MATH 251
Instr. K. Ciesielski
Fall 2015

SAMPLE FINAL TEST
(longer than the actual Final Test)

Solve the following exercises. **Show your work.**

Ex. 1. ST #1 Ex 3: Find the determinant of the matrix. Each time you expand the the matrix, you **must** expand it over a row or column that has the largest number of zeros. If necessary, use the row (or column) reduction method to create additional zeros.

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

Solution: Sol: If we subtract from raw # 4 the raw # 2 and expand by the third column, we get

$$|A| = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 4 & 0 & 3 \end{vmatrix} = (-1)\cdot 1 \begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$$

Next, subtracting from raw # 3 three times the raw # 2 and expanding again by the third column, we get

$$|A| = (-1)\begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 2 & 0 \\ 1 & 2 & 1 \\ -4 & -2 & 0 \end{vmatrix} = (-1)(-1)\begin{vmatrix} -1 & 2 \\ -4 & -2 \end{vmatrix} = 2-(-8) = 10.$$

Ex. 2. ST #1 Ex 4: Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution: Sol: We need to transform $[A;I]$ to $[I;B]$. Then $B = A^{-1}$.

$$[A;I] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ +R_1 \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} \\ \\ \times -\frac{1}{4} \end{matrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{matrix} -R_3 \\ -3R_3 \\ \\ \end{matrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Answer: $A^{-1} = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$.

Ex. 3. ST #1 Ex 6: Let $\mathbf{a} = \langle 0, 1, 2\rangle$, $\mathbf{b} = \langle -1, 0, 7\rangle$, and $\mathbf{c} = \langle 2, 3, -1\rangle$. Evaluate: $2\mathbf{a} - \mathbf{b} + \mathbf{c}$, $|\mathbf{c}|$, and $(\mathbf{a} \cdot \mathbf{b})\ (\mathbf{b} \times \mathbf{c})$. (Do not confuse vectors with numbers. No partial credit for solutions with such errors.)

Solution: Sol:
$2\mathbf{a} - \mathbf{b} + \mathbf{c} = 2\langle 0, 1, 2\rangle - \langle -1, 0, 7\rangle + \langle 2, 3, -1\rangle = \langle 0, 2, 4\rangle + \langle 1, 0, -7\rangle + \langle 2, 3, -1\rangle = \langle 3, 5, -4\rangle$

$|\mathbf{c}| = \sqrt{2^4 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

As $\mathbf{a} \cdot \mathbf{b} = \langle 0, 1, 2\rangle \cdot \langle -1, 0, 7\rangle 0 + 0 + 14 = 14$ and
$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 7 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 7 \\ 3 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 7 \\ 2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} =$
$\mathbf{i}(0 - 21) - \mathbf{j}(1 - 14) + \mathbf{k}(-3 - 0) = \langle -21, 13, -3\rangle$, we have
$(\mathbf{a} \cdot \mathbf{b})\ (\mathbf{b} \times \mathbf{c}) = 14\ \langle -21, 13, -3\rangle = \langle -14 \cdot 21, 14 \cdot 13, -3 \cdot 14\rangle$.

Ex. 4. ST #2 Ex 1: Find a vector equation of the line that passes through the point $P(11, 13, -7)$ and is perpendicular to the plane with the equation: $x - 2z = 17$.

Solution: The direction vector $\mathbf{v}$ of the line coincides with the normal vector of the plane: $\langle 1, 0, -2\rangle$.

Answer: $\langle x, y, z\rangle = \langle 11, 13, -7\rangle + t\langle 1, 0, -2\rangle$, or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ -7 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

Ex. 5. ST #2 Ex 7: Let $\mathbf{v}(t) = \mathbf{i}(t + e)^{-1} + \mathbf{k}\ t^3$ be a velocity of a particle. Find the acceleration vector $\mathbf{a}(t)$ of the particle and its position vector $\mathbf{r}(t)$, where its initial position was $\mathbf{r}(0) = 3\mathbf{i}$.

Solution: $\mathbf{a}(t) = \mathbf{v}'(t) = -(t + e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$.
$\mathbf{r}(t) = \int \mathbf{v}(t)\ dt = \mathbf{i} \ln|t + e| + \mathbf{k}\ t^4/4 + \vec{C}$. To find $\vec{C}$, we calculate $\mathbf{r}(0)$:
$\mathbf{i} \ln|0 + e| + \mathbf{k}\ 0^4/4 + \vec{C} = 3\mathbf{i}$. Since $\ln e = 1$, we get $\mathbf{i} + \vec{C} = 3\mathbf{i}$ and $\vec{C} = 2\mathbf{i}$. Therefore
$\mathbf{r}(t) = \mathbf{i} \ln|t + e| + \mathbf{k}\ t^4/4 + 2\mathbf{i} = (2 + \ln|t + e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$.
Answer: $\mathbf{a}(t) = -(t + e)^{-2}\mathbf{i} + 3t^2\mathbf{k}$ and $\mathbf{r}(t) = (2 + \ln|t + e|)\mathbf{i} + \frac{t^4}{4}\mathbf{k}$.

Ex. 6. ST #2 Ex 10: Sketch and fully describe the domain of the following function, including the name of the surface representing the domain's boundary: $f(x, y, z) = \ln(25 - 4x^2 - 9y^2 - z^2)$.

Solution: Solution: The argument of the logarithm must be positive: $25 - 4x^2 - 9y^2 - z^2 > 0$, that is, $4x^2 + 9y^2 + z^2 < 25$, or $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} < 1$.

Answer: The points inside the ellipsoid $\frac{x^2}{(5/2)^2} + \frac{y^2}{(5/3)^2} + \frac{z^2}{5^2} = 1$. Sketch: to be presented in class.

Ex. 7. ST #3 Ex. 2: Compute the first order partial derivatives of $f(x,y,z) = ze^{x^2}\cos y$.

Solution:

$$\frac{\partial f}{\partial x} = f_x = ze^{x^2}\cos y \cdot 2x = 2xze^{x^2}\cos y$$
$$\frac{\partial f}{\partial y} = f_y = ze^{x^2}(-\sin y) = -ze^{x^2}\sin y$$
$$\frac{\partial f}{\partial z} = f_z = e^{x^2}\cos y$$

Ex. 8. ST #3 Ex. 3: Compute all second order partial derivatives of $g(s,t) = e^{5t} + t\sin(3s)$.

Solution:

$g_s = 3t\cos(3s)$ $\quad g_{ss} = -9t\sin(3s)$ $\quad g_{st} = 3\cos(3s)$

$g_t = 5e^{5t} + \sin(3s)$ $\quad g_{ts} = 3\cos(3s)$ $\quad g_{tt} = 25e^{5t}$

Ex. 9. ST #3 Ex. 4: Find an equation of the plane tangent to the surface $z = x^2 - 5y^3$ at the point $P(2,1,-1)$.

Solution:

$z_x = 2x$; $\quad z_x(P) = 2 \cdot 2 = 4$;

$z_y = -15y^2$; $\quad z_y(P) = -15 \cdot 1^2 = -15$;

Normal vector $\mathbf{n} = \langle z_x(P), z_y(P), -1\rangle = \langle 4, -15, -1\rangle$.

Answer: $4(x-2) - 15(y-1) - 1(z+1) = 0$ or $4x - 15y - z + 6 = 0$.

Ex. 10. ST #3 Ex. 8: Find the point on the cone $z = \sqrt{x^2+y^2}$ which is the closest to the point $(4,-8,0)$.

Solution:

Solution: Distance of (x,y,z) on the surface from $(4,-8,0)$ is $\sqrt{(x-4)^2+(y+8)^2+(z-0)^2}$. Since $z^2 = x^2+y^2$, this is equal to

$f(x,y) = \sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}$.

$f_x(x,y) = \frac{2(x-4)+2x}{2\sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}}$ and $f_y(x,y) = \frac{2(y+8)+2y}{2\sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}}$.

$f_x = 0$ when $2(x-4)+2x = 0$, that is, $4x - 8 = 0$, so $x = 2$.

$f_y = 0$ when $2(y+8)+2y$, that is, $4y+16 = 0$, so $y = -4$.

This gives critical point $(2,-4)$. Since these are the coordinates of a point on the cone, we get $z = \sqrt{2^2+(-4)^2} = \sqrt{20}$.

Answer: Point $(2,-4,\sqrt{20})$.

Ex. 11. ST #3 Ex. 5: Find the absolute maximum and the absolute minimum of the function $f(x, y) = x^3 - xy$ on the region bounded below by parabola $y = x^2 - 1$ and above by line $y = 3$. You will get credit **only** if **all** critical points are found.

Solution: The curves intersect, when $x^2 - 1 = 3$, that is, when $x = \pm 2$.
Thus, we need to consider the region above $x^2 - 1$ and below 3 for x in the interval $[-2, 2]$.

Region's interior: $f_x(x, y) = 3x^2 - y$ and $f_y(x, y) = -x$. This leads to system $3x^2 - y = 0$ and $-x = 0$, with only solution $(x, y) = (0, 0)$. This point belongs to the region. This is our first critical point.

Lower boundary: $y = x^2 - 1$ **and** $-2 \leq x \leq 2$**.** Then
$g(x) = f(x, x^2 - 1) = x^3 - x(x^2 - 1) = x$ and $g'(x) = 1$ is never 0.
So, there are no true critical points, but we need to consider the endpoints of g, $x = \pm 2$.
This give us the critical points $(x, y) = (\pm 2, 3)$.

Upper boundary: $y = 3$ **and** $-2 \leq x \leq 2$**.** Then
$g(x) = f(x, 3) = x^3 - 3x$ and $g'(x) = 3x^2 - 3$, which is 0 when $x = \pm 1 \in [-2, 2]$.
This give us the critical points $(x, y) = (\pm 1, 3)$. (Plus the end points $(x, y) = (\pm 2, 3)$, considered above.)

Checking the critical points: $f(0, 0) = 0$;
$f(2, 3) = 2^3 - 6 = 2$; $f(-2, 3) = (-2)^3 + 6 = -2$;
$f(1, 3) = 1^3 - 3 = -2$; $f(-1, 3) = (-1)^3 + 3 = 2$;

Answer: f has the absolute maximum value 2, at points $(2, 3)$ and $(-1, 3)$.
f has the absolute minimum value -2, at points $(-2, 3)$ and $(1, 3)$.

Ex. 12. ST #4 Ex. 1(a)&(c): Set up the integral formulas, **including the limits of the integrations,** for the following problems. *Do not evaluate the integrals!*

(a) The volume of the solid bounded by $z = x^2 + y^2$, $z = 0$, $x = 0$, $y = 0$, and $x + y = 1$.

Solution: If T is a triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ (i.e., $y = 1 - x$), then $V = \int\int\int_E 1 dV = \int\int_T \int_0^{x^2+y^2} 1\, dz\, dA = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} 1\, dz\, dy\, dx$

(c) The mass of the solid T with the density $\delta(x, y, z) = x^2 + e^z$ bounded by the surfaces: $6x + 2y + z = 12$, $x = 0$, $y = 0$, and $z = 0$.

Solution: The solid is a tetrahedron with a triangular base B on the xy-plane $z = 0$ bounded by $6x + 2y = 12$, $x = 0$, $y = 0$. The upper bound of T is $z = 12 - 6x - 2y$. So, $mass = \int\int\int_T \delta(x, y, z)\, dV = \int\int_B \int_0^{12-6x-2y} (x^2 + e^z)\, dz\, dA$.

Since the triangle side $6x + 2y = 12$ means that $y = 6 - 3x$, which quals 0 for $x = 2$, we get $mass = \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} (x^2 + e^z)\, dz\, dy\, dx$.

Ex. 13. ST #4 Ex. 2: Evaluate the integrals:

(a) $\int_0^1 \int_0^\pi \frac{1}{x+1} + \sin y \; dy \; dx =$

Solution: $int = \int_0^1 \left[\frac{1}{x+1}y - \cos y\right]_0^\pi \; dx = \int_0^1 \left(\frac{1}{x+1}\pi - (\cos\pi - \cos 0)\right) \; dx$. So $int = \int_0^1 \left(\frac{1}{x+1}\pi - (-1-1)\right) \; dx = [\pi \ln|x+1| + 2x]_0^1 = \pi(\ln 2 - \ln 1) + 2 = \pi \ln 2 + 2$

(b) $\int_{-2}^0 \int_0^y (x + 2y^2) \; dx \; dy =$

Solution: $int = \int_{-2}^0 \left[\frac{1}{2}x^2 + 2y^2 x\right]_{x=0}^{x=y} \; dy = \int_{-2}^0 \left(\frac{1}{2}y^2 + 2y^3\right) \; dy = \left[\frac{1}{6}y^3 + \frac{1}{2}y^4\right]_{-2}^0 = 0 - \left(\frac{1}{6}(-8) + \frac{1}{2}16\right) = \frac{4}{3} - 8 = -6\frac{2}{3}$

(c) $\int\int_R \frac{dy \; dx}{\sqrt{9 - x^2 - y^2}}$, where R is the *second quadrant* region bounded by $x^2 + y^2 = 4$.

Solution: We use the polar coordinates, in which the region R is given as $0 \le r \le 2$ and $\pi/2 \le \theta \le \pi$. So, in the second equation using substitution $u = 9 - r^2$,

$int = \int_{\pi/2}^\pi \int_0^2 (9 - r^2)^{-1/2} r \; dr \; d\theta = \int_{\pi/2}^\pi \left[-(9 - r^2)^{1/2}\right]_0^2 \; d\theta =$
$\int_{\pi/2}^\pi \left[-\left((9-4)^{1/2} - 9^{1/2}\right)\right]_0^2 \; d\theta = \left[3 - \sqrt{5}\right]_{\pi/2}^\pi = \frac{3-\sqrt{5}}{2}\pi.$

Ex. 14. ST #4 Ex. 3: Find the mass of the solid bounded by the hemisphere $x^2+y^2+z^2 \le R^2$, $z \ge 0$, with the density $\delta(x, y, z) = x^2 + y^2 + z^2$.

Solution: We use the spherical coordinates. Since the solid, T, is the upper hemisphere, we get
$mass = \int\int\int_T \delta(x, y, z) \; dV = \int\int\int_T (x^2 + y^2 + z^2) \; dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R (\rho^2)\rho^2 \sin\phi \; d\rho \; d\theta \; d\phi =$
$\int_0^{\pi/2} \int_0^{2\pi} \left[\frac{1}{5}\rho^5 \sin\phi\right]_0^R \; d\theta \; d\phi = \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{5}R^5 \sin\phi \; d\theta \; d\phi = \int_0^{\pi/2} \left[\left(\frac{1}{5}R^5 \sin\phi\right)\theta\right]_0^{2\pi} \; d\phi =$
$\int_0^{\pi/2} \frac{2}{5}\pi R^5 \sin\phi \; d\phi = \left[\frac{2}{5}\pi R^5(-\cos\phi)\right]_0^{\pi/2} = -\frac{2}{5}\pi R^5(\cos(\pi/2) - \cos 0) = -\frac{2}{5}\pi R^5(0 - 1) = \frac{2}{5}\pi R^5$

Ex. 15. ST #4 Ex. 4: Find the mass of the plane lamina bounded by $x = 0$ and $x = 9-y^2$ with density $\delta(x, y) = x^2$.

Solution: Notice that $x = 0$ and $x = 9 - y^2$ when $9 - y^2 = 0$ that is, when $y = \pm 3$.
$mass = \int\int_R \delta(x, y) dA = \int_{-3}^3 \int_0^{9-y^2} x^2 \; dx \; dy = \int_{-3}^3 \left[\frac{1}{3}x^3\right]_0^{9-y^2} \; dy = \int_{-3}^3 \frac{1}{3}(9 - y^2)^3 \; dy =$
$\int_{-3}^3 \frac{1}{3}(9^3 - 3 \cdot 9^2(y^2) + 3 \cdot 9(y^2)^2 - (y^2)^3) \; dy = \int_{-3}^3 (3^5 - 3^4 y^2 + 3^2 y^4 - \frac{1}{3}y^6) \; dy =$
$\left[3^5 y - 3^3 y^3 + \frac{3^2}{5}y^5 - \frac{1}{21}y^7\right]_{-3}^3 = 3^5(3+3) - 3^3(3^3 + 3^3) + \frac{3^2}{5}(3^5 + 3^5) - \frac{1}{21}(3^7 + 3^7) =$
$2 \cdot 3^6 - 2 \cdot 3^6 + \frac{2}{5}3^7 - \frac{2}{21}3^7 = 2(\frac{1}{5} - \frac{1}{21})3^7 = 2\frac{21-5}{105}3^7 = 2\frac{16}{35}3^6 = \frac{32}{35}3^6$

Ex. 16. ST #4 Ex. 6: Evaluate the integral, where C is the graph of $y = x^3$ from $(-1,-1)$ to $(1,1)$

$\int_C y^2\,dx + xdy =$

Solution: Clearly x changes from -1 to 1. Put $x = t$. Then $y(t) = t^3$ and $-1 \le t \le 1$ and $\int_C y^2\,dx + xdy = \int_{-1}^{1} (y(t))^2\,x'(t)\;dt + x(t)\,y'(t)\;dt = \int_{-1}^{1} [(t^3)^2\,1 + t\,(3t^2)]\;dt = \int_{-1}^{1} (t^6 + 3t^3)\;dt =$
$\left[\frac{1}{7}t^7 + \frac{3}{4}t^4\right]_{-1}^{1} = \frac{1}{7}(1+1) + \frac{3}{4}(1-1) = \frac{2}{7}$

Ex. 17. ST #4 Ex. 8: Find a potential function of the vector field and use the fundamental theorem for line integrals to evaluate

$\int_{(\pi/2,\pi/2)}^{(\pi,\pi)} (\sin y + y\cos x)\,dx + (\sin x + x\cos y)\,dy =$

Solution: We have $P = \sin y + y\cos x$ and $Q = \sin x + x\cos y$. It is easy to see that $\frac{\partial P}{\partial y} = \cos y + \cos x = \frac{\partial Q}{\partial x}$ so indeed we can find the potential function $f(x,y)$. We have

$f(x,y) = \int P\;dx = \int \sin y + y\cos x\;dx = x\sin y + y\sin x + K(y)$.

Taking partial derivative, in terms of y, of both side we get

$x\cos y + \sin x + K'(y) = \frac{\partial f}{\partial y} = Q = \sin x + x\cos y$, so that $K'(y) = 0$ and $K(y) = C$.

So, the potential function $f(x,y) = x\sin y + y\sin x + C$ and

$int = [f(x,y)]_{(\pi/2,\pi/2)}^{(\pi,\pi)} = [x\sin y + y\sin x]_{(\pi/2,\pi/2)}^{(\pi,\pi)} = (\pi\sin\pi + \pi\sin\pi) - (\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{\pi}{2}\sin\frac{\pi}{2}) =$
$(0+0) - (\frac{\pi}{2} + \frac{\pi}{2}) = -\pi$

Ex. 18. ST #4 Ex. 9: Apply Green's theorem to evaluate the following integral, where the simple closed curve C, with counter clockwise direction, is the boundary of the circle $x^2 + y^2 = 1$.

$\oint_C (\sin x - x^2 y)\,dx + xy^2\,dy =$

Solution: Let D denoted the disk $x^2 + y^2 \le 1$.

By Green's theorem $int = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\;dA$, where $P = \sin x - x^2 y$ and $Q = xy^2$. So,

$int = \int\int_D (y^2 - (-x^2))\;\;dA = \int\int_D (x^2 + y^2)\;\;dA$

Changing to the polar coordinates, we get

$int = \int_0^{2\pi}\int_0^1 r^2\;r\;dr\;d\theta = \int_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^1\;d\theta = \int_0^{2\pi}\frac{1}{4}\;d\theta = \left[\frac{1}{4}\theta\right]_0^{2\pi} = \frac{1}{2}\pi$