NAME (print):

MATH 251 Instr. K. Ciesielski Fall 2017

## SAMPLE TEST # 2 with SOLUTIONS

Solve the following exercises. Show your work.

**Ex. 1.** Find a vector equation of the line that passes through the point P(11, 13, -7) and is perpendicular to the plane with the equation: x - 2z = 17.

Solution: The direction vector  $\mathbf{v}$  of the line coincides with the normal vector of the plane:  $\langle 1, 0, -2 \rangle$ .

Answer: 
$$\langle x, y, z \rangle = \langle 11, 13, -7 \rangle + t \langle 1, 0, -2 \rangle$$
, or  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ -7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ .

**Ex. 2.** Find: (a) the *unit* tangent vector to the curve  $\mathbf{r}(t) = \langle e^t, t, \cos \pi t \rangle$  at the point (1,0,1), and (b) the vector equation of the line tangent to the same curve at the point (e, 1, -1).

Solution:  $\mathbf{r}'(t) = \langle e^t, 1, -\pi \sin \pi t \rangle.$ 

(a) The curve passes through the point (1, 0, 1) at the time t when  $\langle e^t, t, \cos \pi t \rangle = \langle 1, 0, 1 \rangle$ , that is, for t = 0. So,  $\mathbf{r}'(0) = \langle e^0, 1, -\pi \sin 0 \rangle = \langle 1, 1, 0 \rangle$  and  $|\mathbf{r}'(0)| = \sqrt{1 + 1 + 0} = \sqrt{2}$ . Thus, the unit tangent vector is equal  $\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{2}} \langle 1, 1, 0 \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle$ .

(b) The curve passes through the point (e, 1, -1) at the time t when  $\langle e^t, t, \cos \pi t \rangle = \langle e, 1, -1 \rangle$ , that is, for t = 1. So,  $\mathbf{r}'(1) = \langle e^1, 1, -\pi \sin \pi \rangle = \langle e, 1, 0 \rangle$  is the direction vector of the line.

Answer for (b):  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} e \\ 1 \\ 0 \end{bmatrix}.$ 

**Ex. 3.** Find the volume of the pyramid with the vertices: P(3, 2, -1), Q(-2, 5, 1), R(2, 1, 5), and the origin O(0, 0, 0). The volume of a pyramid is equal 1/6th of the volume of parallelepiped spanned by the same vectors.

Solution: We need three vectors indicating the pyramid. For this we can use the vectors  $\mathbf{a} = \vec{OP} = \langle 3, 2, -1 \rangle$ ,  $\mathbf{b} = \vec{OQ} = \langle -2, 5, 1 \rangle$ , and  $\mathbf{c} = \vec{OR} = \langle 2, 1, 5 \rangle$ . Now, the volume of parallelepiped indicated by these vectors is  $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ .

Since  $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{vmatrix} = \mathbf{i}(25-1) - \mathbf{j}(-10-2) + \mathbf{k}(-2-10) = \langle 24, 12, -12 \rangle$ , we have  $V = |\langle 3, 2, -1 \rangle \cdot \langle 24, 12, -12 \rangle| = |3 \cdot 24 + 2 \cdot 12 - 1 \cdot (-12)| = |12(6+2+1)| = 12 \cdot 9$ . Answer: The volume of the pyramid is  $V/6 = 12 \cdot 9/6 = 18$ .

**Ex. 4.** Find an equation of the plane passing through point (1, 11, -13) and parallel to the plane with equation  $2x - 17z + \pi = 0$ .

Solution: The normal of the given equation,  $2x - 17z = -\pi$ , is  $\langle 2, 0, -17 \rangle$ . The plane we seek has the same normal.

Answer: 2(x-1) + 0(y-11) - 17(z+13) = 0.

**Ex. 5.** Describe and sketch the graphs of the surfaces given by the following equations. Name each surface. Give specific informations, like center and radius in the case of a sphere.

- (a)  $2x^2 + 2y^2 + 2z^2 = 7x + 9y + 11z$ Solution: The equation is equivalent to:  $x^2 + y^2 + z^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{11}{2}z = 0$ . Completing to the square is  $x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 + y^2 - \frac{9}{2}y + \left(\frac{9}{4}\right)^2 + z^2 - \frac{11}{2}z + \left(\frac{11}{4}\right)^2 = \left(\frac{7}{4}\right)^2 + \left(\frac{9}{4}\right)^2 + \left(\frac{11}{4}\right)^2$ , that is,  $\left(x - \frac{7}{4}\right)^2 + \left(y - \frac{9}{4}\right)^2 + \left(z - \frac{11}{4}\right)^2 = \frac{49 + 81 + 121}{4^2}$ . Since 49 + 81 + 121 = 251, Answer: Sphere, with the center  $\left(\frac{7}{4}, \frac{9}{4}, \frac{11}{4}\right)$  and radius  $\frac{\sqrt{251}}{4}$ .
- (b)  $4y = x^2 + z^2$

Answer: Circular paraboloid, revolving around the y-axis, opening towards the positive side of the y-axis. Sketch: to be presented in class.

(c)  $4y = z^2$ 

Answer: Cylinder, based on a parabola on yz-plane, opening towards the positive side of the *y*-axis. The lines forming cylinder are parallel to the *x*-axis. Sketch: to be presented in class.

**Ex. 6.** Find the curvature  $\kappa(t)$  of the curve with position vector  $\mathbf{r}(t) = \mathbf{i} \cos t + \mathbf{j} \sin t + 2t \mathbf{k}$ . Solution: Recall that  $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ . Now, we have:

$$\begin{aligned} \mathbf{r}'(t) &= \mathbf{i}(-\sin t) + \mathbf{j}\cos t + 2 \mathbf{k}; \\ \mathbf{r}''(t) &= \mathbf{i}(-\cos t) + \mathbf{j}(-\sin t); \\ |\mathbf{r}'(t)| &= \sqrt{(-\sin t)^2 + (\cos t)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}; \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{i}(0+2\sin t) - \mathbf{j}(0+2\cos t) + \mathbf{k}(\sin^2 t + \cos^2 t); \\ |\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{k}| = \sqrt{4\sin^2 t + 4\cos^2 t + 1^2} = \sqrt{4+1} = \sqrt{5}. \\ \text{Answer: } \kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{5}}{(\sqrt{5})^3} = \frac{1}{5}. \end{aligned}$$

**Ex. 7.** Let  $\mathbf{v}(t) = \mathbf{i}(t+e)^{-1} + \mathbf{k} t^3$  be a velocity of a particle. Find the acceleration vector  $\mathbf{a}(t)$  of the particle and its position vector  $\mathbf{r}(t)$ , where its initial position was  $\mathbf{r}(0) = 3\mathbf{i}$ . Solution:  $\mathbf{a}(t) = \mathbf{v}'(t) = -(t+e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$ .  $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \mathbf{i} \ln |t+e| + \mathbf{k} t^4/4 + \vec{C}$ . To find  $\vec{C}$ , we calculate  $\mathbf{r}(0)$ :  $\mathbf{i} \ln |0+e| + \mathbf{k} 0^4/4 + \vec{C} = 3\mathbf{i}$ . Since  $\ln e = 1$ , we get  $\mathbf{i} + \vec{C} = 3\mathbf{i}$  and  $\vec{C} = 2\mathbf{i}$ . Therefore  $\mathbf{r}(t) = \mathbf{i} \ln |t+e| + \mathbf{k} t^4/4 + 2\mathbf{i} = (2 + \ln |t+e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$ . Answer:  $\mathbf{a}(t) = -(t+e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$  and  $\mathbf{r}(t) = (2 + \ln |t+e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$ . **Ex. 8.** Find the arc length, s, of the curve with position vector  $\mathbf{r}(t) = 2e^t \mathbf{i} + 2t \mathbf{j} + e^{-t} \mathbf{k}$  from t = 0 to t = 1.

Solution:  $s = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 |2e^t \mathbf{i} + 2\mathbf{j} - e^{-t} \mathbf{k}| dt = \int_0^1 \sqrt{(2e^t)^2 + 2^2 + (e^{-t})^2} dt.$ Rearranging, we get  $s = \int_0^1 \sqrt{(2e^t)^2 + 2(2e^t)(e^{-t}) + (e^{-t})^2} dt = \int_0^1 \sqrt{(2e^t + e^{-t})^2} dt.$  Thus,  $s = \int_0^1 (2e^t + e^{-t}) dt = [2e^t - e^{-t}]_0^1 = (2e^1 - e^{-1}) - (2e^0 - e^0) = 2e - e^{-1} - 1.$ Answer:  $s = 2e - \frac{1}{e} - 1.$ 

**Ex. 9.** Sketch and fully describe the graph of a function  $f(x, y) = \sqrt{1 + x^2 + y^2}$ . Solution: Substituting z for f(x, y) we get  $z = \sqrt{1 + x^2 + y^2}$ , or, equivalently,

 $z^2 = 1 + x^2 + y^2$  and  $z \ge 0$ . The equation transforms to  $-x^2 - y^2 + z^2 = 1$ , which is the hyperboloid of two sheets, revolving around z-axis. Since,  $z \ge 0$  we get:

Answer: The graph of a function f(x, y) is the upper half (above xy-plane) of the hyperboloid of two sheets  $-x^2 - y^2 + z^2 = 1$ . Sketch: to be presented in class.

**Ex. 10.** Sketch and fully describe the domain of the following function, including the name of the surface representing the domain's boundary:  $f(x, y, z) = \ln (25 - 4x^2 - 9y^2 - z^2)$ .

Solution: The argument of the logarithm must be positive:  $25 - 4x^2 - 9y^2 - z^2 > 0$ , that is,  $4x^2 + 9y^2 + z^2 < 25$ , or  $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} < 1$ .

Answer: The points inside the ellipsoid  $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} = 1$ . Sketch: to be presented in class.