MATH 261.005
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Partial Differential Equations, PDE, handout

Crucial Boundary Value problem: Find all non-zero solutions of the BVP, where $\lambda > 0$:

$$x''(t) + \lambda x(t) = 0,\ x(0) = 0 \text{ and } x(L) = 0. \tag{1}$$

Solution: Characteristic polynomial $r^2 + \lambda = 0$ has solution $r = \pm\sqrt{\lambda}$. This gives general solution $x(t) = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t$.

$x(0) = 0$ leads to $0 = c_1 \cos 0 + c_2 \sin 0$, so $c_1 = 0$. Therefore, $x(t) = c_2 \sin\sqrt{\lambda}t$.

$x(L) = 0$ leads to $0 = c_2 \sin(\sqrt{\lambda}L)$. If $c_2 = 0$, then $x(t) = 0$, trivial solution we are not interested in. So, we will assume that $c_2 \neq 0$, what implies that

$\sin(\sqrt{\lambda}L) = 0$, that is, $\sqrt{\lambda}L = n\pi$, where n is an integer. So, $\lambda = n^2\pi^2/L^2$, n being an integer.

Answer: (1) has non-zero solution for $\lambda_n = n^2\pi^2/L^2$, where n is an integer. The solution is given by

$$x_n(t) = \sin\left(\frac{n\pi}{L}t\right).$$

Heat PDE: For $\alpha \neq 0$, $t \geq 0$, and a fixed function $f(x)$ defined for $0 < x < L$

$$\alpha^2 x_{xx} = u_t, \quad u(x,0) = f(x) \text{ for } 0 \leq x \leq L; \quad u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for } t > 0. \tag{2}$$

Solution: We hope (guess), that there is a solution of the form $u(x,t) = X(x)T(t)$.

This implies that $\alpha^2 X''T = XT'$, that is, $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$.

We *separated variables*: x's on the left hand side, $\frac{X''}{X}$, t's on the right hand side, $\frac{1}{\alpha^2}\frac{T'}{T}$.

This means that both sides equal to a constant, say $-\lambda$. (We will use only $\lambda > 0$.)

So, $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$ for some constant λ.

We obtained two ordinary differential equations:

(BVP) $X'' + \lambda X = 0, \quad X(0) = 0$ and $X(L) = 0$

(ODE) $T' + \alpha^2 \lambda T = 0$

(BVP) is PDE (1), so it has solutions $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$, each associated with $\lambda_n = n^2\pi^2/L^2$.

Then (ODE) becomes $T' + \alpha^2(n^2\pi^2/L^2)T = 0$ which fundamental solution is $T(t) = e^{-\left(\frac{n^2\pi^2\alpha^2}{L^2}t\right)}$

In particular, $u_n(x,t) = X_n(x)T_n(t) = e^{-\left(\frac{n^2\pi^2\alpha^2}{L^2}t\right)} \sin\left(\frac{n\pi}{L}x\right)$ is a solution for the problem $\alpha^2 x_{xx} = u_t$, $u(0,t) = u(L,t) = 0$ for $t > 0$. So is a linear combination of such solutions:

$$u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n^2\pi^2\alpha^2}{L^2}t\right)} \sin\left(\frac{n\pi}{L}x\right).$$

(Actually, the infinite sum works only for some choice of c_n's, see below.)

Now, Fourier Series result tells us, how to choose c_n's to unsure that $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$ equals $f(x)$ for $0 \leq x \leq L$:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\, dx.$$