Topology 1, Math 581, Fall 2017: Notes and homework

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Class of August 17:

Course and syllabus overview.

Topology is an abstract geometry, sometimes referred to as *Rubber Sheet Geometry.* Material, in this course, will be presented "from abstract definitions and results to specific examples."

Notation:

- Do not confuse $A \in \mathcal{A}$ (which reads "A is an element of $\mathcal{A}$") with $A \subset \mathcal{A}$ (which reads "A is a subset of $\mathcal{A}$" and means "every element of A is also an element of $\mathcal{A}$).

  Notice that $A \subset B \subset C$ implies $A \subset C$, but $A \in B \in C$ does not imply $A \in C$. You will never see in this course a pair A and B, for which we will have simultaneously $A \in B$ and $A \subset B$.

- Notation $f\colon X \to Y$ means that f is a function from a set X, domain of the function, into the set Y. For any set C (usually, $C \subset Y$), the preimage $f^{-1}(C)$ (of C under f) is defined as

$$f^{-1}(C) = \{x \in X\colon f(x) \in C\}.$$

Example 1 *$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ for every A, B, and function f.*

Proof. $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in A \cap B \Leftrightarrow f(x) \in A \;\&\; f(x) \in A$
$\Leftrightarrow x \in f^{-1}(A) \;\&\; x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$. ■

Motivation:

Let $\mathbb{R}$ be the set of real numbers and for $x \in \mathbb{R}$ and $\varepsilon > 0$ let

$$B(x, \varepsilon) = \{r \in \mathbb{R}\colon |x - r| < \varepsilon\}.$$

We will refer to $B(x, \varepsilon)$ as an *open ball*, although for this case it is just an open interval $(x - \varepsilon, x + \varepsilon)$. Let $\mathcal{T}$ be the family of all subsets U of $\mathbb{R}$ such that for every $x \in U$ there is an $\varepsilon > 0$ such that $x \in B(x, \varepsilon) \subset U$:

$$\mathcal{T} = \{U \subset \mathbb{R}\colon \forall x \in U \exists \varepsilon > 0 (B(x, \varepsilon) \subset U)\}.$$

Latter, we will refer to $\mathcal{T}$ as the *standard topology* on $\mathbb{R}$ and its elements $U \in \mathcal{T}$ will be called *open sets*.

Theorem 2 *(Motivational) Let $f: \mathbb{R} \to \mathbb{R}$. The following two definitions of continuity of f are equivalent:*

(a) (Topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.

(b) (ε-δ definition) For every $x \in \mathbb{R}$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in \mathbb{R}$, if $|x - r| < \delta$, then $|f(x) - f(r)| < \varepsilon$.

Proof. Latter today.

For functions $f: X \to Y$ and $g: Y \to Z$ their composition $g \circ f: X \to Z$ is defined via formula: $(g \circ f)(x) = g(f(x))$ for every $x \in X$. Also, if $A \subset X$, then the image $f[A]$ of A under f is defined as $\{f(a): a \in A\}$.

Theorem 3 *We have the following properties:*

(a) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$

(b) $(g \circ f)[A] = g[f[A]]$

Proof. (a) $x \in (g \circ f)^{-1}(C) \Leftrightarrow (g \circ f)(x) \in C \Leftrightarrow g(f(x)) \in C \Leftrightarrow$ $f(x) \in g^{-1}(C) \Leftrightarrow x \in f^{-1}(g^{-1}(C))$.

Proof of (b) is left as an exercise. (Not homework assignment.) ∎

The next theorem gives a motivation of defining continuity of a functions via property (a) of Theorem 2. Note, that the proof is considerably easier than a standard ε-δ proof.

Theorem 4 *If functions $f, g: \mathbb{R} \to \mathbb{R}$ are continuous, then so is their composition $g \circ f: \mathbb{R} \to \mathbb{R}$.*

Proof. Let $U \in \mathcal{T}$. By Theorem 2 it is enough to prove that $(g \circ f)^{-1}(U) \in \mathcal{T}$. By Theorem 3(a), $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Now, $W = g^{-1}(U) \in \mathcal{T}$ by the continuity of g and Theorem 2. Therefore, by the continuity of f (and Theorem 2 used once again), $(g \circ f)^{-1}(U) = f^{-1}(W) \in \mathcal{T}$, as required. ∎

The same proof will work for arbitrary continuous functions defined via a general notion of defined below. (See section 12 in the text.)

Proof of Theorem 2. (a)$\Longrightarrow$(b): Fix an $x \in \mathbb{R}$ and an $\varepsilon > 0$. Using (a), we need to find a δ satisfying (b).

Let $U = B(f(x), \varepsilon) = (f(x) - \varepsilon, f(x) + \varepsilon)$. Notice that $U \in \mathcal{T}$. (This requires checking, that U satisfies the definition of sets in $\mathcal{T}$.) So, by (a),

$f^{-1}(U) \in \mathcal{T}$. Note also, that $x \in f^{-1}(U)$, as $f(x) \in (f(x)-\varepsilon, f(x)+\varepsilon) = U$. Therefore, we have $x \in f^{-1}(U) \in \mathcal{T}$ and, by the definition of $\mathcal{T}$, there is a $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(U)$. We show, that this δ satisfies (b).

Indeed, let $r \in \mathbb{R}$ be such that $|x-r| < \delta$. Then, $r \in (x-\delta, x+\delta) = B(x,\delta) \subset f^{-1}(U)$. Therefore, $f(r) \in U = (f(x)-\varepsilon, f(x)+\varepsilon)$ and so, $|f(x)-f(r)| < \varepsilon$, as required.

(b)$\Longrightarrow$(a): Fix a $U \in \mathcal{T}$. We need to show that $f^{-1}(U)$ is in $\mathcal{T}$. For this, take an $x \in f^{-1}(U)$. We need to find a $\delta > 0$ for which $B(x,\delta) \subset f^{-1}(U)$.

We have $f(x) \in U$, as $x \in f^{-1}(U)$. Since $U \in \mathcal{T}$, there exists an $\varepsilon > 0$ for which $B(f(x),\varepsilon) \subset U$. Using (b) for this x and ε, we can find a $\delta > 0$ such that $|f(x)-f(r)| < \varepsilon$ provided $|x-r| < \delta$. We will show that for this choice of δ we indeed have $B(x,\delta) \subset f^{-1}(U)$.

To see this, take an $r \in B(x,\delta) = (x-\delta, x+\delta)$. We need to show that $r \in f^{-1}(U)$. Since $r \in (x-\delta, x+\delta)$, we have $|x-r| < \delta$. So, by the choice of δ, $|f(x)-f(r)| < \varepsilon$. In particular, $f(r) \in (f(x)-\varepsilon, f(x)+\varepsilon) = B(f(x),\varepsilon) \subset U$. Thus, $r \in f^{-1}(U)$, as required. ∎

Reading assignment: Read Sections 1-7.

It is assumed that you are familiar with the material presented there. Therefore, we will not cover this material in class. (If necessary, we will be reviewing these notion on "as needed" basis.)

Written assignment: Write for the next class:

Exercise 1 *Let $f\colon X \to Y$ be an arbitrary function and $A, B \subset X$. Prove, or give a counterexample, for the following statements:*

(a) $f[A \cup B] = f[A] \cup f[B]$

(b) $f[A \cap B] = f[A] \cap f[B]$

Class of August 22:

What we covered last class: For $x \in \mathbb{R}$ and $\varepsilon > 0$ we define an open ball

$$B(x, \varepsilon) = \{r \in \mathbb{R} \colon |x - r| < \varepsilon\} = (x - \varepsilon, x + \varepsilon).$$

Let $\mathcal{T}$ be the family of all subsets U of $\mathbb{R}$ such that for every $x \in U$ there is an $\varepsilon > 0$ such that $x \in B(x, \varepsilon) \subset U$:

$$\mathcal{T} = \{U \subset \mathbb{R} \colon \forall x \in U \exists \varepsilon > 0 (B(x, \varepsilon) \subset U)\}.$$

We will refer to $\mathcal{T}$ as the *standard topology* on $\mathbb{R}$ and its elements $U \in \mathcal{T}$ will be called *open sets*.

We proved

Let $f \colon \mathbb{R} \to \mathbb{R}$. The following two definitions of continuity of f are equivalent:

(a) (Topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.

(b) (ε-δ definition) For every $x \in \mathbb{R}$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in \mathbb{R}$, if $|x - r| < \delta$, then $|f(x) - f(r)| < \varepsilon$.

New material:

Definition 1 Let X be an arbitrary set having at least two elements. A *topology* on X is any family $\mathcal{T}$ of subsets of X having the following properties:

(1) $\emptyset, X \in \mathcal{T}$.

(2) The union of any subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$.

(3) The intersection of any *finite* subfamily of $\mathcal{T}$ is in $\mathcal{T}$, that is, $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

The pair $\langle X, \mathcal{T} \rangle$ is called a *topological space*. For a fixed topological space $\langle X, \mathcal{T} \rangle$, the sets belonging to the family $\mathcal{T}$ will be refereed to as the *open sets* (with respect to this topology).

In the above definition, we used the following notation:

- Arbitrary unions and intersections of sets: Let $\mathcal{A}$ be a family of sets, say $\mathcal{A} = \{A_t \colon t \in T\}$. Then $\bigcup \mathcal{A} = \bigcup_{t \in T} A_t$ denotes the same set: $\{x \colon \exists A \in \mathcal{A} (x \in A)\}$, that is, $\{x \colon \exists t \in T (x \in A_t)\}$.

- Similarly, $\bigcap \mathcal{A} = \bigcap_{t\in T} A_t$ denotes the same set: $\{x\colon \forall A \in \mathcal{A}(x \in A)\}$, that is, $\{x\colon \forall t \in T(x \in A_t)\}$.

Remark 5 In the definition, condition (3) can be replaced with

(3') The intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$, that is, if $U, V \in \mathcal{T}$, the also $U \cap V \in \mathcal{T}$.

Proof. Easy induction. ∎

Example 6 *Here are some examples of topological spaces $\langle X, \mathcal{T}\rangle$, where X is an arbitrary set.*

- $\mathcal{T} = \mathcal{P}(X)$, *where $\mathcal{P}(X)$ is the* **power set of** X, *that is, the family of all subsets of X. This topology is called the* **discrete topology**.
- $\mathcal{T} = \{\emptyset, X\}$. *This topology is called* **trivial** *or* **indiscrete topology**.
- **The standard topology** $\mathcal{T}$ *on $\mathbb{R}$, defined for Theorem 2.*

More examples:

Example 7 *Examples of topologies on a set X:*

- *For a three elements set $X = \{a, b, c\}$, there are many different possible topologies. (Nine are indicated in Example 1, page76). E.g. $\{\emptyset, \{a\}, \{a, b\}, X\}$. Other examples from the text, section 12.*
- **Finite complement topology** $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F\colon F$ *is finite*$\}$. *Notice that $\langle X, \mathcal{T}_f\rangle$ is discrete, for finite X.*
- **Countable complement topology** $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F\colon F$ *is countable*$\}$. *Notice that $\langle X, \mathcal{T}_C\rangle$ is discrete, for countable X.*

Definition of *finer* and *coarser* topologies.
Solved Ex 1 page 83.
Proof that $f^{-1}(\bigcup_{t\in T} A_t) = \bigcup_{t\in T} f^{-1}(A_t)$:

$$\begin{aligned}
x \in f^{-1}(\bigcup_{t\in T} A_t) \quad &\Leftrightarrow \quad f(x) \in \bigcup_{t\in T} A_t \quad \text{(by the definition of preimage)}\\
&\Leftrightarrow \quad \exists t \in T\ f(x) \in A_t \quad \text{(by the definition of union)}\\
&\Leftrightarrow \quad \exists t \in T\ x \in f^{-1}(x) A_t \quad \text{(by the definition of preimage)}\\
&\Leftrightarrow \quad x \in \bigcup_{t\in T} f^{-1}(A_t) \quad \text{(by the definition of union).}
\end{aligned}$$

Class of August 24:

Recall that a *topology* on X is a family $\mathcal{T}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$;

(2) $\bigcup \mathcal{U} \in \mathcal{T}$ for every $\mathcal{U} \subset \mathcal{T}$;

(3) $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite $\mathcal{U} \subset \mathcal{T}$.

Examples of topological spaces $\langle X, \mathcal{T} \rangle$:

- **Discrete topology** $\mathcal{T} = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the **power set of** X.
- **Trivial** or **indiscrete topology** $\mathcal{T} = \{\emptyset, X\}$.
- **The standard topology** $\mathcal{T}$ on $\mathbb{R}$, defined for Theorem 2.
- **Finite complement topology** $\mathcal{T}_f = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}$. Notice that $\langle X, \mathcal{T}_f \rangle$ is discrete, for finite X.
- **Countable complement topology** $\mathcal{T}_C = \{\emptyset\} \cup \{X \setminus F : F \text{ is countable}\}$. Notice that $\langle X, \mathcal{T}_C \rangle$ is discrete, for countable X.

New material:

Section 13: Basis for a Topology

Definition 2 *Basis — Two related definitions*

FROM A BASIS TO TOPOLOGY — **Basis for a topology:** A collection $\mathcal{B}$ of a subsets of a set X such that

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

[FROM A TOPOLOGY TO ITS BASIS — **Basis for a given topology $\mathcal{T}$:**

Let $\langle X, \mathcal{T} \rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\underline{\mathcal{B} \subset \mathcal{T}}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

The first of these notion is used to create new topologies. The second is used to easier deal with a given, fixed topology $\mathcal{T}$. This second notion is used considerably more often than the first one.

Fact 1 *If $\mathcal{B}$ satisfies (B1) and (B2), then the family*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\} = \left\{\bigcup \mathcal{U} : \mathcal{U} \subset \mathcal{B}\right\}$$

is a topology on X. The family $\mathcal{B}$ is a basis to the topology $\mathcal{T}(\mathcal{B})$.

Fact 2 *(Lemma 13.2) If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}(\mathcal{B})$.*

Discuss examples 1–3.
Go over Lemma 13.3.

There may be more than one basis for a given topology: Example 4 (from Examples 1 and 2).

Example 8 *Two examples of topologies on $\mathbb{R}$:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st} \rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell \rangle$.*

Written assignment for Tuesday, August 29: Exercise 8, page 83. (In part (b), do not forget to prove, that $\mathcal{T}(\mathcal{C})$ is indeed a topology. Do you need to prove, in part (a), that $\mathcal{T}(\mathcal{B})$ is a topology?)

Bonus question to part (b): What if we replace family $\mathcal{C}$ with the family $\mathcal{C}^* = \{[a, b) : a < b,$ and a and b are irrational$\}$? How $\mathcal{T}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C}^*)$ compare to each other?

Be ready for a quiz next class time!

Class of August 29:

Recall that (rephrasing):

Basis for a given topology $\mathcal{T}$: Let $\langle X, \mathcal{T}\rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\underline{\mathcal{B} \subset \mathcal{T}}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Fact 3 *For a collection $\mathcal{B}$ of subsets of X, let*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X \colon \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\}.$$

If $\mathcal{B}$ satisfies the following two conditions:

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

then $\mathcal{T}(\mathcal{B})$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}(\mathcal{B})$.

Restate Lemma 13.3.

(Partially) new material:

Example 9 *Three examples of topologies on $\mathbb{R}$, defined via bases:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) \colon a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st}\rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) \colon a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell\rangle$.*

- **K-topology** *$\mathcal{T}_K$: Let $K = \{1/n \colon n = 1, 2, 3, \ldots\}$. Then $\mathcal{T}_K$ is generated by basis $\mathcal{B}_K = \mathcal{B}_{st} \cup \{(a, b) \setminus K \colon a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_K = \mathcal{T}(\mathcal{B}_K)$. We usually write $\mathbb{R}_K$ for $\langle \mathbb{R}, \mathcal{T}_K\rangle$.*

Fact 4 *(Lemma 13.4) $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are strictly finer than $\mathcal{T}_{st}$.*

Definition of *subbasis* for a topology.

Note that $\mathcal{S} = \{(a, \infty) \colon a \in \mathbb{R}\} \cup \{(-\infty, b) \colon b \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}$ (with the standard topology).

Go over exercises 3 and 6. Possibly, also exercises 4, 5, and/or 7.

Class of September 1:

Recall that (rephrasing):

Basis for a given topology $\mathcal{T}$: Let $\langle X, \mathcal{T}\rangle$ be a fixed topological space. A basis for $\mathcal{T}$ is any collection $\mathcal{B} \subset \mathcal{T}$ such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Fact 5 *For a collection $\mathcal{B}$ of subsets of X, let*

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\}.$$

If $\mathcal{B}$ satisfies the following two conditions:

(B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

(B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

then $\mathcal{T}(\mathcal{B})$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}(\mathcal{B})$.

Restate Lemma 13.3.

(Partially) new material:

Example 10 *Three examples of topologies on $\mathbb{R}$, defined via bases:*

- **Standard topology**, *generated by basis $\mathcal{B}_{st} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, that is, the topology $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st})$. We usually write just $\mathbb{R}$ for $\langle \mathbb{R}, \mathcal{T}_{st}\rangle$.*

  Notice, that this is the same topology that was used in Theorem 2.

- **Lower limit (or Sorgenfrey) topology** *$\mathcal{T}_\ell$ is generated by basis $\mathcal{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_\ell = \mathcal{T}(\mathcal{B}_\ell)$. We usually write $\mathbb{R}_\ell$ for $\langle \mathbb{R}, \mathcal{T}_\ell\rangle$.*

- **K-topology** *$\mathcal{T}_K$: Let $K = \{1/n : n = 1, 2, 3, \ldots\}$. Then $\mathcal{T}_K$ is generated by basis $\mathcal{B}_K = \mathcal{B}_{st} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$, that is, $\mathcal{T}_K = \mathcal{T}(\mathcal{B}_K)$. We usually write $\mathbb{R}_K$ for $\langle \mathbb{R}, \mathcal{T}_K\rangle$.*

Fact 6 *(Lemma 13.4) $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are strictly finer than $\mathcal{T}_{st}$.*

Definition of *subbasis* for a topology.

Note that $\mathcal{S} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}$ (with the standard topology).

Go over exercises 6 and 7, page 83.

Look at home (not homework) at Ex 3. Possibly, also exercises 4 and 5.

Class of August 31:

Next class: Quiz #2. Also, it will be the last day to give corrections to Homework #1, as I will give you my solution.

Go briefly over:

Section 14: Order Topology: For linearly ordered set $\langle X, \leq \rangle$, order topology is generated by subbasis $\mathcal{S} = \{(a, \infty) : a \in X\} \cup \{(-\infty, b) : b \in X\}$.

Describe basis for X. (Definition, page 84.)

Go over examples 1-4.

Section 15: Product Topology on $X \times Y$

Definition 3 For topological spaces $\langle X, \mathcal{T}_1 \rangle$ and $\langle Y, \mathcal{T}_2 \rangle$ let $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ be the family of all open rectangles, that is,

$$\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2} = \{U \times V : U \in \mathcal{T}_1 \ \& \ V \in \mathcal{T}_2\}.$$

Note that $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ satisfies conditions (B1) and (B2) for a topology on $X \times Y$. So, the family $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2})$ is a topology on $X \times Y$.

The topology $\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2})$ is called the *product topology* on $X \times Y$.

Note that, in general,

$$\mathcal{T}(\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}) \neq \mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2},$$

since, $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$ is not closed under unions, as, usually, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is not a rectangle (so, itdoes not belong to $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2}$).

Theorem 11 *If $\mathcal{B}_1$ is a basis for $\langle X, \mathcal{T}_1 \rangle$ and $\mathcal{B}_2$ is a basis for $\langle Y, \mathcal{T}_2 \rangle$, then the family*

$$\mathcal{B}_{\mathcal{B}_1,\mathcal{B}_2} = \{U \times V : U \in \mathcal{B}_1 \ \& \ V \in \mathcal{B}_2\}$$

is a basis for the product topology on $X \times Y$.

Corollary 12 *(Example 1) The family $\mathcal{B} = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$ is a basis for the product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is considered with the standard topology. Thus, the product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the standard topology $\mathcal{T}(\mathcal{B})$ on $\mathbb{R} \times \mathbb{R}$.*

Definition 4 For the Cartesian product $X_1 \times X_2$ define the *projection function* $\pi_1\colon X_1 \times X_2 \to X_1$ onto the first coordinate as $\pi_1(x_1, x_2) = x_1$. Similarly, the projection onto the second coordinate is the function $\pi_2\colon X_1 \times X_2 \to X_2$ defined as $\pi_2(x_1, x_2) = x_2$.

Notice that for $U \subset X_1$ and $V \subset X_2$ we have

$$\pi_1^{-1}(U) = U \times X_2 \quad \text{and} \quad \pi_2^{-1}(V) = X_1 \times V.$$

In particular, for topological spaces $\langle X_1, \mathcal{T}_1 \rangle$ and $\langle X_2, \mathcal{T}_2 \rangle$, the family

$$\mathcal{S} = \{\pi_i^{-1}(W)\colon i \in \{1, 2\} \ \&\ W \in \mathcal{T}_i\}$$

forms a subbasis for the product topology on $X_1 \times X_2$, since we have the identity $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$.

Written assignment for Tuesday, September 5: Exercise 6, page 92.

Class of September 5:

Administer Quiz # 2. Hand solution to Homework #1. Recall that:

- For the topological spaces $\langle X, \mathcal{T}_1\rangle$ and $\langle Y, \mathcal{T}_2\rangle$, the *product topology* on $X \times Y$ is generated by a basis: $\mathcal{B}_{\mathcal{T}_1,\mathcal{T}_2} = \{U \times V : U \in \mathcal{T}_1 \ \& \ V \in \mathcal{T}_2\}$.
- If $\mathcal{B}_1$ is a basis for $\langle X, \mathcal{T}_1\rangle$ and $\mathcal{B}_2$ is a basis for $\langle Y, \mathcal{T}_2\rangle$, then the family $\mathcal{B}_{\mathcal{B}_1,\mathcal{B}_2} = \{U \times V : U \in \mathcal{B}_1 \ \& \ V \in \mathcal{B}_2\}$ is a basis for the product topology on $X \times Y$.

Section 16: Subspace Topology

Definition 5 Let $\langle X, \mathcal{T}\rangle$ be a topological space and Y be any subset of X (containing at least two points). Then the family

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y called the *subspace topology.*

Lemma 13 *If $\mathcal{B}$ is a basis for a topological space $\langle X, \mathcal{T}\rangle$ and $Y \subset X$, then the family*

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

is a basis for $\langle Y, \mathcal{T}_Y\rangle$.

Go over Lemma 16.2 and Example 1.

Discuss briefly Theorem 16.4.

Theorem 14 *(Theorem 16.3) Let $\langle A, \mathcal{T}_A\rangle$ be a subspace of $\langle X, \mathcal{T}_1\rangle$ and $\langle B, \mathcal{T}_B\rangle$ be a subspace of $\langle Y, \mathcal{T}_2\rangle$. Then the following two topologies on $A \times B$ coincide:*

- *$\mathcal{T}_{A\times B}$, the subspace topology of the product topology on $X \times Y$;*
- *$\mathcal{T}(\mathcal{B}_{\mathcal{T}_A,\mathcal{T}_B})$, the product topology for the spaces $\langle A, \mathcal{T}_A\rangle$ and $\langle B, \mathcal{T}_B\rangle$.*

Go over Examples 2 & 3 and discuss Theorem 14.
Go over Exercises 1 and 4.

Written assignment for Tuesday, September 12: Ex. 10, p. 92. You can use in your solution, without a proof, results from Ex. 9, p. 92. Also:

Exercise 2 *For $\theta \in \mathbb{R}$ let L_θ be the line given by equation $y = \theta x$. Describe, for every $\theta \in \mathbb{R}$, the topology L_θ inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.*

Class of September 7:

Next class I will hand you solutions for homework assigned on 8/24 and 8/31.

Recall that:

- If $\langle X, \mathcal{T} \rangle$ is a topological space and $Y \subset X$, then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is the *subspace topology* on Y.
- If $\mathcal{B}$ is a basis for a topological space $\langle X, \mathcal{T} \rangle$ and $Y \subset X$, then the family $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for $\langle Y, \mathcal{T}_Y \rangle$.

New material:

(Ex. 9. p. 92) Show that the dictionary order topology $\mathcal{T}_{\preceq}$ on $\mathbb{R} \times \mathbb{R}$ coincides with the product topology $\mathcal{T}_{ds}$ of $\mathbb{R}_d \times \mathbb{R}$. Compare this topology with the standard topology $\mathcal{T}_{st}$ on $\mathbb{R}^2$.

Proof. In the proof, we will use the following two facts, mentioned many times in class. (For notation, see lecture for Section 13.)

(i) If $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{P}(X)$, then $\mathcal{T}(\mathcal{B}_0) \subset \mathcal{T}(\mathcal{B}_1)$.

(ii) If $\mathcal{T}_0$ is a topology (on X), then $\mathcal{T}(\mathcal{T}_0) = \mathcal{T}_0$.

Property (i) holds, as $\mathcal{T}(\mathcal{B}_0) = \{\bigcup \mathcal{B} : \mathcal{B} \subset \mathcal{B}_0\} \subset \{\bigcup \mathcal{B} : \mathcal{B} \subset \mathcal{B}_1\} = \mathcal{T}(\mathcal{B}_1)$. Property (ii) holds, since the family $\mathcal{B}_0 = \mathcal{T}_0$ is a basis for $\mathcal{T}_0$, and so $\mathcal{T}(\mathcal{T}_0) = \mathcal{T}(\mathcal{B}_0) = \mathcal{T}_0$.

Next, notice that, by Thm 15.1, $\mathcal{B}_{ds} = \{\{x\} \times (p, q) : x, p, q \in \mathbb{R}\}$ is a basis for $\mathcal{T}_{ds}$. Also, $\mathcal{B}_{\preceq} = \{(\langle a, b \rangle, \langle c, d \rangle) : a, b, c, d \in \mathbb{R}\}$ is a basis for $\mathcal{T}_{\preceq}$, where $(\langle a, b \rangle, \langle c, d \rangle) = \{\langle x, y \rangle \in \mathbb{R}^2 : \langle a, b \rangle \prec \langle x, y \rangle \prec \langle c, d \rangle\}$. (Here $\preceq$ is the dictionary order on $\mathbb{R} \times \mathbb{R}$.)

To show $\mathcal{T}_{ds} \subset \mathcal{T}_{\preceq}$, notice that $\mathcal{B}_{ds} \subset \mathcal{B}_{\preceq}$, as $\{x\} \times (p, q) = (\langle x, p \rangle, \langle x, q \rangle) \in \mathcal{B}_{\preceq}$. Therefore, by (i), $\mathcal{T}_{ds} = \mathcal{T}(\mathcal{B}_{ds}) \subset \mathcal{T}(\mathcal{B}_{\preceq}) = \mathcal{T}_{\preceq}$.

To show $\mathcal{T}_{\preceq} \subset \mathcal{T}_{ds}$, first notice that $\mathcal{B}_{\preceq} \subset \mathcal{T}_{ds}$. So, take a non-empty $(\langle a, b \rangle, \langle c, d \rangle) \in \mathcal{B}_{\preceq}$. If $a = c$, then $(\langle a, b \rangle, \langle c, d \rangle) = \{a\} \times (b, d) \in \mathcal{B}_{ds} \subset \mathcal{T}_{ds}$. Otherwise $a < c$ and $(\langle a, b \rangle, \langle c, d \rangle)$ is a union of the following sets from $\mathcal{T}_{ds}$: $\{a\} \times (b, \infty)$, $\{c\} \times (-\infty, d)$, and $\{z\} \times \mathbb{R}$, where $a < z < c$. Therefore, once again, $(\langle a, b \rangle, \langle c, d \rangle) \in \mathcal{B}_{ds} \subset \mathcal{T}_{ds}$. Hence, indeed, $\mathcal{B}_{\preceq} \subset \mathcal{T}_{ds}$.

Now, $\mathcal{T}_{\preceq} \subset \mathcal{T}_{ds}$ follows from (i) and (ii): $\mathcal{T}_{\preceq} = \mathcal{T}(\mathcal{B}_{\preceq}) \subset \mathcal{T}(\mathcal{T}_{ds}) = \mathcal{T}_{ds}$.

To finish the exercise, we will show that $\mathcal{T}_{st} \subsetneq \mathcal{T}_{ds}$. Indeed, to see the inclusion, recall that the family $\mathcal{B}_{st} = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$ is a basis for $\mathcal{T}_{st}$. Also, any set from $\mathcal{B}_{st}$ belongs to the standard basis $\mathcal{B}_{pr}$ for $\mathbb{R}_d \times \mathbb{R}$: $\mathcal{B}_{pr} = \{U \times V : U$ open in $\mathbb{R}_d$ and V open in $\mathbb{R}\}$. Therefore, $\mathcal{B}_{st} \subset \mathcal{B}_{pr} \subset \mathcal{T}_{ds}$ and, by (i) and (ii), $\mathcal{T}_{st} = \mathcal{T}(\mathcal{B}_{st}) \subset \mathcal{T}(\mathcal{T}_{ds}) = \mathcal{T}_{ds}$.

To see that the inclusion is strict, it is enough to notice that, for example, a set $W = \{0\} \times (0, 1)$ belongs to $\mathcal{T}_{ds}$ but it does not belong to $\mathcal{T}_{st}$. ∎

Section 17: Closed sets; Closure and Interior of a Set

Definition 6 A set $A \subset X$ is *closed* in the topological space $\langle X, \mathcal{T}\rangle$ if its complement $X \setminus A$ is open.

Go over Examples 1-5.
Go over Theorem 17.1.
Go over Exercise 1.

Theorem 15 *(Theorem 17.2) Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y iff $A = Y \cap F$ for some closed subset F of X.*

Go over Theorem 17.3.
Go over Exercises 2, 3, and 4.

Class of September 12:

Recall, from the last class:

- A set $A \subset X$ is *closed* in the topological space $\langle X, \mathcal{T} \rangle$ if its complement $X \setminus A$ is open.
- (Theorem 17.2) Let Y be a subspace of X. Then, $A \subset Y$ is closed in Y iff $A = Y \cap F$ for some closed subset F of X.

New material:

Definition 7 Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$.

- The *interior* of A, denoted as $\text{int}(A)$, is defined as a union of all open subsets contained in A, that is, $\text{int}(A) = \bigcup\{U \in \mathcal{T} : U \subset A\}$.

  Notice that $\text{int}(A)$ is open and that it is the largest open subset of A.
- The *closure* of A, denoted either as $\text{cl}(A)$ or as $\bar{A}$, is defined as an intersection of all closed subsets containing in A, that is, $\text{cl}(A) = \bigcap\{F \supset A : F \text{ is closed in } X\}$.

  Notice that $\text{cl}(A)$ is closed and that it is the smallest closed set containing A.

We will sometimes use symbols $\text{int}_X(A)$ and $\text{cl}_X(A)$ in place of $\text{int}(A)$ and $\text{cl}(A)$ to stress that the operation is with respect to the given topology on X.

Go over Exercise 6(a) and (b).

Theorem 16 *(Theorem 17.4) Let Y be a subspace of X and $A \subset Y$. Then $\text{cl}_Y(A) = Y \cap \text{cl}_X(A)$.*

Theorem 17 *(Theorem 17.5) Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T} \rangle$ and $\mathcal{B}$ be a basis for X. Then*

$$x \in \text{cl}(A) \text{ if, and only if, } A \cap B \neq \emptyset \text{ for every } B \in \mathcal{B} \text{ with } x \in B.$$

In particular, the result is true with $\mathcal{B} = \mathcal{T}$.

Read Examples 6 and 7. Try to solve Ex 9.

Let $A = K \cup (2, 3)$, where $K = \{1/n : n \in \{1, 2, 3, \ldots\}\}$. Find the closures of A in: $\mathbb{R}$ (i.e., $\mathbb{R}$ with the standard topology), $\mathbb{R}_\ell$, $\mathbb{R}_d$ (i.e., $\mathbb{R}$ with the discrete topology), and $\mathbb{R}_K$.

Answer: $\text{cl}_{\mathbb{R}}(A) = \{0\} \cup K \cup [2, 3]$; $\text{cl}_{\mathbb{R}_\ell}(A) = \{0\} \cup K \cup [2, 3)$; $\text{cl}_{\mathbb{R}_d}(A) = A$; $\text{cl}_{\mathbb{R}_K}(A) = K \cup [2, 3]$;

Written assignment due Thursday, Sept. 14: Ex 8(b) & 17, p. 101.

Class of September 14:

Administer Q #3. Recall, from the last class:

- The *interior* of A is $\operatorname{int}(A) = \bigcup\{U \in \mathcal{T} \colon U \subset A\}$.
- The *closure* of A is $\operatorname{cl}(A) = \bigcap\{F \supset A \colon F \text{ is closed in } X\}$.
- If Y be is subspace of X and $A \subset Y$, then $\operatorname{cl}_Y(A) = Y \cap \operatorname{cl}_X(A)$.
- If $A \subset X$ and $\mathcal{B}$ is a basis for X, then

$$x \in \operatorname{cl}(A) \text{ if, and only if, } A \cap B \neq \emptyset \text{ for every } B \in \mathcal{B} \text{ with } x \in B.$$

New material: State Ex 9.

Section 17, Limit Points

Definition 8 Let $A \subset X$ be a subset of a topological space $\langle X, \mathcal{T}\rangle$. A point $x \in X$ is a *limit point* (or *accumulation point*) of A provided $x \in \operatorname{cl}(A \setminus \{x\})$. The set of all limit points of A is denoted as A'.

Go over Example 8.

Theorem 18 *(Theorem 17.6) Let A be a subset of a topological space $\langle X, \mathcal{T}\rangle$. Then $\operatorname{cl}(A) = A \cup A'$.*

Theorem 19 *(Theorem 17.7) Let A be a subset of a topological space $\langle X, \mathcal{T}\rangle$. Then A is closed in X if, and only if, $A' \subset A$.*

Section 17: Hausdorff spaces

Definition 9 Let $\langle X, \mathcal{T}\rangle$ be a topological space. We say that:

- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).

Notice that if X is T_2 then it is also T_1, and if X is T_1 then it is also T_0.

Examples:

- A space X with a trivial topology $\mathcal{T} = \{\emptyset, X\}$ is not T_0.
- $X = \{0, 1\}$ with a topology $\mathcal{T} = \{\emptyset, \{0\}, X\}$ is T_0 but not T_1.
- $X = \mathbb{R}$ with a cofinite topology $\mathcal{T} = \{\emptyset\} \cup \{X \setminus F \colon F \text{ is finite}\}$ is T_1 but not T_2.
- The following spaces are T_2: any space with the discrete topology, $\mathbb{R}$ with the standard topology, $\mathbb{R}_\ell$, $\mathbb{R}_K$.

Theorem 20 *(Exercise 15) A space X is T_1 if, and only if, every finite subset of X is closed.*

Corollary 21 *(Theorem 17.8) Every finite subset in a Hausdorff space is closed.*

Theorem 22 *(Theorem 17.9) Let X be a T_1 topological space and $A \subset X$. Then $x \in A'$ if, and only if, $U \cap A$ is infinite for every open U containing x.*

Definition 10 Let X be a topological. We say that $x \in X$ is an *isolated point* provided $\{x\}$ is open in X.

Remark 23 If X is T_1 and an open set U is finite, then every $x \in U$ is isolated.

Definition 11 Let X be a topological. We say that a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to an $x \in X$ provided for every open set U containing x there exists an N such that $x_n \in U$ for every $n \geq N$.

If this is the case, we say also, that x is a *limit* of a sequence $\langle x_n \rangle_{n=1}^{\infty}$.

Theorem 24 *(Theorem 17.10) If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X* converges *to at most one point in X.*

Example: (Exercise 14) Theorem 22 (17.10) is false for T_1 spaces. For example, if $X = \mathbb{R}$ is considered with the cofinite topology (which is T_1) and $x_n = 1/n$ for every n, then every real number is a limit of $\langle x_n \rangle_{n=1}^{\infty}$.

Ex. 11. p. 101: Show that the product of two Hausdorff spaces is Hausdorff.

SOLUTION: Let X and Y be Hausdorff. Let $p_1 = \langle x_1, y_1 \rangle$ and $p_2 = \langle x_2, y_2 \rangle$ be distinct points from $X \times Y$. We need to find disjoint open subsets W_1 and W_2 of $X \times Y$ such that $p_1 \in W_1$ and $p_2 \in W_2$.

If $x_1 \neq x_2$, then, since X is Hausdorff, there are disjoint open subsets U_1 and U_2 of X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then, $W_1 = U_1 \times Y$ and $W_2 = U_2 \times Y$ are as desired.

If $x_1 = x_2$, then $y_1 \neq y_2$, since $p_1 \neq p_2$. Then, since Y is Hausdorff, there are disjoint open subsets V_1 and V_2 of Y such that $y_1 \in V_1$ and $y_2 \in V_2$. Then, $W_1 = X \times V_1$ and $W_2 = X \times V_2$ are as desired. ∎

Class of September 19:

Administer Quiz #4.
Recall that for a topological space $\langle X, \mathcal{T} \rangle$ and $A \subset X$:

- $A' = \{x \in X : x \in \operatorname{cl}(A \setminus \{x\})\}$.
- X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$. Equivalently, X is T_1 if, and only if, every singleton is closed in X.
- If X is a T_1 space and $A \subset X$, then

  $x \in A'$ if, and only if, $U \cap A$ is infinite for every open U containing x.
- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- A sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to an $x \in X$ provided for every open $U \ni x$ there exists an N such that $x_n \in U$ for every $n \geq N$.
- (Theorem 17.10) If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.
- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.

New material
Ex. 13 page 101: Show that X is a Hausdorff space if, and only if, the **diagonal** $\Delta = \{\langle x, x \rangle : x \in X\}$ *is closed in* $X^2 = X \times X$.

Solution: It is enough to prove that

- X is a Hausdorff if, and only if, $\Delta^c = X^2 \setminus \Delta$ is open in X^2.

"$\Longrightarrow$" Let $z = \langle x, y \rangle \in \Delta^c$. It is enough to show that there exists an open $W \subset X^2$ such that $z \in W \subset \Delta^c$.

Indeed, $x \neq y$, since $\langle x, y \rangle \in \Delta^c$. So, by Hausdorff property, there exists disjoint open sets $U \ni x$ and $V \ni y$. Let $W = U \times V$. Then, W is open and $z = \langle x, y \rangle \in W$. Moreover, if $\langle a, b \rangle \in W = U \times V$, then $a \neq b$, as $U \cap V = \emptyset$. In particular, $\langle a, b \rangle \in \Delta^c$. Therefore, $z \in W \subset \Delta^c$, as required.

"$\Longleftarrow$" Choose distinct $x, y \in X$. Then, $\langle x, y\rangle \in \Delta^c$. Since Δ^c is open, there exists a basic open set $U \times V$ (i.e., U and V open in X) such that $\langle x, y\rangle \in U \times V \subset \Delta^c$. Clearly $x \in U$ and $y \in V$. It is enough to prove that $U \cap V = \emptyset$.

Indeed, if $U \cap V \neq \emptyset$, then there exists an $a \in U \cap V$. However, this is impossible, since then $\langle a, a\rangle \in (U \times V) \cap \Delta$, contradicting the fact that $U \times V \subset \Delta^c$. ■

Go over Exercise 10.

Section 18: Continuous functions

Definition 12 Let X and Y be the topological spaces. A function $f\colon X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.

Notice, that the definition agrees with (a) from Theorem 2.

Theorem 25 *Let X and Y be the topological spaces and $\mathcal{B}$ a basis for Y. Then $f\colon X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.*

Similarly, if $\mathcal{S}$ is a subbasis for Y, then $f\colon X \to Y$ is continuous if, and only if, $f^{-1}(S)$ is open in X for every $S \in \mathcal{S}$.

Example 3:

- $f\colon \mathbb{R} \to \mathbb{R}_\ell$, $f(x) = x$, is discontinuous, as $f^{-1}([0,1)) = [0,1)$ is not open in $\mathbb{R}$.
- $f\colon \mathbb{R}_\ell \to \mathbb{R}$ is continuous, as $f^{-1}(U) = U \in \mathcal{T}_{st} \subset \mathcal{T}_\ell$ for every $U \in \mathcal{T}_{st}$.

Go over Exercise 3(a).
Go over Theorem 18.1. (Very important!)

Class of September 21:

Hand out solutions for all remaining homework assignments.
Recall that:

- If $\mathcal{B}$ a basis for Y, then $f: X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.
- Restate conditions (1)–(4) from Theorem 18.1. Recall we proved that (1)-(3) are equivalent.

New material

Finish the proof of Theorem 18.1: (4) equivalent to other conditions.
Stress continuity at a point, (4)
Go over Exercises 2 and 6.

Section 18: Homeomorphisms

Definition 13 Let X and Y be the topological spaces and let $f: X \to Y$ be a bijection (i.e., one-to-one and onto). Then f is a *homeomorphism* (from X onto Y) provided both f and $f^{-1}: Y \to X$ are continuous.

Topological spaces X and Y are *homeomorphic* provided there is a homeomorphism from X onto Y.

Fact. *If $f: X \to Y$ is homeomorphism, then $U \subset X$ is open in X, if, and only if, $f[U]$ is open in Y. In particular, if τ is a topology on X and $\mathcal{T}$ is a topology on Y, then $\mathcal{T} = \{f[U]: U \in \tau\}$ and $\tau = \{f^{-1}[V]: V \in \mathcal{T}\}$.*

Proof. Notice that $(f^{-1})^{-1} = f$.

If $U \in \tau$, then, since $f^{-1}: Y \to X$ is continuous, $f[U] = (f^{-1})^{-1}(U) \in \mathcal{T}$.
If $f[U] \in \tau$, then, since $f: X \to Y$ is continuous, $U = f^{-1}(f[U]) \in \mathcal{T}$. ■

Go over Examples 4-6.

A mapping $f: X \to Y$ is an *imbedding* provided f is injective (i.e., one-to-one), continuous, and $f^{-1}: f[X] \to X$ is also continuous. In such a case a mapping $f': X \to f[X]$, $f'(x) = f(x)$, is a homeomorphism (from X onto $f[X]$.

Go over Exercises 4 and 5.

Class of September 26:

In class mid term test will be on Tuesday, October 3

If everybody agrees, we will start test 15-30 minutes earlier, that is, between 6:00 and 6:15pm. You will be able to take up to 2 hours to complete it.

Key elements to review for the test: (1) Any homework assignment. (2) All definitions. (3) All theorems, with special emphasis on: continuous functions, closure and interior operations, Hausdorff and T_1 spaces, as well as subspaces and product spaces. (4) The exercises from the text.

Next class we will have review preparing for the mid term test. I plan to go over one of my previous mid term tests for this course.

Recall that:

- Spaces X and Y are *homeomorphic* provided there exists a *homeomorphism* $f\colon X \to Y$, that is, a bijection such that both f and $f^{-1}\colon Y \to X$ are continuous.
- **Fact.** *If $f\colon \langle X, \tau\rangle \to \langle Y, \mathcal{T}\rangle$ is a homeomorphism, then $U \in \tau$, if, and only if, $f[U] \in \mathcal{T}$. In particular, $\mathcal{T} = \{f[U]\colon U \in \tau\}$ and $\tau = \{f^{-1}[V]\colon V \in \mathcal{T}\}$.*
- $f\colon X \to Y$ is an *imbedding* provided f is injective (i.e., one-to-one), continuous, and $f^{-1}\colon f[X] \to X$ is also continuous.

Section 18: Constructing Continuous Functions

Go over Theorem 18.2.

Go over Theorem 18.3 (The pasting Lemma).

Go over Example 8; Theorem 18.4, and Exercise 11.

Variant of Exercise 12, with $f(x, y) = \frac{xy^2}{x^2+y^4}$ for $\langle x, y\rangle \neq \langle 0, 0\rangle$ and $f(0, 0) = 0$. Show that it is discontinuous (on curve $y^2 = x$), but $f \restriction L$ is continuous for every straight line L.

Try to solve at home (not for credit) Exercises 10 and 13 page 112.

Class of September 28:

Review for the mid term test: Go over the following test.

Topology Math 581 NAME (print): __________
Instr. K. Ciesielski
Fall 2016

Mid Term Test

Solve each of the following exercises, each on a separate page.

Ex. 1. Let X and Y be topological spaces and $f\colon X \to Y$. Consider the following two conditions (each of which is equivalent to the continuity of f). Provide a *direct* proof that (a) implies (b). (By *direct* I mean: do not use any theorems about continuous functions, unless you prove them.)

(a) $f^{-1}(U)$ is open in X for every open set U in Y.

(b) $f^{-1}(F)$ is closed in X for every closed set F in Y.

PROOF. Let F be closed in Y. Then $Y \setminus F$ is open in Y. So, by (a), $f^{-1}(Y \setminus F)$ is open in X. But $f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$. So, $X \setminus f^{-1}(F)$ is open in X, that is, $f^{-1}(F)$ is closed in X.

Ex. 2. Show that if X is a T_1 topological space, then so is X^2, where $X^2 = X \times X$ is considered with the product topology. Include definition of a T_1 topological space.

PROOF. Essentially the same as the proof for T_2 topological spaces.

Ex. 3. Let $\langle X, \mathcal{T}\rangle$ be a topological space and, for $A \subset X$, let $\operatorname{int}(A)$ denote the interior of A in X.

(a) Show that $A \subset B \subset X$ implies that $\operatorname{int}(A) \subset \operatorname{int}(B)$. Include the definition of $\operatorname{int}(A)$.

(b) Prove, using (a), that $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$ for every $A, B \subset X$.

(c) Give an example showing that, in general, the equation $\operatorname{int}(A \cup B) = \operatorname{int}(A) \cup \operatorname{int}(B)$ need not hold. (Specify sets A and B and a topological space of your example.)

Proof. (a): straight from the definition; (b): easy.

For (c) consuder $\mathbb{R}$ with the standard topology and its subsets: $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus A$.

Ex. 4. Let f be a function from a topological space X into a topological space Y. Let $\mathcal{F} = \{F_i : i = 1, \ldots, n\}$ be a finite family of closed subsets of X such that $X = \bigcup_{i=1}^n F_i$. Show that if the restriction $f \restriction F_i : F_i \to Y$ of f to F_i is continuous for every $i \in \{1, \ldots, n\}$, then f is continuous.

Proof. Let C be an arbitrary closed subset of Y. Then

$$f^{-1}(C) = f^{-1}(C) \cap \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} (F_i \cap f^{-1}(C)) = \bigcup_{i=1}^{n} (F_i \cap (f \restriction F_i)^{-1}(C))$$

is closed in X, since each set $F_i \cap (f \restriction F_i)^{-1}(C)$ is closed in F_i, and so, also in X, as F_i is closed in X.

Ex. 5. Let X be a Hausdorff topological space. Show that if $f : X \to X$ is continuous, then $G(f) = \{\langle x, f(x) \rangle \in X \times X : x \in X\}$, the graph of f, is a closed subset of X^2, where $X^2 = X \times X$ is considered with the product topology.

Proof. Let $W = X^2 \setminus G(f)$. It is enough to show that W is open in X^2. So, choose $\langle x, y \rangle \in W$. It is enough to show that there exists an open set B in X^2 such that $\langle x, y \rangle \in B \subset W$.

Indeed, since $\langle x, y \rangle \notin G(f)$, we have $y \neq f(x)$. Since X is a Hausdorff space, there are open disjoint subsets V_0 and V_1 of X, with $f(x) \in V_0$ and $y \in V_1$. Since f is continuous, $U_0 = f^{-1}(V_0)$ is open in X. Clearly, $x \in f^{-1}(V_0) = U_0$, so $\langle x, y \rangle$ is in $B = U_0 \times V_1$. Since B is clearly open in X^2, to finish the proof it is enough to show that $B \subset W$, that is, that $B = f^{-1}(V_0) \times V_1$ is disjoint from $G(f)$.

Indeed, if $\langle a, b \rangle \in f^{-1}(V_0) \times V_1$, then $f(a) \in V_0$. As, $b \in V_1$ and $V_1 \cap V_1 = \emptyset$, we conclude that $b \neq f(a)$. Therefore, $\langle a, b \rangle \notin G(f)$, as required.

Class of October 3: In class Mid Term Test.

Class of October 5:

Hand the results of the test and the solutions for its exercises.

Discuss test results and, in general, the course standings.

Discuss, in details, solutions of test problems.

Class of October 10:

New material:

Section 19: The product topology

Definition 14 For sets J and X let X^J denotes the family of all functions $f\colon J \to X$.

Let $\{A_\alpha\}_{\alpha\in J}$ be an arbitrary indexed family of sets and let $X = \bigcup_{\alpha\in J} A_\alpha$. (Notice that the index set J may be uncountable!) The *Cartesian product* of the family $\{A_\alpha\}_{\alpha\in J}$, denoted by $\prod_{\alpha\in J} A_\alpha$, is defined as

$$\prod_{\alpha\in J} A_\alpha = \{f \in X^J\colon f(j) \in A_j \text{ for all } j \in J\}.$$

Elements of $\{A_\alpha\}_{\alpha\in J}$ will be also sometimes denotes as $\langle a_\alpha\rangle_{\alpha\in J}$ and referred to as *J-tuples*.

Notice that $X^J = \prod_{\alpha\in J} A_\alpha$, where $A_\alpha = X$ for every $\alpha \in J$.

Notice, that this definition agrees the definition of the finite cartesian product (over the set $J = \{1,\dots,n\}$) $\prod_{i=1}^n A_i = A_1 \times \cdots \times A_n$ as the set of all sequences $\langle a(1),\dots,a(n)\rangle$ with $a(i) \in A_i$, since any such sequence can be considerd as a function $a\colon \{1,\dots,n\} \to X$. Similar agreement is also for $J = \{1,2,3,\dots\}$.

Definition 15 Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Then, on the product space $X = \prod_{\alpha \in J} X_\alpha$, we define the following two kinds of topologies.

box topology $\mathcal{T}_{box}$: Generated by a basis $\mathcal{B}_{box}$ formed by all sets of the form

$$\prod_{\alpha \in J} U_\alpha \quad \text{where each } U_\alpha \text{ is open in } X_\alpha.$$

product topology $\mathcal{T}_{prod}$: Generated by a subbasis $\mathcal{S}$ formed by all sets of the form

$$\pi_\beta^{-1}(U_\beta) \text{ for all } \beta \in J \text{ and open subsets } U_\beta \text{ of } X_\beta,$$

where $\pi_\beta \colon X \to X_\beta$ is the *projection* onto βth coordinate, that is, defined as $\pi_\beta(x) = x(\beta)$.

Notice that $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} U_\alpha$, where $U_\alpha = X_\alpha$ for all $\alpha \neq \beta$.

A natural basis, $\mathcal{B}_{prod}$ associated with $\mathcal{S}$ is formed by finite intersections of sets from $\mathcal{S}$, that is, all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α and *the set* $\{\alpha \in J \colon U_\alpha \neq X_\alpha\}$ *is finite.*

Go over Theorem 19.6:

Theorem 26 *Let $X = \prod_{\alpha \in J} X_\alpha$. If $f_\alpha \colon A \to X_\alpha$ and $f \colon A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*

- *continuity of f implies the continuity of each f_α;*
- *continuity of all f_α's implies the continuity of $f \colon A \to \langle X, \mathcal{T}_{prod} \rangle$.*

Go over Example 2.

Go over Theorem 19.2: bases for $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$ in term of basis for X_α's.

State Theorem 19.3: subspace topology on $A = \prod_{\alpha \in J} A_\alpha \subset X$.

Theorem 19.4: product of Hausdorff spaces is Hausdorff ($\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$).

Class of October 12:

Recall that for $X = \prod_{\alpha \in J} X_\alpha$, each X_α being a topological space,

- *Box topology* $\mathcal{T}_{box}$ on X is generated by basis $\mathcal{B}_{box} = \{\prod_{\alpha \in J} U_\alpha\colon$ each U_α is open in $X_\alpha\}$.
- *Product topology* $\mathcal{T}_{prod}$ on X is generated by subbasis $\mathcal{S}_{prod} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$ or, equivalently, by a basis $\mathcal{B}_{prod} = \{\prod_{\alpha \in J} U_\alpha \in \mathcal{B}_{box}\colon\ U_\alpha = X_\alpha$ for all but finitely many $\alpha\}$.
- $\mathcal{T}_{prod} \subset \mathcal{T}_{box}$; equation holds when J is finite (or all but finitely many spaces X_α have trivial topology $\{\emptyset, X_\alpha\}$);
- *If $f_\alpha\colon A \to X_\alpha$ and $f\colon A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*

  continuity of f implies the continuity of each f_α;

  continuity of all f_α's implies the continuity of $f\colon A \to \langle X, \mathcal{T}_{prod}\rangle$;

  continuity of all f_α's does not imply continuity of $f\colon A \to \langle X, \mathcal{T}_{box}\rangle$

  (as $f\colon \mathbb{R} \to \langle \mathbb{R}^\omega, \mathcal{T}_{box}\rangle$, $f(x) = \langle x, x, x, \ldots\rangle$ is discontinuous).

New material

Go over Theorem 19.5: $\prod_{\alpha \in J} \mathrm{cl}(A_\alpha) = \mathrm{cl}\left(\prod_{\alpha \in J} A_\alpha\right)$ (in $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$).

Solve Exercise 7.

Written assignment for Tuesday, October 17: Exercise 8, page 118: *Given sequences $\langle a_1, a_2, \ldots\rangle$ and $\langle b_1, b_2, \ldots\rangle$ of real numbers with $a_i > 0$ for all i, define $h\colon \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation*

$$h(\langle x_1, x_2, \ldots\rangle) = \langle a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots\rangle. \tag{1}$$

Show that if $\mathbb{R}^\omega$ is given the product topology, then h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Section 20: The Metric Topology

Define a *metric* (*distance*) on X as a function $d\colon X \times X \to [0, \infty)$.

A *metric space* is a pair $\langle X, d \rangle$, where d is a metric on X.

In a metric space $\langle X, d \rangle$, define an *open ball* (centered at $x \in X$ with radius $\varepsilon > 0$) as $B_d(x, \varepsilon) = \{y \in X\colon d(x, y) < \varepsilon\}$.

Prove that a family $\mathcal{B}_d = \{B(x, \varepsilon)\colon x \in X \ \& \ \varepsilon > 0\}$ is a basis for a topology on X.

Define a metric topology for a metric space $\langle X, d \rangle$ as $\mathcal{T}(\mathcal{B}_d)$, that is, as a topology generated by the family of all open balls in $\langle X, d \rangle$.

Go over Example 1 (discrete metric) and 2 (standard metric on $\mathbb{R}$).

Definition 16 A topological space $\langle X, \tau \rangle$ is *metrizable* provided there exists a metric d on X such that $\tau = \mathcal{T}(\mathcal{B}_d)$.

Class of October 17:

Recall that:

- A *metric space* is a pair $\langle X, d\rangle$, where d is a metric on X.
- $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ is an *open ball* in $\langle X, d\rangle$.
- $\mathcal{B}_d = \{B(x,\varepsilon) : x \in X \ \&\ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).
- $\langle X, \tau\rangle$ is *metrizable* provided $\tau = \mathcal{T}(\mathcal{B}_d)$ for some metric d on X.

New material

Go over Exercise 3(a): $d : X \times X \to \mathbb{R}$ is continuous in X^2, where X is considered with the metric topology.

Proof. Let $B = (a,b)$ be basic open set in $\mathbb{R}$. Need to prove that $d^{-1}(B)$ is open in X^2.

Fix $\langle x, y\rangle \in d^{-1}(B)$. So, $d(x,y) \in B$. We need to find an open set U in X^2 with $\langle x, y\rangle \in U \subset d^{-1}(B)$. Let $\varepsilon > 0$ be such that $(d(x,y)-\varepsilon, d(x,y)+\varepsilon) \subset B$. Define $U = B(x, \varepsilon/2) \times B(y, \varepsilon/2)$. It is open in X^2 and contains $\langle x, y\rangle$.

So, fix $\langle z, t\rangle \in U$. Then $d(x,z) < \varepsilon/2$ and $d(y,t) < \varepsilon/2$. By the triangle inequality we get $d(z,x) + d(x,y) + d(y,t) \geq d(z,t)$, so

$$d(z,x) + d(y,t) \geq d(z,t) - d(x,y).$$

Similarly, $d(x,z) + d(z,t) + d(t,y) \geq d(x,y)$, so

$$d(x,z) + d(t,y) \geq d(x,y) - d(z,t).$$

Hence, $|d(z,t) - d(x,y)| \leq d(x,z) + d(t,y) < \varepsilon/2 + \varepsilon/2$ and so we have $d(z,t) \in (d(x,y) - \varepsilon, d(x,y) + \varepsilon) \subset B$, as required.

Define: bounded set and its diameter.

Go over Theorem 20.1. (So, boundedness is not a topological property! Recall *topological property*, see page 105.)

Define Euclidean metric and square metric on $\mathbb{R}^n$.

Go over Theorem 20.3, using Lemma 20.2.

Class of October 19:

Recall

- Product and metric topologies on $\mathbb{R}^n$ coincide. (Review the proof.)

New material

Define uniform metric on $\mathbb{R}^J$.

State Theorem 20.4 (on relations between box, uniform, and product topologies on $\mathbb{R}^J$).

Go over Exercise 5, page 127. Note, that this implies that, on $\mathbb{R}^\omega$, box, uniform, and product topologies are distinct.

Class of October 24:

Recall

- uniform metric on $\mathbb{R}^J$ is defined as: $\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \colon \alpha \in J\}$, where $\bar{d}(x,y) = \min\{|x-y|, 1\}$
- uniform topology on $\mathbb{R}^J$: generated by $\bar{\rho}$.

New material

Restate and prove Theorem 20.4 (on relations between box, uniform, and product topologies on $\mathbb{R}^J$).

Go over Exercise 6.

Suggested to solve at home (no homework, yet): Exercise 4 page 127.

Go over Theorem 20.5 (countable product of metric spaces is metrizable).

Class of October 26:

Administer Quiz # 6. Announce and give material for Test # 2.

Section 21: The Metric Topology continued

- Subspace of a metric space is metric.
- No relation between ordered topologies and metric topologies.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.

State Theorem 21.1: for metric spaces, ε-δ definition of continuity is equivalent to topological definition of continuity. (This is an obvious generalization of Theorem 2.)

Definition 17 Let $\langle X, \tau \rangle$ be a topological space.

- A family $\mathcal{B}_x \subset \tau$ is a *basis (for X) at x* provided for every open set $U \ni x$ there is a $B \in \mathcal{B}_x$ with $x \in B \subset U$.
- A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Proposition 27 *Every metrizable space is first countable.*

Note that for first countable spaces, a countable basis $\{B_n : n = 1, 2, 3, \ldots\}$ can be chosen monotone: $B_1 \supset B_2 \supset B_3 \supset \cdots$.

Go over Lemma 21.2, version for first countable spaces:

Lemma 28 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

State Theorem 21.3, version for first countable spaces.

Bonus written assignment for no latter than Thursday, Nov. 2: Exercise 4 page 127.

Class of October 31:

Test #2 will be given on Tuesday, November 7. We can start at 6pm. The material for the test will be given on Thursday, November 2.

Administer Quiz # 7. Recall that:

- A topological space $\langle X, \mathcal{T} \rangle$ is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x, that is, $\mathcal{B}_x \subset \mathcal{T}$ and for every open set $U \ni x$ there is a $B \in \mathcal{B}_x$ with $x \in B \subset U$.

- **Lemma (21.2)** *Let X be a first countable topological space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

- **Theorem (21.3)** *Let X and Y topological spaces and let $f\colon X \to Y$. Assume also that X is first countable. Then f is continuous if, and only if, for every sequence $\langle x_n \rangle_n$ in X converging to an $x \in X$, $\langle f(x_n) \rangle_n$ converges to $f(x)$. Moreover, the implication "$\Longrightarrow$" does not require the assumption of first countability.*

New material

Prove Theorem 21.3, version for first countable spaces.

Go over Lemma 21.4 (no proof) and Theorem 21.5.

Definition 18 Let $\langle Y, d \rangle$ be a metric space, X any set, and $f_n\colon X \to Y$ be a sequence of functions. We say that the sequence $\langle f_n \rangle_n$ *converges uniformly* to an $f\colon X \to Y$ provided for every $\varepsilon > 0$ there exists an N (independent of x) such that for every $x \in X$

$$d(f_n(x), f(x)) < \varepsilon \quad \text{for all } n > N.$$

State Theorem 21.6: uniform limit of continuous functions is continuous.

Go over Exercise 6: *uniform* convergence assumption in Thm 21.6 is essential.

Prove Theorem 21.6.

Discuss Exercise 9: the implication in Theorem 21.6 cannot be reversed.

Go over Example 1: $\mathbb{R}^\omega$ with the box topology is not first countable. In particular, it is not metrizable.

Go over Example 2: uncountable product $\mathbb{R}^J$, considered with the product topology, is not first countable. In particular, it is not metrizable.

Class of November 2:

Administer Quiz # 8.

Test #2 will be given on Tuesday, November 7. We can start at 6pm. Hand the material for the test.

Suggested to solve at home (no homework, not difficult, but interesting): Exercise 7, p. 134.

Solve Ex. 4b page 127.

Is additional review needed?

Skip the rest of Chapter 2, that is, section 22.

Chapter 3: Connectedness and Compactness

Stress usability of these notions to the proofs of three classical calculus theorems: *Intermediate Value Theorem*, *Maximum Value Theorem*, and *Uniform Continuity Theorem*.

Intermediate Value Theorem is a consequence of *connectedness* property.

The other two theorems are the consequences of *compactess* property.

Class of November 7:

Administer Test #2. We will start at 6pm.

Class of November 9:

Discuss results of Test #2.

Section 23: Connected spaces

Definition 19 Let X be a topological space. A *separation* of X is any pair $\langle U, V\rangle$ of open, non-empty, disjoint sets for which $X = U \cup V$. A topological space X is *connected* provided it **does not** exist a separation of X.

Example 1: Any X with indiscrete topology is connected.

Any X with discrete topology is *disconnected*, that is, not connected.

Fact: A space is connected, when $\emptyset$ and X are its only subsets that are simultaneously closed and open.

Definition 20 Let Y be a subspace of X. A *separation* of Y is any pair $A, B \subset Y$ non-empty sets such that $Y = A \cup B$ and $\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset$.

Lemma 29 *A subspace Y of X is connected is, and only if, there is no separation of Y.*

Go over Examples 2, 3, 4, and 5.

Lemma 30 *Assume that sets C and D forms separation of X. If a subspace Y of X is connected, then either $Y \subset C$ or $Y \subset D$.*

Theorem 31 (Star Lemma) *Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.*

Theorem 32 (Theorem 23.4) *Let A be a connected subspace of X. If $A \subset B \subset \mathrm{cl}(A)$, then B is connected.*

Theorem 33 (Theorem 23.5) *Continuous image of connected space is connected.*

This, together with the fact that intervals are connected, is the Intermediate Value Theorem.

Class of November 14:

Recall

- A topological space X is *connected* provided it **does not** exist a separation of X, where a *separation* of X is any pair U, V of open, non-empty disjoint sets with $X = U \cup V$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.
- A closure of a connected space is connected.
- Continuous image of connected space is connected.

New material

Theorem 34 (Thm 23.6) *Finite product of connected spaces is connected.*

Actually, arbitrary product of connected spaces, considered with the product topology, is connected. We show this only for $\mathbb{R}^\omega$, Example 7. (In general, this is Exercise 10.)

Example 6: $\mathbb{R}^\omega$ with the box topology is disconnected.
Go over Exercises 2, 7, and 8, page 152.
Suggestion to students: Look over Exercises 3, 4, and 9, page 152.

Section 24: Connected spaces of the Real Line

Recall that $\mathbb{R}$ has the *least upper bound property* provided every non-empty bounded above subset A of $\mathbb{R}$ has an upper bound $\sup(A) \in \mathbb{R}$.

Theorem 35 (Theorem 24.1, for $\mathbb{R}$ only) *A subset A of $\mathbb{R}$ (considered with the standard topology) is connected if, and only if, A is an interval (possible degenerated).*

Class of November 16:

Recall

- Continuous image of connected space is connected.
- Finite product of connected spaces is connected.
- $\mathbb{R}^\omega$ with the box topology is disconnected, while with the product topology is connected.
- $A \subset \mathbb{R}$ is connected if, and only if, A is convex (an interval).

New material

Finish discussion of

(Theorem 24.1, for $\mathbb{R}$ only) *A subset A of $\mathbb{R}$ (considered with the standard topology) is connected if, and only if, A is an interval (possible degenerated).*

Go over the Intermediate Value Theorem, Theorem 24.3.

Define *path connectedness*.

Note that every path connected space is connected.

Go over Exercises 2 and 1.

Go over Examples 3, 4, and 5.

Go over Examples 7, *topologists sine curve*: it is connected but not path connected.

Class of November 28:

Recall

- Definition of *path connectedness*.
- Every path connected space is connected.
- *Topologists sine curve* is connected but not path connected.

Discuss again *topologists sine curve.*

Section 25: Define components and path components.
Explain Theorem 25.1.
Go over Examples 1 and 2.
Define locally connected spaces and locally path connected spaces.
Go over Example 3.
Go over Theorems 25.3 and 25.4.
Briefly discuss Exercise 10: quasi components.

Class of November 30:

Give material for the final test, which will be administered in class, on *Tuesday, December 5*, 2017, in the usual place. We will start at 6pm and you will have 2 hours to complete the test.

Review for the final test. Start with:
Exercises 8, **10**, and 11 page 158.

Class of December 5:

Administer the final test.