lemma C** (Lemma 9, page 106): If $f$ is bounded, then $F'(x) = f(x)$ a.e.

# 3/9/05: TEST

**Homework:** Solve the remaining two Test exercises at home.

**Solutions to midterm test exercises**

**Ex. 1.** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$ (not necessarily measurable) such that $A_n \subseteq A_{n+1}$ for every $n$. Show that

$$\lim_{n\to\infty} m^*(A_n) = m^*\left(\bigcup_{k=1}^{\infty} A_k\right).$$

Solution. Certainly for every natural $n$ we have $m^*(A_n) \le m^*\left(\bigcup_{k=1}^{\infty} A_k\right)$ as $A_n \subset \bigcup_{k=1}^{\infty} A_k$. Thus $\lim_{n\to\infty} m^*(A_n) \le m^*\left(\bigcup_{k=1}^{\infty} A_k\right)$. So, it is enough to prove the converse inequality.

If there is an $n$ such that $m^*(A_n) = \infty$ the inequality is obvious. So assume that $m^*(A_n) < \infty$ for every $n$ and fix an $\varepsilon > 0$. By induction we will construct a sequence $\langle U_n : n = 1, 2, \ldots\rangle$ of open sets such that for every $n$ we have

$$A_n \subset U_n \subset U_{n+1} \quad \text{and} \quad m(U_n) - m^*(A_n) < \varepsilon.$$

It is easy to find a required open set $U_1 \supset A_1$ by using the definition of $m^*$. So, assume that we already have constructed $U_n$. We need to find an open $U_{n+1}$ containing $A_{n+1} \cup U_n$ such that $m(U_{n+1}) - m^*(A_{n+1}) < \varepsilon$. Let $\delta = \varepsilon - (m(U_n) - m^*(A_n)) > 0$ and, by the definition of $m^*$, find an open $V_n$ containing $A_{n+1} \setminus U_n$ such that $m(V_n) - m^*(A_{n+1} \setminus U_n) < \delta$. We will show that $U_{n+1} = U_n \cup V_n$ is as desired.

Indeed, clearly it is open and contains $A_{n+1} \cup U_n$. The inequality is justified by

$$\begin{aligned}
m(U_{n+1}) \;-\;& m^*(A_{n+1}) \\
=\;& m(U_n \cup V_n) - m^*(A_{n+1}) && \text{(subadditivity of } m\text{)} \\
\le\;& m(U_n) + m(V_n) - m^*(A_{n+1}) && \text{(measurability of } U_n\text{)} \\
=\;& m(U_n) + m(V_n) - [m^*(A_{n+1} \cap U_n) + m^*(A_{n+1} \setminus U_n)] \\
=\;& [m(U_n) - m^*(A_{n+1} \cap U_n)] + [m(V_n) - m^*(A_{n+1} \setminus U_n)] \\
<\;& [m(U_n) - m^*(A_{n+1} \cap U_n)] + \delta && \text{(as } A_n \subset A_{n+1} \cap U_n\text{)} \\
\le\;& [m(U_n) - m^*(A_n)] + \delta \\
=\;& [m(U_n) - m^*(A_n)] + \varepsilon - (m(U_n) - m^*(A_n)) = \varepsilon.
\end{aligned}$$

This finishes the inductive construction.

Now, since $\varepsilon + m^*(A_n) > m(U_n)$ and $U_1 \subset U_2 \subset \cdots$ are measurable from a result proved in class (related to proposition 14) we get

$$\varepsilon + \lim_{n\to\infty} m^*(A_n) \geq \lim_{n\to\infty} m(U_n) = m^*\left(\bigcup_{k=1}^{\infty} U_k\right) \geq m^*\left(\bigcup_{k=1}^{\infty} A_k\right).$$

Since this holds for any $\varepsilon > 0$, we get $\lim_{n\to\infty} m^*(A_n) \geq m^*\left(\bigcup_{k=1}^{\infty} A_k\right)$. ∎

**Ex. 2.** For $A \subset \mathbb{R}$ and $b \in \mathbb{R}$ let $b\,A = \{b\,a: a \in A\}$. Show that

(a) $m^*(b\,A) = |b|\, m^*(A)$;

(b) if $A$ is measurable, then so is $b\,A$.

Solution. (a) If $b = 0$, then $m^*(b\,A) = m^*(\{0\}) = 0 = 0\, m^*(A)$ (unless $A = \emptyset$, in which case the equation is obvious). So, assume that $b \neq 0$.

First notice that if $\{I_n\}_n$ is a countable sequence of intervals covering $A$, then $\{b\,I_n\}_n$ covers $b\,A$. In particular, $m^*(b\,A) \leq \sum_n \ell(b\,I_n) = |b| \sum_n \ell(I_n)$. So

$$\frac{1}{|b|} m^*(b\,A) \leq \inf_{A\subset\bigcup_n I_n} \sum_n \ell(I_n) = m^*(A).$$

Thus $m^*(b\,A) \leq |b|\, m^*(A)$.

Applying this inequality to $b\,A$ in place of $A$ and to $b^{-1}$ in place of $b$ we get also

$$m^*(A) = m^*(b^{-1}\,(b\,A)) \leq |b|^{-1}\, m^*(b\,A)$$

justifying the converse inequality.

(b) Once again, it is obvious if $b = 0$. So, assume $b \neq 0$.

Let $\varepsilon > 0$. We will find an open set $U \supset b\,A$ with $m^*(U \setminus b\,A) < \varepsilon$. Since $A$ is measurable, there exists an open $V \supset A$ such that $m^*(V \setminus A) < \varepsilon/|b|$. But then, $U = b\,V$ is an open set containing $b\,A$ and, by part (a),

$$m^*(U \setminus b\,A) = m^*(b\,V \setminus b\,A) = m^*(b\,(V \setminus A)) = |b|\, m^*(V \setminus A) < \varepsilon$$

finishing the proof. ∎

**Ex. 3.** Let $f: \mathbb{R} \to [-\infty, \infty]$ be integrable and define function $F: \mathbb{R} \to \mathbb{R}$ by a formula $F(x) = \int_{-\infty}^{x} f$ for every $x \in \mathbb{R}$. Show that $F$ is continuous.

Solution. Note that $|f|$ is integrable. (Exercise 10 page 93.) Fix an $\varepsilon > 0$. We need to find a $\delta > 0$ such that $|F(x) - F(y)| < \varepsilon$ for every $y$ with $|x - y| < \delta$.

By Proposition 14, page 88, there exists a $\delta > 0$ such that $\int_A |f| < \varepsilon$ for every $A \in \mathcal{M}$ with $m(A) < \delta$. We will show that such $\delta$ is as desired. Indeed, if $y \in (x - \delta, x]$ and we put $A = [y, x)$, then

$$|F(x) - F(y)| = \left|\int_{(-\infty,x)} f - \int_{(-\infty,y)} f\right| = \left|\int_{[y,x)} f\right| \leq \int_{[y,x)} |f| = \int_A |f| < \varepsilon.$$

Similarly, if $y \in [x, x + \delta)$ and we put $A = [x, y)$, then

$$|F(x) - F(y)| = \left|\int_{(-\infty,x)} f - \int_{(-\infty,y)} f\right| = \left|\int_{[x,y)} f\right| \leq \int_{[x,y)} |f| = \int_A |f| < \varepsilon.$$

This finishes the argument. ■

**Ex. 4.** Let $E \subset \mathbb{R}$ be Lebesgue measurable and let $f: E \to [-\infty, \infty]$ be a measurable function such that $\int_E [f(x)]^2\, dx = 0$. Show that $f(x) = 0$ for almost all $x \in E$.

Solution. Let $A = \{x \in E: f(x) \neq 0\}$ and by way of contradiction assume that $m(A) > 0$. Since $A$ is a union of sets $A_n = \{x \in E: [f(x)]^2 > 1/n\}$, there is an $n$ for which $m(A_n) > 0$. But then $[f(x)]^2 \geq 1/n \cdot \chi_{A_n}$ and so $\int_E [f(x)]^2 \geq \int_E 1/n \cdot \chi_{A_n} = 1/n \cdot m(A_n) > 0$ contradicting our assumption. ■

**Ex. 5.** Is it true that the product $f \cdot g$ of two integrable functions $f$ and $g$ must be integrable? Prove it, or give a counterexample.

Solution. The answer is **NO**. Indeed, let $f: (0, 1] \to \mathbb{R}$, $f(x) = \frac{1}{2\sqrt{x}}$ and notice that $f$ is integrable since $\int f < \infty$. Indeed, if $f_n = f\chi_{[1/n,1]}$, then by the Lebesque monotone convergence theorem we have

$$\int f = \lim_n \int f_n = \lim_n \int_{1/n}^1 f = \lim_n [\sqrt{x}]_{1/n}^1 = \lim_n (1 - 1/\sqrt{n}) = 1.$$

However, for $g = f$ the product $(f \cdot g)(x) = \frac{1}{4x}$ is not integrable, since

$$\int f \cdot g = \lim_n \int fg\chi_{[1/n,1]} = \lim_n \int_{1/n}^1 \frac{1}{4x} = \lim_n \frac{1}{4}[\ln x]_{1/n}^1 = \lim_n \frac{1}{4}(1 - \ln n^{-1}) = \infty.$$

# 3/10/05: sec. 5.1–5.3

**Theorem** (Theorem 10, page 107): $F'(x) = f(x)$ a.e. for any integrable $f$.

PROOF. Use Lemma C and Theorems A and B.

**Section 5.2:** Prove **Lemma 4** and **Theorem 5** (our Theorem A) from page 103.

**Read at home Lemma 1 (Vitali)** from page 98.

# 3/16/05 and 3/16/05: No classes, Spring Break

# 3/23/05: sec. 5.1

**Homework** Solve Exercise 10, page 104.

**Section 5.1:** A collection of nondegenerate intervals $\mathcal{J}$ *covers* $E \in \mathcal{M}$ *in sense of Vitali* provided for every $x \in E$ and $\varepsilon > 0$ there exists an $I \in \mathcal{J}$ such that $x \in I$ and $\ell(I) < \varepsilon$.

**Vitali Lemma (Lemma 1, page 98).** *Let $E \subset \mathbb{R}$ be such that $m^*(E) < \infty$ and let $\mathcal{J}$ be a cover of $E$ in sense of Vitali. Then for every $\varepsilon > 0$ there is a finite collection $\{I_1, \ldots, I_N\}$ of disjoint intervals from $\mathcal{J}$ such that $m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) < \varepsilon$.*

PROOF. It is enough to prove the lemma for $\mathcal{J}$ consisting of closed intervals. Indeed, if this is the case, that we can apply this form of the lemma to $\bar{\mathcal{J}} = \{\mathrm{cl}(I) \colon I \in \mathcal{J}\}$ to find $I_1, \ldots, I_N \in \mathcal{J}$ with disjoint closures and such that $m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) = m^*\left(E \setminus \bigcup_{n=1}^{N} \mathrm{cl}(I_n)\right) < \varepsilon$. Thus, we can assume that $\mathcal{J}$ contains only closed intervals.

Also notice that if $U \supset E$ is open, then $\mathcal{J}_U = \{I \in \mathcal{J} \colon I \subset U\}$ is still a cover of $E$ in sense of Vitali.

For the rest of the proof follow the text.

Define *derivates* $D^+f(x)$, $D^-f(x)$, $D_+f(x)$, and $D_-f(x)$. E.g.,

$$D_-f(x) = \liminf_{h\to 0^+} \frac{f(x) - f(x-h)}{h}.$$

$f$ is differentiable at $x$ iff $D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x) \in \mathbb{R}$.

Prove **Theorem 3** (our Theorem B) from page 100.

# 3/24/05: sec. 5.4

**Homework (bonus)** Solve Exercise 4, page 102, that is, prove Proposition 2, page 99.

Prove **Theorem 3** (our Theorem B) from page 100.

This completes the proof that for any integrable $f\colon [a,b] \to \mathbb{R}$

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x) \text{ for almost all } x.$$

Problem of this section: *For which functions $f\colon [a,b] \to \mathbb{R}$ we have*

$$\int_a^x f'(t)\,dt = f(x) - f(a) \textit{ for every } x \in [a,b]?$$

Clearly such $f$ must be differentiable a.e. We will show that this holds if and only if $f$ is *absolutely continuous*, that is, when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every finite sequence $\{(x_i, x_i')\colon i = 1, \dots, n\}$ of pairwise disjoint subintervals of $[a,b]$

$$\sum_{i=1}^n (x_i' - x_i) < \delta \implies \sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon.$$

Note that the sum and the difference of two absolutely continuous functions is absolutely continuous.

Note that Cantor ternary function (see Ex. 15) is not absolutely continuous.

# 3/30/05: sec. 5.4

**Lemma** (essentially Lemma 11 page 108) *If $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous, then it is continuous and of bounded variation. In particular, it is differentiable a.e.*

**Lemma 13 page 109** *If $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous and $f'(x) = 0$ a.e., then $f$ is constant.*

**Theorem** (essentially Theorem 14 and Corollary 15, page 110) *The following conditions are equivalent for every $f\colon [a,b] \to \mathbb{R}$.*

(a) *$f$ is an indefinite integral, that is, $f(x) = C + \int_a^x g(t)\,dt$ for some integrable function $g\colon [a,b] \to \mathbb{R}$.*

(b) *$f$ is absolutely continuous.*

(c) *$f$ is differentiable a.e., $f'$ is integrable, and $f(x) = f(a) + \int_a^x f'(t)\,dt$ for every $x \in [a,b]$.*

**Proof** of (a)$\Longrightarrow$(b). Use Proposition 4.14 from page 62. Note that if the intervals $\{(x_i, x_i')\colon i = 1,\ldots,n\}$ are pairwise disjoint and $A = \bigcup_{i=1}^n (x_i, x_i')$, then $\sum_{i=1}^n |f(x_i') - f(x_i)| = \sum_{i=1}^n \left|\int_{(x_i,x_i')} g(t)\,dt\right| \le \sum_{i=1}^n \int_{(x_i,x_i')} |g(t)|\;dt = \int_A |g(t)|\;dt$.

Exercises for section 5.4 that you should try to go over: 12, 14, 16–20.

# 3/31/05: sec. 5.5 and 6.1

Go *briefly* over sec. 5.5.

Define *convex* and *strictly convex* function on $(a, b)$.

State and prove Lemma 16, page 113.

State part of Proposition 17, page 113. Prove that for every convex function all one sided derivatives exist.

State Proposition 18, page 114. Prove it for differentiable function.

State and prove Corollary 19, page 114. Prove it for differentiable function.

State and prove Proposition 20, page 115 — **Jensen Inequality**. Stress it for $\varphi(x) = e^x$ and $\varphi(x) = x^p$ for $p \geq 1$.

For measurable $f\colon [0, 1] \to [-\infty, \infty]$, $p \in (0, \infty)$, and $q \in (0, \infty]$ we put

- $||f||_p = \left(\int_0^1 |f|^p\right)^{1/p}$;
- $||f||_\infty = \text{esssup}\, |f| \stackrel{\text{def}}{=} \inf\{b\colon |f| \leq b \;\; a.e.\}$;
- $L^q[0, 1] = \{f\colon ||f||_q < \infty\}$.

Solve Exercises 1, 3, and 4, page 119.

# 4/6/05: sec. 6.1 and 6.2

For measurable $f\colon [0,1] \to [-\infty, \infty]$, $p \in (0, \infty)$, and $q \in (0, \infty]$ we put

- $||f||_p = \left(\int_0^1 |f|^p\right)^{1/p}$;
- $||f||_\infty = \text{esssup}\,|f| \stackrel{\text{def}}{=} \inf\{b\colon |f| \le b \ \ a.e.\}$;
- $L^q[0,1] = \{f\colon ||f||_q < \infty\}$.

Note that $||f||_q = 0$ if and only if $f = 0$ a.e. So, $||f - g||_q = 0$ if and only if $f = g$ a.e. Such functions will be identified!

Note that $||\alpha f||_q = |\alpha|\ ||f||_q$ for every $\alpha \in \mathbb{R}$. Next goal to show that $||\cdot||_q$ is a norm for $q \ge 1$, that is, that $||f + g||_q \le ||f||_q + ||g||_q$.

Solve Exercise 2 page 119: $\lim_{p\to\infty} ||f|| = ||f||_\infty$. Note that $||f||_p \le ||f||_\infty$. Also for $0 < a < b < ||f||_\infty$ and $E = \{x\colon |f(x)| \ge b\}$ we have

$$||f||_p = \left(\int_0^1 |f|^p\right)^{1/p} \ge \left(\int_E b^p\right)^{1/p} = m(E)^{1/p} \cdot b > a$$

for $p$ large enough, since $\lim_{p\to\infty} m(E)^{1/p} = 1$.

Note that $L^q[0,1] \subset L^p[0,1]$ for $0 < p < q \le \infty$ as for $E = \{x\colon |f(x)| < 1\}$

$$\int_0^1 |f|^p = \int_E |f|^p + \int_{\tilde{E}} |f|^p \le 1 + \int_{\tilde{E}} |f|^q \le 1 + \int_0^1 |f|^q.$$

Note that the inclusion is strict, as justified by $f(x) = \frac{1}{x^r}$ for $q^{-1} < r < p^{-1}$.

Prove **Minkowski Inequality** (for $1 \le p \le \infty$).

Note that this implies that $L^p[0,1]$, for $1 \le p \le \infty$, is closed under addition and constant multiplication. So, it is a linear space.

State Minkowski Inequality for $0 < p < 1$.

Suggested exercise: Ex. 5 page 122.

State **Hölder Inequality**, page 121. Read the proof at home.

# 4/7/05: sec. 6.3

Recall the definition of *normed linear space*. Define *sequence convergent in norm*, *Cauchy sequence*, and *Banach space*.

Goal: to prove **Riesz-Ficher Them** that $L^p$ is complete for $1 \le p \le \infty$.

Prove Proposition 5, page 124, and Theorem 6 (Riesz-Ficher), page 125.

**Homework** Solve Exercises 10 and 11, page 126.

Exercises for Section 6.3 that you should try to go over: 16–18.

# 4/13/05 and 4/14/05(with extra lecture time): sec. 6.4 and 6.5

**Proposition 8, page 128.** *For every* $1 \le p \le \infty$*,* $f \in L^p$*, and* $\varepsilon > 0$ *there is a step function* $\varphi$ *such that* $||f - \varphi||_p < \varepsilon$*.*

The proof is essentially identical to the proof of Proposition 22, page 69, which you proved in solving Exercise 23, page 71.

**Remark** Notice that by Hölder inequality used with $g \equiv 1 \in L^q$ for every $f \in L^p$ we have $||f||_1 = \int |f| \le ||f||_p$.

For a normed space $\langle X, ||\cdot||\rangle$ any linear function $F\colon X \to \mathbb{R}$ is called a *linear functional*. We say that $F$ is *bounded* provided the number,

$$||F|| \stackrel{\text{def}}{=} \sup_{x \in X} \frac{|F(x)|}{||x||},$$

its *norm*, is finite.

**Next Goal — prove the Riesz Representation theorem:** *For every* $1 \le p \le \infty$*, a linear functional on* $L^p$ *is bounded if and only if there is a* $g \in L^q$ *such that for every* $f \in L^p$*.*

$$F(f) = \int fg.$$

*Moreover, in such case* $||F|| = ||g||_q$*.*

Proof: follow Section 6.5.

**Solutions to exercises 16–18, pages 126–127.**

Notice that for $1 \le p < \infty$ convexity of $x^p$ implies that for every $a, b > 0$

$$(a+b)^p \le 2^{p-1}(a^p + b^p) \tag{1}$$

as $(a+b)^p = 2^p(.5a + .5b)^p \le 2^p(.5a^p + .5b^p) = 2^{p-1}(a^p + b^p)$.

**Solution to Ex. 16, page 126.** Note that $0 \le |\,||f_n||_p - ||f||_p\,| \le ||f_n - f||_p$, so $||f_n - f||_p \to 0$ implies $||f_n||_p \to ||f||_p$. Conversely, by (1), we have $|f_n - f|^p \le 2^{p-1}(|f_n|^p + |f|^p)$. Let $g_n = 2^{p-1}(|f_n|^p + |f|^p)$. Then each $g_n$ is integrable and the sequence $\langle g_n \rangle$ converges a.e. to $g = 2^{p-1}(|f|^p + |f|^p)$. Moreover, since $||f_n||_p \to ||f||_p$,

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} 2^{p-1}\left( ||f_n||_p^p + \int |f|^p \right) = 2^{p-1}\left( ||f||_p^p + \int |f|^p \right) = \int g.$$

So, by Th. 17, page 92, $\lim_{n\to\infty} \int |f_n - f|^p = \int \lim_{n\to\infty} |f_n - f|^p = 0$.

**Solution to Ex. 18, page 126.** Notice that, by (1),

$$\begin{aligned} |g_n f_n - gf|^p &\le (|g_n f_n - g_n f| + |g_n f - gf|)^p \\ &\le 2^{p-1}\left(|g_n f_n - g_n f|^p + |g_n f - gf|^p\right) \\ &\le 2^{p-1}\left(|g_n|^p |f_n - f|^p + |f|^p |g_n - g|^p\right) \\ &\le 2^{p-1}\left(M^p |f_n - f|^p + |f|^p |g_n - g|^p\right). \end{aligned}$$

Thus, $\int |g_n f_n - gf|^p \le 2^{p-1}\left(M^p \int |f_n - f|^p + \int |f|^p |g_n - g|^p\right)$ and so it is enough to show that $\lim_{n\to\infty} \int |f_n - f|^p = 0$ and $\lim_{n\to\infty} \int |f|^p |g_n - g|^p = 0$. The first of this is just $f_n \to f$ in $L^p$. The second follows from the Lebesgue dominated convergence theorem, since $|f|^p |g_n - g|^p$ converges to 0 a.e. and $|f|^p |g_n - g|^p \le |f|^p (|g_n| + |g|)^p \le |f|^p (2M)^p$, with the bounding function $|f|^p (2M)^p$ is integrable, as $\int |f|^p (2M)^p = ||f|||_p^p (2M)^p < \infty$.

**Ex. 17, page 127.** Notice that for every $1 \le p < \infty$, $h \in L^p$, and a measurable set $E \subset [0,1]$

$$||f \cdot \chi_E||_p = \left( \int_E |f|^p \right)^{1/p}.$$

In particular, $f \cdot \chi_E \in L^p$.

We need to show that $\left|\int fg - \int f_n g\right| = \left|\int (f-f_n)g\right|$ converges to 0. For this note that for every measurable set $E \subset [0,1]$, by Hölder inequality,

$$\begin{aligned}
\left|\int (f-f_n)g\right| &= \left|\int (f-f_n)g\right| \\
&= \left|\int_E (f-f_n)g + \int_{\tilde{E}} (f-f_n)g\right| \\
&\leq \left|\int [(f-f_n)\cdot \chi_E]g\right| + \left|\int (f-f_n)[g\cdot \chi_{\tilde{E}}]\right| \\
&\leq \int (|f-f_n|\cdot \chi_E)|g| + \left|\int (f-f_n)[g\cdot \chi_{\tilde{E}}]\right| \\
&\leq \int |g| \sup_{x\in E} |f(x)-f_n(x)| + \left|\int (f-f_n)[g\cdot \chi_{\tilde{E}}]\right| \\
&\leq \sup_{x\in E} |f(x)-f_n(x)| \int |g| + ||(f-f_n)||_p ||g\cdot \chi_{\tilde{E}}||_q \\
&\leq ||g||_1 \cdot \sup_{x\in E} |f(x)-f_n(x)| + (||f||_p + ||f_n)||_p) \left(\int_{\tilde{E}} |g|^q\right)^{1/q} \\
&\leq ||g||_q \cdot \sup_{x\in E} |f(x)-f_n(x)| + (||f||_p + M) \left(\int_{\tilde{E}} |g|^q\right)^{1/q}.
\end{aligned}$$

Fix an $\varepsilon > 0$. We like to choose $E$ such that $(||f||_p + M)\left(\int_{\tilde{E}} |g|^q\right)^{1/q} \leq \varepsilon/2$, that is, that $\int_{\tilde{E}} |g|^q \leq \left(\frac{\varepsilon}{2(||f||_p+M)}\right)^q$. Since function $|g|^q$ is integrable, by Proposition 14, page 88, there exists $\delta$ such that this inequality holds for any $E$ with $m(\tilde{E}) < \delta$.

By Egoroff's Theorem, there exists measurable $E \subset [0,1]$ such that $m(\tilde{E}) < \delta$ and $f_n$ converges to $f$ uniformly on $E$. Using this inequality with this $E$, we can find an $N$ such that foe every $n \geq N$ we have $||g||_q \cdot \sup_{x\in E} |f(x)-f_n(x)| \leq \varepsilon/2$. Thus, $\left|\int (f-f_n)g\right| \leq \varepsilon$ for every $n \geq N$.

**Exercise is false for $p = 1$.** The argument above fails, since then $||g||_q$ is not expressible by an integral. More precisely, $g \equiv 1$ belongs to $L^\infty$, the statement of the exercise would in particular imply that $\int f = \lim \int f_n$ for every sequence $f_n$ of $L^1$ functions converging a.e. to $f \equiv 0$, as long as $||f_n||_1 \leq 1$. However, this is false for the functions $f_n = c_n \chi_{J_n}$, where open intervals $J_n$ are pairwise disjoint and $c_n$ is such that $||f_n||_1 = c_n \cdot \ell(J_n) = 1$.

# 4/27/05 and 4/28/05 (with extra lecture time)

Prove **Hölder Inequality**, page 121.

Short discussion of Lebesgue measure on $\mathbb{R}^n$ with $n \geq 2$. Mention Fubini Theorem.

Give student evaluations

Solve the exercises from the Ph.D. entrance exam, see below.

**Ph.D. Entrance Exam — Real Analysis**

April 2005

Choose six of the following:

1. For a bounded set $E$, define
$$m_*(E) = b - a - m^*([a,b] \setminus E),$$
where $[a,b]$ is an interval containing $E$, and $m^*$ denotes the usual outer measure. Prove the following statements.
   (a) If $E$ be the set of all irrational numbers in $[0,1]$, then $m_*(E) = 1$.
   (b) $m_*(E)$ is independent of the choice of $[a,b]$, as long as it contains $E$.
   (c) $m_*(E) \le m^*(E)$.

2. Let $E$ be a measurable set in $[0,1]$ with $mE = c$ ($\frac{1}{2} < c < 1$). Let $E_1 = E + E = \{x+y; x, y \in E\}$. Show that there exists a measurable set $E_2 \subset E_1$ such that $mE_2 = 1$.

3. Let $f(x)$ be monotone increasing on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. If the set $\{f(x); x \in [0,1]\}$ is dense in $[0,1]$, show that $f$ is a continuous function on $[0,1]$. Is it absolutely continuous on $[0,1]$? Prove your conclusion.

4. Let $f_n(x)$ be a sequence of continuous functions on [0,1] and $f_n(x) \ge f_{n+1}(x)$ $(n = 1, 2, \cdots)$. For every $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) < 0$. Determine and prove if there is a $\delta > 0$ such that
$$\lim_{n\to\infty} f_n(x) \le -\delta \quad \forall x \in [0,1].$$

5. Let $f \in L^1(R^1)$ and define
$$F(t) \equiv \int_{-\infty}^{\infty} f(x)\sin(xt)dx.$$
Prove $F(t)$ is continuous in $R$. Is $F(t)$ uniformly continuous in $R$? Prove your conclusion.

6. Let $\{f_n\}$ be a sequence of measurable functions on $(a,b)$, and
$$E = \{x \in (a,b) : \ f_n(x) \text{ is convergent}\}.$$
Show that $E$ is measurable.

7. (a) State Fatou's Lemma.
   (b) Show by an example that the strict inequality in Fatou's Lemma is possible.
   (c) Show that Fatou's Lemma can be derived from the Monotone Convergence Theorem.

8. Suppose $f$ is a non-negative integrable function on $[0,1]$. If
$$\int_0^1 f^n = \int_0^1 f \quad \text{for all } n = 1, 2, \cdots,$$
then $f(x)$ must be the characteristic function of some measurable set $E \subset [0,1]$.