

SAMPLE TEST # 3

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1. Show that the following limit does not exist

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y}{x^4 + 7y^4}$$

Solution:

$$\text{On } x\text{-axis, } y = 0: L_1 = \lim_{x \rightarrow 0} \frac{x^3 \cdot 0}{x^4 + 0} = 0.$$

$$\text{On the line } y = x: L_2 = \lim_{x \rightarrow 0} \frac{x^3 \cdot x}{x^4 + 7x^4} = \lim_{x \rightarrow 0} \frac{x^4}{8x^4} = \frac{1}{8}.$$

Answer: Limit does not exist as $L_1 \neq L_2$.

Ex. 2. Compute the first order partial derivatives of $f(x, y, z) = ze^{x^2} \cos y$.

Solution:

$$\frac{\partial f}{\partial x} = f_x = ze^{x^2} \cos y \cdot 2x = 2xze^{x^2} \cos y$$

$$\frac{\partial f}{\partial y} = f_y = ze^{x^2} (-\sin y) = -ze^{x^2} \sin y$$

$$\frac{\partial f}{\partial z} = f_z = e^{x^2} \cos y$$

Ex. 3. Compute all second order partial derivatives of $g(s, t) = e^{5t} + t \sin(3s)$.

Solution:

$$g_s = 3t \cos(3s) \quad g_{ss} = -9t \sin(3s) \quad g_{st} = 3 \cos(3s)$$

$$g_t = 5e^{5t} + \sin(3s) \quad g_{ts} = 3 \cos(3s) \quad g_{tt} = 25e^{5t}$$

Ex. 4. Find an equation of the plane tangent to the surface $z = x^2 - 5y^3$ at the point $P(2, 1, -1)$.

Solution:

$$z_x = 2x; \quad z_x(P) = 2 \cdot 2 = 4;$$

$$z_y = -15y^2; \quad z_y(P) = -15 \cdot 1^2 = -15;$$

$$\text{Normal vector } \mathbf{n} = \langle z_x(P), z_y(P), -1 \rangle = \langle 4, -15, -1 \rangle.$$

$$\text{Answer: } 4(x - 2) - 15(y - 1) - 1(z + 1) = 0 \quad \text{or} \quad 4x - 15y - z + 6 = 0.$$

Ex. 5. Find the absolute maximum and the absolute minimum of the function $f(x, y) = x^3 - xy$ on the region bounded below by parabola $y = x^2 - 1$ and above by line $y = 3$. You will get credit **only** if **all** critical points are found.

Solution: The curves intersect, when $x^2 - 1 = 3$, that is, when $x = \pm 2$.

Thus, we need to consider the region above $x^2 - 1$ and below 3 for x in the interval $[-2, 2]$.

Region's interior: $f_x(x, y) = 3x^2 - y$ and $f_y(x, y) = -x$. This leads to system $3x^2 - y = 0$ and $-x = 0$, with only solution $(x, y) = (0, 0)$. This point belongs to the region. This is our first critical point.

Lower boundary: $y = x^2 - 1$ and $-2 \leq x \leq 2$. Then

$$g(x) = f(x, x^2 - 1) = x^3 - x(x^2 - 1) = x \text{ and } g'(x) = 1 \text{ is never } 0.$$

So, there are no true critical points, but we need to consider the endpoints of g , $x = \pm 2$.

This give us the critical points $(x, y) = (\pm 2, 3)$.

Upper boundary: $y = 3$ and $-2 \leq x \leq 2$. Then

$$g(x) = f(x, 3) = x^3 - 3x \text{ and } g'(x) = 3x^2 - 3, \text{ which is } 0 \text{ when } x = \pm 1 \in [-2, 2].$$

This give us the critical points $(x, y) = (\pm 1, 3)$. (Plus the end points $(x, y) = (\pm 2, 3)$, considered above.)

Checking the critical points: $f(0, 0) = 0$;

$$f(2, 3) = 2^3 - 6 = 2; f(-2, 3) = (-2)^3 + 6 = -2;$$

$$f(1, 3) = 1^3 - 3 = -2; f(-1, 3) = (-1)^3 + 3 = 2;$$

Answer: f has the absolute maximum value 2, at points $(2, 3)$ and $(-1, 3)$.

f has the absolute minimum value -2 , at points $(-2, 3)$ and $(1, 3)$.

Ex. 6. Find the volume of the solid bounded above by the surface $z = 28xy$, bounded below by xy -plane, and which is above the region bounded by $y = x^6$ and $y = x$.

Solution: The curves intersect, when $x^6 = x$, that is, when $x = 0$ and $x = 1$.

Thus, we need to find an integral above x^6 and below x , on the interval $[0, 1]$:

$$\int_0^1 \int_{x^6}^x 28xy \, dy \, dx = \int_0^1 [14xy^2]_{y=x^6}^{y=x} \, dx = \int_0^1 [14x^3 - 14x^{13}]_{y=x^6}^{y=x} \, dx = \left[\frac{14}{4}x^4 - x^{14} \right]_{x=0}^{x=1} = 2.5$$

Ex. 7. Evaluate $\int_0^1 \int_0^x 4e^{x^2} dy dx$

Solution: $\int_0^1 \int_0^x 4e^{x^2} dy dx = \int_0^1 [4e^{x^2} y]_{y=0}^{y=x} dx = \int_0^1 4(e^{x^2} x - e^{x^2} 0) dx = \int_0^1 4e^{x^2} x dx$

Using substitution $v = x^2$, we obtain that it is equal

$$\left[2e^{x^2} \right]_{x=0}^{x=1} = 2(e^1 - e^0) = 2(e - 1).$$

Ex. 8. Find the point on the cone $z = \sqrt{x^2 + y^2}$ which is the closest to the point $(4, -8, 0)$.

Solution: Distance of (x, y, z) on the surface from $(4, -8, 0)$ is $\sqrt{(x-4)^2 + (y+8)^2 + (z-0)^2}$.

Since $z^2 = x^2 + y^2$, this is equal to

$$f(x, y) = \sqrt{(x-4)^2 + (y+8)^2 + (x^2 + y^2)}.$$

$$f_x(x, y) = \frac{2(x-4)+2x}{2\sqrt{(x-4)^2 + (y+8)^2 + (x^2 + y^2)}} \text{ and } f_y(x, y) = \frac{2(y+8)+2y}{2\sqrt{(x-4)^2 + (y+8)^2 + (x^2 + y^2)}}.$$

$$f_x = 0 \text{ when } 2(x-4) + 2x = 0, \text{ that is, } 4x - 8 = 0, \text{ so } x = 2.$$

$$f_y = 0 \text{ when } 2(y+8) + 2y, \text{ that is, } 4y + 16 = 0, \text{ so } y = -4.$$

This gives critical point $(2, -4)$. Since these are the coordinates of a point on the cone, we get $z = \sqrt{2^2 + (-4)^2} = \sqrt{20}$.

Answer: Point $(2, -4, \sqrt{20})$.

Ex. 9. Find the directional derivative of $f(x, y) = 10e^y \sin x$ at the point $P(\pi/4, 0)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

Solution: The unit vector in the direction of \mathbf{v} is equal

$$\mathbf{u} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{1}{\sqrt{4^2 + (-3)^2}} \mathbf{v} = \frac{1}{5} \langle 4, -3 \rangle = \langle .8, -.6 \rangle.$$

$$f_x(x, y) = 10e^y \cos x; f_x(P) = 10e^0 \cos(\pi/4) = 5\sqrt{2}.$$

$$f_y(x, y) = 10e^y \sin x; f_y(P) = 10e^0 \sin(\pi/4) = 5\sqrt{2}.$$

$$\nabla f(P) = \langle f_x(P), f_y(P) \rangle = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle.$$

$$D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = \langle 5\sqrt{2}, 5\sqrt{2} \rangle \cdot \langle .8, -.6 \rangle = (5\sqrt{2})(.8) + (5\sqrt{2})(-.6) = \sqrt{2}.$$

Ex. 10. Find the gradient of $g(x, y, z) = x^2 + e^{yz} + \cos(x + 2y)$.

Solution: $\nabla g(x, y, z) = \langle g_x, g_y, g_z \rangle = \langle 2x - \sin(x + 2y), ze^{yz} - 2 \sin(x + 2y), ye^{yz} \rangle.$