

**SAMPLE FINAL TEST**  
 (longer than the actual Final Test)

Solve the following exercises. **Show your work.**

**Ex. 1. ST #1 Ex 3:** Find the determinant of the matrix. Each time you expand the the matrix, you **must** expand it over a row or column that has the largest number of zeros. If necessary, use the row (or column) reduction method to create additional zeros.

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

**Solution:** If we subtract from row # 4 the row # 2 and expand by the third column, we get

$$|A| = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 4 & 0 & 3 \end{vmatrix} = (-1) \cdot 1 \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$$

Next, subtracting from row # 3 three times the row # 2 and expanding again by the third column, we get

$$|A| = (-1) \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -4 & -2 & 0 \end{vmatrix} = (-1)(-1) \begin{vmatrix} -1 & 2 \\ -4 & -2 \end{vmatrix} = 2 - (-8) = 10.$$

**Ex. 2. ST #1 Ex 4:** Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

**Solution:** We need to transform  $[A; I]$  to  $[I; B]$ . Then  $B = A^{-1}$ .

$$[A; I] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{+R_1} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{\times -\frac{1}{4}} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \xrightarrow{-R_3, -3R_3} \begin{bmatrix} 1 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\text{Answer: } A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

**Ex. 3. ST #1 Ex 6:** Let  $\mathbf{a} = \langle 0, 1, 2 \rangle$ ,  $\mathbf{b} = \langle -1, 0, 7 \rangle$ , and  $\mathbf{c} = \langle 2, 3, -1 \rangle$ . Evaluate:  $2\mathbf{a} - \mathbf{b} + \mathbf{c}$ ,  $|\mathbf{c}|$ , and  $(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{c})$ . (Do not confuse vectors with numbers. No partial credit for solutions with such errors.)

**Solution:**

$$2\mathbf{a} - \mathbf{b} + \mathbf{c} = 2\langle 0, 1, 2 \rangle - \langle -1, 0, 7 \rangle + \langle 2, 3, -1 \rangle = \langle 0, 2, 4 \rangle + \langle 1, 0, -7 \rangle + \langle 2, 3, -1 \rangle = \langle 3, 5, -4 \rangle$$

$$|\mathbf{c}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

As  $\mathbf{a} \cdot \mathbf{b} = \langle 0, 1, 2 \rangle \cdot \langle -1, 0, 7 \rangle = 0 + 0 + 14 = 14$  and

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 7 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 7 \\ 3 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 7 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} =$$

$$\mathbf{i}(0 - 21) - \mathbf{j}(1 - 14) + \mathbf{k}(-3 - 0) = \langle -21, 13, -3 \rangle, \text{ we have}$$

$$(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{c}) = 14 \langle -21, 13, -3 \rangle = \langle -14 \cdot 21, 14 \cdot 13, -3 \cdot 14 \rangle.$$

**Ex. 4. ST #2 Ex 1:** Find a vector equation of the line that passes through the point  $P(11, 13, -7)$  and is perpendicular to the plane with the equation:  $x - 2z = 17$ .

**Solution:** The direction vector  $\mathbf{v}$  of the line coincides with the normal vector of the plane:  $\langle 1, 0, -2 \rangle$ .

$$\text{Answer: } \langle x, y, z \rangle = \langle 11, 13, -7 \rangle + t\langle 1, 0, -2 \rangle, \text{ or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ -7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

**Ex. 5. ST #2 Ex 7:** Let  $\mathbf{v}(t) = \mathbf{i}(t + e)^{-1} + \mathbf{k} t^3$  be a velocity of a particle. Find the acceleration vector  $\mathbf{a}(t)$  of the particle and its position vector  $\mathbf{r}(t)$ , where its initial position was  $\mathbf{r}(0) = 3\mathbf{i}$ .

**Solution:**  $\mathbf{a}(t) = \mathbf{v}'(t) = -(t + e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$ .

$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \mathbf{i} \ln |t + e| + \mathbf{k} t^4/4 + \vec{C}$ . To find  $\vec{C}$ , we calculate  $\mathbf{r}(0)$ :

$\mathbf{i} \ln |0 + e| + \mathbf{k} 0^4/4 + \vec{C} = 3\mathbf{i}$ . Since  $\ln e = 1$ , we get  $\mathbf{i} + \vec{C} = 3\mathbf{i}$  and  $\vec{C} = 2\mathbf{i}$ . Therefore

$\mathbf{r}(t) = \mathbf{i} \ln |t + e| + \mathbf{k} t^4/4 + 2\mathbf{i} = (2 + \ln |t + e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$ .

Answer:  $\mathbf{a}(t) = -(t + e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$  and  $\mathbf{r}(t) = (2 + \ln |t + e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$ .

**Ex. 6. ST #2 Ex 10:** Sketch and fully describe the domain of the following function, including the name of the surface representing the domain's boundary:  $f(x, y, z) = \ln(25 - 4x^2 - 9y^2 - z^2)$ .

**Solution:** The argument of the logarithm must be positive:  $25 - 4x^2 - 9y^2 - z^2 > 0$ , that is,  $4x^2 + 9y^2 + z^2 < 25$ , or  $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} < 1$ .

Answer: The points inside the ellipsoid  $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} = 1$ . Sketch: to be presented in class.

**Ex. 7. ST #3 Ex. 2:** Compute the first order partial derivatives of  $f(x, y, z) = ze^{x^2} \cos y$ .

**Solution:**

$$\frac{\partial f}{\partial x} = f_x = ze^{x^2} \cos y \cdot 2x = 2xze^{x^2} \cos y$$

$$\frac{\partial f}{\partial y} = f_y = ze^{x^2} (-\sin y) = -ze^{x^2} \sin y$$

$$\frac{\partial f}{\partial z} = f_z = e^{x^2} \cos y$$

**Ex. 8. ST #3 Ex. 3:** Compute all second order partial derivatives of  $g(s, t) = e^{5t} + t \sin(3s)$ .

**Solution:**

$$g_s = 3t \cos(3s) \quad g_{ss} = -9t \sin(3s) \quad g_{st} = 3 \cos(3s)$$

$$g_t = 5e^{5t} + \sin(3s) \quad g_{ts} = 3 \cos(3s) \quad g_{tt} = 25e^{5t}$$

**Ex. 9. ST #3 Ex. 4:** Find an equation of the plane tangent to the surface  $z = x^2 - 5y^3$  at the point  $P(2, 1, -1)$ .

**Solution:**

$$z_x = 2x; \quad z_x(P) = 2 \cdot 2 = 4;$$

$$z_y = -15y^2; \quad z_y(P) = -15 \cdot 1^2 = -15;$$

$$\text{Normal vector } \mathbf{n} = \langle z_x(P), z_y(P), -1 \rangle = \langle 4, -15, -1 \rangle.$$

$$\text{Answer: } 4(x - 2) - 15(y - 1) - 1(z + 1) = 0 \quad \text{or} \quad 4x - 15y - z + 6 = 0.$$

**Ex. 10. ST #3 Ex. 8:** Find the point on the cone  $z = \sqrt{x^2 + y^2}$  which is the closest to the point  $(4, -8, 0)$ .

**Solution:**

Solution: Distance of  $(x, y, z)$  on the surface from  $(4, -8, 0)$  is  $\sqrt{(x - 4)^2 + (y + 8)^2 + (z - 0)^2}$ . Since  $z^2 = x^2 + y^2$ , this is equal to

$$f(x, y) = \sqrt{(x - 4)^2 + (y + 8)^2 + (x^2 + y^2)}.$$

$$f_x(x, y) = \frac{2(x-4)+2x}{2\sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}} \quad \text{and} \quad f_y(x, y) = \frac{2(y+8)+2y}{2\sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}}.$$

$$f_x = 0 \text{ when } 2(x - 4) + 2x = 0, \text{ that is, } 4x - 8 = 0, \text{ so } x = 2.$$

$$f_y = 0 \text{ when } 2(y + 8) + 2y, \text{ that is, } 4y + 16 = 0, \text{ so } y = -4.$$

This gives critical point  $(2, -4)$ . Since these are the coordinates of a point on the cone, we get  $z = \sqrt{2^2 + (-4)^2} = \sqrt{20}$ .

$$\text{Answer: Point } (2, -4, \sqrt{20}).$$

**Ex. 11. ST #3 Ex. 5:** Find the absolute maximum and the absolute minimum of the function  $f(x, y) = x^3 - xy$  on the region bounded below by parabola  $y = x^2 - 1$  and above by line  $y = 3$ . You will get credit **only** if **all** critical points are found.

**Solution:** The curves intersect, when  $x^2 - 1 = 3$ , that is, when  $x = \pm 2$ .

Thus, we need to consider the region above  $x^2 - 1$  and below 3 for  $x$  in the interval  $[-2, 2]$ .

**Region's interior:**  $f_x(x, y) = 3x^2 - y$  and  $f_y(x, y) = -x$ . This leads to system  $3x^2 - y = 0$  and  $-x = 0$ , with only solution  $(x, y) = (0, 0)$ . This point belongs to the region. This is our first critical point.

**Lower boundary:**  $y = x^2 - 1$  and  $-2 \leq x \leq 2$ . Then

$$g(x) = f(x, x^2 - 1) = x^3 - x(x^2 - 1) = x \text{ and } g'(x) = 1 \text{ is never } 0.$$

So, there are no true critical points, but we need to consider the endpoints of  $g$ ,  $x = \pm 2$ .

This give us the critical points  $(x, y) = (\pm 2, 3)$ .

**Upper boundary:**  $y = 3$  and  $-2 \leq x \leq 2$ . Then

$$g(x) = f(x, 3) = x^3 - 3x \text{ and } g'(x) = 3x^2 - 3, \text{ which is } 0 \text{ when } x = \pm 1 \in [-2, 2].$$

This give us the critical points  $(x, y) = (\pm 1, 3)$ . (Plus the end points  $(x, y) = (\pm 2, 3)$ , considered above.)

**Checking the critical points:**  $f(0, 0) = 0$ ;

$$f(2, 3) = 2^3 - 6 = 2; f(-2, 3) = (-2)^3 + 6 = -2;$$

$$f(1, 3) = 1^3 - 3 = -2; f(-1, 3) = (-1)^3 + 3 = 2;$$

**Answer:**  $f$  has the absolute maximum value 2, at points  $(2, 3)$  and  $(-1, 3)$ .

$f$  has the absolute minimum value  $-2$ , at points  $(-2, 3)$  and  $(1, 3)$ .

**Ex. 12. ST #4 Ex. 1(a)&(c):** Set up the integral formulas, **including the limits of the integrations**, for the following problems. *Do not evaluate the integrals!*

- (a) The volume of the solid bounded by  $z = x^2 + y^2$ ,  $z = 0$ ,  $x = 0$ ,  $y = 0$ , and  $x + y = 1$ .

**Solution:** If  $T$  is a triangle bounded by  $x = 0$ ,  $y = 0$ , and  $x + y = 1$  (i.e.,  $y = 1 - x$ ), then  $V = \int \int \int_E 1 dV = \int \int_T \int_0^{x^2+y^2} 1 dz dA = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} 1 dz dy dx$

- (c) The mass of the solid  $T$  with the density  $\delta(x, y, z) = x^2 + e^z$  bounded by the surfaces:  $6x + 2y + z = 12$ ,  $x = 0$ ,  $y = 0$ , and  $z = 0$ .

**Solution:** The solid is a tetrahedron with a triangular base  $B$  on the  $xy$ -plane  $z = 0$  bounded by  $6x + 2y = 12$ ,  $x = 0$ ,  $y = 0$ . The upper bound of  $T$  is  $z = 12 - 6x - 2y$ . So,  $mass = \int \int \int_T \delta(x, y, z) dV = \int \int_B \int_0^{12-6x-2y} (x^2 + e^z) dz dA$ .

Since the triangle side  $6x + 2y = 12$  means that  $y = 6 - 3x$ , which quals 0 for  $x = 2$ , we get  $mass = \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} (x^2 + e^z) dz dy dx$ .

**Ex. 13. ST #4 Ex. 2:** Evaluate the integrals:

$$(a) \int_0^1 \int_0^\pi \frac{1}{x+1} + \sin y \, dy \, dx =$$

**Solution:**  $\int_0^1 \left[ \frac{1}{x+1}y - \cos y \right]_0^\pi dx = \int_0^1 \left( \frac{1}{x+1}\pi - (\cos \pi - \cos 0) \right) dx$ . So

$$\int_0^1 \left( \frac{1}{x+1}\pi - (-1 - 1) \right) dx = [\pi \ln |x+1| + 2x]_0^1 = \pi(\ln 2 - \ln 1) + 2 = \pi \ln 2 + 2$$

$$(b) \int_{-2}^0 \int_0^y (x + 2y^2) \, dx \, dy =$$

**Solution:**  $\int_{-2}^0 \left[ \frac{1}{2}x^2 + 2y^2x \right]_{x=0}^{x=y} dy = \int_{-2}^0 \left( \frac{1}{2}y^2 + 2y^3 \right) dy = \left[ \frac{1}{6}y^3 + \frac{1}{2}y^4 \right]_{-2}^0 = 0 - \left( \frac{1}{6}(-8) + \frac{1}{2}16 \right) = \frac{4}{3} - 8 = -6\frac{2}{3}$

$$(c) \int \int_R \frac{dy \, dx}{\sqrt{9 - x^2 - y^2}}, \text{ where } R \text{ is the second quadrant region bounded by } x^2 + y^2 = 4.$$

**Solution:** We use the polar coordinates, in which the region  $R$  is given as  $0 \leq r \leq 2$  and  $\pi/2 \leq \theta \leq \pi$ . So, in the second equation using substitution  $u = 9 - r^2$ ,

$$\int_{\pi/2}^\pi \int_0^2 (9 - r^2)^{-1/2} r \, dr \, d\theta = \int_{\pi/2}^\pi \left[ -(9 - r^2)^{1/2} \right]_0^2 d\theta = \int_{\pi/2}^\pi \left[ -\left( (9 - 4)^{1/2} - 9^{1/2} \right) \right] d\theta = \left[ 3 - \sqrt{5} \right]_{\pi/2}^\pi = \frac{3 - \sqrt{5}}{2} \pi.$$

**Ex. 14. ST #4 Ex. 3:** Find the mass of the solid bounded by the hemisphere  $x^2 + y^2 + z^2 \leq R^2$ ,  $z \geq 0$ , with the density  $\delta(x, y, z) = x^2 + y^2 + z^2$ .

**Solution:** We use the spherical coordinates. Since the solid,  $T$ , is the upper hemisphere, we get

$$\begin{aligned} \text{mass} &= \int \int \int_T \delta(x, y, z) \, dV = \int \int \int_T (x^2 + y^2 + z^2) \, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R (\rho^2) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \\ &= \int_0^{\pi/2} \int_0^{2\pi} \left[ \frac{1}{5} \rho^5 \sin \phi \right]_0^R d\theta \, d\phi = \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{5} R^5 \sin \phi \, d\theta \, d\phi = \int_0^{\pi/2} \left[ \left( \frac{1}{5} R^5 \sin \phi \right) \theta \right]_0^{2\pi} d\phi = \\ &= \int_0^{\pi/2} \frac{2}{5} \pi R^5 \sin \phi \, d\phi = \left[ \frac{2}{5} \pi R^5 (-\cos \phi) \right]_0^{\pi/2} = -\frac{2}{5} \pi R^5 (\cos(\pi/2) - \cos 0) = -\frac{2}{5} \pi R^5 (0 - 1) = \frac{2}{5} \pi R^5 \end{aligned}$$

**Ex. 15. ST #4 Ex. 4:** Find the mass of the plane lamina bounded by  $x = 0$  and  $x = 9 - y^2$  with density  $\delta(x, y) = y^2$ .

**Solution:** Notice that  $x = 0$  and  $x = 9 - y^2$  when  $9 - y^2 = 0$  that is, when  $y = \pm 3$ .

$$\begin{aligned} \text{mass} &= \int \int_R \delta(x, y) \, dA = \int_{-3}^3 \int_0^{9-y^2} y^2 \, dx \, dy = \int_{-3}^3 [xy^2]_0^{9-y^2} dy = \int_{-3}^3 y^2(9 - y^2) \, dy = \\ &= \int_{-3}^3 (9y^2 - y^4) \, dy = \left[ 3y^3 - \frac{1}{5}y^5 \right]_{-3}^3 = 3(27 + 27) - \frac{1}{5}(243 + 243) = 162 - 97.2 = 64.8 \end{aligned}$$

**Ex. 16. ST #4 Ex. 6:** Evaluate the integral, where  $C$  is the graph of  $y = x^3$  from  $(-1, -1)$  to  $(1, 1)$

$$\int_C y^2 dx + x dy =$$

**Solution:** Clearly  $x$  changes from  $-1$  to  $1$ . Put  $x = t$ . Then  $y(t) = t^3$  and  $-1 \leq t \leq 1$  and  $\int_C y^2 dx + x dy = \int_{-1}^1 (y(t))^2 x'(t) dt + x(t) y'(t) dt = \int_{-1}^1 [(t^3)^2 1 + t(3t^2)] dt = \int_{-1}^1 (t^6 + 3t^3) dt = \left[ \frac{1}{7} t^7 + \frac{3}{4} t^4 \right]_{-1}^1 = \frac{1}{7}(1 + 1) + \frac{3}{4}(1 - 1) = \frac{2}{7}$

**Ex. 17. ST #4 Ex. 8:** Find a potential function of the vector field and use the fundamental theorem for line integrals to evaluate

$$\int_{(\pi/2, \pi/2)}^{(\pi, \pi)} (\sin y + y \cos x) dx + (\sin x + x \cos y) dy =$$

**Solution:** We have  $P = \sin y + y \cos x$  and  $Q = \sin x + x \cos y$ . It is easy to see that  $\frac{\partial P}{\partial y} = \cos y + \cos x = \frac{\partial Q}{\partial x}$  so indeed we can find the potential function  $f(x, y)$ . We have

$$f(x, y) = \int P dx = \int \sin y + y \cos x dx = x \sin y + y \sin x + K(y).$$

Taking partial derivative, in terms of  $y$ , of both side we get

$$x \cos y + \sin x + K'(y) = \frac{\partial f}{\partial y} = Q = \sin x + x \cos y, \text{ so that } K'(y) = 0 \text{ and } K(y) = C.$$

So, the potential function  $f(x, y) = x \sin y + y \sin x + C$  and

$$\text{int} = [f(x, y)]_{(\pi/2, \pi/2)}^{(\pi, \pi)} = [x \sin y + y \sin x]_{(\pi/2, \pi/2)}^{(\pi, \pi)} = (\pi \sin \pi + \pi \sin \pi) - \left( \frac{\pi}{2} \sin \frac{\pi}{2} + \frac{\pi}{2} \sin \frac{\pi}{2} \right) = (0 + 0) - \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = -\pi$$

**Ex. 18. ST #4 Ex. 9: Apply Green's theorem** to evaluate the following integral, where the simple closed curve  $C$ , with counter clockwise direction, is the boundary of the circle  $x^2 + y^2 = 1$ .

$$\oint_C (\sin x - x^2 y) dx + x y^2 dy =$$

**Solution:** Let  $D$  denoted the disk  $x^2 + y^2 \leq 1$ .

By Green's theorem  $\text{int} = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ , where  $P = \sin x - x^2 y$  and  $Q = x y^2$ . So,

$$\text{int} = \int \int_D (y^2 - (-x^2)) dA = \int \int_D (x^2 + y^2) dA$$

Changing to the polar coordinates, we get

$$\text{int} = \int_0^{2\pi} \int_0^1 r^2 r dr d\theta = \int_0^{2\pi} \left[ \frac{1}{4} r^4 \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \left[ \frac{1}{4} \theta \right]_0^{2\pi} = \frac{1}{2} \pi$$