Solutions to the SAMPLE TEST # 2

Ex. 1(a): $y'' + 10y' + 25 = 0$. This is first order linear ODE in terms of y'. That is, using $u = y'$, we get $u' + 10u = -25$.

Then $\mu = \exp\left(\int 10\ dt\right) = e^{10t}$. Also,
$u = \frac{1}{\mu}\int \mu(t)g(t)\ dt = e^{-10t}\int e^{10t}(-25)\ dt = e^{-10t}(\frac{1}{10}e^{10t}(-25) + C) = -2.5 + Ce^{-10t}$.
Thus, $y = \int u(t)\ dt = \int(-2.5 + Ce^{-10t})\ dt = -2.5t + C\frac{1}{-10}e^{-10t} + C_2$.
Replacing constant $C\frac{1}{-10}$ with a constant C_1 leads to
answer: $y(t) = -2.5t + C_1e^{-10t} + C_2$.

Ex. 1(b): $y'' + 10y' + 25y = 0$. This is second order linear homogeneous ODE with the characteristic equation $r^2 + 10r + 25 = 0$.

The equation is equivalent to $(r + 5)^2 = 0$, so it has only one double root $r = -5$. Thus, a fundamental pair of solution of our ODE is: e^{-5t} and te^{-5t}, and its general solution
answer: $y(t) = C_1e^{-5t} + C_2te^{-5t}$.

Ex. 1(c): $y'' + 10y' + 29y = 0$. This is second order linear homogeneous ODE with the characteristic equation $r^2 + 10r + 29 = 0$. By quadratic formula its roots are
$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-10\pm\sqrt{100-4\cdot 29}}{2} = \frac{-10\pm\sqrt{-16}}{2} = \frac{-10\pm 4\mathbf{i}}{2} = -5 \pm 2\mathbf{i}$,
i.e., the equation has two complex roots, with real part $\lambda = -5$ and imaginary part $\mu = 2$.
Thus, a fundamental pair of solution of our ODE is: $e^{-5t}\cos(2t)$ and $e^{-5t}\sin(2t)$,
and its general solution
answer: $y(t) = C_1e^{-5t}\cos(2t) + C_2e^{-5t}\sin(2t)$.

Ex. 1(d): $y'' + 10y' + 24y = 0$. This is second order linear homogeneous ODE with the characteristic equation $r^2 + 10r + 24 = 0$.

The equation is equivalent to $(r + 4)(r + 6) = 0$,
so it has two real roots: $r = -4$ and $r = -6$. Thus,
a fundamental pair of solution of our ODE is: e^{-4t} and e^{-6t}, and its general solution
answer: $y(t) = C_1e^{-4t} + C_2e^{-6t}$.

Ex. 1(e): $2y'' + 3y' + y = t^2 + 3\sin t$. This is second order linear non-homogeneous ODE.

Its solution is of the form $y = y_0 + y_1 + y_2$, where
y_0 is a general solution of the homogeneous ODE $2y'' + 3y' + y = 0$,
y_1 is a particular solution of the non-homogeneous ODE $2y'' + 3y' + y = t^2$, and
y_2 is a particular solution of the non-homogeneous ODE $2y'' + 3y' + y = 3\sin t$.
Thus, this exercise splits into three separate problems, for finding y_0, y_1, and y_2.

Finding y_0: The ODE $2y'' + 3y' + y = 0$ has the characteristic equation $2r^2 + 3r + 1 = 0$,
that is, $(2r + 1)(r + 1) = 0$. So it has two real roots $r = -1$ and $r = -0.5$.
This gives a fundamental pair of solution for $2y'' + 3y' + y = 0$ as e^{-t} and $e^{-t/2}$ and
$y_0 = C_1e^{-t} + C_2e^{-t/2}$.

Finding y_1: The method of undetermined coefficients says that the ODE $2y'' + 3y' + y = t^2$
has solution of the form $y_1 = t^s(at^2 + bt + c)$. We can take $s = 0$, as $y_1 = at^2 + bt + c$
is not of the form of y_0, the solution of the corresponding homogeneous ODE.
Then $y_1' = 2at + b$ and $y_1'' = 2a$. Substituting this to $2y'' + 3y' + y = t^2$ gives

$2(2a) + 3(2at + b) + (at^2 + bt + c)1 = t^2$, that is,
$at^2 + (6a + b)t + (4a + 3b + c) = 1t^2 + 0t + 0 \cdot 1$.
The coefficients of these two polynomials must be equal, that is,
$a = 1$, $6a + b = 0$, and $4a + 3b + c = 0$.
This gives $a = 1$, $b = -6a = -6$, and $c = -4a - 3b = -4 + 18 = 14$ and leads to
$y_1 = t^2 - 6t + 14$.

Finding y_2: The method of undetermined coefficients says that the ODE $2y''+3y'+y = 3\sin t$ has solution of the form $y_2 = t^s(A\sin t + B\cos t)$. We can take $s = 0$, as $y_2 = A\sin t + B\cos t$ is not of the form of y_0.
Then $y_2' = A\cos t - B\sin t$ and $y_2'' = -A\sin t - B\cos t$.
Substituting this to $2y'' + 3y' + y = 3\sin t$ gives
$2(-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + (A\sin t + B\cos t)1 = 3\sin t$, that is,
$(-2A - 3B + A)\sin t + (-2B + 3A + B)\cos t = 3\sin t + 0\cos t$.
The coefficients of sin and cos must be equal, that is,
$-A - 3B = 3$ and $3A - B = 0$.
Hence, $B = 3A$, $-A - 3(3A) = 3$, so $A = -0.3$ and $B = 3A = -0.9$. So,
$y_2 = -0.3\sin t - 0.9\cos t$.

Therefore $y = y_0 + y_1 + y_2$ leads to the
final answer: $y(t) = C_1e^{-t} + C_2e^{-t/2} + t^2 - 6t + 14 - 0.3\sin t - 0.9\cos t$.

Ex. 2: $y'' + y' - 2y = 2t$, $y(0) = 0$, $y'(0) = 1$.
This is initial value problem for second order linear non-homogeneous ODE.
Its solution is of the form $y = y_0 + y_1$, where
y_0 is a general solution of the homogeneous ODE $y'' + y' - 2y = 0$ and
y_1 is a particular solution of the non-homogeneous ODE $y'' + y' - 2y = 2t$.
Thus, this exercise splits into three separate problems: finding y_0, finding y_1, finding in $y = y_0 + y_1$ the constants satisfying the initial condition.

Finding y_0: The ODE $y'' + y' - 2y = 0$ has the characteristic equation $r^2 + r - 2 = 0$, that is, $(r + 2)(r - 1) = 0$. So it has two real roots $r = -2$ and $r = 1$.
This gives a fundamental pair of solution for $2y'' + 3y' + y = 0$ as e^{-2t} and e^t and
$y_0 = C_1e^{-2t} + C_2e^t$.

Finding y_1: The method of undetermined coefficients says that the ODE $y'' + y' - 2y = 2t$ has solution of the form $y_1 = t^s(at + b)$. We can take $s = 0$, as $y_1 = at + b$ is not of the form of y_0, the solution of the corresponding homogeneous ODE.
Then $y_1' = a$ and $y_1'' = 0$. Substituting this to $y'' + y' - 2y = 2t$ gives
$0 + a - 2(at + b) = 2t$, that is, $-2at + (a - 2b) = 2t + 0 \cdot 1$.
The coefficients of these two polynomials must be equal, that is,
$-2a = 2$ and $a - 2b = 0$. Hence $a = -1$ and $b = a/2 = -1/2$,
giving $y_1 = -t - 0.5$. Thus, the general solution of ODE, $y = y_0 + y_1$, is
$y = C_1e^{-2t} + C_2e^t - t - 0.5$.

Solving initial value problem: Taking derivative of y we get $y' = -2C_1e^{-2t} + C_2e^t - 1$.
$y(0) = 0$ leads to $C_1e^0 + C_2e^0 - 0 - 0.5 = 0$, that is, $C_1 + C_2 = 0.5$.
$y'(0) = 1$ leads to $-2C_1e^0 + C_2e^0 - 1 = 1$, that is, $-2C_1 + C_2 = 2$.
Subtracting first equation from the second leads to $-3C_1 = 1.5$, hence $C_1 = -0.5$.
So, $C_2 = 0.5 - C_1 = 1$ and we get
final solution: $y(t) = -0.5e^{-2t} + e^t - t - 0.5$.

Ex. 3: Find a particular solution of the equation $y'' + 3y = 3\sin 2t$.
The method of undetermined coefficients says that the ODE
has solution of the form $y = t^s(a\sin 2t + b\cos 2t)$.
The corresponding homogeneous ODE has the characteristic equation $r^2 + 3 = 0$,
so it has two complex roots $\pm\sqrt{3}\mathbf{i}$ and fundamental solutions $\sin\sqrt{3}t$, $\cos\sqrt{3}t$.
These are not of the form of $y = a\sin 2t + b\cos 2t$, so we can take $s = 0$.
Then we calculate $y'' = -4a\sin 2t - 4b\cos 2t$.
Substitution of y and y'' to our ODE gives
$(-4a\sin 2t - 4b\cos 2t) + 3(a\sin 2t + b\cos 2t) = 3\sin 2t$, that is,
$-a\sin 2t - b\cos 2t = 3\sin 2t + 0\cos 2t$.
The coefficients of sin and cos must be equal, that is,
$-a = 3$ and $-b = 0$, so $a = -3$ and $b = 0$. This gives
a final solution: $y = -3\sin 2t$.

Ex. 4: Find a second solution of $(x-1)y'' - xy' + y = 0$, if $y_1(x) = e^x$ is its solution.
The method of reduction of order, which we must use, says to use substitution
$y = vy_1$, that is (in our case), $y = ve^x$. We need to find v. First, we calculate
$y' = v'e^x + ve^x = e^x(v' + v)$ and $y'' = (v''e^x + v'e^x) + (v'e^x + ve^x) = e^x(v'' + 2v' + v)$
Substituting these back to our ODE leads to
$(x-1)e^x(v'' + 2v' + v) - xe^x(v' + v) + ve^x = 0$, and after simplification,
$e^x[(x-1)(v'' + 2v' + v) - x(v' + v) + v] = 0$,
$e^x[(x-1)v'' + (2(x-1) - x)v' + ((x-1) - x + 1)v] = 0$,
$e^x[(x-1)v'' + (x-2)v'] = 0$. Dividing this by e^x and putting $u = v'$ we get ODE
$(x-1)u' + (x-2)u = 0$. By further algebra we get
$(x-1)\frac{du}{dx} = -(x-2)u$, so $\frac{du}{u} = -\frac{x-2}{x-1}dx$ and $\int\frac{du}{u} = -\int(1 - \frac{1}{x-1})dx$.
Hence $\ln|u| = -[x - \ln|x-1| + C]$. Thus
$|u| = \exp(-[x - \ln|x-1| + C]) = e^{-x}e^{\ln|x-1|}e^{-C} = e^{-x}|x-1|e^{-C}$. Since $x > 1$, we get
$|u| = e^{-x}(x-1)e^{-C}$ and $u = K(x-1)e^{-x}$, where $K = \pm e^{-C}$ is a constant.
From here $v = \int u\,dx = K\int(x-1)e^{-x}\,dx$. Then, by integration by parts,
$v = \int u\,dx = K[(x-1)(-e^{-x}) - \int 1(-e^{-x})\,dx] = K[e^{-x} - xe^{-x} - e^{-x} + c] = K[-xe^{-x} + c]$
Taking $c = 0$ and $K = -1$, we get $v = xe^{-x}$ and a second solution for our ODE:
$y = ve^x = xe^{-x}e^x = x$.
Answer: $y = x$.

Ex. 5: $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$; use the variation of parameters method.
The homogeneous part of our ODE has the characteristic equation
$r^2 + 4r + 4 =$, that is, $(r+2)^2 = 0$. There is only one, double root $r = -2$,
so $y'' + 4y' + 4y = 0$ has solutions $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$.
The method of the variation of parameters says that a particular solution is of the form
$y = u_1y_1 + u_2y_2$, where $u_1 = -\int\frac{y_2g(t)}{W}\,dt$ and $u_2 = \int\frac{y_1g(t)}{W}\,dt$.
Here $g(t) = t^{-2}e^{-2t}$ and W is Wronskian:
$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = (e^{-2t})^2[(1-2t) - (-2t)] = e^{-4t} \neq 0$. So
$u_1 = -\int\frac{y_2g(t)}{W}\,dt = -\int\frac{(te^{-2t})t^{-2}e^{-2t}}{e^{-4t}}\,dt = -\int t^{-1}\,dt = -\ln t$ and
$u_2 = \int\frac{y_1g(t)}{W}\,dt = \int\frac{(e^{-2t})t^{-2}e^{-2t}}{e^{-4t}}\,dt = \int t^{-2}\,dt = -t^{-1}$. As $y = u_1y_1 + u_2y_2$, we get
Answer: $y = (-\ln t)e^{-2t} + (-t^{-1})te^{-2t}$, that is, $y = (-\ln t)e^{-2t} - e^{-2t}$,

or just $y = (-\ln t)e^{-2t}$, as e^{-2t} is a solution to the homogeneous part.

Ex. 6. The ODE is of the form $mu''(t) + \gamma u'(t) + ku(t) = F(t)$.
Here there is no external force, that is, $F(t) = 0$.
We have the weight $w = 2lb$ and so mass $m = \frac{w}{g} = \frac{2lb}{32ft/sec^2} = \frac{1}{16}\frac{lb\ sec^2}{ft}$.
$\gamma = \frac{\text{resisting force}}{\text{velocity}} = \frac{8lb}{11ft/sec} = \frac{8}{11}\frac{lb\ sec}{ft}$
$k = \frac{\text{stretch weight}}{\text{displacement}} = \frac{2lb}{0.5ft} = 4lb/ft$.
Clearly, initial position and velocity are $u(0) = 3in = 0.25ft$ and $u'(0) = -0.7ft/sec$. (Negative, as our positive direction is down.)
Answer: $\frac{1}{16}u'' + \frac{8}{11}u' + 4u = 0$, $u(0) = 0.25$, $u'(0) = -0.7$.