MATH 261.005 Instr. K. Ciesielski Fall 2011

Solutions to the SAMPLE TEST # 3

Ex. 1. Find the general solution for the following differential equations:

(a) $y^{(4)} + 8y'' + 16y = 0$

Solution: $r^4 + 8r^2 + 16 = 0$; $(r^2 + 4)^2 = 0$; $[(r + 2i)(r - 2i)]^2 = 0$;

Double roots $\pm 2i$;

Answer: $y = c_1 \sin 2t + c_2 \cos 2t + c_3 t \sin 2t + c_4 t \cos 2t$.

(b) $y^{(4)} - 8y'' + 16y = 0$

Solution: $r^4 - 8r^2 + 16 = 0$; $(r^2 - 4)^2 = 0$; $[(r + 2)(r - 2)]^2 = 0$;

Double roots ± 2;

Answer: $y = c_1 e^{2t} + c_2 e^{-2t} + c_3 t e^{2t} + c_4 t e^{-2t}$.

(c) $y''' - 3y'' + 2y' = e^{-t}$

Solution: $r^3 - 3r^2 + 2r = 0$; $r(r^2 - 3r + 2) = 0$; $r(r - 1)(r - 2) = 0$;

Solution to homogenuous part: $c_1 e^{0t} + c_2 e^{1t} + c_3 e^{2t} = c_1 + c_2 e^t + c_3 e^{2t}$.

Particular solution guess: $y = Ae^{-t}$. Then $y' = -Ae^{-t}$, $y'' = Ae^{-t}$, $y''' = -Ae^{-t}$.

So $y''' - 3y'' + 2y' = -Ae^{-t} - 3Ae^{-t} + 2(-Ae^{-t}) = -6Ae^{-t}$, and $-6Ae^{-t} = e^{-t}$.

Thus, $A = -\frac{1}{6}$, giving a particular solution: $y = -\frac{1}{6}e^{-t}$.

Answer: $y = c_1 + c_2 e^t + c_3 e^{2t} - \frac{1}{6}e^{-t}$.

(d) $y''' - 3y'' + 2y' = t$

Solution: $r^3 - 3r^2 + 2r = 0$; $r(r^2 - 3r + 2) = 0$; $r(r - 1)(r - 2) = 0$;

Solution to homogeneous part: $c_1 e^{0t} + c_2 e^{1t} + c_3 e^{2t} = c_1 + c_2 e^t + c_3 e^{2t}$.

Particular solution guess: $y = t^s(at + b)$.

Since $1 = e^{0t}$ is a solution to homogeneous part, $s > 1$.

Take $s = 1$, leading to a particular solution guess two: $y = t(at + b) = at^2 + bt$.

Then $y' = 2at + b$, $y'' = 2a$, $y''' = 0$.

So $y''' - 3y'' + 2y' = 0 - 3(2a) + 2(2at + b) = 4at + (2b - 6a)$, and $4at + (2b - 6a) = t$.

Hence, $4a = 1$, leading to $a = 1/4$, and $2b - 6a = 0$, giving $b = 3a = 3/4$.

Thus, a particular solution: $y = \frac{1}{4}t^2 + \frac{3}{4}t$.

Answer: $y = c_1 + c_2 e^t + c_3 e^{2t} + \frac{1}{4}t^2 + \frac{3}{4}t$.

Ex. 2. Find the interval of convergence of the following series. Check the endpoints for extra credit. $\sum_{n=1}^{\infty} \frac{(4x+3)^n}{9n^2}$.

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4x+3)^{n+1}/9(n+1)^2}{(4x+3)^n/9n^2}\right| = \lim_{n\to\infty}\left|\frac{(4x+3)^{n+1}}{(4x+3)^n}\right|\left[\frac{9(n+1)^2}{9n^2}\right]^{-1}$$

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |4x+3|\lim_{n\to\infty}\left[\frac{(n+1)^2}{n^2}\right]^{-1} = |4x+3|\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^{-2} = |4x+3|$.

Converges when $|4x+3| < 1$, that is, $-1 < 4x+3 < 1$.

So that $-4 < 4x < -2$ and $-1 < x < -1/2$.

At the endpoints $4x+3 = \pm 1$ we get $\sum_{n=1}^{\infty}\frac{(4x+3)^n}{9n^2} = \sum_{n=1}^{\infty}\frac{(\pm 1)^n}{9n^2}$,

which converges absolutely by the p-series test.

Answer: Interval $[-1, -1/2]$, that is, $-1 \leq x \leq -1/2$.

Ex. 3. Use power series with $x_0 = 1$ to solve $y'' + xy' + y = 0$. Find the recurrence formula and use it to find the first two non-zero terms in each of *two independent solutions*.

Solution: $y = \sum_{n=0}^{\infty} a_n(x-1)^n$.

So, $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^k$ and

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k$.

Since

$$xy' = [1+(x-1)]\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n + \sum_{k=1}^{\infty} k a_k (x-1)^k = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n + \sum_{k=0}^{\infty} k a_k(x-1)^k,$$

expression $y'' + xy' + y$ is equal to

$$\left(\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n\right) +$$

$$\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} n a_n (x-1)^n\right) + \left(\sum_{n=0}^{\infty} a_n(x-1)^n\right).$$

So $(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + n a_n + a_n = 0$ for $n \geq 0$.

Thus, $a_{n+2} = -\frac{a_{n+1}+a_n}{n+2}$ for $n \geq 0$.

In particular, $a_2 = -\frac{1}{2}a_1 - \frac{1}{2}a_0$, $a_3 = -\frac{1}{3}(-\frac{1}{2}a_1 - \frac{1}{2}a_0 + a_1) = -\frac{1}{6}a_1 + \frac{1}{6}a_0$, and

$y = a_0\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots\right] + a_1\left[(x-1) - \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \cdots\right]$.

Ex. 4. Calculate the Laplace transform, $\mathcal{L}[f(t)]$, of the function $f(t) = 5t^2$. Show the details and use limits to evaluate any improper integrals.

Solution:

$\mathcal{L}[5t^2](s) = \int_0^\infty e^{-st} 5t^2 \, dt = 5 \lim_{A\to\infty} \int_0^A t^2 e^{-st} \, dt$

Integrating by parts twice we get

$\int t^2 e^{-st} \, dt = t^2 \, \frac{-1}{s} e^{-st} - \int 2t \frac{-1}{s} e^{-st} \, dt = \frac{-1}{s} t^2 e^{-st} + \frac{2}{s} \int t e^{-st} \, dt =$

$= \frac{-1}{s} t^2 e^{-st} + \frac{2}{s} \left(t \, \frac{-1}{s} e^{-st} - \int 1 \frac{-1}{s} e^{-st} \, dt \right) =$

$= \frac{-1}{s} t^2 e^{-st} - \frac{2}{s^2} t e^{-st} + \frac{2}{s} \frac{1}{s} \frac{-1}{s} e^{-st} + C = e^{-st} \left(\frac{-1}{s} t^2 - \frac{2}{s^2} t + \frac{-2}{s^3} \right) + C$

So, for $s > 0$,

$\mathcal{L}[5t^2](s) = 5 \lim_{A\to\infty} \left[-\frac{1}{s^3} \frac{s^2t^2+2st+2}{e^{st}} \right]_{t=0}^{t=A} = 5 \lim_{A\to\infty} -\frac{1}{s^3} \left(\frac{s^2A^2+2sA+2}{e^{sA}} - \frac{2}{e^0} \right) = 5 \frac{2}{s^3},$

where the last equation follows from L'Hospital Rule.

Answer: $\mathcal{L}[5t^2](s) = \frac{10}{s^3}$ for $s > 0$.

Ex. 5. Use **Laplace transforms** to solve $y'' + 5y' - 6y = 3$, $y(0) = 0$, $y'(0) = 1$. Recall that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$. (For $a = 0$ this gives $\mathcal{L}[1] = \frac{1}{s}$.)

Solution:

$\mathcal{L}[y''] + 5\mathcal{L}[y'] - 6\mathcal{L}[y] = 3\mathcal{L}[1]$; so

$(s^2\mathcal{L}[y] - sy(0) - y'(0)) + 5(\mathcal{L}[y] - y(0)) - 6\mathcal{L}[y] = 3\frac{1}{s}$; thus

$(s^2 + 5s - 6)\mathcal{L}[y] - sy(0) - y'(0) - 5y(0) = \frac{3}{s}$; i.e.,

$(s-1)(s+6)\mathcal{L}[y] - 1 = \frac{3}{s}$. Therefore,

$\mathcal{L}[y] = \frac{3+s}{s(s-1)(s+6)} = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+6} = \frac{a(s^2+5s-6)+b(s^2+6s)+c(s^2-s)}{s(s-1)(s+6)} = \frac{(a+b+c)s^2+(5a+6b-c)s-6a}{s(s-1)(s+6)}$. So,

$-6a = 3, \quad 5a + 6b - c = 1, \quad a + b + c = 0;$

$a = -0.5, \quad 6b - c = 3.5, \quad b + c = 0.5$; adding two equations, $7b = 4$; so $b = 4/7$.

Hence $c = \frac{1}{2} - \frac{4}{7} = -\frac{1}{14}$ and $\mathcal{L}[y] = -\frac{1}{2}\frac{1}{s} + \frac{4}{7}\frac{1}{s-1} - \frac{1}{14}\frac{1}{s+6}$, so

$y = -\frac{1}{2} + \frac{4}{7}e^t - \frac{1}{14}e^{-6t}.$