MATH 251

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SAMPLE TEST # 3

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1. Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4 + 7y^4}$$

Solution:

On x-axis,
$$y = 0$$
: $L_1 = \lim_{x \to 0} \frac{x^3 \cdot 0}{x^4 + 0} = 0$.

On the line
$$y = x$$
: $L_2 = \lim_{x \to 0} \frac{x^3 \cdot x}{x^4 + 7x^4} = \lim_{x \to 0} \frac{x^4}{8x^4} = \frac{1}{8}$.

Answer: Limit does not exist as $L_1 \neq L_2$.

Ex. 2. Compute the first order partial derivatives of $f(x, y, z) = ze^{x^2} \cos y$.

Solution:

$$\frac{\partial f}{\partial x} = f_x = ze^{x^2}\cos y \cdot 2x = 2xze^{x^2}\cos y$$

$$\frac{\partial f}{\partial y} = f_y = ze^{x^2}(-\sin y) = -ze^{x^2}\sin y$$

$$\frac{\partial f}{\partial z} = f_z = e^{x^2} \cos y$$

Ex. 3. Compute all second order partial derivatives of $g(s,t) = e^{5t} + t\sin(3s)$.

Solution:

$$g_s = 3t\cos(3s)$$
 $g_{ss} = -9t\sin(3s)$ $g_{st} = 3\cos(3s)$

$$g_t = 5e^{5t} + \sin(3s)$$
 $g_{ts} = 3\cos(3s)$ $g_{tt} = 25e^{5t}$

Ex. 4. Find an equation of the plane tangent to the surface $z = x^2 - 5y^3$ at the point P(2, 1, -1).

Solution:

$$z_x = 2x;$$
 $z_x(P) = 2 \cdot 2 = 4;$

$$z_y = -15y^2;$$
 $z_y(P) = -15 \cdot 1^2 = -15;$

Normal vector
$$\mathbf{n} = \langle z_x(P), z_y(P), -1 \rangle = \langle 4, -15, -1 \rangle$$
.

Answer:
$$4(x-2) - 15(y-1) - 1(z+1) = 0$$
 or $4x - 15y - z + 6 = 0$.

Ex. 5. Find the absolute maximum and the absolute minimum of the function f(x,y) = $x^3 - xy$ on the region bounded below by parabola $y = x^2 - 1$ and above by line y = 3. You will get credit **only** if **all** critical points are found.

Solution: The curves intersect, when $x^2 - 1 = 3$, that is, when $x = \pm 2$.

Thus, we need to consider the region above x^2-1 and below 3 for x in the interval [-2,2].

Region's interior: $f_x(x,y) = 3x^2 - y$ and $f_y(x,y) = -x$. This leads to system $3x^2 - y = 0$ and -x=0, with only solution (x,y)=(0,0). This point belongs to the region. This is our first critical point.

Lower boundary: $y = x^2 - 1$ and $-2 \le x \le 2$. Then

$$g(x) = f(x, x^2 - 1) = x^3 - x(x^2 - 1) = x$$
 and $g'(x) = 1$ is never 0.

So, there are no true critical points, but we need to consider the endpoints of q, $x = \pm 2$. This give us the critical points $(x, y) = (\pm 2, 3)$.

Upper boundary: y = 3 and $-2 \le x \le 2$. Then

$$g(x) = f(x,3) = x^3 - 3x$$
 and $g'(x) = 3x^2 - 3$, which is 0 when $x = \pm 1 \in [-2,2]$.

This give us the critical points $(x,y)=(\pm 1,3)$. (Plus the end points $(x,y)=(\pm 2,3)$, considered above.)

Checking the critical points: f(0,0) = 0;

$$f(2,3) = 2^3 - 6 = 2$$
; $f(-2,3) = (-2)^3 + 6 = -2$; $f(1,3) = 1^3 - 3 = -2$; $f(-1,3) = (-1)^3 + 3 = 2$;

$$f(1,3) = 1^3 - 3 = -2; f(-1,3) = (-1)^3 + 3 = 2;$$

Answer: f has the absolute maximum value 2, at points (2,3) and (-1,3).

f has the absolute minimum value -2, at points (-2,3) and (1,3).

Ex. 6. Find the volume of the solid bounded above by the surface z=28xy, bounded below by xy-plane, and which is above the region bounded by $y = x^6$ and y = x.

Solution: The curves intersect, when $x^6 = x$, that is, when x = 0 and x = 1.

Thus, we need to find an integral above x^6 and below x, on the interval [0,1]:

$$\int_0^1 \int_{x^6}^x 28xy \ dy \ dx = \int_0^1 \left[14xy^2 \right]_{y=x^6}^{y=x} \ dx = \int_0^1 \left[14x^3 - 14x^{13} \right]_{y=x^6}^{y=x} \ dx = \left[\frac{14}{4}x^4 - x^{14} \right]_{x=0}^{x=1} = 2.5$$

Ex. 7. Evaluate $\int_0^1 \int_0^x 4e^{x^2} dy dx$

Solution: $\int_0^1 \int_0^x 4e^{x^2} dy dx = \int_0^1 \left[4e^{x^2}y \right]_{y=0}^{y=x} dx = \int_0^1 4(e^{x^2}x - e^{x^2}0) dx = \int_0^1 4e^{x^2}x dx$ Using substitution $v = x^2$, we obtain that it is equal $\left[2e^{x^2}\right]_{x=0}^{x=1} = 2(e^1 - e^0) = 2(e-1).$

Ex. 8. Find the point on the cone $z = \sqrt{x^2 + y^2}$ which is the closest to the point (4, -8, 0).

Solution: Distance of (x, y, z) on the surface from (4, -8, 0) is $\sqrt{(x-4)^2 + (y+8)^2 + (z-0)^2}$.

Since $z^2 = x^2 + y^2$, this is equal to

 $f(x,y) = \sqrt{(x-4)^2 + (y+8)^2 + (x^2+y^2)}.$ $f_x(x,y) = \frac{2(x-4)+2x}{2\sqrt{(x-4)^2 + (y+8)^2 + (x^2+y^2)}} \text{ and } f_y(x,y) = \frac{2(y+8)+2y}{2\sqrt{(x-4)^2 + (y+8)^2 + (x^2+y^2)}}.$ $f_x = 0 \text{ when } 2(x-4) + 2x = 0, \text{ that is, } 4x - 8 = 0, \text{ so } x = 2.$

 $f_y = 0$ when 2(y+8) + 2y, that is, 4y + 16 = 0, so y = -4.

This gives critical point (2, -4). Since these are the coordinates of a point on the cone, we get $z = \sqrt{2^2 + (-4)^2} = \sqrt{20}$.

Answer: Point $(2, -4, \sqrt{20})$

Ex. 9. Find the directional derivative of $f(x,y) = 10e^y \sin x$ at the point $P(\pi/4,0)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

Solution: The unit vector in the direction of \mathbf{v} is equal

 $\mathbf{u} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{1}{\sqrt{4^2 + (-3)^2}} \mathbf{v} = \frac{1}{5} \langle 4, -3 \rangle = \langle .8, -.6 \rangle.$

 $f_x(x,y) = 10e^y \cos x$; $f_x(P) = 10e^0 \cos(\pi/4) = 5\sqrt{2}$.

 $f_y(x,y) = 10e^y \sin x$; $f_x(P) = 10e^0 \sin(\pi/4) = 5\sqrt{2}$.

 $\nabla f(P) = \langle f_x(P), f_y(P) \rangle = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle.$

 $D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u} = \langle 5\sqrt{2}, 5\sqrt{2} \rangle \cdot \langle .8, -.6 \rangle = (5\sqrt{2})(.8) + (5\sqrt{2})(-.6) = \sqrt{2}.$

Ex. 10. Find the gradient of $g(x, y, z) = x^{2} + e^{yz} + \cos(x + 2y)$.

Solution: $\nabla g(x,y,z) = \langle g_x, g_y, g_z \rangle = \langle 2x - \sin(x+2y), ze^{yz} - 2\sin(x+2y), ye^{yz} \rangle$.