MATH 261.005 NAME (print): ____________________

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SAMPLE TEST # 4

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1. Transform the following system of equations into a single second order equation in terms of x_1. Then give the initial condition for the resulted equation that corresponds to the given initial conditions. Do not solve.
$x_1' = -0.5x_1 + 2x_2; \quad x_2' = -2x_1 - 0.5x_2; \quad x_1(0) = -2, \quad x_2(0) = 2.$

Solution: From the first equation we get $x_2 = 0.5x_1' + 0.25x_1$. Substituting this to the second equation gives $(0.5x_1' + 0.25x_1)' = -2x_1 - 0.5(0.5x_1' + 0.25x_1)$, which in turn leads to $0.5x_1'' + 0.25x_1' = -2x_1 - 0.25x_1' - 0.125x_1$. Multiplying this equation by 8 produces $4x_1'' + 2x_1' = -16x_1 - 2x_1' - x_1$, so $4x_1'' = -4x_1' - 17x_1$.

Clearly $x_1(0) = -2$. To calculate $x_1'(0)$ we put $t = 0$ to the first equation and use given boundary values: $x_1'(0) = -0.5x_1(0) + 2x_2(0) = -0.5(-2) + 2 \cdot 2 = 5$.

Answer: $4x_1'' = -4x_1' - 17x_1; \quad x_1(0) = -2; \quad x_1'(0) = 5.$

Ex. 2. Use eigenvalues and eigenvectors to find the general solution of the given systems of differential equations. The solution must be expressed in terms of real-valued functions.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

Solution: The eigenvalues are obtained as roots of the equation
$\det \begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix} = 0$, that is, $(1-r)(-4-r) + 6 = 0$, or $r^2 + 3r - 4 + 6$. Hence, $(r+2)(r+1) = 0$, leading to the eigenvalues -1 and -2.

The eigenvalue $r = -1$ leads to the eigenvector equation $\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $2\xi_1 - 2\xi_2 = 0$. Thus, $\xi_2 = \xi_1$, $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenvalue $r = -2$ leads to the eigenvector equation $\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $3\xi_1 - 2\xi_2 = 0$. Thus, $\xi_2 = 1.5\xi_1$, $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 1.5\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$, and the eigenvector is $\begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$.

Answer: $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} e^{-2t}$.

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

Solution: The eigenvalues are obtained as roots of the equation $\det \begin{pmatrix} 1-r & 2 \\ -5 & -1-r \end{pmatrix} = 0$, that is, $(1-r)(-1-r)+10 = 0$, or $r^2 - 1 + 10 = 0$. Hence, we have two complex the eigenvalues: $\pm 3i$.

Eigenvalue $r = 3i$ leads to eigenvector equation $\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $(1-3i)\xi_1 + 2\xi_2 = 0$. Thus, $\xi_2 = (1.5i - 0.5)\xi_1$,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ (1.5i-0.5)\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1.5i - 0.5 \end{pmatrix},$$

and the eigenvector is $\begin{pmatrix} 1 \\ 1.5i - 0.5 \end{pmatrix}$. Since the real part of the eigenvalue is 0, this leads to the complex solution $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1.5i - 0.5 \end{pmatrix} e^{0t}(\cos 3t + i \sin 3t)$. Therefore $\mathbf{x}^{(1)}(t) = \begin{pmatrix} \cos 3t + i \sin 3t \\ (-0.5\cos 3t - 1.5 \sin 3t) + i(1.5\cos 3t - 0.5 \sin 3t) \end{pmatrix}$ and

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} \cos 3t \\ -0.5\cos 3t - 1.5\sin 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ 1.5\cos 3t - 0.5\sin 3t \end{pmatrix}.$$

Answer: $\mathbf{x}(t) = c_1 \begin{pmatrix} \cos 3t \\ -0.5\cos 3t - 1.5\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} \sin 3t \\ 1.5\cos 3t - 0.5\sin 3t \end{pmatrix}$.

(c) $\mathbf{x}' = \begin{pmatrix} 6 & -3 \\ 3 & 0 \end{pmatrix} \mathbf{x}$

Solution: The eigenvalues are obtained as roots of the equation $\det \begin{pmatrix} 6-r & -3 \\ 3 & -r \end{pmatrix} = 0$, that is, $(6-r)(-r)+9 = 0$, $r^2 - 6r + 9 = 0$, or $(r-3)^2 = 0$. Hence, we have one double eigenvalue: $r = 3$.

The first eigenvector equation is $\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $3\xi_1 - 3\xi_2 = 0$. Thus, $\xi_2 = \xi_1$, $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the eigenvector is $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This gives the first fundamental solution $\mathbf{x}^{(1)} = \xi e^{3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$. Recall that the second fundamental solution is of the form $\mathbf{x}^{(2)} = \xi t e^{3t} + \eta e^{3t}$, where η is one of the solutions of the system $\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \xi$, that is, $\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence, $3\eta_1 - 3\eta_2 = 1$, that is, $\eta_1 = \eta_2 + 1/3$. So, $\eta = \begin{pmatrix} \eta_2 + 1/3 \\ \eta_2 \end{pmatrix} = \eta_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$. One of these solutions is when $\eta_2 = 0$, giving $\eta = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$.

Answer: $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{3t} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} e^{3t} \right].$

Ex. 3. Solve the following boundary value problem or show that it does not have a solution. $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 0$.

Solution: The characteristic function of our equation is $r^2 + 4 = 0$ and has solution $r = \pm 2i$. Thus, the general solution of the equation is of the form $y(t) = c_1 \cos 2t + c_2 \sin 2t$. For $t = 0$ this leads to the equation $y(0) = c_1 \cos 0 + c_2 \sin 0$, that is, $0 = c_1 1 + c_2 0$. Thus $c_1 = 0$ and $y(t) = c_2 \sin 2t$. For $t = \pi$ this leads to the equation $y(\pi) = c_2 \sin 2\pi$, that is, $0 = c_2 0$, which holds for any value of c_2. Thus, the boundary value problem has infinitely many solutions, each of the form $y(t) = c \sin 2t$, where c is an arbitrary constant.

Ex. 4. Determine whether the method of separation of variables can be used to replace the partial differential equation $u_{xx} + u_{xt} + u_t = 0$ by a pair of ordinary differential equations. If so, find the ordinary differential equations. Do not solve them.

Solution: We assume that u is of the form $u(x,t) = X(x)T(t)$. Then we have $u_{xx} = X''T$, $u_{xt} = X'T'$, and $u_t = XT'$. Substituting this to the equation gives $X''T + X'T' + XT' = 0$. So, $X''T + (X' + X)T' = 0$ and $X''T = -(X' + X)T'$. Thus, $\frac{X''}{X'+X} = -\frac{T'}{T}$ and this quantity is equal to a constant, which we denote by λ. Thus the method of separation of variables can be used to our partial differential equation and it leads to the pair of ordinary differential equations $X'' = \lambda(X' + X)$ and $T' = -\lambda T$.

Ex. 5. Solve the heat equation: $u_t = 9u_{xx}$, $u(0,t) = u(2,t) = 0$, $u(x,0) = 13$ for $0 < x < 2$.

Solution: The general solution of the heat equation

$$u_t = \alpha^2 u_{xx},\ u(0,t) = u(L,t) = 0,\ u(x,0) = f(x) \text{ for } 0 < x < L$$

is given by $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin \frac{n\pi x}{L}$, where $c_n = \frac{2}{L}\int_o^L f(x) \sin \frac{n\pi x}{L} dx$. In our case $L = 2$ and $\alpha^2 = 9$, reducing the solution to $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2(9\pi^2/4)t} \sin \frac{n\pi x}{2}$, where

$$\begin{aligned} c_n = \frac{2}{2}\int_0^2 u(x,0) \sin \frac{n\pi x}{2} dx &= \int_0^2 13 \sin \frac{n\pi x}{2} dx \\ &= -13 \frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_0^2 \\ &= -\frac{26}{n\pi}(\cos n\pi - \cos 0) \\ &= \frac{26}{n\pi}(\cos 0 - \cos n\pi) = \begin{cases} 0 & n \text{ is even} \\ 52/(n\pi) & n \text{ is odd} \end{cases}. \end{aligned}$$

This leads us to a final solution

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{52}{n\pi} e^{-n^2(9\pi^2/4)t} \sin \frac{n\pi x}{2} = \sum_{k=0}^{\infty} \frac{52}{(2k+1)\pi} e^{-(2k+1)^2(9\pi^2/4)t} \sin \frac{(2k+1)\pi x}{2},$$

where either of the two formats is acceptable as an answer.