MATH 251 NAME (print): ____________________

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SAMPLE TEST # 2 with SOLUTIONS

Solve the following exercises. **Show your work.**

Ex. 1. Find a vector equation of the line that passes through the point $P(11, 13, -7)$ and is perpendicular to the plane with the equation: $x - 2z = 17$.

Solution: The direction vector $\mathbf{v}$ of the line coincides with the normal vector of the plane: $\langle 1, 0, -2\rangle$.

Answer: $\langle x, y, z\rangle = \langle 11, 13, -7\rangle + t\langle 1, 0, -2\rangle$, or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ -7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

Ex. 2. Find: (a) the *unit* tangent vector to the curve $\mathbf{r}(t) = \langle e^t, t, \cos \pi t\rangle$ at the point $(1, 0, 1)$, and (b) the vector equation of the line tangent to the same curve at the point $(e, 1, -1)$.

Solution: $\mathbf{r}'(t) = \langle e^t, 1, -\pi \sin \pi t\rangle$.

(a) The curve passes through the point $(1, 0, 1)$ at the time t when $\langle e^t, t, \cos \pi t\rangle = \langle 1, 0, 1\rangle$, that is, for $t = 0$. So, $\mathbf{r}'(0) = \langle e^0, 1, -\pi \sin 0\rangle = \langle 1, 1, 0\rangle$ and $|\mathbf{r}'(0)| = \sqrt{1+1+0} = \sqrt{2}$. Thus, the unit tangent vector is equal $\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\rangle$.

(b) The curve passes through the point $(e, 1, -1)$ at the time t when $\langle e^t, t, \cos \pi t\rangle = \langle e, 1, -1\rangle$, that is, for $t = 1$. So, $\mathbf{r}'(1) = \langle e^1, 1, -\pi \sin \pi\rangle = \langle e, 1, 0\rangle$ is the direction vector of the line.

Answer for (b): $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} e \\ 1 \\ 0 \end{bmatrix}$.

Ex. 3. Find the volume of the pyramid with the vertices: $P(3, 2, -1)$, $Q(-2, 5, 1)$, $R(2, 1, 5)$, and the origin $O(0, 0, 0)$. The volume of a pyramid is equal 1/6th of the volume of parallelepiped spanned by the same vectors.

Solution: We need three vectors indicating the pyramid. For this we can use the vectors $\mathbf{a} = \vec{OP} = \langle 3, 2, -1\rangle$, $\mathbf{b} = \vec{OQ} = \langle -2, 5, 1\rangle$, and $\mathbf{c} = \vec{OR} = \langle 2, 1, 5\rangle$. Now, the volume of parallelepiped indicated by these vectors is $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Since $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{vmatrix} = \mathbf{i}(25 - 1) - \mathbf{j}(-10 - 2) + \mathbf{k}(-2 - 10) = \langle 24, 12, -12\rangle$, we have $V = |\langle 3, 2, -1\rangle \cdot \langle 24, 12, -12\rangle| = |3 \cdot 24 + 2 \cdot 12 - 1 \cdot (-12)| = |12(6 + 2 + 1)| = 12 \cdot 9$.

Answer: The volume of the pyramid is $V/6 = 12 \cdot 9/6 = 18$.

Ex. 4. Find an equation of the plane passing through point $(1, 11, -13)$ and parallel to the plane with equation $2x - 17z + \pi = 0$.

Solution: The normal of the given equation, $2x - 17z = -\pi$, is $\langle 2, 0, -17\rangle$. The plane we seek has the same normal.

Answer: $2(x - 1) + 0(y - 11) - 17(z + 13) = 0$.

Ex. 5. Describe and sketch the graphs of the surfaces given by the following equations. Name each surface. Give specific informations, like center and radius in the case of a sphere.

(a) $2x^2 + 2y^2 + 2z^2 = 7x + 9y + 11z$

Solution: The equation is equivalent to: $x^2 + y^2 + z^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{11}{2}z = 0$. Completing to the square is $x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 + y^2 - \frac{9}{2}y + \left(\frac{9}{4}\right)^2 + z^2 - \frac{11}{2}z + \left(\frac{11}{4}\right)^2 = \left(\frac{7}{4}\right)^2 + \left(\frac{9}{4}\right)^2 + \left(\frac{11}{4}\right)^2$, that is, $\left(x - \frac{7}{4}\right)^2 + \left(y - \frac{9}{4}\right)^2 + \left(z - \frac{11}{4}\right)^2 = \frac{49+81+121}{4^2}$. Since $49 + 81 + 121 = 251$,

Answer: Sphere, with the center $\left(\frac{7}{4}, \frac{9}{4}, \frac{11}{4}\right)$ and radius $\frac{\sqrt{251}}{4}$.

(b) $4y = x^2 + z^2$

Answer: Circular paraboloid, revolving around the y-axis, opening towards the positive side of the y-axis. Sketch: to be presented in class.

(c) $4y = z^2$

Answer: Cylinder, based on a parabola on yz-plane, opening towards the positive side of the y-axis. The lines forming cylinder are parallel to the x-axis. Sketch: to be presented in class.

Ex. 6. Find the curvature $\kappa(t)$ of the curve with position vector $\mathbf{r}(t) = \mathbf{i}\cos t + \mathbf{j}\sin t + 2t\,\mathbf{k}$.

Solution: Recall that $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. Now, we have:

$\mathbf{r}'(t) = \mathbf{i}(-\sin t) + \mathbf{j}\cos t + 2\,\mathbf{k}$;

$\mathbf{r}''(t) = \mathbf{i}(-\cos t) + \mathbf{j}(-\sin t)$;

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$;

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{i}(0 + 2\sin t) - \mathbf{j}(0 + 2\cos t) + \mathbf{k}(\sin^2 t + \cos^2 t);$$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{k}| = \sqrt{4\sin^2 t + 4\cos^2 t + 1^2} = \sqrt{4+1} = \sqrt{5}$.

Answer: $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{5}}{(\sqrt{5})^3} = \frac{1}{5}$.

Ex. 7. Let $\mathbf{v}(t) = \mathbf{i}(t+e)^{-1} + \mathbf{k}\,t^3$ be a velocity of a particle. Find the acceleration vector $\mathbf{a}(t)$ of the particle and its position vector $\mathbf{r}(t)$, where its initial position was $\mathbf{r}(0) = 3\mathbf{i}$.

Solution: $\mathbf{a}(t) = \mathbf{v}'(t) = -(t+e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \mathbf{i}\ln|t+e| + \mathbf{k}\,t^4/4 + \vec{C}$. To find $\vec{C}$, we calculate $\mathbf{r}(0)$:

$\mathbf{i}\ln|0+e| + \mathbf{k}\,0^4/4 + \vec{C} = 3\mathbf{i}$. Since $\ln e = 1$, we get $\mathbf{i} + \vec{C} = 3\mathbf{i}$ and $\vec{C} = 2\mathbf{i}$. Therefore

$\mathbf{r}(t) = \mathbf{i}\ln|t+e| + \mathbf{k}\,t^4/4 + 2\mathbf{i} = (2 + \ln|t+e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$.

Answer: $\mathbf{a}(t) = -(t+e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$ and $\mathbf{r}(t) = (2 + \ln|t+e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$.

Ex. 8. Find the arc length, s, of the curve with position vector $\mathbf{r}(t) = 2e^t\ \mathbf{i} + 2t\ \mathbf{j} + e^{-t}\ \mathbf{k}$ from $t = 0$ to $t = 1$.

Solution: $s = \int_0^1 |\mathbf{r}'(t)|\ dt = \int_0^1 |2e^t\ \mathbf{i} + 2\ \mathbf{j} - e^{-t}\ \mathbf{k}|\ dt = \int_0^1 \sqrt{(2e^t)^2 + 2^2 + (e^{-t})^2}\ dt$. Rearranging, we get $s = \int_0^1 \sqrt{(2e^t)^2 + 2(2e^t)(e^{-t}) + (e^{-t})^2}\ dt = \int_0^1 \sqrt{(2e^t + e^{-t})^2}\ dt$. Thus, $s = \int_0^1 (2e^t + e^{-t})\ dt = [2e^t - e^{-t}]_0^1 = (2e^1 - e^{-1}) - (2e^0 - e^0) = 2e - e^{-1} - 1$.

Answer: $s = 2e - \frac{1}{e} - 1$.

Ex. 9. Sketch and fully describe the graph of a function $f(x, y) = \sqrt{1 + x^2 + y^2}$.

Solution: Substituting z for $f(x, y)$ we get $z = \sqrt{1 + x^2 + y^2}$, or, equivalently, $z^2 = 1 + x^2 + y^2$ and $z \geq 0$. The equation transforms to $-x^2 - y^2 + z^2 = 1$, which is the hyperboloid of two sheets, revolving around z-axis. Since, $z \geq 0$ we get:

Answer: The graph of a function $f(x, y)$ is the upper half (above xy-plane) of the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$. Sketch: to be presented in class.

Ex. 10. Sketch and fully describe the domain of the following function, including the name of the surface representing the domain's boundary: $f(x, y, z) = \ln(25 - 4x^2 - 9y^2 - z^2)$.

Solution: The argument of the logarithm must be positive: $25 - 4x^2 - 9y^2 - z^2 > 0$, that is, $4x^2 + 9y^2 + z^2 < 25$, or $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} < 1$.

Answer: The points inside the ellipsoid $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} = 1$. Sketch: to be presented in class.