Chapter XV: Characterizations of Integrability

Our characterizations of integrability are concerned with continuity. The characterizations require a shift in perspective from the way we thought about continuity previously: Instead of considering various points at which a function is continuous, we consider the entire *set* of points of continuity; actually, we will focus on the set of points of discontinuity. The following discussion motivates the shift in perspective in general terms.

Continuous functions are integrable (Theorem 12.33). On the other hand, we have seen examples of integrable functions that are not continuous; two simple examples are in Example 12.11 and Exercise 12.14. In fact, an integrable function on $[a, b]$ can fail to be continuous at each rational number in $[a, b]$; this is the case for the function f defined on $[0, 1]$ by (as in Exercise 12.21)

$$f(x) = \begin{cases} 0 & \text{, if } x \text{ is irrational} \\ 1 & \text{, if } x = 0 \\ \frac{1}{n} & \text{, if } x \in \mathsf{Q} - \{0\} \text{ and } x = \frac{m}{n} \text{ in lowest terms.} \end{cases}$$

The example we just gave shows that an integrable function can be discontinuous at infinitely many points between any two points of its domain (recall Theorem 1.26). This seems to suggest that, in general, there is no connection between the notions of integrability and continuity.

On the other hand, recall Theorem 12.15: *A function f is integrable over $[a, b]$ if and only if for each $\epsilon > 0$, there is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that*

$$\Sigma_{i=1}^{n}[M_i(f) - m_i(f)]\Delta x_i < \epsilon.$$

Roughly, the theorem says that the integrability of a function f over $[a, b]$ is equivalent to being able to subdivide $[a, b]$ into small intervals (the intervals $[x_{i-1}, x_i]$) on each of which f does not oscillate very much in comparison with the length Δx_i of the interval. Therefore, if we can find an appropriate notion of the length of a set, we may be able to obtain the following type of theorem: *A function f on $[a, b]$ is integrable over $[a, b]$ if and only if the "length" of the set of points at which f is not continuous is zero.* In order to have such a theorem, we see from the example above that our definition of "length" must have the property that the "length" of the set of all rational numbers in any closed and bounded interval is zero; also, since the function in Example 12.12 is not integrable and is not continuous at any point, the "length" of any closed and bounded interval $[a, b]$ itself must *not* be zero (when $a \neq b$).

In section 3, we give a natural definition for a set to have "length" zero, called measure zero. Then we prove that a function is integrable over $[a, b]$ if and only if the set of points at which the function is not continuous has measure zero (Theorem 15.33). Thus, we uncover a close connection between continuity and integrability after all.

1. Background

We discuss four general topics: Countable sets; series; open sets and closed sets; and covers. We primarily focus on the aspects of each topic that we use to get to the characterization theorems in the last section of the chapter. However, we include some examples and results merely for the purpose of providing insight.

Countable Sets

A *one-to-one correspondence* between two sets is a one-to-one function from one of the sets onto the other set.

A *finite set* is a set that can be placed in one-to-one correspondence with the set $\mathsf{N}_n = \{1, 2, ..., n\}$ for some n; we also consider the empty set to be a finite set.

A *countable set* is a set that can be placed in one-to-one correspondence with a subset of the natural numbers. Thus, any finite set is a countable set; we use the term *countably infinite* to refer to a countable set that is not finite. A set that is not countable is called *uncountable*.

Theorem 15.1: A nonempty set X is countable if and only if there is a function from the set N of all natural numbers onto X.

Proof: Assume that there is a function f from N onto X. Then, by the Well Ordering Principle (1.18), there is a least natural number ℓ_x in $f^{-1}(x)$ for each $x \in X$. Let

$$M = \{\ell_x : x \in X\},$$

and define a function $g : M \to X$ by letting $g(\ell_x) = x$ for each $\ell_x \in M$. It is easy to check that g is a one-to-one function from M onto X. Therefore, X is countable.

Conversely, assume that X is countable. Then there is a one-to-one function h from a subset S of the natural numbers onto X. Note that S is nonempty since X is nonempty (by assumption); hence, we can obtain a function r from N onto S as follows: Simply fix a point $m \in S$, and define r by letting

$$r(n) = \begin{cases} n & \text{, if } n \in S \\ m & \text{, if } n \in \mathsf{N} - S. \end{cases}$$

It follows easily that the composition $h \circ r$ is a function from N onto X. ¥

Exercise 15.2: A countable union of countable sets is a countable set; in other words, if A_i is a countable set for each $i = 1, 2, ...,$ then $\cup_{i=1}^{\infty} A_i$ is a countable set.

(*Hint:* Consider the set $\mathsf{N} \times \mathsf{N}$ of all points in the plane whose coordinates are natural numbers; it is easy to describe geometrically (without a formula) a function from N onto $\mathsf{N} \times \mathsf{N}$. What next?)

Series

We defined sequences and limits of sequences in section 8 of Chapter IV. A *series* $\Sigma_{i=1}^{\infty} a_i$, where $a_i \in \mathsf{R}^1$ for each $i = 1, 2, \ldots$, is defined to be the sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n = \Sigma_{i=1}^{n} a_i$ for each n. The finite sums s_n are called the *partial sums* of the series $\Sigma_{i=1}^{\infty} a_i$, and $\{s_n\}_{n=1}^{\infty}$ is called the *sequence of partial sums* of the series $\Sigma_{i=1}^{\infty} a_i$ (the terminology is merely descriptive since, by definition, the sequence of partial sums *is* the series). The numbers a_i are called the *terms of the series* $\Sigma_{i=1}^{\infty} a_i$.

In view of our definition of series and our definition of convergence of sequences in section 8 of Chapter IV, we already know what it means for a series $\Sigma_{i=1}^{\infty} a_i$ to converge to a number p: A series $\Sigma_{i=1}^{\infty} a_i$ converges to p provided that the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums converges to p, which we signify by writing $\Sigma_{i=1}^{\infty} a_i = p$. When $\Sigma_{i=1}^{\infty} a_i = p$, we call p the *sum of the series.* We say that a series *diverges* provided that the series does not converge.

It will be convenient sometimes to use the term *series* to include finite sums. When we want to emphasize that a series may be a finite sum, we use the phrase *a finite or infinite series*. Generally speaking, however, the word series all by itself should be taken to mean an infinite series.

A series of the form $\Sigma_{i=1}^{\infty} ar^i$, where a and r are fixed real numbers, is called *geometric series*; r is called the *common ratio of the series* (r is the ratio of the $(i+1)^{\text{st}}$ term of the series to the i^{th} term).

We determine when a geometric series converges and obtain a simple formula for its sum. First, we prove a lemma that we use many times later as well.

Lemma 15.3: If $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$.

Proof: Since $\lim_{i\to\infty} r^n = 0$ if and only if $\lim_{i\to\infty} |r^n| = 0$, we assume for the proof that $0 < r < 1$. Then $\frac{1}{r} > 1$; hence, letting $c = \frac{1}{r} - 1$, we have that $c > 0$. Thus, since $\frac{1}{r^{n+1}} = (1+c)^{n+1}$ for each n, we see by an easy induction (Theorem 1.20) that for each n,

$$\tfrac{1}{r^{n+1}} = 1 + (n+1)c + [\text{sum of positive terms}] > 1 + (n+1)c.$$

Therefore, since $r > 0$ and $c > 0$, we have

$$(*)\ 0 < r^{n+1} < \tfrac{1}{1+(n+1)c} \quad \text{for each } n.$$

To complete the proof, recall that theorems about limits of functions apply to limits of sequences by Theorem 4.38. Thus,

$$\lim_{n\to\infty} \tfrac{1}{1+(n+1)c} \overset{4.38,\ 4.20}{=} \tfrac{1}{\lim_{n\to\infty} 1+(n+1)c} \overset{1.22}{=} 0.$$

Therefore, by (*) and Theorem 4.38, we can apply the the Squeeze Theorem (Theorem 4.34) to see that $\lim_{n\to\infty} r^{n+1} = 0$. ¥

Theorem 15.4: If $-1 < r < 1$, then the geometric series $\Sigma_{i=1}^{\infty} ar^i$ converges and $\Sigma_{i=1}^{\infty} ar^i = \frac{ar}{1-r}$. If $|r| \geq 1$ and $a \neq 0$, then the series $\Sigma_{i=1}^{\infty} ar^i$ diverges.

Proof: Assume that $-1 < r < 1$. Let $s_n = \Sigma_{i=1}^{n} ar^i$ (the n^{th} partial sum of $\Sigma_{i=1}^{\infty} ar^i$) for each n. We show that $\lim_{n\to\infty} s_n = \frac{ar}{1-r}$, which will prove the first part of the theorem.

Note that

$$s_n - rs_n = \Sigma_{i=1}^{n} ar^i - \Sigma_{i=1}^{n} ar^{i+1} = ar - ar^{n+1}.$$

Hence, $s_n = \frac{a(r-r^{n+1})}{1-r}$. Therefore, $\lim_{n\to\infty} s_n = \frac{ar}{1-r}$ by Lemma 15.3 (and by applying Theorem 4.38 to Theorems 4.2 and 4.9). This proves the first part of the theorem.

We leave the proof of second part of the theorem to the reader. ¥

We discuss series in depth in several later chapters; until then, the material we have presented is for the most part the only information about series we need.

Open Sets and Closed Sets

We discuss the notion of open set and its companion notion, closed set. Open sets and closed sets are the basic notions in the field of mathematics called topology. We give a very brief introduction to these types of sets in R^1.

Open sets can be defined as a generalization of open intervals: A subset of R^1 is an *open set* provided that it is a union of open intervals. This is a good working definition of open set (it can be used right away to prove many theorems); however, it is a terrible *definition*! – it conceals the inherent geometrical idea behind the notion. I have mentioned before that, in my opinion, a definition should convey the fundamental idea behind the notion being defined. Thus, I prefer the following definition:

Definition. A subset U of R^1 is an *open set* provided that no point of U is arbitrarily close to the complement $\mathsf{R}^1 - U$ of U.

The definition of open set shows that open sets are stable in the following sense: A point is in an open set U if and only if all points sufficiently close to the point are points of U (see Theorem 2.3). This stability of open sets is important in many areas of mathematics and in applications (e.g., to dynamical sysyems).

Example 15.5: The following sets are open: $\cup_{n=1}^{\infty}(n, n+1)$; $\mathsf{R}^1 - \{p\}$ for any $p \in \mathsf{R}^1$; $\mathsf{R}^1 - \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$. The following sets are not open: $\{p\}$ for any $p \in \mathsf{R}^1$; N (the set of natural numbers); $[0, 1]$; $[0, 1)$; Q (the set of rationals); $\mathsf{R}^1 - \{1, \frac{1}{2}, \frac{1}{3}, ...\}$.

Our initial (rejected) definition of open set now becomes a theorem:

Theorem 15.6: A subset U of R^1 is an open set if and only if U is a union of open intervals.

Proof: Assume that U is an open set. Let $x \in U$. Then, by the definition of open set, $x \not\to \mathsf{R}^1 - U$. Hence, by Theorem 2.3, there is an open interval I_x such that $x \in I_x$ and $I_x \cap (\mathsf{R}^1 - U) = \emptyset$, which says $I_x \subset U$. Hence, selecting one such interval I_x for each $x \in U$, it is clear that

$$U = \cup_{x \in U} I_x.$$

Therefore, we have written U as a union of open intervals.

Conversely, assume that U is a union of open intervals. Let $p \in U$. Then p is a point of an open interval I such that $I \subset U$; in other words, $p \in I$ and $I \cap (\mathsf{R}^1 - U) = \emptyset$. Hence, by Theorem 2.3, $p \not\frown \mathsf{R}^1 - U$. This proves that no point of U is arbitrarily close to $\mathsf{R}^1 - U$. Therefore, by definition, U is an open set. ¥

Definition: A subset A of R^1 is a *closed set* provided that A contains all points that are arbitrarily close to A.

Exercise 15.7: Which of the sets in Exercise 15.5 are closed sets?

Exercise 15.8: True or false: A subset A of R^1 is a closed set if and only if A is a union of closed intervals.

Exercise 15.9: A subset A of R^1 is a closed set if and only if $\mathsf{R}^1 - A$ is an open set.

Exercise 15.10: A subset A of R^1 is a closed set if and only if A contains all its limit points. (Limit point is defined in section 4 of Chapter II.)

Covers

It is convenient to have a name for a collection of sets whose union contains a given set. The term *cover* is descriptive (and standard):

Definition: Let X be a set, let $A \subset X$, and let $\mathcal{C}$ be a collection of subsets of X.

- $\mathcal{C}$ *covers* A, or $\mathcal{C}$ *is a cover of* A, provided that $\cup\mathcal{C} \supset A$.
- A *subcover of a cover* $\mathcal{C}$ of A is a subcollection $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathcal{C}'$ covers A.

It is important to keep in mind that a subcover of a cover $\mathcal{C}$ of A is not just a subcollection of $\mathcal{C}$: A subcover of $\mathcal{C}$ must also cover A. For example, let $\mathcal{C}$ be the collection of two closed intervals given by

$$\mathcal{C} = \{[0,3],\ [2,5]\};$$

then, considering $\mathcal{C}$ as a cover of the interval $[0,2]$, we see that $\mathcal{C}' = \{[0,3]\}$ is a subcover of $\mathcal{C}$; however, considering $\mathcal{C}$ as a cover of the interval $[1,4]$, we see that $\mathcal{C}$ itself is the only subcover of $\mathcal{C}$.

An important covering property of closed and bounded sets in R^1 is in Exercise 15.13 below. The following exercise shows that being a closed set is necessary in Exercise 15.13:

Exercise 15.11: Give an example of a cover by open sets of the open interval $(0,1)$ that has no finite subcover.

Exercise 15.12: If $\mathcal{C}$ is a cover of $[a, b]$ by open sets, then $\mathcal{C}$ has a finite subcover.

(*Hint:* Use the Nested Interval Property (Theorem 5.11) in conjunction with the bisecting process illustrated by the proof of Theorem 5.13.)

Exercise 15.13: If X is a closed and bounded subset of R^1, then any cover of X by open sets has a finite subcover.

(*Hint:* Use Exercise 15.9 and Exercise 15.12.)

2. Oscillation

For our purpose, the concept of oscillation provides a useful way to describe the points at which a function is not continuous (Exercise 15.19). Oscillation plays a central role in the proofs of the characterization theorems in section 4.

Definition: Let $X \subset \mathsf{R}^1$, and let $f : X \to \mathsf{R}^1$ be a bounded function.

- The *oscillation of f on a nonempty subset A of X*, denoted by $\mathcal{O}_f(A)$, is defined by

$$\mathcal{O}_f(A) = lub\, f(A) - glb\, f(A).$$

- The *oscillation of f at a point p of X*, denoted by $\mathcal{O}_f(p)$, is defined by

$$\mathcal{O}_f(p) = glb\, \{\mathcal{O}_f([p-\delta, p+\delta] \cap X) : \delta > 0\};$$

  from now on, we write $\mathcal{O}_f(p) = glb_{\delta>0}\mathcal{O}_f([p-\delta, p+\delta])$ for ease in notation.

Note the difference between the oscillation of f on the set consisting of a single point p and the oscillation of f at the point p: $\mathcal{O}_f(\{p\}) = 0$, whereas $\mathcal{O}_f(p)$ may well not be zero.

Exercise 15.14: True or false: If $X \subset \mathsf{R}^1$, $f : X \to \mathsf{R}^1$ is a bounded function, and $p \in Y \subset X$, then $\mathcal{O}_f(p) \leq \mathcal{O}_f(Y)$.

Exercise 15.15: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a bounded function, and let A be a nonempty subset of X. If $A \subset B \subset X$, then $\mathcal{O}_f(A) \leq \mathcal{O}_f(B)$.

Exercise 15.16: Let $f : \mathsf{R}^1 \to \mathsf{R}^1$ be the greatest-integer function (i.e., for each $x \in \mathsf{R}^1$, $f(x)$ is the greatest integer that is less than or equal to x). Determine the oscillation of f at various points of R^1.

Exercise 15.17: Let

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0. \end{cases}$$

Compute the oscillation of f on $[0, 1]$ and the oscillation of f at $x = 0$.

Exercise 15.18: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a bounded function, and let $p \in X$. Then f is continuous at p if and only if $\mathcal{O}_f(p) = 0$.

Notation: Let $X \subset \mathsf{R}^1$, and let $f : X \to \mathsf{R}^1$ be a bounded function.

- $\mathcal{D}_f = \{x \in X : f$ is not continuous at $x\}$.
- For each $\eta > 0$, $\mathcal{E}_f(\eta) = \{x \in X : \mathcal{O}_f(x) \geq \eta\}$.

Exercise 15.19: If $X \subset \mathsf{R}^1$ and $f : X \to \mathsf{R}^1$ is a bounded function, then

$$\mathcal{D}_f = \cup_{n=1}^{\infty} \mathcal{E}_f(\tfrac{1}{n}).$$

Exercise 15.20: If $f : [a,b] \to \mathsf{R}^1$ is a bounded function, then $\mathcal{E}_f(\eta)$ is a closed set for all $\eta > 0$.

3. Content Zero and Measure Zero

The characterizations of integrability in the next section are in terms of content zero and measure zero. Intuitively, content zero and measure zero say that a set has small length (content zero says this in a stronger way than measure zero does). As might be expected, we will define a set to have content zero or to have measure zero in terms of sums of lengths of intervals that cover the set; the difference in the two concepts is a matter of the number of sets allowed in the covers.

The *length of a bounded interval* ($[a,b]$, (a,b), $[a,b)$, or $(a,b]$) is $b-a$. For a countable collection of intervals, we use the term *length sum* to refer to the sum of the (finite or infinite) series whose terms are the lengths of the intervals in the collection (length sum may be infinite). We note that length sum is independent of the order in which the terms of the series are written down (see Theorem 22.35).

Definition: A subset A of R^1 is said to have *content zero* provided that for each $\epsilon > 0$, there are finitely many open intervals covering A such that the length sum of the intervals is less than ϵ.

Exercise 15.21: Every finite set has content zero. The interval $[0,1]$ does not have content zero; on the other hand, the set $A = \{\frac{1}{n} : n = 1, 2, ...\}$ is an infinite set that has content zero. Does the set N of all natural numbers have content zero?

Exercise 15.22: Any subset of a set that has content zero has content zero.

Exercise 15.23: The set of all rational numbers in $[0,1]$ does not have content zero.

Definition: A subset A of R^1 is said to have *measure zero* provided that for each $\epsilon > 0$, there are countably many open intervals covering of A such that the length sum of the intervals is less than ϵ.

Exercise 15.24: Any countable subset of R^1 has measure zero. In particular, the set Q of all rational numbers has measure zero (but not content zero by Exercise 15.23).

Exercise 15.25: Any subset of a set of measure zero has measure zero.

Exercise 15.26: The countable union of sets of measure zero has measure zero. (You can use Theorem 22.35.)

Exercise 15.27: Does the set of all irrational numbers in $[0, 1]$ have measure zero?

Theorem 15.28: If a subset of R^1 has content zero, then it has measure zero; conversely, if a closed and bounded subset of R^1 has measure zero, then it has content zero.

Proof: The first part of the theorem is obvious from definitions.

To prove the second part, assume that A is a closed and bounded subset of R^1 such that A has measure zero. Let $\epsilon > 0$. Then there is a countable cover $\mathcal{C}$ of A by open intervals whose length sum is $< \epsilon$. By Exercise 15.13, $\mathcal{C}$ has a finite subcover $\mathcal{F}$. Since $\mathcal{F} \subset \mathcal{C}$, it is clear that the length sum of the intervals in $\mathcal{F}$ is $< \epsilon$. Thus, we have proved that for each $\epsilon > 0$, there are finitely many open intervals covering A such that the length sum of the intervals is less than ϵ. Therefore, A has content zero. ¥

Corollary 15.29: Let A_n be a closed and bounded subset of R^1 for each $n = 1, 2, \ldots$. Then $\cup_{n=1}^{\infty} A_n$ has measure zero if and only if A_n has content zero for each n.

Proof: If $\cup_{n=1}^{\infty} A_n$ has measure zero, then each set A_n has measure zero (by Exercise 15.25); therefore, by the second part of Theorem 15.28, each set A_n has content zero.

Conversely, if each set A_n has content zero, then each set A_n has measure zero (by the first part of Theorem 15.28); therefore, $\cup_{n=1}^{\infty} A_n$ has measure zero (by Exercise 15.26). ¥

4. Characterizations of Integrability

We obtain two characterizations of integrability. The first characterization says that f is integrable over $[a, b]$ if and only if the set of points at which the oscillation of f is $\geq \eta$ has content zero for each $\eta > 0$ (Theorem 15.31). The second characterization says that f is integrable over $[a, b]$ if and only if the set of points at which f is not continuous has measure zero (Theorem 15.33). The two characterizations are obviously related; we use the first characterization to prove the second. We feel that the second characterization is by far the better of the two – after all, it is easier to visualize the points at which a function is not continuous than it is to find the points at which the oscillation of the function is $\geq \eta$ for each η. We give applications of the second characterization in some exercises at the end of the section; included is the theorem about the integrability of quotients (Exercise 15.34) which we promised at the beginning of section 5 of Chapter XIII.

We use the following lemma to construct a special partition in the proof of Theorem 15.31.

Lemma 15.30: Let $f : [a,b] \to \mathsf{R}^1$ be a bounded function such that for some $\epsilon > 0$, $\mathcal{O}_f(x) < \epsilon$ for all $x \in [a,b]$. Then there is a $\delta > 0$ such that

$$\mathcal{O}_f([c,d]) < \epsilon \ \text{ for all } [c,d] \subset [a,b] \text{ such that } d - c < \delta.$$

Proof: Since $\mathcal{O}_f(x) < \epsilon$ for all $x \in [a,b]$, there is a $\delta_x > 0$ for each $x \in [a,b]$ such that

(1) $\mathcal{O}_f([p - \delta_x, p + \delta_x]) < \epsilon$.

Let $\mathcal{C}$ be the following cover of $[a,b]$ by open intervals:

$$\mathcal{C} = \{(x - \tfrac{\delta_x}{2}, x + \tfrac{\delta_x}{2}) : x \in [a,b]\}.$$

Then, by Exercise 15.13, $\mathcal{C}$ has a finite subcover $\mathcal{C}'$, say

$$\mathcal{C}' = \{(x_i - \tfrac{\delta_{x_i}}{2}, x_i + \tfrac{\delta_{x_i}}{2}) : i = 1, 2, ..., n\}.$$

Now, let

$$\delta = \min\{\tfrac{\delta_{x_i}}{2} : i = 1, 2, ..., n\}.$$

We show that this choice of δ satisfies the conclusion of the lemma. Let $[c,d] \subset [a,b]$ such that $0 < d - c < \delta$. Since $c \in [a,b]$ and $\mathcal{C}'$ covers $[a,b]$,

$$c \in (x_k - \tfrac{\delta_{x_k}}{2}, x_k + \tfrac{\delta_{x_k}}{2}) \ \text{ for some } k.$$

Thus, since $d - c < \delta \leq \frac{\delta_{x_k}}{2}$,

$$[c,d] \subset (x_k - \delta_{x_k}, x_k + \delta_{x_k}).$$

Therefore,

$$\mathcal{O}_f([c,d]) \overset{15.15}{\leq} \mathcal{O}_f([x_k - \delta_{x_k}, x_k + \delta_{x_k}]) \overset{(1)}{<} \epsilon. \ ¥$$

We prove our first characterization theorem. Recall from the notation preceding Exercise 15.19 that $\mathcal{E}_f(\eta) = \{x \in [a,b] : \mathcal{O}_f(p) \geq \eta\}$.

Theorem 15.31: Let $f : [a,b] \to \mathsf{R}^1$ be a bounded function. Then f is integrable over $[a,b]$ if and only if $\mathcal{E}_f(\eta)$ has content zero for each $\eta > 0$.

Proof: In the proof, the notation $M_i(f)$, $m_i(f)$, $U_P(f)$ and $L_P(f)$ is from section 2 of Chapter XII.

Assume that there is an $\eta > 0$ such that $\mathcal{E}_f(\eta)$ does not have content zero. We prove that this assumption implies that f is not integrable over $[a,b]$. We prove this by showing that the set of differences $U_P - L_P$ for all partitions P of $[a,b]$ is bounded away from zero and then applying Theorem 12.15.

By our assumption that $\mathcal{E}_f(\eta)$ does not have content zero, there is a $\delta > 0$ satisfying the following:

(1) The length sum of any finitely many
open intervals covering $\mathcal{E}_f(\eta)$ is $\geq \delta$.

Now, let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[a,b]$. Let

$$\mathcal{A} = \{(x_{i-1}, x_i) : (x_{i-1}, x_i) \cap \mathcal{E}_f(\eta) \neq \emptyset\}.$$

Let λ denote the length sum of the intervals in $\mathcal{A}$; in other words,

(2) $\lambda = \Sigma\{\Delta x_i : (x_{i-1}, x_i) \in \mathcal{A}\}$.

We show that

(3) $\lambda \geq \delta$.

Proof of (3)*:* Note that if $\mathcal{E}_f(\eta) \subset \cup\mathcal{A}$, then $\lambda \geq \delta$ by (1). Hence. to prove (3), we can assume that $\mathcal{E}_f(\eta) \not\subset \cup\mathcal{A}$. Thus, $\mathcal{E}_f(\eta) - \cup\mathcal{A}$ is a nonempty subset of P, say

$$\mathcal{E}_f(\eta) - \cup\mathcal{A} = \{x_{i_1}, x_{i_2}, ..., x_{i_k}\}.$$

Now, suppose by way of contradiction that $\lambda < \delta$. Then, letting

$$\mathcal{B} = \{(x_{i_j} - \tfrac{\delta-\lambda}{3k}, x_{i_j} - \tfrac{\delta-\lambda}{3k}) : j = 1, 2, ..., k\},$$

we see that $\mathcal{A} \cup \mathcal{B}$ is a cover of $\mathcal{E}_f(\eta)$ by open intervals whose length sum is $\lambda + k(2\frac{\delta-\lambda}{3k})$. Therefore, since

$$\lambda + k(2\tfrac{\delta-\lambda}{3k}) = \tfrac{1}{3}\lambda + \tfrac{2}{3}\delta < \delta,$$

we have a contradiction to (1). This completes the proof of (3).

For any interval $(x_{i-1}, x_i) \in \mathcal{A}$ and any point $p \in (x_{i-1}, x_i) \cap \mathcal{E}_f(\eta)$,

$$M_i(f) - m_i(f) \geq \mathcal{O}_f(p);$$

thus, we have that

(4) $M_i(f) - m_i(f) \geq \eta$ for any interval $(x_{i-1}, x_i) \in \mathcal{A}$.

Therefore,

$$\begin{aligned} U_P(f) - L_P(f) &= \Sigma_{i=1}^{n}[M_i(f) - m_i(f)]\Delta x_i \\ &\geq \Sigma\{[M_i(f) - m_i(f)]\Delta x_i : (x_{i-1}, x_i) \in \mathcal{A}\} \\ &\overset{(4)}{\geq} \eta\Sigma\{\Delta x_i : (x_{i-1}, x_i) \in \mathcal{A}\} \overset{(2)}{=} \eta\lambda \overset{(3)}{\geq} \eta\delta. \end{aligned}$$

Therefore, since η and δ are independent of P and since P was an arbitrary partition of $[a, b]$, we see from Theorem 12.15 that f is not integrable over $[a, b]$.

Conversely, assume that $\mathcal{E}_f(\eta)$ has content zero for each $\eta > 0$. We prove that this assumption implies that f is integrable over $[a, b]$ by using Theorem 12.15.

Let

$$M = lub\, f([a, b]), \quad m = glb\, f([a, b]).$$

If $M = m$, then f is a constant function and, therefore, f is integrable. Hence, we can assume that

(5) $M \neq m$.

Now, let $\epsilon > 0$. Then, by our assumption for this part of the proof, $\mathcal{E}_f(\frac{\epsilon}{2(b-a)})$ has content zero. Hence, there are finitely many open intervals $I_1, I_2, ..., I_r$ satisfying (6) and (7) below:

(6) $\mathcal{E}_f(\frac{\epsilon}{2(b-a)}) \subset \cup_{i=1}^r I_i$

and (noting that $M - m > 0$ by (5))

(7) the length sum of the intervals $I_1, I_2, ..., I_r$ is $< \frac{\epsilon}{2(M-m)}$.

For use later, we let Q denote the set of endpoints of the intervals $I_1, I_2, ..., I_r$.

Next, let

$$K = [a, b] - \cup_{j=1}^r I_j.$$

If $K = \emptyset$, we disregard K in what follows. Since the intervals $I_1, I_2, ..., I_r$ cover $\mathcal{E}_f(\frac{\epsilon}{2(b-a)})$, we have that

(8) $\mathcal{O}_f(x) < \frac{\epsilon}{2(b-a)}$ for all $x \in K$.

It is easy to see that K is the union of finitely many mutually disjoint closed intervals $L_1, L_2, ..., L_s$. Note that an interval L_j may be of the form $[x, x]$; this could possibly occur if x is a common endpoint of two of the intervals $I_1, I_2, ..., I_r$ or if $x = a$ or b is an endpoint of one of the intervals $I_1, I_2, ..., I_r$ (see Figure 15.31 below – intervals I_j are indicated with parentheses of various sizes; dots represent endpoints of the intervals L_j except the dot for b).

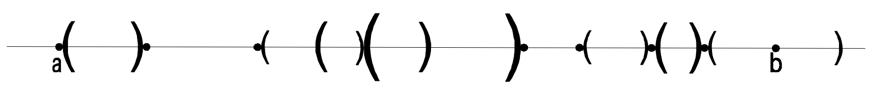

Figure 15.31

By (8) and Lemma 15.30, we see that, for each j, there is a partition $P_j = \{y_0, y_1, ..., y_{n_j}\}$ of L_j such that

(9) $\mathcal{O}_f([y_{i-1}, y_i]) < \frac{\epsilon}{2(b-a)}$ for all $y_{i-1}, y_i \in P_j$.

Now, let $P = (\cup_{j=1}^s P_j) \cup Q$, the points of the partitions $P_1, P_2, ..., P_s$ together with the endpoints of the intervals $I_1, I_2, ..., I_r$. Consider P as a partition of $[a, b]$,

$$P = \{x_0, x_1, ..., x_n\}.$$

There are two mutually distinct types of intervals $[x_{i-1}, x_i]$: Those for which $x_{i-1}, x_i \in P_j$ for some j, and those for which $x_{i-1}, x_i \in Q$. Hence, for each i, either $x_{i-1}, x_i \in P_j$ for some j or $x_{i-1}, x_i \in Q$, and not both. Thus, since

$$U_P(f) - L_P(f) = \Sigma_{i=1}^n [M_i(f) - m_i(f)]\Delta x_i,$$

we have

(10) $U_P(f) - L_P(f) = \Sigma\{[M_i(f) - m_i(f)]\Delta x_i : x_{i-1}, x_i \in Q\}$
$+ \Sigma\{[M_i(f) - m_i(f)]\Delta x_i : x_{i-1}, x_i \in P_j \text{ for some } j\}.$

Now, note that

(11) $\Sigma\{[M_i(f) - m_i(f)]\Delta x_i : x_{i-1}, x_i \in Q\}$

$$\leq \Sigma\{[M - m]\Delta x_i : x_{i-1}, x_i \in Q\} \overset{(7)}{<} [M - m]\frac{\epsilon}{2(M-m)} = \frac{\epsilon}{2}$$

and that

(12) $\Sigma\{[M_i(f) - m_i(f)]\Delta x_i : x_{i-1}, x_i \in P_j \text{ for some } j\}$

$$= \Sigma\{[\mathcal{O}_f([x_{i-1}, x_i])]\Delta x_i : x_{i-1}, x_i \in P_j \text{ for some } j\}$$

$$\overset{(9)}{<} \Sigma\{\frac{\epsilon}{2(b-a)}\Delta x_i : x_{i-1}, x_i \in P_j \text{ for some } j\} \leq \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2}.$$

By (10), (11) and (12),

$$U_P(f) - L_P(f) < \epsilon.$$

We have shown that for any $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \epsilon$. Therefore, f is integrable over $[a, b]$ by Theorem 12.15. ¥

For ready use in the proof of our next theorem, we reformulate the theorem we just proved as follows:

Corollary 15.32: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function. Then f is integrable over $[a, b]$ if and only if $\mathcal{E}_f(\frac{1}{n})$ has content zero for each $n = 1, 2, \ldots$.

Proof: Note from the definition of $\mathcal{E}_f(\eta)$ (above Exercise 15.19) that

$$\mathcal{E}_f(\eta) \subset \mathcal{E}_f(\eta') \quad \text{when } 0 < \eta' \leq \eta;$$

also, recall from Exercise 15.22 that a subset of a set of content zero has content zero. Hence, it follows using the Archimedean Property (Theorem 1.22) that $\mathcal{E}_f(\eta)$ has content zero for each $\eta > 0$ if and only if $\mathcal{E}_f(\frac{1}{n})$ has content zero for each $n = 1, 2, \ldots$. Therefore, our corollary now follows immediately from Theorem 15.31. ¥

We are ready to prove our main characterization of intergability. The proof is very short because the theorem is an easy consequence of Corollary 15.32 and three previous results.

Theorem 15.33: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function. Then f is integrable over $[a, b]$ if and only if the set $\mathcal{D}_f$ of points of discontinuity of f has measure zero.

Proof: By Exercise 15.19,

$$\mathcal{D}_f = \cup_{n=1}^{\infty} \mathcal{E}_f(\tfrac{1}{n});$$

furthermore, for each $n = 1, 2, \ldots,$ $\mathcal{E}_f(\frac{1}{n})$ is a closed set (by Exercise 15.20) and $\mathcal{E}_f(\frac{1}{n})$ is bounded (since $\mathcal{E}_f(\frac{1}{n}) \subset [a, b]$). Thus, by Corollary 15.29, $\mathcal{D}_f$ has measure zero if and only if $\mathcal{E}_f(\frac{1}{n})$ has content zero for each n. Therefore, our theorem follows from Corollary 15.32. ¥

Exercise 15.34: Illustrate the applicability of Theorem 15.33 by using it to prove the following result, which provides full generality for Theorem 13.36 (the result applies to functions not covered by Theorem 13.36 in view of Exercise 13.38):

If f and g are integrable over $[a, b]$, $g(x) \neq 0$ for any x, and $\frac{f}{g}$ is bounded, then $\frac{f}{g}$ is integrable over $[a, b]$.

Exercise 15.35: Illustrate the applicability of Theorem 15.33 by using it to give a very short, simple proof of Theorem 13.26. (The proof we gave for Theorem 13.26 was reasonably short, but only because the proof depended on several previous lemmas and theorems in Chapter XIII.)

Exercise 15.36: Illustrate the applicability of Theorem 15.33 by using it to work Exercise 12.21.

Exercise 15.37: If X is a set and A is a nonempty subset of X, then the *characteristic function for* A, denoted by χ_A, is defined by

$$\chi_A = \begin{cases} 1 & \text{, if } x \in A \\ 0 & \text{, if } x \in X - A. \end{cases}$$

Prove the following result:

If $A \subset [a, b]$ and A is a nonempty closed set, then A has measure zero if and only if χ_A is integrable over $[a, b]$ and $\int_a^b \chi_A = 0$.

Is it necessary for A to be a closed set in the result?

Exercise 15.38: Give an example of an integrable function on $[0, 1]$ that is not continuous at uncountably many points.

Exercise 15.39: True or false: Let f and g be bounded functions on $[a, b]$ such that $f(x) = g(x)$ except for those points x in a set of measure zero; if f is integrable over $[a, b]$, then g is integrable over $[a, b]$.

Exercise 15.40: True or false: If f is integrable over $[a, b]$ and $f(x) \geq 0$ except for those points x in a set of measure zero, then $\int_a^b f \geq 0$.

Chapter XVI: The Natural Logarithm and Exponential Functions

Consider the very simple function $f(t) = \frac{1}{t}$ on an interval $[a, b]$, $a > 0$. This is the simplest rational function that is not a polynomial. Nevertheless, we do not know a formula for a function F whose derivative is f even though F exists (by part (1) of the Fundamental Theorem of Calculus (Theorem 14.2)). On the other hand, we *do* know how to find such formulas for some more complicated, closely related functions: For example, consider the function f_q for any rational number $q \neq 1$ given by

$$f_q(x) = \frac{1}{x^q} \quad \text{for all } x \in [a, b];$$

then $F_q(x) = \frac{x^{1-q}}{1-q}$ is a function whose derivative is f_q (by Theorem 8.16); furthermore, the function f_q can be chosen as close to f on $[a, b]$ as we like by letting q be close enough to 1. What is so unusual here is that we know more about the complicated functions which approximate the simple function f than we know about the simple function. There is an obvious conclusion to draw – in the context of what we are discussing, the function $f(t) = \frac{1}{t}$ is the function that is complicated, not the functions f_q, and our original opinion that f is simpler than f_q is actually an optical illusion.

We will *not* find a formula, as the term is commonly used, for a function whose derivative is f, where $f(t) = \frac{1}{t}$. Instead, we will discover that the function $F(x) = \int_1^x f$, whose derivative is f (by Theorem 14.2), has some surprising algebraic properties; the properties will enable us to know that F is a special function, called the natural logarithm function, that was originally studied from a completely different point of view.

We devote the chapter to studying the natural logarithm function and its inverse (the natural exponential function). The chapter lays the foundation for studying logarithm and exponential functions in general, which we do in the next chapter.

1. Preliminary: Reversing the Limits of Integration

Throughout most of the chapter, we investigate the specific function f defined on the positive reals by $f(x) = \int_1^x \frac{1}{t}$ for each $x > 0$. Note that there is a problem with writing $\int_1^x \frac{1}{t}$ when $x < 1$; the problem is that we have only considered integrals $\int_a^b f$ when $a \leq b$. In other words, we need a reasonable definition for $\int_b^a f$ when $a < b$. We determine the proper defintion in this section.

The most obvious definition for $\int_b^a f$ when $a < b$ is to simply define $\int_b^a f = \int_a^b f$, thereby ignoring the order in which the limits of integration are written. However, there is a significant drawback to doing this: A natural and useful extension of the Fundamental Theorem of Calculus would fail for no good reason, but simply because of our arbitrary definition. We will present the extension of the Fundamental Theorem of Calculus we have in mind after we arrive at a reasonable definition for $\int_b^a f$ when $a < b$. We first prove a theorem and a corollary; these results lead us in a natural way to the definition we want.

Recall that part (1) of the Fundamental Theorem of Calculus (Theorem 14.2) says that if f is continuous on $[a, b]$, then

$$\left(\int_a^x f\right)' = f(x) \quad \text{for all } x \in [a, b].$$

The following theorem determines the derivative of the function obtained by fixing the upper limit of integration rather than the lower limit.

Theorem 16.1: If $a < b$ and $f : [a, b] \to \mathsf{R}^1$ is a continuous function, then

$$\left(\int_x^b f\right)' = -f(x) \quad \text{for all } x \in [a, b].$$

Proof: Let $h(x) = \int_x^b f$ for all $x \in [a, b]$. Recall from Theorem 12.33 that f is integrable over $[a, b]$; hence, by Theorem 13.40, $(\int_a^x f) + h(x) = \int_a^b f$. Thus,

$$h(x) = \int_a^b f - \int_a^x f \quad \text{for all } x \in [a, b].$$

Therefore, since $\int_a^b f$ is a constant and $\left(\int_a^x f\right)' = f(x)$ by part (1) of the Fundamental Theorem of Calculus (Theorem 14.2), we see that h is differentiable and that

$$h'(x) \overset{7.3}{=} 0 - f(x) = -f(x). \text{ ¥}$$

The following corollary is an extension of the first part of the Fundamental Theorem of Calculus:

Corollary 16.2: Assume that $a < b$ and that $f : [a, b] \to \mathsf{R}^1$ is a continuous function. Let $c \in [a, b]$, and define $F : [a, b] \to \mathsf{R}^1$ as follows:

$$F(x) = \begin{cases} \int_c^x f & , \text{ if } x \geq c \\ -\int_x^c f & , \text{ if } x < c. \end{cases}$$

Then $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof: If $x \geq c$, then part (1) of the Fundamental Theorem of Calculus (Theorem 14.2) shows that $F'(x) = f(x)$. If $x < c$, then Theorem 16.1 shows that

$$\left(\int_x^c f\right)' = -f(x);$$

therefore, since $F(x) = -\int_x^c f$ when $x < c$, we have by Theorem 7.4 that $F'(x) = -(-f(x)) = f(x)$. ¥

We use Corollary 16.2 often, so we want to avoid writing the formula for the function F in the corollary in two parts. This is easily accomplished by extending our definition of the integral as follows:

Definition: For any function f that is integrable over $[a, b]$, $a \leq b$, we define

$$\int_b^a f = -\int_a^b f;$$

we call $\int_b^a f$ a *negatively oriented integral*.[9]

We can now state Corollary 16.2 in a more succinct way that makes it an obvious direct extension of part (1) of the Fundamental Theorem of Calculus:

Corollary 16.3: Assume that $a < b$, $f : [a, b] \to \mathsf{R}^1$ is a continuous function, and $c \in [a, b]$. Then

$$\left(\int_c^x f\right)' = f(x) \quad \text{for all } x \in [a, b].$$

Proof: In view of the definition we just gave, the corollary is the same as Corollary 16.2. ¥

Exercise 16.4: For any $a > 0$, $\frac{a-1}{a} \leq \int_1^a \frac{1}{t} \leq a - 1$.

We point out that with obvious changes in sign or directions of inequalities, or sometimes with no changes at all, results about integrals in previous chapters are valid for negatively oriented integrals.

2. Algebraic Properties of $\int_1^x \frac{1}{t}$

From the definition in the preceding section, the notation $\int_1^x \frac{1}{t}$ now has meaning when $0 < x < 1$ and, hence, for all $x > 0$. In other words, $f(x) = \int_1^x \frac{1}{t}$ defines a function for all $x > 0$.

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[9] In books, negatively oriented integrals – although they are not given a name – are included at about the same time that integrals are defined, with the following implicit or sometimes explicit "explanation":

If $a < b$ and $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, b]$, then the quantities $\Delta x_i = x_i - x_{i-1}$ in upper and lower sums from a to b are positive ; the quantities Δx_i change to $x_{i-1} - x_i$ in upper and lower sums from b to a and, thus, they are negative, so $\int_a^b f = -\int_b^a f$.

It is therefore implanted in our minds that integrals depend on the two orientations on $[a, b]$ – left to right, right to left – and, hence, that there are two integrals of a single function over $[a, b]$ – one from a to b and the other from b to a. However, *we* have never said such a thing; in fact, we have purposely always used the phrase *integral over* $[a, b]$ to avoid any suggestion that there are two integrals of the same function. Furthermore, we have done the basic theory of integration without being concerned with negatively oriented integrals: We have only introduced negatively oriented integrals at the time in our development when they are needed. Most importantly, we have explained the definition of negatively oriented integrals in an appropriate way for calculus of a single real variable – as a notion that comes from a natural, desirable generalization of the Fundamental Theorem of Calculus (rather than as an arcane consequence of the definition of upper and lower sums).

Oriented integrals do actually arise naturally when integrals are initially defined, but this occurs in multidimensional calculus when defining integrals around oriented curves. To suggest that this geometry can be seen and understood for integrals over intervals is, at best, wishful thinking.

We do not want to continually write $f(x) = \int_1^x \frac{1}{t}$ when studying the function f; as a matter of convenience, we also want to give the function f a name. Thus, we are faced with a dilemma: We can use a distinctive letter for the function and not name the function – which results in stating important results without distinguished, proper notation and terminology – or we can use the relevant notation and terminology – which opens us up to the accusation of getting ahead of our story about the function and introducing terminology that seems to make no sense. We choose the latter and ask the reader not to worry for now about where the notation and terminology come from – it will be explained later (in section 6), after we have laid the proper foundation.

Definition: The function $f : (0, \infty) \to \mathsf{R}^1$ defined by

$$f(x) = \int_1^x \frac{1}{t} \quad \text{for all } x > 0$$

is called the *natural logarithm function* and is denoted by ln; in other words,

$$\ln(x) = \int_1^x \frac{1}{t} \quad \text{for all } x > 0.$$

We prove in Theorem 16.7 that the natural logarithm function converts multiplication to addition. We use this result, and some general results, to derive other algebraic properties of the natural logarithm function (Corollaries 16.9 and 16.12).

Lemma 16.5: $\ln(1) = 0$ and $\ln'(x) = \frac{1}{x}$ for all $x > 0$.

Proof: The first part of the lemma is true simply because $\int_a^a f = 0$ for any function f (recall definition of the integral in section 3 of Chapter XII). The second part is true by Corollary 16.3. ¥

Having just given a formula for the derivative of the natural logarithm function, we note that we obtain a formula for the integral of the natural logarithm function in section 5 (Theorem 16.29). At this time you can read section 5 with complete understanding: The thought process in section 5 that leads to the formula for $\int_a^b \ln(x)$ is natural and is independent of all material in this chapter except the lemma we just proved.

Exercise 16.6: Find a function whose derivative is $\tan(x)$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Repeat the exercise for the interval $[\frac{3\pi}{4}, \frac{5\pi}{4}]$.

Theorem 16.7: $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y > 0$.

Proof: Fix $a > 0$. Let $f : (0, \infty) \to \mathsf{R}^1$ be the function defined by letting $f(y) = ay$ for all $y > 0$.

Note that for all $y > 0$, $\ln'(y) = \frac{1}{y}$ (by Lemma 16.5) and $f'(y) = a$ (Example 6.2). Hence, by the Chain Rule (Theorem 7.23),

$$(\ln \circ f)'(y) \overset{7.23}{=} \ln'(f(y))f'(y) = \frac{1}{ay}(a) = \frac{1}{y} = \ln'(y) \quad \text{for all } y > 0.$$

Thus, by Theorem 10.8, we have that

(1) $(\ln \circ f)(y) - \ln(y) = C$ for all $y > 0$, C fixed.

The constant C in (1) is $\ln(a)$ since

$$C \overset{(1)}{=} \ln(f(1)) - \ln(1) = \ln(a) - \ln(1) \overset{16.5}{=} f(a).$$

Hence, by (1), $(\ln \circ f)(y) - \ln(y) = \ln(a)$ for all $y > 0$; in other words,

$$\ln(ay) = \ln(a) + \ln(y) \quad \text{for all } y > 0.$$

Therefore, since $a > 0$ was arbitrary, we have proved the theorem. ¥

Theorem 16.8: If $f : (0, \infty) \to \mathsf{R}^1$ is a function such that

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y > 0,$$

then

$$f(\tfrac{x}{y}) = f(x) - f(y) \quad \text{for all } x, y > 0.$$

Proof: First, note that $f(1) = f(1 \cdot 1) = f(1) + f(1)$ and, hence, $f(1) = 0$. Thus, for any $y > 0$,

$$0 = f(1) = f(y\tfrac{1}{y}) = f(y) + f(\tfrac{1}{y});$$

hence,

(1) $f(\frac{1}{y}) = -f(y)$ for all $y > 0$.

Finally, for any $x, y > 0$,

$$f(\tfrac{x}{y}) = f(x\tfrac{1}{y}) = f(x) + f(\tfrac{1}{y}) \overset{(1)}{=} f(x) - f(y). \text{ ¥.}$$

Corollary 16.9: $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$ for all $x, y > 0$.

Proof: The corollary follows from Theorem 16.8 since ln satisfies the hypothesis of Theorem 16.8 by Theorem 16.7. ¥

Exercise 16.10: If $f : (0, \infty) \to \mathsf{R}^1$ is a function such that

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y > 0,$$

then, for any integer k,

$$f(x^k) = kf(x) \quad \text{for all } x > 0.$$

For the next theorem, recall that we defined x^r when r is rational in the third bullet in the definition above Theorem 8.15.

Theorem 16.11: If $f : (0, \infty) \to \mathsf{R}^1$ is a function such that

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y > 0,$$

then, for any rational number r,

$$f(x^r) = rf(x) \quad \text{for all } x > 0.$$

Proof: Fix a rational number $r = \frac{m}{n}$, where m is and n are integers ($n \neq 0$). Fix $x > 0$. Then

$$f(x) = f((x^{\frac{1}{n}})^n) \overset{16.10}{=} nf(x^{\frac{1}{n}});$$

hence,

(1) $f(x^{\frac{1}{n}}) = \frac{1}{n} f(x)$.

Since $x^r = (x^{\frac{1}{n}})^m$ (by the definition above Theorem 8.15), we have that

$$f(x^r) = f((x^{\frac{1}{n}})^m) \overset{16.10}{=} mf(x^{\frac{1}{n}}) \overset{(1)}{=} m[\tfrac{1}{n} f(x)] = rf(x). \ ¥$$

Corollary 16.12: For any rational number r, $\ln(x^r) = r\ln(x)$ for all $x > 0$.

Proof: The corollary follows from Theorem 16.11 since ln satisfies the hypothesis of Theorem 16.11 by Theorem 16.7. ¥

We could ask if Corollary 16.12 is true for all real numbers r, but there is a problem with asking this question at this time: We have not defined x^r for all real numbers r. We will soon be able to give a reasonable definition of x^r for any real number r and $x > 0$ (section 4); then we will see that $\ln(x^r) = r\ln(x)$ for all real numbers r and $x > 0$ (Exercise 16.22).

Exercise 16.13: Prove Corollary 16.12 in a manner similar to the way we proved Theorem 16.7 by considering the functions $f(x) = \ln(x^r)$ and $g(x) = r\ln(x)$.

3. The Graph of $\ln(\mathsf{x}) = \int_1^{\mathsf{x}} \frac{1}{\mathsf{t}}$

We prove three corollaries and then use the corollaries to obtain a picture of the graph of the natural logarithm function ln.

Corollary 16.14: ln is strictly increasing.

Proof: By Lemma 16.5, $\ln'(x) = \frac{1}{x} > 0$ for all $x > 0$. Therefore, ln is strictly increasing by part (1) of Theorem 10.17. ¥

Corollary 16.15: ln is concave down.

Proof: By Lemma 16.5, $\ln'(x) = \frac{1}{x}$ for all $x > 0$; hence, by Lemma 7.5,

$$\ln''(x) = \frac{-1}{x^2} \quad \text{for all } x > 0.$$

Thus, $\ln''(x) < 0$ for all $x > 0$. Therefore, ln is concave down by part (2) of Corollary 10.31. ¥

The two corollaries we just proved, and the fact that $\ln(1) = 0$ (Lemma 16.5), give us a lot of information about the graph of ln. However, two issues still need to be addressed: What happens as x approaches 0 (from the right), and what happens as x approaches ∞? Specifically, does ln have a vertical asymptote at $x = 0$ (meaning $\lim_{x\to 0} \ln(x) = \infty$), and does ln have a horizontal asymptote (meaning $\lim_{x\to\infty} \ln(x) = c$)? The answers are *yes* and *no*, respectively, and come from combining Corollary 16.14 with the following corollary:

Corollary 16.16: ln maps $(0, \infty)$ onto R^1.

Proof: We make note of the following fact (from Lemma 16.5):

(1) $\ln(1) = 0$.

Now, let $y \in \mathsf{R}^1$. We prove the corollary by showing that y is a value of ln.

Assume first that $y \geq 0$. By (1) and the fact that ln is strictly increasing (Corollary 16.14), we see that $\ln(2) > 0$. Hence, by the Archimedean Property (Theorem 1.22), there is a natural number n_0 such that $y < n_0 \ln(2)$. Thus, since $n_0 \ln(2) = \ln(2^{n_0})$ (by Corollary 16.12), we have that

(2) $\ln(2^{n_0}) > y$.

Since the function ln is differentiable (by Lemma 16.5), ln is continuous (by Theorem 6.14). Therefore, by (1), (2) and the Intermediate Value Theorem (Theorem 5.2), $y = \ln(x)$ for some x such that $1 \leq x < 2^{n_0}$.

Finally, assume that $y < 0$. By (1) and the fact that ln is strictly increasing (Corollary 16.14), we see that $\ln(\frac{1}{2}) < 0$; that is, $-\ln(\frac{1}{2}) > 0$. Hence, by the Archimedean Property (Theorem 1.22), there is a natural number n_1 such that $-y < n_1[-\ln(\frac{1}{2})]$. Thus, $n_1 \ln(\frac{1}{2}) < y$. Thus, since $n_1 \ln(\frac{1}{2}) = \ln((\frac{1}{2})^{n_1})$ (by Corollary 16.12), we have that

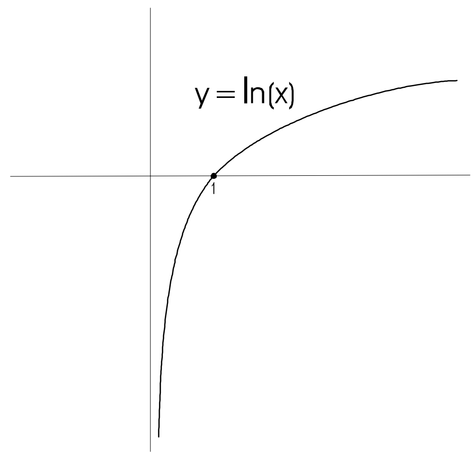
(3) $\ln((\frac{1}{2})^{n_1}) < y$.

By (1), (3) and the Intermediate Value Theorem (Theorem 5.2), $y = \ln(x)$ for some x such that $(\frac{1}{2})^{n_1} < x < 1$. ¥

Now, putting the three preceding corollaries together, we see that the figure on the next page is a correct picture of the graph of ln. To fully appreciate the significance of what we have accomplished, note that we have graphed a function without knowing an algebraic formula for the function and only knowing one value of the function!

4. The Inverse of $\ln(x) = \int_1^x \frac{1}{t}$

The natural logarithm function ln has a (unique) inverse $\ln^{-1}$ by Corollary 16.14. We establish various properties of the inverse function $\ln^{-1}$. The algebraic properties of $\ln^{-1}$ in Theorem 16.19 lead us to a reasonable definition of a^t for any $a > 0$ and *any* real number t; we address the appropriateness of the



definition in Exercises 16.20 and 16.21. We then find the number e that shows that the inverse of ln should be called an exponential function (Theorem 16.23). Corollary 16.24 and the result in Exercise 16.25 are of particular interest in connection with the discussion at the beginning of section 2 of Chapter X.

Since we have graphed ln in the preceding section, we know what the graph of $\ln^{-1}$ looks like: it is simply the reflection of the graph of ln about the line $y = x$ (for the reason given in section 1 of Chapter VIII). Nevertheless, we include the following theorem for convenient reference.

Theorem 16.17: The function ln has a (unique) inverse function $\ln^{-1}$ which is defined on all of R^1 and which maps R^1 onto $(0, \infty)$.

Proof: Since ln is strictly increasing (Corollary 16.14), ln is obviously one-to-one; hence, ln has a (unique) inverse function $\ln^{-1}$. The rest of the theorem follows at once from Corollary 16.16. ¥

Exercise 16.18: $\ln^{-1}$ is differentiable and $(\ln^{-1})'(x) = \ln^{-1}(x)$ for all $x \in \mathsf{R}^1$.

Our next theorem gives algebraic properties of $\ln^{-1}$; the property in part (4) will be extended to all real numbers r in Exercise 16.22 after we define a^r for any real number r.

Theorem 16.19: The inverse function of ln has the following properties:

(1) $\ln^{-1}(0) = 1$.

(2) $\ln^{-1}(x)\ln^{-1}(y) = \ln^{-1}(x + y)$ for all $x, y \in \mathsf{R}^1$.

(3) $\frac{\ln^{-1}(x)}{\ln^{-1}(y)} = \ln^{-1}(x - y)$ for all $x, y \in \mathsf{R}^1$.

(4) $[\ln^{-1}(x)]^r = \ln^{-1}(rx)$ for all $x \in \mathsf{R}^1$ and all rational numbers r.

Proof: Part (1) is by Lemma 16.5. Parts (2), (3) and (4) follow from "corresponding" results for ln in section 2, as we show below.

Fix $x, y \in \mathsf{R}^1$. Then, for part (2),

$$\ln\left(\ln^{-1}(x)\ln^{-1}(y)\right) \overset{16.7}{=} \ln\left(\ln^{-1}(x)\right) + \ln\left(\ln^{-1}(y)\right)$$

$$= x + y = \ln\left(\ln^{-1}(x+y)\right),$$

and, therefore, part (2) now follows from the one-to-oneness of ln; similarly, for part (3),

$$\ln\left(\frac{\ln^{-1}(x)}{\ln^{-1}(y)}\right) \overset{16.9}{=} \ln\left(\ln^{-1}(x)\right) - \ln\left(\ln^{-1}(y)\right)$$

$$= x - y = \ln\left(\ln^{-1}(x-y)\right),$$

and, thus, part (3) now follows from the one-to-oneness of ln; finally, for part (4), fixing a rational number r,

$$\ln\left([\ln^{-1}(x)]^r\right) \overset{16.12}{=} r\ln\left(\ln^{-1}(x)\right) = rx = \ln\left(\ln^{-1}(rx)\right)$$

and, therefore, part (4) now follows from the one-to-oneness of ln. ¥

The properties in Theorem 16.19 look a lot like many of the familiar laws of exponents: If we agree to write a^t for $\ln^{-1}(t)$, then the properties of $\ln^{-1}$ in Theorem 16.19 become

$$a^0 = 1,\ \ a^x a^y = a^{x+y},\ \ \tfrac{a^x}{a^y} = a^{x-y},\ \ (a^x)^r = a^{rx}\ (r \text{ rational}).$$

But this is meaningless formal symbolic manipulation – we have *not even defined* a^t and, when we do, we will need to see if there is a particular number a for which $a^t = \ln^{-1}(t)$ for all t. The next definition and Theorem 16.23 settle the matter. (Regarding the appropriateness of the definition, see Exercises 16.20 and 16.21).

Definition: For any $a > 0$ and $t \in \mathsf{R}^1$, we define the t^{th} *power of* a, written a^t, by

$$a^t = \ln^{-1}[t\ln(a)],$$

which is defined for each t since $\ln^{-1}$ is defined on all of R^1 (see Theorem 16.17).

The definition of a^t may seem somewhat mysterious. Nevertheless, the first two exercises below indicate why the definition is appropriate, even "correct", by relating the definition to rational powers as we have known them ever since studying arithmetic.

Exercise 16.20: Let $a > 0$. Show that when r is a rational number, the definition of a^r above agrees with the definition of a^r in section 4 of Chapter VIII (third bullet in the definition above Theorem 8.15).

Exercise 16.21: Show that the function $f(t) = a^t$ is continuous on R^1 and that *no other* continuous function on R^1 agrees with f on the rational numbers. (Thus, the definition of a^t above is the only way to define powers so that taking powers is continuous and so that, by the preceding exercise, the definition for rational powers agrees with the definition in section 4 of Chapter VIII.)

Exercise 16.22: The definition of a^t allows us to remove the restriction that r is rational in Corollary 16.12 and in part (4) of Theorem 16.19:

$$\ln(x^r) = r\ln(x) \quad \text{for all } x > 0 \text{ and all } r \in \mathsf{R}^1;$$
$$[\ln^{-1}(x)]^r = \ln^{-1}(rx) \text{ for all } x, r \in \mathsf{R}^1.$$

Theorem 16.23: Let $e = \ln^{-1}(1)$. Then $e^t = \ln^{-1}(t)$ for all $t \in \mathsf{R}^1$.

Proof: Let $t \in \mathsf{R}^1$. By the definition above, $e^t = \ln^{-1}[t\ln(e)]$; also, by the choice of e in our theorem, $\ln(e) = 1$. Therefore, $e^t = \ln^{-1}(t)$. ¥

For more about the number e, see the discussion following the proof of Corollary 16.26.

We mentioned the following corollary and exercise in the discussion at the beginning of section 2 of Chapter X.

Corollary 16.24: $(e^x)' = e^x$ for all $x \in \mathsf{R}^1$ (where $e = \ln^{-1}(1)$).

Proof: The corollary is due to Theorem 16.23 and Exercise 16.18. ¥

Exercise 16.25: We can now determine all differentiable functions whose derivatives are themselves: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is a differentiable function, then $f' = f$ if and only if $f(x) = ce^x$ for some constant c (and all $x \in \mathsf{R}^1$).
(*Hint:* Rewrite $f'(x) = f(x)$ as $e^{-x}f'(x) - e^{-x}f(x) = 0$.)

The simple formula for the derivative of e^x allows us to easily integrate e^x :

Corollary 16.26: The function e^x is integrable over any closed and bounded interval $[a, b]$ and $\int_a^b e^x = e^b - e^a$.

Proof: The corollary follows at once from Corollary 16.24 and part (2) of the Fundamental Theorem of Calculus (Theorem 14.2). ¥

Since the function e^x is the inverse of the natural logarithm function (by Theorem 16.23), we call e^x the *natural exponential function*. We also use the term *natural* in connection with the function e^x for another reason, which we discuss at the beginning of section 4 of Chapter XVII.

The use of the letter e dates back to Leonhard Euler (1707 - 1783). He used e to stand for exponential (*not* because e was the first letter of his last name!). The number e can be expressed as a limit of a specific sequence of numbers rather than in terms of the natural logarithm function:

Exercise 16.27: Prove that $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$ and, hence, that

$$e = \lim_{n\to\infty}\left(1 + \tfrac{1}{n}\right)^n, \text{ where } n = 1, 2, \dots .$$

(*Hint:* $1 = \ln'(1) = \lim_{x\to 0}\frac{\ln(1+x) - \ln(1)}{x}$.)

In Chapter XXI we use Exercise 16.27 to represent e as the sum of a series (Theorem 21.42).

Exercise 16.28: Let $f(x) = x^x$ for all $x > 0$. At what points x (if any) does f have local or global extrema?

5. Integrating the Natural Logarithm Function

We have formulas for the derivative of the natural logarithm function (Lemma 16.5) and for the derivative and the integral of the natural exponential function (Corollaries 16.24 and 16.26). In this section we complete the basic calculus of these two functions by finding a formula for the integral of the natural logarithm function. Note that since the natural logarithm function is continuous (by Lemma 16.5 and Theorem 6.14), the natural logarithm function is, indeed, integrable over any closed and bounded interval of positive reals (by Exercise 5.3 and Theorem 12.33); however, this does not give us a formula for the integral.

Part (2) of the Fundamental Theorem of Calculus (Theorem 14.2) gives us a way to find a formula for the integral of any continuous function. To apply this theorem to ln, we need to find a function g such that $g' = \ln$; that is, such that

$$g'(x) = \int_1^x \frac{1}{t}.$$

To understate the situation, it is not immediately obvious how to find such a function g. Nevertheless, a little thought – a naive approach – leads to the answer. We first discuss the approach in general.

Suppose we are given a function f and we want to find a function g whose derivative is f. Suppose further that f is differentiable. There are two natural (although outrageously simplistic) ways to try to find g: start by trying $g(x) = \frac{[f(x)]^2}{2}$ or by trying $g(x) = xf(x)$. Both of these attempts are natural since, on differentiating g, we are sure to get an expression with f in it: In the first case,

$$(*)\ g'(x) \overset{7.23}{=} f(x)f'(x)$$

and, in the second case,

$$(**)\ g'(x) \overset{7.4}{=} xf'(x) + f(x).$$

We analyze each case in turn.

Consider (*) when $f(x) = \ln(x)$. Then, since $f'(x) = \frac{1}{x}$ (by Lemma 16.5), we must find g such that $g'(x) = \frac{\ln(x)}{x}$. This seems more complicated than finding g such that $g'(x) = \ln(x)$.

So, let us turn to (**). If we knew a function h whose derivative is $xf'(x)$, we would be done since

$$(\#)\ (g-h)'(x) \overset{7.2}{=} g'(x) - h'(x) = xf'(x) + f(x) - xf'(x) = f(x).$$

In practice, we can almost never find such a function h – the function $xf'(x)$ is just too complicated. However, when $f(x) = \ln(x)$, the function $xf'(x)$ is about as simple as possible: $x\ln'(x) = 1$ (by Lemma 16.5)! Hence, $h(x) = x$ is a function whose derivative is $x\ln'(x)$. Therefore, recalling that $g(x) = xf(x)$ was the choice that led to (**), we have found a function whose derivative is ln; namely, by (#), $g - h$ is such a function, where $g(x) = x\ln(x)$ and $h(x) = x$. In other words,

$$(\#\#)\ \Big(x\ln(x) - x\Big)' = \ln(x).$$

The discussion above explains how we arrive at the formula in the following theorem:

Theorem 16.29: The function ln is integrable over any closed and bounded interval $[a, b]$, where $a > 0$, and

$$\int_a^b \ln(x) = b\ln(b) - a\ln(a) + a - b.$$

Proof: Since ln is differentiable (by Lemma 16.5), ln is continuous (by Theorem 6.14); hence, ln is continuous on $[a, b]$ (by Exercise 5.3). Therefore, by the formula in (##) preceding the theorem, we can apply part (2) of the Fundamental Theorem of Calculus (Theorem 14.2) to obtain that

$$\int_a^b \ln(x) = b\ln(b) - a\ln(a) + a - b.\ ¥$$

6. What Have We Accomplished?

We have accomplished a lot, more than may be immediately apparent. Let us see what we have accomplished other than our specific results.

First, we can now state and prove the laws of exponents in complete generality (which justifies using the notation a^t in the definition in section 4):

Theorem 16.30: Fix $a > 0$.

(1) $a^0 = 1$.

(2) $a^x a^y = a^{x+y}$ for all $x, y \in \mathsf{R}^1$.

(3) $\frac{a^x}{a^y} = a^{x-y}$ for all $x, y \in \mathsf{R}^1$.

(4) $(a^x)^y = a^{xy}$ for all $x, y \in \mathsf{R}^1$.

(5) $(ab)^x = a^x b^x$ for any $b > 0$ and all $x \in \mathsf{R}^1$.

Proof: We prove each part in turn. Fix $x, y \in \mathsf{R}^1$ for use after the proof of part (1).

Proof of (1): $a^0 = \ln^{-1}[0\ln(a)] = \ln^{-1}(0) \overset{16.19}{=} 1$.

Proof of (2): $a^x a^y = \ln^{-1}[x\ln(a)]\ln^{-1}[y\ln(a)] \overset{16.19}{=} \ln^{-1}[x\ln(a) + y\ln(a)]$
$= \ln^{-1}[(x+y)\ln(a)] = a^{x+y}$.

Proof of (3): $\frac{a^x}{a^y} = \frac{\ln^{-1}[x\ln(a)]}{\ln^{-1}[y\ln(a)]} \overset{16.19}{=} \ln^{-1}[x\ln(a) - y\ln(a)]$
$= \ln^{-1}[(x-y)\ln(a)] = a^{x-y}$.

Proof of (4): $(a^x)^y = \ln^{-1}[y\ln(a^x)] \overset{16.22}{=} \ln^{-1}[yx\ln(a)] = a^{yx} = a^{xy}$.

Proof of (5): For any given $b > 0$,

$$(ab)^x = \ln^{-1}[x\ln(ab)] \overset{16.7}{=} \ln^{-1}[x\left(\ln(a)+\ln(b)\right)]$$
$$= \ln^{-1}[x\ln(a)+x\ln(b)] \overset{16.19}{=} \ln^{-1}[x\ln(a)]\ln^{-1}[x\ln(b)] = a^x b^x. \text{ ¥}$$

We will call the function $f(x) = a^x$, for $a > 0$, the *exponential function with respect to a*.

Second, we can now explain clearly why we called ln the natural logarithm function back in section 2. The common meaning of the word *logarithm* is "the exponent that indicates the power to which a number is raised to produce a given number" (*Merriam Webster's Collegiate Dictionary*, Merriam-Webster, Inc., Springfield, Massachusetts, Tenth Edition, 1993). We see that $\ln(t)$ fits the definition just quoted since, by Theorem 16.23, $\ln(t)$ is the exponent that the number e is raised to produce a given number t. The term *natural* is used in connection with the logarithm function ln for two reasons: ln and its inverse ($f(x) = e^x$) arise naturally in connection with many physical phenomena, and ln and its inverse are the simplest and most convenient of all logarithm and exponential functions (we study other logarithm and exponential functions in the next chapter).

Next, we can at long last resolve the two issues we raised at the end of section 4 of Chapter VIII; namely, we now know what x^p means for all $x > 0$ and all real numbers p (e.g., $2^{\sqrt{2}} = \ln^{-1}[\sqrt{2}\ln(2)]$, and we can prove that Theorem 8.16 extends to the function $f(x) = x^p$ for all $x > 0$:

Theorem 16.31: Fix $p \in \mathsf{R}^1$, and define $f : (0,\infty) \to \mathsf{R}^1$ by $f(x) = x^p$. Then f is differentiable at each $x > 0$ and

$$f'(x) = px^{p-1}.$$

Proof: Assume that $x > 0$. Let $g = p \cdot \ln$. Then, by the definition of x^p (above Exercise 16.20), we see that $f = \ln^{-1} \circ g$. Hence, by the Chain Rule (Theorem 7.23),

$$f'(x) = [\left(\ln^{-1}\right)'(g(x))]\,[g'(x)];$$

furthermore,

$$\left(\ln^{-1}\right)'(g(x)) \overset{16.18}{=} \ln^{-1}(g(x)) = \ln^{-1}(p\ln(x)) \overset{16.22}{=} \ln^{-1}(\ln(x^p)) = x^p$$

and

$$g'(x) \overset{7.4}{=} p\ln'(x) \overset{16.5}{=} \tfrac{p}{x}.$$

Therefore,

$$f'(x) = x^p\tfrac{p}{x} \overset{16.30\,(3)}{=} px^{p-1}. \text{ ¥}$$

Finally, we have uncovered many new and interesting functions whose derivatives and integrals are worth investigating. We define and discuss some of these functions in the next chapter (we have already considered one such function in Exercise 16.28).

Chapter XVII: General Logarithm and Exponential Functions

We studied the natural logarithm function and the natural exponential function in the preceding chapter. It is now time to examine logarithm functions and exponential functions in general.

We define the logarithm and exponential functions in section 2. We obtain a simple but important algebraic relationship between logarithm functions and the natural logarithm function in section 3. This result shows that the functions that are commonly called logarithms, and which historically came from other considerations, are nothing more or less than the functions of the form

$$F(x) = c\int_1^x \tfrac{1}{t}, \text{ for some constant } c \neq 0 \text{ and all } x > 0.$$

We systematically present the properties of logarithm functions in section 3 and the properties of exponential functions in section 4; the properties include the derivatives and the integrals of the functions, as well as other information directly concerned with the graphs of the functions.

In the final section, we concern ourselves with functions that have the principal algebraic property of logarithm functions or of exponential functions. For logarithm functions, the property is that they change multiplication to addition; for exponential functions, the property is that they change addition to multiplication. We show that any function with the first or second property that is also bounded on some interval must, in fact, be a logarithm function or an exponential function, respectively. The result shows, for example, that each principal algebraic property, all by itself, almost implies differentiability.

1. Definitions of Logarithm and Exponential Functions

We will see that it is convenient to define exponential functions before we define logarithm functions.

Fix $a > 0$. Recall (from above Exercise 16.20) that we have defined a^x to be $\ln^{-1}[x\ln(a)]$ for any $x \in \mathsf{R}^1$. We now consider a^x to be the value of a function at x; we denote the function by $\exp_a$, often writing $\exp_a(x)$ instead of a^x. We note the following theorem before we formally define the exponential functions and the logarithm functions.

Theorem 17.1: For any $a > 0$ such that $a \neq 1$, the function $\exp_a$ is one-to-one and maps R^1 onto $(0, \infty)$.

Proof: Recall from part (1) of Theorem 16.19 that $\ln^{-1}(0) = 1$; thus, since $a \neq 1$, $\ln(a) \neq 0$. Hence the linear function f given by $f(x) = x\ln(a)$ for all $x \in \mathsf{R}^1$ is a one-to-one function from R^1 onto R^1. Also, $\ln^{-1}$ is one-to-one and maps R^1 onto $(0, \infty)$ (by Theorem 16.17). Therefore, since

$$\exp_a = \ln^{-1} \circ f,$$

$\exp_a$ is one-to-one and maps R^1 onto $(0, \infty)$. ¥

With Theorem 17.1 in mind with respect to the domain and range of $\exp_a$, we give the following formal definition:

Definition: Let $a > 0$. The *exponential function with respect to a* is the function $\exp_a : \mathsf{R}^1 \to (0, \infty)$ defined above; that is,

$$\exp_a(x) = a^x = \ln^{-1}[x \ln(a)] \quad \text{for all } x \in \mathsf{R}^1.$$

The basic algebraic properties of exponential functions are enumerated in Theorem 16.30.

Note that the exponential function with respect to 1 is not very interesting since $1^x = 1$ for all $x \in \mathsf{R}^1$ (by the definition of a^t above Exercise 16.20).

We now define the logarithm functions. Regarding the use of the word *logarithm*, see the discussion in the second paragraph following the proof of Theorem 16.30.

Definition: Let $b > 0$ such that $b \neq 1$. The *logarithm to the base b of a number* $x > 0$, denoted by $\log_b(x)$, is $\exp_b^{-1}(x)$ (which exists and is unique by Theorem 17.1); in other words, $\log_b(x)$ is the power to which b must be raised to obtain x.

The notation $\log_b$ stands for the *logarithm function with base b*; that is, $\log_b = \exp_b^{-1}$.

We note that the general definition of logarithm does, indeed, include our old friend the natural logarithm:

Theorem 17.2: $\ln = \log_e$ (where $e = \ln^{-1}(1)$).

Proof: By Theorem 16.23, $\exp_e = \ln^{-1}$. Hence, $\ln = \exp_e^{-1}$. Therefore, by the definition of logarithm above, $\ln = \log_e$. ¥

2. The Algebraic Relation between Logarithm Functions and ln

We show that each logarithm function is a constant multiple of the natural logarithm function and, conversely, that every nonzero constant multiple of the natural logarithm function is a logarithm function.

The formula in the following theorem makes it easy for us to prove that logarithm functions are differentiable and to obtain formulas for their derivatives (Theorem 17.10 in the next section).

Theorem 17.3: For any $b > 0$ such that $b \neq 1$,

$$\log_b(x) = \frac{\ln(x)}{\ln(b)} \quad \text{for all } x > 0.$$

Proof: Note that the right-hand side of the formula is defined since $\ln(b) \neq 0$ (because $b \neq 1$; recall part (1) of Theorem 16.19).

Fix $x > 0$. Let $y = \log_b(x)$. Then, by the definition of $\log_b(x)$ in section 1, $b^y = x$. Therefore,

$$y\ln(b) \overset{16.22}{=} \ln(b^y) = \ln(x).$$ ¥

Theorem 17.4: Every logarithm function is a constant multiple of ln and, conversely, every nonzero constant multiple of ln is a logarithm function. In other words, a function f is a logarithm function if and only if $f = c \cdot \ln$ for some constant $c \neq 0$.

Proof: Every logarithm function is a constant multiple of ln by Theorem 17.3.

Conversely, assume that $c \neq 0$. We show that $c \cdot \ln$ is a logarithm function. Let $b = e^{\frac{1}{c}}$. By Theorem 17.1, $b > 0$ and $b \neq 1$ (since $e^0 = 1$ by Theorem 16.30). Hence, b is a permissible base for a logarithm function, so $\log_b$ is a logarithm function. Now, for any $x > 0$,

$$\log_b(x) \overset{17.3}{=} \frac{\ln(x)}{\ln(e^{\frac{1}{c}})} \overset{16.23}{=} \frac{\ln(x)}{\frac{1}{c}} = c\ln(x).$$

Therefore, $c \cdot \ln$ is a logarithm function. ¥

Exercise 17.5: For any $a, b, c > 0$ such that $a \neq 1$ and $b \neq 1$,

$$\log_a(b) \cdot \log_b(c) = \log_a(c).$$

Exercise 17.6: Any two logarithm functions are constant multiples of one another; that is, $\log_b = c\log_{b'}$ for some constant c, which depends on b and b'. Find c. (Once you find c, this change-in-base result allows us to express a logarithm to one base in terms of a logarithm to any other base.)

For a given real number c, we know from Theorem 16.31 that the function $f(x) = \exp_x(c)$ is differentiable and that $f'(x) = cx^{c-1}$. In the following exercise, you are asked to prove the analogous result for logarithms.

Exercise 17.7: Fix $c > 0$, and let f be the change-in-base function given by $f(x) = \log_x(c)$ for all $x > 0$ such that $x \neq 1$.

Find $f'(x)$, thereby showing f is differentiable.

For what values of c is f one-to-one? Find a formula for f^{-1} when f is one-to-one.

3. Properties of Logarithm Functions

We present the algebraic properties and the basic calculus of logarithm functions.

The algebraic properties of logarithm functions are in the following theorem:

Theorem 17.8: Fix $b > 0$ such that $b \neq 1$.

(1) $\log_b(1) = 0$.

(2) $\log_b(xy) = \log_b(x) + \log_b(y)$ for all $x, y > 0$.

(3) $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$ for all $x, y > 0$.

(4) $\log_b(x^y) = y\log_b(x)$ for all $x > 0$ and all $y \in \mathsf{R}^1$.

Proof: By Theorem 17.3, $\log_b = \frac{1}{\ln(b)} \cdot \ln$. Therefore, (1), (2), (3) and (4) follow easily from the corresponding results for ln (namely, 16.5, 16.7, 16.9 and 16.22, respectively). ¥

Our next theorem concerns the simplest functional properties of logarithm functions.

Theorem 17.9: For any $b > 0$ such that $b \neq 1$, the function $\log_b$ is one-to-one and maps $(0, \infty)$ onto R^1.

Proof: Since $\log_b = \exp_b^{-1}$ (by definition of $\log_b$), it is obvious that $\log_b$ is one-to-one and that the rest of the theorem follows from Theorem 17.1. ¥

In the next several results, we examine the calculus of logarithm functions.

Theorem 17.10: For any $b > 0$ such that $b \neq 1$, the function $\log_b$ is differentiable and

$$\log_b'(x) = \frac{1}{x \ln(b)} \quad \text{for all } x > 0.$$

Proof: By Theorem 17.3, $\log_b = \frac{1}{\ln(b)} \cdot \ln$. Therefore, for any $x > 0$,

$$\log_b'(x) \overset{7.4}{=} \frac{1}{\ln(b)} \cdot \ln'(x) \overset{16.5}{=} \frac{1}{\ln(b)} \frac{1}{x} = \frac{1}{x \ln(b)}. \text{ ¥}$$

We obtain several corollaries to Theorem 17.10. First, however, we note a lemma that we use several times; the lemma is simply an observation based on two results in Chapter XVI.

Lemma 17.11: $\ln(b) < 0$ when $0 < b < 1$, and $\ln(b) > 0$ when $b > 1$.

Proof: The lemma follows immediately from the fact that $\ln(1) = 0$ (by Lemma 16.5) and the fact that ln is strictly increasing (by Corollary 16.14). ¥

We now present three corollaries to Theorem 17.11. The first corollary is less interesting than the others, but we include it for convenient reference in the proofs of other results here and in the next section

Corollary 17.12: For all $x > 0$,

$$\log_b'(x) < 0 \text{ when } 0 < b < 1, \ \log_b'(x) > 0 \text{ when } b > 1.$$

Proof: The corollary follows from Theorem 17.10 and Lemma 17.11. ¥

Corollary 17.13: The function $\log_b$ is strictly decreasing if $0 < b < 1$ and is strictly increasing if $b > 1$.

Proof: The corollary follows from Corollary 17.12 and Theorem 10.17. ¥

Corollary 17.14: The function $\log_b$ is concave up when $0 < b < 1$ and is concave down when $b > 1$.

Proof: For any $b > 0$ such that $b \neq 1$ and for all $x > 0$,

$$\log_b''(x) \overset{17.10}{=} \left(\frac{1}{x \ln(b)}\right)' \overset{7.5}{=} \frac{-(x \ln(b))'}{(x \ln(b))^2} \overset{6.2}{=} \frac{-\ln(b)}{(x \ln(b))^2}.$$

Hence, for all $x > 0$, we see from Lemma 17.11 that

$$\log_b''(x) > 0 \text{ when } 0 < b < 1, \;\; \log_b''(x) < 0 \text{ when } b > 1.$$

Therefore, our corollary now follows by applying Corollary 10.31. ¥

Exercise 17.15: Draw the graphs of $\log_b$ for some values of $b > 1$ and for some values of b such that $0 < b < 1$. Draw the graphs all in the same figure (for the purpose of comparison); be sure to indicate what happens as x approaches zero (from the right) and as x goes to infinity.

In Theorem 16.29, we found a formula for the integral of the natural logarithm function over any closed and bounded interval of positive reals. Because $\log_b = \frac{1}{\ln(b)} \cdot \ln$ (by Theorem 17.3), we can use Theorem 16.29 to integrate all logarithm functions:

Theorem 17.16: For any $b > 0$ such that $b \neq 1$, the function $\log_b$ is integrable over any closed and bounded interval $[c, d]$ with $c > 0$ and

$$\int_c^d \log_b(x) = d \log_b(d) - c \log_b(c) + \frac{c-d}{\ln(b)}$$

Proof: Fix $b > 0$ such that $b \neq 1$, and fix c and d such that $0 < c \leq d$. Since ln is integrable over $[c, d]$ (by Theorem 16.27) and since $\log_b = \frac{1}{\ln(b)} \cdot \ln$ (by Theorem 17.3), we have by Theorem 13.11 that $\log_b$ is integrable over $[c, d]$ and

$$\begin{aligned} \int_c^d \log_b(x) &\overset{17.3}{=} \int_c^d \frac{1}{\ln(b)} \cdot \ln(x) \overset{13.11}{=} \frac{1}{\ln(b)} \int_c^d \ln(x) \\ &\overset{16.29}{=} \frac{1}{\ln(b)} \Big(d\ln(d) - c\ln(c) + c - d. \Big) = d\frac{\ln(d)}{\ln(b)} - c\frac{\ln(c)}{\ln(b)} + \frac{c-d}{\ln(b)} \\ &\overset{17.3}{=} d \log_b(d) - c \log_b(c) + \frac{c-d}{\ln(b)}. \end{aligned}$$ ¥

The formula in Theorem 17.16 can be stated entirely in terms of the natural logarithm:

Exercise 17.17: Under the assumptions in Theorem 17.16,

$$\int_c^d \log_b(x) = \frac{1}{\ln(b)} \Big(\ln(\tfrac{d^d}{c^c}) + c - d \Big).$$

4. Properties of Exponential Functions

We already know the algebraic properties and the most elementary functional properties of exponential functions (Theorems 16.30 and 17.1). Thus, we turn our attention to the calculus of exponential functions.

The following theorem says that the rate of change of an exponential function is proportional to the function itself. We see from the theorem that $\exp_e$ is the only exponential function that is its own rate of change; in this sense, $\exp_e$ is *natural as an exponential function* – however, in view of Exercise 16.25, $\exp_e$ should be considered *unusual as a function*!

Theorem 17.18: For any $a > 0$, the function $\exp_a$ is differentiable and

$$\exp_a'(x) = a^x \ln(a) \quad \text{for all } x \in \mathsf{R}^1.$$

Proof: Fix $a > 0$ and $x \in \mathsf{R}^1$. Note that

(1) $\exp_a(x) = a^x \overset{16.23}{=} e^{\ln(a^x)} \overset{16.22}{=} e^{x\ln(a)}$.

By (1), $\exp_a'(x) = \left(e^{x\ln(a)}\right)'$. Therefore, using the Chain Rule (Theorem 7.23) and Corollary 16.24 for the first equality below,

$$\exp_a'(x) = \left(e^{x\ln(a)}\right)(x\ln(a))' \overset{7.4}{=} \left(e^{x\ln(a)}\right)\ln(a) \overset{(1)}{=} a^x \ln(a). \ ¥$$

The following corollaries are analogous to the corollaries for logarithm functions in the preceding section.

Corollary 17.19: For all $x \in \mathsf{R}^1$,

$$\exp_a'(x) < 0 \text{ when } 0 < a < 1, \ \ \exp_a'(x) > 0 \text{ when } a > 1.$$

Proof: Since $a^x > 0$ for all $x \in \mathsf{R}^1$ (by Theorem 17.1), the corollary follows from Lemma 17.11 and Theorem 17.18. ¥

Corollary 17.20: The function $\exp_a$ is strictly decreasing if $0 < a < 1$ and is strictly increasing if $a > 1$.

Proof: The corollary follows from Corollary 17.19 and Theorem 10.17. ¥

Recall that the concavity of $\log_b$ depends on whether $b < 1$ or $b > 1$ (Corollary 17.14). Nevertheless, the following corollary shows that the concavity of the inverse functions of the logarithm functions is always the same.

Corollary 17.21: For any $a > 0$ such that $a \neq 1$, the function $\exp_a$ is concave up.

Proof: For all $x \in \mathsf{R}^1$,

$$\exp_a''(x) \overset{17.18}{=} \Big(a^x \ln(a)\Big)' \overset{7.4}{=} (a^x)' \ln(a) \overset{17.18}{=} a^x[\ln(a)]^2 > 0.$$

Therefore, $\exp_a$ is concave up by Corollary 10.31. ¥

Exercise 17.22: Draw the graphs of $\exp_a$ for some values of $a > 1$ and for some values of a such that $0 < a < 1$. Draw the graphs all in the same figure (for the purpose of comparison); be sure to indicate what happens as x approaches $\pm\infty$.

Next, we integrate the exponential functions. We found it easy to integrate the logarithm functions (Theorem 17.16). This was because we already had a formula for the integral of ln and the logarithm functions are constant multiples of ln (Theorem 17.3). Similarly, it is now easy to integrate the exponential functions – by just looking at the formula in Theorem 17.18, we see that $\frac{\exp_a}{\ln(a)}$

is a function whose derivative is a^x. However, we must remember that substantial work is behind the proof of Theorem 17.18; aside from results specifically mentioned in the proof, the Inverse Function Theorem (Theorem 8.7) is used to prove Corollary 16.24 (see Exercise 16.18).

Theorem 17.23: For any $a > 0$ such that $a \neq 1$, the function $\exp_a$ is integrable over any closed and bounded interval $[c, d]$ and

$$\int_c^d a^x = \frac{1}{\ln(a)}(a^d - a^c).$$

Proof: Fix $a > 0$ such that $a \neq 1$, and fix $c \leq d$.

Since $a \neq 1$, $\ln(a) \neq 0$ (by part (1) of Theorem 16.19). Thus, $\frac{a^x}{\ln(a)}$ is defined for all x (the expression $\frac{a^x}{\ln(a)}$ comes from the discussion above). Now,

$$(*) \quad \left(\frac{a^x}{\ln(a)}\right)' \overset{7.4}{=} \frac{1}{\ln(a)}\,(a^x)' \overset{17.18}{=} a^x \quad \text{for all } x \in [c, d],$$

Since $\exp_a$ is differentiable (by Theorem 17.18), $\exp_a$ is continuous (by Theorem 6.14). Hence, $\exp_a$ is continuous on $[c, d]$ (by Exercise 5.3). Therefore, by (*) and part (2) of the Fundamental Theorem of Calculus (Theorem 14.2),

$$\int_c^d a^x = \frac{a^d}{\ln(a)} - \frac{a^c}{\ln(a)} = \frac{1}{\ln(a)}(a^d - a^c). \ ¥$$

5. Logarithm and Exponential Types of Functions

The algebraic properties of logarithm functions are listed in Theorem 17.8. The property in part (2) of Theorem 17.8 says that logarithm functions convert multiplication (of positive numbers) to addition; this property implies the other properties in Theorem 17.8 (e.g., recall Theorems 16.8 and 16.11). Thus, we are led to call a function $f : (0, \infty) \to \mathsf{R}^1$ a *logarithm type of function*, abbreviated *L - function*, provided that

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y > 0.$$

Similarly, since $a^{x+y} = a^x a^y$ (by Theorem 16.30), we call a function $f : \mathsf{R}^1 \to (0, \infty)$ an *exponential type of function*, abbreviated *E - function*, provided that

$$f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathsf{R}^1.$$

The question arises as to whether there is an L - function or an E - function other than the logarithm functions and the exponential functions (respectively). We show that the answer is *no* under the assumtion that the L - function and the E - function are bounded above on some interval (Theorem 17.29 and Theorem 17.33). In particular, the only L - functions or E - functions that are continuous at some point are the logarithm functions and the exponential functions, respectively.

On the other hand, there are L - functions and E - functions that are not continuous at any point; the methods for constructing such functions require set - theoretic techniques that we will not discuss.[10] We note that discontinuous L - functions and E - functions are strange – they must be unbounded on every nondegenerate interval (by our main theorems).

We accomplish our proofs in the following way: First, we introduce the notion of additive functions, which play a prominent role in the proofs of our main results; we prove that all additive functions that are bounded above on some interval are linear (i.e., of the form $f(x) = cx$); then we characterize all L - functions and E - functions as compositions of additive functions with ln and $\exp_e$, respectively; finally, we prove our main theorems by applying the theorem about additive functions to the general characterizations of L - functions and E - functions.

Additive Functions

We prove Theorem 17.24, which we use in the proofs of our main theorems about L - functions and E - functions.

Definition: A function $f : \mathsf{R}^1 \to \mathsf{R}^1$ is said to be *additive* provided that

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathsf{R}^1.$$

Obviously, all linear functions are additive. We show, conversely, that all additive functions that are bounded above on some interval are linear.

Theorem 17.24: Let $f : \mathsf{R}^1 \to \mathsf{R}^1$ be an additive function that is bounded above on some interval $[a, b]$, $a < b$. Then there is a constant c such that

$$f(x) = cx \quad \text{for all } x \in \mathsf{R}^1.$$

Proof: Since $f(0) = f(0+0) = f(0) + f(0)$, we see that $f(0) = 0$. Hence, for all $x \in \mathsf{R}^1$,

$$0 = f(0) = f(x - x) = f(x) + f(-x).$$

Thus,

(1) $f(-x) = -f(x)$ for all $x \in \mathsf{R}^1$.

Since f is bounded above on $[a, b]$ (by assumption), there exists $M > 0$ such that

(2) $f(x) < M$ for all $x \in [a, b]$.

Let $d = b - a$. Note that if $x \in [0, d]$, then $x + a \in [a, b]$. Hence, by (2), we have that

(3) $f(x + a) < M$ for all $x \in [0, d]$.

Next, note that for any $x \in [0, d]$,

[10] Discontinuous L - functions and discontinuous E - functions can be obtained from Lemmas 17.28 and 17.32 by using a discontinuous additive function g. A brief history of the discovery and significance of discontinuous additive functions is on pages 503 - 505 of the paper by J. W. Green and W. Gustin, *Quasiconvex sets*, Canadian Journal of Math. **2**(1950), 489 - 507. For a construction of a discontinuous additive function, see, for example, the proof of Theorem 3 of the paper by F. B. Jones, *Connected and disconnected plane sets and the functional equation* $\mathsf{f}(\mathsf{x}) + \mathsf{f}(\mathsf{y}) = \mathsf{f}(\mathsf{x} + \mathsf{y})$, Bull. Amer. Math. Soc. **48**(1942), 115 - 120.

$$f(x) = f(x+a-a) \overset{(1)}{=} f(x+a) - f(a) \overset{(3)}{<} M - f(a);$$

hence,

(4) f is bounded above on $[0, d]$.

Now, let $c = \frac{f(d)}{d}$. Define a function $g : \mathsf{R}^1 \to \mathsf{R}^1$ as follows:

$$g(x) = f(x) - cx \quad \text{for all } x \in \mathsf{R}^1.$$

Since f is bounded above on $[0, d]$ (by (4)), g is the difference of two functions that are bounded above on $[0, d]$; hence,

(5) g is bounded above on $[0, d]$.

Since g is the difference of two additive functions, it is obvious that

(6) g is additive.

Note that since $c = \frac{f(d)}{d}$, $g(d) = f(d) - cd = 0$; hence, we see from (6) that g is periodic with period d, which means

(7) $g(x + d) = g(x)$ for all $x \in \mathsf{R}^1$.

Now, it follows easily from (5) and (7) that

(8) g is bounded above on all of R^1.

Finally, we prove that $f(x) = cx$ for all $x \in \mathsf{R}^1$ by proving that $g(x) = 0$ for all $x \in \mathsf{R}^1$.

Suppose by way of contradiction that $g(p) \neq 0$ for some $p \in \mathsf{R}^1$. By (1), either $g(p) > 0$ or $g(-p) > 0$; hence, we can assume without loss of generality that $g(p) > 0$. Using (6) and a simple induction (Theorem 1.20), we see that

$$g(np) = ng(p) \quad \text{for all } n = 1, 2, \ldots .$$

Thus, since $g(p) > 0$, we see that g is not bounded above (by the Archimedean Property (Theorem 1.22)). This contradicts (8). Hence, we have proved that $g(x) = 0$ for all $x \in \mathsf{R}^1$; therefore, by the formula defining g,

$$f(x) = cx \quad \text{for all } x \in \mathsf{R}^1. \text{ ¥}$$

Exercise 17.25: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is an additive function, then $f(qx) = qf(x)$ for all rationals q and all $x \in \mathsf{R}^1$.

Exercise 17.26: Use Exercise 17.25 to prove directly the following special case of Theorem 17.24:

If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is a continuous additive function, then there is a constant c such that $f(x) = cx$ for all $x \in \mathsf{R}^1$.

L - Functions

We gave the definition of an L-function at the beginning of the section. Concerning the assumption in the definition that the domain of an L-function is $(0, \infty)$, we offer the following exercise:

Exercise 17.27: Let $X \subset \mathsf{R}^1$ and let $f : X \to \mathsf{R}^1$ be a function such that

$$f(xy) = f(x) + f(y) \quad \text{whenever } x, y \in X,$$

(1) If $0 \in X$, $f(x) = 0$ for all $x \in X$.

(2) If $1 \in X$ and X is symmetric about the origin (i.e., $x \in X$ implies $-x \in X$), then $f(-x) = f(x)$ for all $x \in X$.

Our main result about L-functions is Theorem 17.29. The proof of the theorem uses the following lemma.

Lemma 17.28: A function $f : (0, \infty) \to \mathsf{R}^1$ is an L-function if and only if there is an additive function $g : \mathsf{R}^1 \to \mathsf{R}^1$ such that $f = g \circ \ln$.

Proof: Assume that f is an L-function; that is,

(1) $f(xy) = f(x) + f(y)$ for all $x, y > 0$.

Define $g : \mathsf{R}^1 \to \mathsf{R}^1$ as follows:

(2) $g(t) = f(e^t)$ for all $t \in \mathsf{R}^1$ $(e = \ln^{-1}(1))$.

We see that g is additive since, for all $x, y \in \mathsf{R}^1$,

$$g(x+y) \overset{(2)}{=} f(e^{x+y}) \overset{16.30}{=} f(e^x e^y) \overset{(1)}{=} f(e^x) + f(e^y) \overset{(2)}{=} g(x) + g(y).$$

In addition, $f = g \circ \ln$ since, for all $x > 0$,

$$f(x) \overset{16.23}{=} f(e^{\ln(x)}) \overset{(2)}{=} g(\ln(x)).$$

Conversely, assume that $f = g \circ \ln$ for some additive function $g : \mathsf{R}^1 \to \mathsf{R}^1$. Then f is an L-function since, for all $x, y > 0$,

$$f(xy) = g(\ln(xy)) \overset{16.7}{=} g\Big(\ln(x) + \ln(y)\Big)$$

$$= g(\ln(x)) + g(\ln(y)) = f(x) + f(y). \text{ ¥}$$

Theorem 17.29: Assume that $f : (0, \infty) \to \mathsf{R}^1$ is bounded above on some interval $[a, b]$, $0 < a < b$. Then f is an L-function if and only if f is a logarithm function or f is the zero function.

Proof: If f is a logarithm function, then f is an L-function by part (2) of Theorem 17.8. Clearly, the zero function is an L-function.

Conversely, assume f is an L-function that is bounded above on $[a, b]$, where $0 < a < b$. By Lemma 17.28, there is an additive function $g : \mathsf{R}^1 \to \mathsf{R}^1$ such that $f = g \circ \ln$.

We find an interval $[s,t]$, $s < t$, on which g is bounded above. Since ln is differentiable (by Lemma 16.5), ln is continuous (by Theorem 6.14). Thus, by Theorem 5.13, $\ln([a,b]) = [s,t]$, where $s < t$ since ln is one-to-one (by Theorem 16.17). Hence, $\ln^{-1}([s,t]) = [a,b]$ (using again that ln is one-to-one). Thus, since $f = g \circ \ln$,

$$g([s,t]) = f(\ln^{-1}([s,t])) = f([a,b]).$$

Therefore, since f is bounded above on $[a,b]$ (by assumption), clearly g is bounded above on $[s,t]$.

Since g is an additive function bounded above on $[s,t]$, where $s < t$, we see from Theorem 17.24 that there is a constant c such that

$$g(x) = cx \quad \text{for all } x \in \mathsf{R}^1.$$

Thus, since $f = g \circ \ln$,

$$f(x) = g(\ln(x)) = c\ln(x) \quad \text{for all } x \in \mathsf{R}^1,$$

which proves that $f = c \cdot \ln$. Therefore, by Theorem 17.4, f is a logarithm function if $c \neq 0$. ¥

Exercise 17.30: State and prove the analogue of Theorem 17.29 for functions f defined $\mathsf{R}^1 - \{0\}$ that satisfy the equation for L-functions (i.e., $f(xy) = f(x) + f(y)$ for all $x, y \in \mathsf{R}^1 - \{0\}$).

(*Hint:* Recall Exercise 17.27.)

E - Functions

We defined E-functions at the beginning of the section.

Exercise 17.31: The requirement in the definition of an E-function that f have only positive values only eliminates the function that is constantly zero from consideration: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is a function such that

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathsf{R}^1,$$

then either $f(x) = 0$ for all $x \in \mathsf{R}^1$ or $f(x) > 0$ for all $x \in \mathsf{R}^1$.

Our main result about E-functions is Theorem 17.33. First, we prove a lemma.

Lemma 17.32: A function $f : \mathsf{R}^1 \to (0,\infty)$ is an E-function if and only if there is an additive function $g : \mathsf{R}^1 \to \mathsf{R}^1$ such that $f = \exp_e \circ g$ $(e = \ln^{-1}(1))$.

Proof: Assume that f is an E-function. Then, since $f(x) > 0$ for all $x \in \mathsf{R}^1$, the function $g = \ln \circ f$ is defined on all of R^1. We see that g is additive since, for all $x, y \in \mathsf{R}^1$,

$$\begin{aligned} g(x+y) &= \ln\Big(f(x+y)\Big) = \ln\Big(f(x)f(y)\Big) \\ &\overset{16.7}{=} \ln(f(x)) + \ln(f(y)) = g(x) + g(y). \end{aligned}$$

Also, $f = \exp_e \circ g$ since

$$f = \ln^{-1} \circ \ln \circ f \overset{16.23}{=} \exp_e \circ (\ln \circ f) = \exp_e \circ g.$$

Conversely, assume $f = \exp_e \circ g$ for some additive function $g : \mathsf{R}^1 \to \mathsf{R}^1$. Then f is an E-function since, for all $x, y \in \mathsf{R}^1$,

$$\begin{aligned} f(x+y) &= \exp_e \Big(g(x+y)\Big) = \exp_e \Big(g(x) + g(y)\Big) \\ &\overset{16.30}{=} \exp_e(g(x)) \exp_e(g(y)) = f(x)f(y). \end{aligned}$$ ¥

Theorem 17.33: Assume that $f : \mathsf{R}^1 \to (0, \infty)$ is bounded on some interval $[a, b]$, $a < b$. Then f is an E-function if and only if f is an exponential function.

Proof: If f is an exponential function, then f is an E-function by part (2) of Theorem 16.30.

Conversely, assume f is an E-function that is bounded above on $[a, b]$, where $a < b$. By Lemma 17.32, there is an additive function $g : \mathsf{R}^1 \to \mathsf{R}^1$ such that $f = \exp_e \circ g$.

We show that g is bounded above on $[a, b]$. Since f maps R^1 to $(0, \infty)$ and f is bounded above on $[a, b]$,

$$f([a, b]) \subset (0, m) \quad \text{for some } m > 0.$$

Thus, since $f = \exp_e \circ g$,

$$\exp_e \Big(g([a, b])\Big) \subset (0, m).$$

Therefore, since $\exp_e$ is not bounded above on any set that is not bounded above, it follows that $g([a, b])$ must be bounded above on $[a, b]$.

Since g is and additive function bounded above on $[a, b]$, where $a < b$, we have by Theorem 17.24 that there is a constant c such that

$$g(x) = cx \quad \text{for all } x \in \mathsf{R}^1.$$

Thus, since $f = \exp_e \circ g$,

$$f(x) = \exp_e(g(x)) = \exp_e(cx) = e^{cx} \quad \text{for all } x \in \mathsf{R}^1.$$

Therefore, by part (4) of Theorem 16.30, f is the exponential function given by $f(x) = (e^c)^x$. ¥

Chapter XVIII: L'Hôpital's Rules

We know from Theorem 4.20 that the limit of the quotient, $\frac{f}{g}$, of two functions exists and is the quotient of the limits of the functions when the limit of each function exists and the limit for g is not 0. It is evident that if the limit of f exists and is not 0 and the limit of g is 0, then the limit of $\frac{f}{g}$ does not exist or may be $+\infty$ or $-\infty$. We are left with the case when the limit of f and the limit of g are both 0. This case occurs throughout calculus, and we have dealt with it before – finding derivatives is evaluating limits of this type (by definition of the derivative).

When the limit of f and the limit of g are both 0, the limit of $\frac{f}{g}$ may not exist, may be $+\infty$ or $-\infty$, or may be any real number. We illustrate with simple examples: $\lim_{x\to 0} \frac{x}{x^2}$ does not exist, $\lim_{x\to 0} \frac{|x|}{x^2} = +\infty$ (see definition in section 1 of this chapter), $\lim_{x\to 0} \frac{-|x|}{x^2} = -\infty$, and for any fixed real number t, $\lim_{x\to 0} \frac{tx}{x} = t$.

We obtain a systematic method for investigating limits of quotients of differentiable functions when the functions separately have limit equal to 0, $+\infty$, or $-\infty$. In general terms, the method says that the limit of such a quotient is the limit of the ratio of the rates of speed (derivatives) of the functions provided that the latter limit exists. (For a simple illustration, consider limits of $\frac{ax}{bx}$ as x approaches $0, +\infty$, or $-\infty$).

The method we are concerned with is called *l'Hôpital's rules*, named after the Marquis de l'Hôpital (1661 - 1704). However, the method was actually due to John Bernoulli (1667 - 1748), who was l'Hôpital's tutor. The method originally appeared in the first *textbook* ever published on differential calculus, entitled *Analyse des infiniment petits* and written by l'Hôpital in 1696; this is undoubtedly the reason the method bears his name. An example from l'Hôpital's book is in Exercise 18.14.

We use nothing about integration theory in this chapter, so the chapter could have been placed immediately after Chapter X (we will use a result from Chapter X). We chose to postpone presenting l'Hôpital's rules until now in order that the rules be followed closely by applications that illustrate the rules in another context – sequences and series, which we study in the next two chapters.

We mention some notation and terminology:

We write ∞ instead of $+\infty$; $\pm\infty$ means ∞ or $-\infty$ (i.e., one or the other).

The symbol $\frac{\infty}{\infty}$ signifies what is called the *indeterminate form infinity over infinity as x approaches p* ($x \to p$), which refers to limits of the form $\lim_{x\to p} \frac{f(x)}{g(x)}$, where $\lim_{x\to p} |f(x)| = \infty = \lim_{x\to p} |g(x)|$. Similarly, the symbol $\frac{0}{0}$ is called the *indeterminate form zero over zero as* $x \to p$, and is shorthand for limits of the form $\lim_{x\to p} \frac{f(x)}{g(x)}$, where $\lim_{x\to p} f(x) = 0 = \lim_{x\to p} g(x)$.

We organize the chapter as follows: We give definitions for limits involving infinity in section 1, we present l'Hôpital's Rule for $\frac{\infty}{\infty}$ in section 2, and we present l'Hôpital's Rule for $\frac{0}{0}$ in section 3.

The indeterminate forms $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are the main forms for l'Hôpital's rules; these two forms can be applied in other situations that we discuss near the end of section 2 (above Exercise 18.5).

1. Definitions for Limits Involving Infinity

We give the definitions for infinite limits as x approaches a number p or as x approaches ∞ or $-\infty$, and we give definitions for finite limits as x approaches ∞ or $-\infty$. We conclude with two theorems concerning finite limits as x approaches ∞ or $-\infty$.

We will be confronted with a number of definitions involving limits and infinity all at once. This should not tax the reader's ability to understand and remember the definitions since the definitions are natural and each definition is a variation of the others. Moreover, let us imagine for a moment that ∞ and $-\infty$ are limit points of R^1 and that an interval about ∞ or $-\infty$ is an interval of the form $(a, \infty]$ or $[-\infty, a)$, respectively; then the definitions below become natural analogues of the definition of finite limits in section 1 of Chapter III – thus, the definitions do not involve substantially new ideas.

Definition ($\lim_{x\to p} f(x) = \pm\infty$)**:** Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a limit point of X.

- *The limit of f as x approaches p is ∞*, $\lim_{x\to p} f(x) = \infty$, provided that for each real number M, there is a real number $\delta > 0$ such that

$$f(x) > M \quad \text{for all } x \in X - \{p\} \text{ such that } |x - p| < \delta.$$

- *The limit of f as x approaches p is $-\infty$*, $\lim_{x\to p} f(x) = -\infty$, provided that for each real number M, there is a real number $\delta > 0$ such that

$$f(x) < M \quad \text{for all } x \in X - \{p\} \text{ such that } |x - p| < \delta.$$

- The definitions for one-sided limits being ∞ or $-\infty$ ($\lim_{x\to p^+} f(x) = \pm\infty$, $\lim_{x\to p^-} f(x) = \pm\infty$) are obtained from the definitions just given by analogy with the definitions for one-sided limits in section 5 of Chapter III.

Definition ($\lim_{x\to\infty} f(x)$)**:** Let $X \subset \mathsf{R}^1$ such that X has no upper bound, and let $f : X \to \mathsf{R}^1$ be a function.

- *The limit of f as x approaches ∞ is the real number L*, $\lim_{x\to\infty} f(x) = L$, provided that for each $\epsilon > 0$, there is a real number N such that

$$|f(x) - L| < \epsilon \quad \text{for all } x \in X \text{ such that } x > N.$$

- *The limit of f as x approaches ∞ is ∞*, $\lim_{x\to\infty} f(x) = \infty$, provided that for each real number M, there is a real number N such that

$$f(x) > M \quad \text{for all } x \in X \text{ such that } x > N.$$

- *The limit of f as x approaches ∞ is $-\infty$*, $\lim_{x\to\infty} f(x) = -\infty$, provided that for each real number M, there is a real number N such that

$$f(x) < M \quad \text{for all } x \in X \text{ such that } x > N.$$

Definition ($\lim_{x\to-\infty} f(x)$): Let $X \subset \mathsf{R}^1$ such that X has no lower bound, and let $f : X \to \mathsf{R}^1$ be a function. (The only change here is to write $x < N$ instead of $x > N$ as above.)

- *The limit of f as x approaches $-\infty$ is the real number L*, $\lim_{x\to-\infty} f(x) = L$, provided that for each $\epsilon > 0$, there is a real number N such that

$$|f(x) - L| < \epsilon \quad \text{for all } x \in X \text{ such that } x < N.$$

- *The limit of f as x approaches $-\infty$ is ∞*, $\lim_{x\to-\infty} f(x) = \infty$, provided that for each real number M, there is a real number N such that

$$f(x) > M \quad \text{for all } x \in X \text{ such that } x < N.$$

- *The limit of f as x approaches $-\infty$ is $-\infty$*, $\lim_{x\to-\infty} f(x) = -\infty$, provided that for each real number M, there is a real number N such that

$$f(x) < M \quad \text{for all } x \in X \text{ such that } x < N.$$

We note the following standard terminology for many of the limits defined above:

Definition: The line $x = p$ in the plane ($p \in \mathsf{R}^1$ fixed) is a *vertical asymptote for a function* f provided that at least one of the following is true: $\lim_{x\to p^+} f(x) = \infty$, $\lim_{x\to p^-} f(x) = \infty$, $\lim_{x\to p^+} f(x) = -\infty$, $\lim_{x\to p^-} f(x) = -\infty$.

The line $y = L$ in the plane ($L \in \mathsf{R}^1$ fixed) is a *horizontal asymptote for a function* f provided that $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Exercise 18.1: Find $\lim_{x\to\infty} \frac{x}{\sin(x)-3}$ (if the limit is finite or $\pm\infty$).

Exercise 18.2: Find $\lim_{x\to 0} \frac{|\sin(x)|}{1-\cos(x)}$ (if the limit is finite or $\pm\infty$).

Exercise 18.3: Let $X \subset \mathsf{R}^1$ such that X has no upper bound, and let $f : X \to \mathsf{R}^1$ be a positive function. Then $\lim_{x\to\infty} f(x) = \infty$ if and only if $\lim_{x\to\infty} \frac{1}{f(x)} = 0$.

Exercise 18.4: Let $X \subset \mathsf{R}^1$ such that X has no upper bound, and let $f : X \to \mathsf{R}^1$ be a function. Let $Y = \{\frac{1}{x} : x \in X \text{ and } x \neq 0\}$, and define $F : Y \to \mathsf{R}^1$ by

$$F(\tfrac{1}{x}) = f(x), \ \text{ all } \tfrac{1}{x} \in Y.$$

Then $\lim_{\frac{1}{x}\to 0^+} F(\frac{1}{x}) = \lim_{x\to\infty} f(x)$ if either limit exists.

The analogous result for $-\infty$ holds when X has no lower bound.

Two Limit Theorems As x Approaches $\pm\infty$

The two theorems below place many limit theorems we obtained earlier in the setting of limits as x approaches $\pm\infty$. The theorems will be needed for exercises in this chapter and will be used in the next chapter.

Theorem 18.5: Let $X \subset \mathsf{R}^1$ such that X has no upper bound, and let $f, g : X \to \mathsf{R}^1$ be functions such that

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\infty} g(x) = M, \quad \text{where } L, M \in \mathsf{R}^1.$$

Then the analogues of Theorems 3.1 (on uniqueness), 4.1 (on sums), 4.2 (on differences), 4.9 (on products), 4.20 (on quotients), and 4.34 (Squeeze Theorem) hold for limits as x approaches ∞. Also, the analogues of the theorems hold when X has no lower bound and x approaches $-\infty$.

Proof: Let $Y = \{\frac{1}{x} : x \in X \text{ and } x \neq 0\}$, and define $F, G : Y \to \mathsf{R}^1$ by

$$F(\tfrac{1}{x}) = f(x), \quad G(\tfrac{1}{x}) = g(x), \quad \text{all } \tfrac{1}{x} \in Y.$$

Then, our theorem follows by using Exercise 18.4 to apply any theorem listed in our theorem to F and G. ¥

Our next theorem is the variant of the Substitution Theorem (Theorem 4.29) for limits as x approaches $\pm\infty$.

Theorem 18.6 (Substitution Theorem for Infinity): Let $X, Y, Z \subset \mathsf{R}^1$ such that X has no upper bound, and let $f : X \to Y$ and $g : Y \to Z$ be functions. If $\lim_{x\to\infty} f(x) = L$, where $L \in \mathsf{R}^1$, and g is continuous at L, then

$$\lim_{x\to\infty}(g \circ f)(x) = g(L).$$

The analogous result when X has no lower bound and x approaches $-\infty$ holds.

Proof: Let $\epsilon > 0$. Since g is continuous at L, we have by Corollary 3.13 that there is a $\delta > 0$ such that

$$|g(y) - g(L)| < \epsilon \quad \text{for all } y \in Y \text{ such that } |y - L| < \delta.$$

Since $\lim_{x\to\infty} f(x) = L$, there is a real number N such that

$$|f(x) - L| < \delta \quad \text{for all } x \in X \text{ such that } x > N.$$

It follows easily that

$$|(g(f(x)) - g(L)| < \epsilon \quad \text{for all } x \in X \text{ such that } x > N.$$

Therefore, we have proved that $\lim_{x\to\infty}(g \circ f)(x) = g(L)$.

The proof of the result for $-\infty$ is similar. ¥

Exercise 18.7: Find $\lim_{x\to\infty}(\sqrt{x^2 + 8x} - x)$ (if the limit exists or is $\pm\infty$).

2. L'Hôpital's Rule for $\frac{\infty}{\infty}$

We prove l'Hôpital's rule for the indeterminate form $\frac{\infty}{\infty}$. We then discuss exponential indeterminate forms to which the theorem can be applied.

The proofs of l'Hôpital's rules in this section and the next use a generalization of the Mean Value Theorem due to Cauchy:

Theorem 18.8 (Cauchy Mean Value Theorem): Assume that f and g are continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $p \in (a, b)$ such that

$$f'(p)[g(b) - g(a)] = g'(p)[f(b) - f(a)].$$

Proof: The theorem is in Exercise 10.6. In case you did not work Exercise 10.6, and in the interest of completeness, we note that the theorem follows easily by considering the function h given by

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)], \quad \text{all } x \in [a, b]$$

and applying Rolle's Theorem (Lemma 10.1). ¥

Theorem 18.9 (L'Hôpital's Rule for $\frac{\infty}{\infty}$): Let I be an open interval, let $p \in I$, and let $f, g : I - \{p\} \to \mathsf{R}^1$ be functions such that

$$\lim_{x \to p} |f(x)| = \infty = \lim_{x \to p} |g(x)|.$$

Assume that f and g are differentiable on $I - \{p\}$ and that $g(x) \neq 0$ and $g'(x) \neq 0$ for any $x \in I - \{p\}$. If

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = L \text{ (including } L = \pm\infty),$$

then $\lim_{x \to p} \frac{f(x)}{g(x)} = L$. In addition, the analogous result holds when p is an end point of I or when $p = \pm\infty$ (and $I = (a, \infty)$ or $(-\infty, a)$ for some $a \in \mathsf{R}^1$).

Proof: We divide the proof into four cases. The proof for the first case is substantially longer than the proofs for the other three cases.

Case 1: $p \in I$ and $L \in \mathsf{R}^1$. By Theorem 6.14, we have that

(1) f and g are continuous on $I - \{p\}$.

Throughout the proof we fix a point $t \in I$ such that $p < t$. Let $s \in I$ such that $p < s < t$. By (1) and Exercise 5.3, g is continuous on $[s, t]$. Thus, if $g(s) = g(t)$, then the Mean Value Theorem (Theorem 10.2) shows that $g'(x) = 0$ for some point $x \in (s, t)$; however, this contradicts the assumption in our theorem that $g'(x) \neq 0$ for any $x \in I - \{p\}$. Therefore, we have proved that

(2) $g(s) \neq g(t)$.

By Theorem 18.8, there is a point $p_s \in (s, t)$ such that

$$f'(p_s)[g(t) - g(s)] = g'(p_s)[f(t) - f(s)].$$

Thus, by (2) and since $g'(p_s) \neq 0$ (by assumption in our theorem), we have

$$\frac{f'(p_s)}{g'(p_s)} = \frac{f(t)-f(s)}{g(t)-g(s)}.$$

Therefore, since $g(s) \neq 0$ (by assumption in our theorem), we can divide the numerator and denominator of the right-hand side by $-g(s)$, which shows we have proved the following:

(3) For each $s \in (p,t)$, there is a point $p_s \in (s,t)$ such that $\frac{f'(p_s)}{g'(p_s)} = \frac{\frac{f(s)}{g(s)} - \frac{f(t)}{g(s)}}{1-\frac{g(t)}{g(s)}}$.

Let $\epsilon > 0$. Since $\lim_{x\to p} \frac{f'(x)}{g'(x)} = L$ and $L < \infty$ (by assumption in Case 1), we can assume that t was chosen above close enough to p so that

$$L - \tfrac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < L + \tfrac{\epsilon}{2} \quad \text{for all } x \in (p,t).$$

Then, since the points p_s in (3) are in (p,t), we have by (3) that

(4) $L - \frac{\epsilon}{2} < \frac{\frac{f(s)}{g(s)} - \frac{f(t)}{g(s)}}{1-\frac{g(t)}{g(s)}} < L + \frac{\epsilon}{2}$ for all $s \in (p,t)$.

Since $\lim_{x\to p} |g(x)| = \infty$ (by assumption in our theorem), we can assume that t was chosen above close enough to p so that

$$\frac{g(t)}{g(s)} < 1 \quad \text{for all } s \in (p,t).$$

Hence, by multiplying through (4) by the positive quantity $1 - \frac{g(t)}{g(s)}$, we obtain

(5) $(L - \frac{\epsilon}{2})(1 - \frac{g(t)}{g(s)}) < \frac{f(s)}{g(s)} - \frac{f(t)}{g(s)} < (1 - \frac{g(t)}{g(s)})(L + \frac{\epsilon}{2})$ for all $s \in (p,t)$.

The last major step in the proof for Case 1 is to use (5) to prove the following:

(*) There exists $q \in (p,t)$ such that for all $s \in (p,q)$,

$$L - \epsilon < \frac{f(s)}{g(s)} < L + \epsilon.$$

Proof of (*): We prove (7) and (9) and then obtain the point q.
By the first inequality in (5), we have

(6) $L - \frac{\epsilon}{2} < (L - \frac{\epsilon}{2})\frac{g(t)}{g(s)} + \frac{f(s)}{g(s)} - \frac{f(t)}{g(s)}$ for all $s \in (p,t)$.

Since $\lim_{x\to p} |g(x)| = \infty$ (by assumption in our theorem), we see that

$$\lim_{s\to p^+} \left(-(L - \tfrac{\epsilon}{2})\frac{g(t)}{g(s)} + \frac{f(t)}{g(s)}\right) = 0;$$

hence, there is a $\delta_1 > 0$ such that

$$-\tfrac{\epsilon}{2} < -(L - \tfrac{\epsilon}{2})\frac{g(t)}{g(s)} + \frac{f(t)}{g(s)} \quad \text{for all } s \in (p,t) \text{ such that } s - p < \delta_1.$$

Thus, writing $L - \epsilon = (L - \frac{\epsilon}{2}) - \frac{\epsilon}{2}$ and applying (6), we have that

(7) $L - \epsilon < \frac{f(s)}{g(s)}$ for all $s \in (p, t)$ such that $s - p < \delta_1$.

Next, we prove (9) (the proof is similar to the proof of (7)). By the second inequality in (5), we have

(8) $\frac{f(s)}{g(s)} - \frac{f(t)}{g(s)} + (L + \frac{\epsilon}{2})\frac{g(t)}{g(s)} < L + \frac{\epsilon}{2}$ for all $s \in (p, t)$.

Since $\lim_{x \to p} |g(x)| = \infty$ (by assumption in our theorem),

$$\lim_{s \to p^+} \left(-(L + \tfrac{\epsilon}{2})\tfrac{g(t)}{g(s)} + \tfrac{f(t)}{g(s)} \right) = 0;$$

hence, there is a $\delta_2 > 0$ such that

$$-(L + \tfrac{\epsilon}{2})\tfrac{g(t)}{g(s)} + \tfrac{f(t)}{g(s)} < \tfrac{\epsilon}{2} \text{ for all } s \in (p, t) \text{ such that } s - p < \delta_2.$$

Thus, writing $L + \epsilon = (L + \frac{\epsilon}{2}) + \frac{\epsilon}{2}$ and applying (8), we have that

(9) $\frac{f(s)}{g(s)} < L + \epsilon$ for all $s \in (p, t)$ such that $s - p < \delta_2$.

Finally, let $q \in (p, t)$ such that $q - p < \min\{\delta_1, \delta_2\}$. Then (7) and (9) both hold for all $s \in (p, q)$. Therefore, q satisfies (*).

Since (*) holds for any t sufficiently close to p, we see that $\lim_{x \to p^+} \frac{f(x)}{g(x)} = L$. A similar argument shows that $\lim_{x \to p^-} \frac{f(x)}{g(x)} = L$. Therefore, we have proved the theorem for the case when $p \in I$ and L is a real number (the assumptions in Case 1).

Case 2*:* p is an end point of I and $L \in \mathsf{R}^1$. The proof for this case follows from the proof for Case 1.

Case 3*:* $p = \pm\infty$ (and $I = (a, \infty)$ or $(-\infty, a)$ for some $a \in \mathsf{R}^1$) and $L \in \mathsf{R}^1$. We only consider the case when $p = \infty$ (the proof when $p = -\infty$ is similar). Thus, $I = (a, \infty)$ for some point $a \in \mathsf{R}^1$.

We assume without loss of generality that $a > 0$.

Let $J = (0, \frac{1}{a})$. Define functions $F, G : J \to \mathsf{R}^1$ as follows (the functions are, indeed, defined on all of I since if $t \in I$, then $\frac{1}{t} > a$):

$$F(t) = f(\tfrac{1}{t}), \;\; G(t) = g(\tfrac{1}{t}), \;\; \text{all } t \in J.$$

We will apply the theorem in the setting of Case 2 to the functions F and G, the interval J, and the end point 0 of J. We first show that F and G satisfy the required assumptions.

Since $\lim_{x \to \infty} |f(x)| = \infty = \lim_{x \to \infty} |g(x)|$ (by assumption in the theorem), we see easily that $\lim_{t \to 0} |F(t)| = \infty = \lim_{t \to 0} |G(t)|$. By the Chain Rule (Theorem 7.23) and Lemma 7.5, F and G are differentiable on J and $G'(t) \neq 0$ for any $t \in J$ (since, by assumption, $g'(x) \neq 0$ for any $x \in (a, \infty)$). Since $g(x) \neq 0$ for any $x \in (a, \infty)$, $G(t) \neq 0$ for any $t \in J$. Finally, since $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$ (by assumption in the theorem), we see that

$$\lim_{t\to 0}\frac{F'(t)}{G'(t)} \overset{7.23,\ 7.5}{=} \lim_{t\to 0}\frac{f'(\frac{1}{t})[\frac{-1}{t^2}]}{g'(\frac{1}{t})[\frac{-1}{t^2}]} = \lim_{t\to 0}\frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L.$$

Thus, applying Case 2, we have that

$$\lim_{t\to 0}\frac{F(t)}{G(t)} = L.$$

Therefore, since $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0}\frac{F(t)}{G(t)}$, we have that $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$.

Case 4: $L = \pm\infty$. The proof for this case follows from simple adjustments in the proofs for the preceding three cases to take into account the definition of an infinite limit. We leave the details to the reader. ¥

Exercise 18.10: Find $\lim_{x\to\infty}\frac{\ln(x)}{x^t}$ for $t > 0$ (if the limit is finite or $\pm\infty$).

Exercise 18.11: Find $\lim_{x\to\infty}\frac{(\ln(x))^n}{x}$ for each natural number n (if the limit is finite or $\pm\infty$).

The Indeterminate Forms 0^0, ∞^0 and 1^∞

Sometimes l'Hôpital's Rule for $\frac{\infty}{\infty}$ can be used to evaluate limits of expressions of the form $f(x)^{g(x)}$ (where, of course, $f(x) > 0$ for all x). Assume that $f(x)^{g(x)}$ has one of the indeterminate forms 0^0 or ∞^0 as $x \to p$, which means $\lim_{x\to p} f(x) = 0$ or ∞ and $\lim_{x\to p} g(x) = 0$, where p is a real number or $p = \pm\infty$. Then the idea that leads to applying l'Hôpital's rules is to rewrite $f(x)^{g(x)}$ as $e^{g(x)\ln[f(x)]}$ and to rewrite $g(x)\ln[f(x)]$ as $\frac{\ln[f(x)]}{\frac{1}{g(x)}}$; the expression $\frac{\ln[f(x)]}{\frac{1}{g(x)}}$ has the indeterminate form $\frac{\infty}{\infty}$ (since $x = 0$ is a vertical asymptote of natural logarithm function and $\lim_{x\to\infty}\ln(x) = \infty$). Thus, if the functions $\ln[f(x)]$ and $\frac{1}{g(x)}$ satisfy the other assumptions of Theorem 18.9 and L in (the assumption of) Theorem 18.9 is finite, then

$$\lim_{x\to p} f(x)^{g(x)} = e^L$$

(by Theorem 18.9 and the Substitution Theorems 4.29 or 18.6).

We can apply the procedure to the quotient $\frac{g(x)}{\frac{1}{\ln[f(x)]}}$; since this quotient has the indeterminate form $\frac{0}{0}$ when $f(x)^{g(x)}$ has the form 0^0 or ∞^0, we would use Theorem 18.18 in the next section.

We can also apply the procedure when we have the indeterminate form 1^∞ (meaning $\lim_{x\to p} f(x) = 1$ and $\lim_{x\to p}|g(x)| = \infty$).

Many of the exercises below in this section and in the next section illustrate the procedure.

Exercise 18.12: Find $\lim_{x\to 0} x^x$ (if the limit is finite or $\pm\infty$).

Exercise 18.13: Discuss $\lim_{x\to\infty}\frac{e^x}{x^p}$ for $p > 0$ ($e = \ln^{-1}(1)$).

Exercise 18.14: Find $\lim_{x\to\infty} x^{\frac{1}{x}}$ (if the limit is finite or $\pm\infty$).

Exercise 18.15: Find $\lim_{x\to 0}(\frac{1}{x})^x$ (if the limit is finite or $\pm\infty$).

Exercise 18.16: Find $\lim_{x\to\infty} \frac{x-\sin(x)}{x}$ (if the limit is finite or $\pm\infty$).

Exercise 18.17: Is "$\infty^0 = 1$", by which we mean if $f(x)^{g(x)}$ has the indeterminate form ∞^0 as $x \to p$ and $\lim_{x\to p} f(x)^{g(x)}$ exists, then must the limit be equal to 1? (Surely, it is natural to ask: If the usual law of exponents for real numbers carries over to always give $\infty^0 = 1$, then why use a limit method each time to work a problem?)

3. L'Hôpital's Rule for $\frac{0}{0}$

We prove the analogue of Theorem 18.9 for the indeterminate form $\frac{0}{0}$.

Theorem 18.18 (L'Hôpital's Rule for $\frac{0}{0}$): Let I be an open interval, let $p \in I$, and let $f, g : I - \{p\} \to \mathsf{R}^1$ be functions such that

$$\lim_{x\to p} f(x) = 0 = \lim_{x\to p} g(x).$$

Assume that f and g are differentiable on $I - \{p\}$ and that $g(x) \neq 0$ and $g'(x) \neq 0$ for any $x \in I - \{p\}$. If

$$\lim_{x\to p} \frac{f'(x)}{g'(x)} = L \text{ (including } L = \pm\infty),$$

then $\lim_{x\to p} \frac{f(x)}{g(x)} = L$. In addition, the analogous result holds when p is an end point of I or when $p = \pm\infty$ (and $I = (a, \infty)$ or $(-\infty, a)$ for some $a \in \mathsf{R}^1$).

Proof: We divide the proof of the theorem into the same four cases into which we divided the proof of Theorem 18.9.

Case 1*:* $p \in I$ and $L \in \mathsf{R}^1$. We start by extending the domain of f and g to all of I as follows (we use the same notation, f and g, for the extended functions): Let

(1) $f(p) = 0 = g(p)$.

By Theorem 6.14, the functions f and g are continuous on $I - \{p\}$. Thus, since $\lim_{x\to p} f(x) = 0 = \lim_{x\to p} g(x)$ (by assumption in our theorem), we have by (1) and Theorem 3.12 that

(2) f and g are continuous on I.

We prove that

(3) $g(x) \neq 0$ for any $x \in I - \{p\}$.

Proof of (3)*:* Suppose by way of contradiction that $g(q) = 0$ for some $q \in I - \{p\}$. Then, by (1), $g(q) = g(p)$; also, by (2) and Exercise 5.3, g is continuous on the closed interval with end points p and q. Hence, by the Mean Value Theorem (Theorem 10.2), $g'(x) = \frac{g(p)-g(q)}{p-q} = 0$ for some point x between q and p; however, this contradicts the assumption in our theorem that $g'(x) \neq 0$ for any $x \in I - \{p\}$. Therefore, we have proved (3).

Now, fix a point $z \in I - \{p\}$. Assume that $z < p$. The assumptions in Theorem 18.8 are satisfied by the functions $f|[z, p]$ and $g|[z, p]$ (the functions are continuous on $[z,p]$ by (2) and Exercise 5.3, and the functions are differentiable on (z,p) by assumption in our theorem). Hence, by Theorem 18.8, there is a point $p_z \in (z,p)$ such that

$$f'(p_z)[g(p)-g(z)] = g'(p_z)[f(p)-f(z)].$$

Hence, by (1), $f'(p_z)g(z) = g'(p_z)f(z)$. Thus, since $g'(p_z) \neq 0$ by assumption in our theorem and since $g(z) \neq 0$ by (3),

$$\tfrac{f'(p_z)}{g'(p_z)} = \tfrac{f(z)}{g(z)}.$$

A similar argument when $z > p$ shows that there is a point $p_z \in (p,z)$ such that $\frac{f'(p_z)}{g'(p_z)} = \frac{f(z)}{g(z)}$. Therefore, we have proved the following:

(4) For each $x \in I - \{p\}$, there is a point p_x between p and x such that $\frac{f'(p_x)}{g'(p_x)} = \frac{f(x)}{g(x)}$.

Since the points p_x in (4) lie between p and x, $\lim_{x\to p} p_x = p$. In addition, by assumption in our theorem, $\lim_{x\to p} \frac{f'(x)}{g'(x)} = L$. Hence,

$$\lim_{x\to p} \tfrac{f'(p_x)}{g'(p_x)} = L.$$

Therefore, by (4), $\lim_{x\to p} \frac{f(x)}{g(x)} = L$. This proves the theorem under the assumptions in Case 1.

Case 2*:* p is an end point of I and $L \in \mathsf{R}^1$. The proof for this case follows from the proof for Case 1.

Case 3*:* $p = \pm\infty$ (and $I = (a,\infty)$ or $(-\infty,a)$ for some $a \in \mathsf{R}^1$) and $L \in \mathsf{R}^1$. The proof for this case is the same as the proof for Case 3 of Theorem 8.4 (with the obvious changes).

Case 4*:* $L = \pm\infty$. The proof for this case follows from simple adjustments in the proofs for the preceding three cases to take into account the definition of an infinite limit. We leave the details to the reader. ¥

We comment at some length about part of the proof of Theorem 18.18.

After (4) in the proof of Theorem 18.18, we stated that $\lim_{x\to p} p_x = p$. Technically, this limit makes no sense unless we specify *one* such p_x for each x, thereby obtaining a function of x (we take limits of functions, not of sets of points). Of course, we infer from the statement in (4) and the notation p_x that we have chosen one point p_x for each x. But, What allows us to do this?

The answer is the Axiom of Choice, which says that that there is a choice function for any infinite collection $\mathcal{C}$ of nonempty sets (a *choice function for* $\mathcal{C}$ is a function $\varphi : \mathcal{C} \to \cup\mathcal{C}$ such that $\varphi(A) \in A$ for all $A \in \mathcal{C}$). Without the Axiom of Choice, it is not always obvious how to define a choice function for an infinite collection $\mathcal{C}$ of nonempty sets and, in fact, it may not even be possible to do so:

Try to define a choice function for the collection of all nonempty subsets of the reals.

Thus, to be precise in the proof of Theorem 18.4, we should have said the following after (4): For each $x \in I - \{p\}$, let

$$A_x = \{p_x : p_x \text{ satisfies } (4)\}$$

and let $\mathcal{C} = \Big\{\{A_x : x \in I - \{p\}\Big\}$; then, by the Axiom of Choice, there is a choice function φ for $\mathcal{C}$. The rest of the proof of Theorem 18.4 then proceeds as before, replacing p_x with $\varphi(A_x)$.

Invoking the Axiom of Choice and showing how it is used in such a situation is probably more trouble than it is worth. Therefore, as is customary, we write proofs without mentioning the Axiom of Choice. Nevertheless, we can now keep our eyes open for when the Axiom of Choice is being used – it's good for us to know when something is going on. We will specifically make note of our use of the Axiom of Choice only one more time (after the proof of Theorem 19.39).

Our unexpressed use of the Axiom of Choice in the future will not bother you: You probably used the Axiom of Choice, perhaps without knowing you were using it, in working some exercises in previous chapters (for example, I'll bet you used the Axiom of Choice in working Exercise 10.5).

Finally, we mention that the Axiom of Choice is independent of and consistent with the other usual axioms of set theory (see Thomas J. Jech, *The Axiom of Choice*, North-Holland Publ. Co., Amsterdam and London, and American Elsevier Publ. Co., Inc., New York, 1973).

Exercise 18.19: Find $\lim_{x\to 1} \frac{1+\cos(\pi x)}{1-x+\ln(x)}$ (if the limit is finite or $\pm\infty$).

Exercise 18.20: Find $\lim_{x\to 0}(\frac{1}{\sin(x)} - \frac{1}{x})$ (if the limit is finite or $\pm\infty$).

Exercise 18.21: Find $\lim_{x\to p} \frac{\sqrt{2p^3x-x^4}-p\sqrt[3]{p^2x}}{p-\sqrt[4]{px^3}}$. (This is the example in l'Hôpital's book that we referred to in the introduction.)

Exercise 18.22: Find $\lim_{x\to 0}(e^x + x)^{\frac{1}{x}}$ (if the limit is finite or $\pm\infty$; $e = \ln^{-1}(1)$).

Exercise 18.23: Find $\lim_{x\to 0} \frac{2^x-5^x}{\sin(x)}$ (if the limit is finite or $\pm\infty$).

Exercise 18.24: Find $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^3)\,dt$ (if the limit is finite or $\pm\infty$).

Exercise 18.25: Find $\lim_{x\to 0} \frac{1}{x^{16}}\int_0^{x^5} \sin(t^2)\,dt$ (if the limit is finite or $\pm\infty$).

Exercise 18.26: Assuming that the second derivative f'' of f is continuous on an open interval I and $p \in I$, find

$$\lim_{h\to 0} \frac{f(p+h)-2f(p)+f(p-h)}{h^2}.$$

The limit in Exercise 18.26 reminds us of limits in Exercises 6.11, 612 and 6.13. The result in Exercise 6.13 can be worked using l'Hôpital's Rule for $\frac{0}{0}$

under the additional assumptions that f has a continuous derivative on I, φ is differentiable on the open interval J about 0, and $\varphi'(h) \neq 0$ for all $h \in J$. Even this special case of Exercise 6.13 is useful.

Exercise 18.27: Find $\lim_{h \to 0} \frac{(p+h^3)^7 - p^5}{h^3}$ (if the limit is finite or $\pm\infty$).

Exercise 18.28: Assume that $f : I \to \mathsf{R}^1$ has a continuous first derivative on I, and let $p \in I$. Find $\lim_{h \to 0} \frac{f(p+h^2)f(p+h^2) - f(p)f(p)}{2h^2}$ (if the limit is finite or $\pm\infty$).

Exercise 18.29: In Exercise 16.27, you were asked to prove that $e = \lim_{x \to 0}(1+x)^{\frac{1}{x}}$ using (as a hint) the limit definition of $\ln'(1)$. Work Exercise 16.27 again, this time using l'Hôpital's Rule for $\frac{0}{0}$ (Theorem 18.18) instead of the limit definition of $\ln'(1)$.

Moreover, prove that $e^x = \lim_{t \to \infty}(1 + \frac{x}{t})^t$ for each $x \in \mathsf{R}^1$.

Exercise 18.30: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0. \end{cases}$$

Find $f'(0)$ and $f''(0)$ using l'Hôpital's Rule for $\frac{0}{0}$ (Theorem 18.18). Do you think that the n^{th} derivative of f at $x = 0$ exists (we return to this later)?

Exercise 18.31: What limits (finite or $\pm\infty$) can be obtained when $f(x)^{g(x)}$ has the indeterminate form 0^0?

Chapter XIX: Numerical Sequences

Recall that we defined and briefly discussed the notions of sequence and limit of a sequence in section 8 of Chapter IV. We study these notions carefully in this chapter and in the next chapter.

The reader should refresh his memory concerning the definitions of a sequence and convergence of a sequence in section 8 of Chapter IV (we do not repeat these definitions here).

Throughout the chapter, the term *sequence* means a sequence of points in R^1 (i.e., a *numerical sequence*).

In section 1 we present the most fundamental results for studying sequences – limits of combinations of sequences (sums, products, and so on), a squeeze theorem and a theorem that allows us to use the methods of calculus to study sequences. In section 2, we prove that bounded increasing (or decreasing) sequences converge; this important theorem provides the theoretical foundation for studying convergence of inductively defined sequences in section 3. Of particular note in section 3 is the graphical method that we introduce in the latter part of the section; the graphical method is a tool that can be used to gain intuition regarding numerous inductively defined sequences.

Finally, in section 4, we recast arbitrary closeness and continuity as defined in Chapter II in terms of sequences.

Regarding notation, $\lim_{n\to\infty} s_n$ implicitly signifies that $\{s_n\}_{n=1}^{\infty}$ is a sequence and, thus, that the values of n are limited to the natural numbers; the same is true when n is replaced by any of the letters i, j, k, ℓ, or m; when we are concerned with functions that are not (necessarily) sequences, we use letters near the end of the alphabet in the limit notation (e.g., $\lim_{x\to p} f(x)$).

1. The Algebra of Sequences

We discuss (finite) sums, differences, products and quotients for limits of sequences. We also include the Squeeze Theorem for Sequences as well as a very simple but extremely useful theorem for determining whether specific sequences converge (Theorem 19.7).

The results in this section are simple consequences of results we proved earlier for limits in general. In particular, Theorem 4.38 says, without being specific, that results about limits in Chapter III and Chapter IV hold (with obvious modifications) for limits of sequences. Theorem 4.38 can now be replaced by Theorem 18.5, for which sequences are obviously a special case. We want to avoid having to refer to *both* Theorem 18.5 *and* a relevant previous theorem about limits listed in Theorem 18.5 each time we give the reason for convergence of a sequence; thus, we list the pertinent results separately and in terms of sequences here.

Theorem 19.1: If $\lim_{n\to\infty} s_n = p$ and $\lim_{n\to\infty} s_n = q$, then $p = q$.

Proof: Apply Theorem 18.5 for the case of Theorem 3.1. ¥

Theorem 19.2: If $\lim_{n\to\infty} s_n = p$ and $\lim_{n\to\infty} t_n = q$, then the sequence $\{s_n + t_n\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty}(s_n + t_n) = p + q$.

Proof: Apply Theorem 18.5 for the case of Theorem 4.1. ¥

Theorem 19.3: If $\lim_{n\to\infty} s_n = p$ and $\lim_{n\to\infty} t_n = q$, then the sequence $\{s_n - t_n\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty}(s_n - t_n) = p - q$.

Proof: Apply Theorem 18.5 for the case of Theorem 4.2. ¥

Theorem 19.4: If $\lim_{n\to\infty} s_n = p$ and $\lim_{n\to\infty} t_n = q$, then the sequence $\{s_n t_n\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} s_n t_n = pq$.

Proof: Apply Theorem 18.5 for the case of Theorem 4.9. ¥

Theorem 19.5: If $\lim_{n\to\infty} s_n = p$ and $\lim_{n\to\infty} t_n = q \neq 0$ and if $t_n \neq 0$ for any n, then the sequence $\{\frac{s_n}{t_n}\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{p}{q}$.

Proof: Apply Theorem 18.5 for the case of Theorem 4.20. ¥

Theorem 19.6 (Squeeze Theorem for Sequences): If $s_n \leq t_n \leq u_n$ for each $n = 1, 2, ...$ and

$$\lim_{n\to\infty} s_n = p = \lim_{n\to\infty} u_n,$$

then $\lim_{n\to\infty} u_n = p$.

Proof: Apply Theorem 18.5 for the case of Theorem 4.34. ¥

Our next theorem is trivial to prove but provides a very important method for studying sequences. Sequences, being functions defined only on the natural numbers, are in and of themselves not suited to the methods of calculus. The theorem we now present overcomes this obstacle for many sequences. We will illustrate how to use the theorem in the example that follows the theorem.

Theorem 19.7: Let $\{s_n\}_{n=1}^{\infty}$ be a sequence, and let $f : [1, \infty) \to \mathsf{R}^1$ be a function such that $f(n) = s_n$ for each $n \in \mathsf{N}$. If $\lim_{x\to\infty} f(x) = L$ (including $L = \pm\infty$), then $\lim_{n\to\infty} s_n = L$.

Proof: The theorem is obvious from definitions. ¥

We note that the converse of Theorem 19.7 is false even when f is continuous (Exercise 19.17).

Example 19.8: We show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

With Theorem 19.7 in mind, consider the function f defined by $f(x) = x^{\frac{1}{x}}$ for all $x \geq 1$. If you worked Exercise 18.14, then you already know that $\lim_{x\to\infty} f(x) = 1$. We include the details for completeness.

By Theorem 16.23,

$$f(x) = e^{\ln(x^{\frac{1}{x}})} \text{ for all } x \geq 1.$$

By Exercise 16.22, we can write the exponent for e as $\frac{\ln(x)}{x}$ which, since we are considering the limit as $x \to \infty$, has the indeterminate form $\frac{\infty}{\infty}$. Furthermore, since $\ln'(x) = \frac{1}{x}$ (by Lemma 16.5),

$$\lim_{x\to\infty} \frac{\ln'(x)}{x'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Hence, by l'Hôpital's Rule for $\frac{\infty}{\infty}$ (Theorem 18.9), $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$. In other words,

$$\lim_{x\to\infty} \ln(x^{\frac{1}{x}}) = 0.$$

Thus, since $f(x) = e^{\ln(x^{\frac{1}{x}})}$, we have by Theorem 18.6 that

$$\lim_{x\to\infty} f(x) = e^0 = 1.$$

Therefore, since $f(n) = n^{\frac{1}{n}}$ for each $n \in \mathsf{N}$, it follows immediately that the sequence $\{n^{\frac{1}{n}}\}_{n=1}^{\infty}$ converges to 1.

We will use the symbol $n!$, which we define below. We note that $n!$ is the number of ways to order n different objects in a list (the elementary proof of this fact is in Lemma 21.39).

Definition: For any natural number n, n *factorial*, written $n!$, is the product $n(n-1)(n-2)\cdots(2)(1)$, and $0! = 1$.

Exercise 19.9: Find $\lim_{n\to\infty} \frac{\sqrt[3]{n}+\sqrt[5]{n}}{\sqrt[2]{n}+\sqrt[5]{n}}$ (if the limit exists).

Exercise 19.10: Find $\lim_{n\to\infty} \frac{n^4}{n!}$ (if the limit exists).

Exercise 19.11: Find $\lim_{n\to\infty} \frac{n!}{10^n}$ (if the limit exists).

Exercise 19.12: Find $\lim_{n\to\infty} \frac{n^2\sin(n)}{n^3+1}$ (if the limit exists).

Exercise 19.13: Find $\lim_{n\to\infty}(\frac{n-1}{n+1})^n$ (if the limit exists).

Exercise 19.14: Find $\lim_{n\to\infty} \frac{\ln(3+e^n)}{2n}$ (if the limit exists).

Exercise 19.15: Find $\lim_{n\to\infty} \ln(8n+3) - \ln(2n)$ (if the limit exists).

Exercise 19.16: In Lemma 15.3 we used some previous results (including induction and the Squeeze Theorem) to show that $\lim_{n\to\infty} r^n = 0$ for each r such that $-1 < r < 1$. Give a short proof that the limit is 0 based on Theorem 19.7.

Also, give a short proof that $\lim_{n\to\infty} r^n = \infty$ when $r > 1$. Prove that $\lim_{n\to\infty} r^n$ does not exist when $r < -1$.

Exercise 19.17: Show that the converse of Theorem 19.7 is false even when the function f is continuous.

Exercise 19.18: If $\{s_n\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} s_{2n} = L$ and $\lim_{n\to\infty} s_{2n-1} = L$, then $\{s_n\}_{n=1}^{\infty}$ converges to L.

Exercise 19.19: True or false: If $\{s_n\}_{n=1}^{\infty}$ is a sequence such that $s_{n+1} - s_n \geq \epsilon$ for all n and some $\epsilon > 0$, then $\{s_n\}_{n=1}^{\infty}$ is not bounded.

2. Bounded Monotonic Sequences

The term *monotonic* refers to a function that is either increasing or decreasing on its domain. A sequence $\{s_n\}_{n=1}^{\infty}$ is *eventually monotonic* provided that there exists N such that the sequence $\{s_n\}_{n=N}^{\infty}$ is monotonic.[11] We also use the more specific terms *eventually increasing* and *eventually decreasing*.

We prove the following important fundamental result:

Theorem 19.20 (Bounded Monotonic Sequence Property): Any bounded eventually monotonic sequence converges; moreover, any bounded increasing sequence converges to the least upper bound of its values, and any bounded decreasing sequence converges to the greatest lower bound of its values.

Proof: We begin by proving the second part of the theorem for the case of bounded increasing sequences.

Assume that $\{s_n\}_{n=1}^{\infty}$ is a bounded increasing sequence. By the Completeness Axiom (section 1 of Chapter I), $\{s_n : n \in \mathsf{N}\}$ has a least upper bound ℓ.

We show that $\lim_{n\to\infty} s_n = \ell$. Let $\epsilon > 0$. Then, since ℓ is the *least* upper bound for $\{s_n : n \in \mathsf{N}\}$, there exists N such that

$$\ell - \epsilon < s_N;$$

thus, since $\{s_n\}_{n=1}^{\infty}$ is increasing and ℓ is an upper bound for $\{s_n : n \in \mathsf{N}\}$,

$$\ell - \epsilon < s_N \leq s_n \leq \ell \quad \text{for all } n \geq N.$$

Therefore,

$$|s_n - \ell| < \epsilon \quad \text{for all } n \geq N.$$

This proves that $\lim_{n\to\infty} s_n = \ell$.

The second part of the theorem for the case when $\{s_n\}_{n=1}^{\infty}$ is a bounded decreasing sequence can be proved similarly (using the Greatest Lower Bound Axiom) or, even better, by applying the first part of the theorem to the sequence $\{-s_n\}_{n=1}^{\infty}$.

The first part of the theorem follows easily from the second part. ¥

We make three comments about Theorem 19.20. Then we give some examples.

First, in comparing the first part of Theorem 19.20 with the rest of the theorem, we note that a bounded eventually increasing sequence may not converge to the least upper bound of its values (e.g., the sequence whose terms are $2, 0, 0, 0, \ldots$).

[11] Technically, $\{s_n\}_{n=N}^{\infty}$ is not a sequence unless $N = 1$, but when we say $\{s_n\}_{n=N}^{\infty}$ is a sequence, the meaning is obvious – we are referring to the sequence $\{s_{N-1+i}\}_{i=1}^{\infty}$. We continue to use the notation $\{s_n\}_{n=N}^{\infty}$ as shorthand for $\{s_{N-1+i}\}_{i=1}^{\infty}$ since it will cause no confusion.

Second, the part of Theorem 19.20 that says every bounded increasing sequence converges is so basic that it is actually equivalent to the Completeness Axiom in section 1 of Chapter I; we leave the proof to the reader (Exercise 19.26).

Finally, it is clear that Theorem 19.20 is an existence theorem, just as were the Intermediate Value Theorem (Theorem 5.2) and the Maximum - Minimum Theorem (Theorem 5.13). In particular, it may not be easy to find the value of the limit of a bounded monotonic sequence – see Example 19.21.

We illustrate Theorem 19.20 with two examples in this section and with another example in the next section. In the first example, we are only able to show that the sequence converges – we do not know what the limit of the sequence is. In the second example, we are able to find the limit of the sequence after we use Theorem 19.20 to know the limit exists. In the example in the next section, we illustrate how Theorem 19.20 is used in connection with inductively defined sequences.

Example 19.21: For each $n \in \mathsf{N}$, let $s_n = \frac{(1)(3)(5)\cdots(2n-1)}{(2)(4)(6)\cdots(2n)}$. The sequence $\{s_n\}_{n=1}^{\infty}$ is bounded since $0 < s_n < 1$ for all n. The sequence is decreasing since $s_{n+1} = \frac{2n+1}{2n+2} s_n$ for all n. Therefore, by Theorem 19.20, $\{s_n\}_{n=1}^{\infty}$ converges.

Example 19.22: Fix r such that $0 < r < 1$. We show using Theorem 19.20 that $\lim_{n\to\infty} nr^n = 0$.

First, we prove that the sequence $\{nr^n\}_{n=1}^{\infty}$ is eventually decreasing (the entire sequence is not decreasing: try $r = \frac{3}{4}$). Since $r < 1$ and since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, there exists N such that

$$\tfrac{n}{n+1} > r \quad \text{for all } n \geq N.$$

Hence, for all $n \geq N$, $(n+1)r < n$ which, since $r > 0$, gives $(n+1)r^{n+1} < nr^n$. This proves that the sequence $\{nr^n\}_{n=1}^{\infty}$ is decreasing for all $n \geq N$.

Thus, since the sequence $\{nr^n\}_{n=1}^{\infty}$ is bounded (below by 0), we now know from Theorem 19.20 that the the sequence $\{nr^n\}_{n=1}^{\infty}$ converges, say

$$\lim_{n\to\infty} nr^n = L.$$

Hence, $\lim_{n\to\infty} r(nr^n) \overset{19.4}{=} rL$; also, $\lim_{n\to\infty} r^{n+1} \overset{15.3}{=} 0$. Using these two facts for the last equality below, we have

$$L = \lim_{n\to\infty} nr^n = \lim_{n\to\infty}(n+1)r^{n+1} = \lim_{n\to\infty}(rnr^n + r^{n+1}) \overset{19.2}{=} rL.$$

Hence, $(1-r)L = 0$. Therefore, since $r \neq 1$, $L = 0$.

Exercise 19.23: Rework Example 19.22 using Theorem 19.7 instead of Theorem 19.20. Don't give up if your first attempt fails!

Exercise 19.24: For each $n \in \mathsf{N}$, let $s_n = \frac{\ln(2)\ln(4)\cdots\ln(2n)}{\ln(3)\ln(5)\cdots\ln(2n+1)}$. Determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges.

Exercise 19.25: Show using Theorem 19.20 that the series $\Sigma_{i=1}^{\infty} \frac{1}{i!}$ converges.

(*Hint:* Recall Theorem 15.4.)

Exercise 19.26: Prove that the statement *every bounded increasing sequence converges* implies the Completeness Axiom in section 1 of Chapter I. (We proved the converse in the proof of Theorem 19.20.)

3. Inductively Defined Sequences

The sequences in Examples 19.21 and 19.22 could have been defined by induction (Theorem 1.20): For the sequence in Example 19.21, define $s_1 = \frac{1}{2}$ and, assuming we have defined s_n, define $s_{n+1} = \frac{2n+1}{2n+2} s_n$; for the sequence in Example 19.22, define $s_1 = r$ and, assuming we have defined s_n, define $s_{n+1} = r s_n + r^{n+1}$. In general, defining sequences by an inductive formula leads to surprisingly interesting sequences. The surprise is due to the fact that the inductive formula relating s_{n+1} to s_n can be extremely simple, yet the properties of the resulting sequence can be very complicated (and can take some effort to uncover).

For all sequences we have seen, their terms have been defined by formulas; however, we may not even be able to *find* a formula that tells us the exact value of the n^{th} term of many inductively defined sequences. Thus, Theorem 19.20 plays a central role in investigating inductively defined sequences, and the existential nature of Theorem 19.20 is even more directly visible here than it was in the preceding section.

The example below illustrates everything we have said. In addition, the verifications for the example illustrate three important working principles: First, looking at the first several terms of a sequence may lead you to detect a pattern; second, grouping various terms of a sequence together can be effective in proving convergence (or divergence); third, proving only the existence of the limit of a sequence can sometimes be turned into finding the exact value of the limit (we saw this before, in the verifications for Example 19.22).

Example 19.27: Let $s_1 = 1$ and, assuming we have defined s_n, let $s_{n+1} = 1 + \frac{1}{s_n}$. The sequence $\{s_n\}_{n=1}^{\infty}$ is neither increasing nor decreasing:

$$1, 2, \tfrac{3}{2}, \tfrac{5}{3}, \tfrac{8}{5}, \tfrac{13}{8}, \tfrac{21}{13}, \tfrac{34}{21}, \tfrac{55}{34}, \ldots .$$

On examining the pattern, we are led to conjecture that the odd terms s_{2n-1} are increasing and the even terms s_{2n} are decreasing. We prove the conjecture by induction:

From the terms listed above, $s_1 < s_3$. Assume inductively that $s_{2k-1} < s_{2k+1}$ for some given $k \in \mathsf{N}$. Then, since all terms of the sequence are positive (by an easy induction), $\frac{1}{s_{2k+1}} < \frac{1}{s_{2k-1}}$; hence, $1 + \frac{1}{s_{2k+1}} < 1 + \frac{1}{s_{2k-1}}$ which, by the inductive formula for the sequence $\{s_n\}_{n=1}^{\infty}$, says $s_{2k+2} < s_{2k}$; then, repeating the steps starting with $s_{2k+2} < s_{2k}$, we obtain $1 + \frac{1}{s_{2k}} < 1 + \frac{1}{s_{2k+2}}$, which says $s_{2k+1} < s_{2k+3}$. Therefore, by the Induction Principle (Theorem

1.20), we have proved that all the odd terms s_{2n-1} are increasing. The proof that the even terms s_{2n} are decreasing is similar: From the terms listed above, $s_2 > s_4$. Assume inductively that $s_{2k} > s_{2k+2}$ for some given $k \in \mathsf{N}$. Then, $\frac{1}{s_{2k+2}} > \frac{1}{s_{2k}}$ and, hence, $1 + \frac{1}{s_{2k+2}} > 1 + \frac{1}{s_{2k}}$, which says $s_{2k+3} > s_{2k+1}$; thus, $1 + \frac{1}{s_{2k+1}} > 1 + \frac{1}{s_{2k+3}}$, which says $s_{2k+2} > s_{2k+4}$. Therefore, by the Induction Principle, all the even terms s_{2n-1} are decreasing.

Next, we show that the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded. It is easy to see (by induction) that $s_n \geq 1$ for all $n \in \mathsf{N}$. Thus, since $s_{n+1} = 1 + \frac{1}{s_n}$ for all $n \in \mathsf{N}$, $s_{n+1} \leq 2$ for all $n \in \mathsf{N}$. Hence, we have shown that

$$1 \leq s_n \leq 2 \quad \text{for all } n \in \mathsf{N}.$$

This proves that the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded.

Now, we can apply Theorem 19.20 to the sequences $\{s_{2n-1}\}_{n=1}^{\infty}$ and $\{s_{2n}\}_{n=1}^{\infty}$ separately to obtain that they each converge, say

$$\lim_{n\to\infty} s_{2n-1} = L, \ \ \lim_{n\to\infty} s_{2n} = M.$$

Note that since $1 \leq s_n \leq 2$ for all $n \in \mathsf{N}$, $L > 0$ and $M > 0$; we will use this fact several times (usually without saying so).

Finally, we determine the exact value for L and for M; as a consequence, we obtain that $L = M$. Therefore, we can conclude that $\lim_{n\to\infty} s_n$ is $L = M$ by Exercise 19.18.

Using the inductive formula that defined $\{s_n\}_{n=1}^{\infty}$, we see that for each $n \in \mathsf{N}$,

$$s_{2n+1} = 1 + \frac{1}{s_{2n}} = 1 + \frac{1}{1+\frac{1}{s_{2n-1}}} = \frac{1+2s_{2n-1}}{1+s_{2n-1}};$$

thus, since $\lim_{n\to\infty} s_{2n-1} = L$ and, hence, $\lim_{n\to\infty} s_{2n+1} = L$, we have

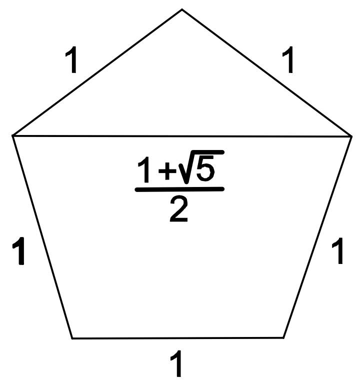
$$L = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} \frac{1+2s_{2n-1}}{1+s_{2n-1}} \overset{19.2,\ 19.5}{=} \frac{1+2L}{1+L}.$$

Thus, $L^2 - L - 1 = 0$. Hence, by the quadratic formula, $L = \frac{1\pm\sqrt{5}}{2}$. Therefore, since $L > 0$, we conclude that

$$L = \frac{1+\sqrt{5}}{2}.$$

Similarly, $M = \frac{1+\sqrt{5}}{2}$. Therefore, by Exercise 19.18, $\lim_{n\to\infty} s_n = \frac{1+\sqrt{5}}{2}$, the golden ratio(!), which we briefly discuss.

The limit $\frac{1+\sqrt{5}}{2}$ of the sequence in Example 19.27 is called the golden ratio because, in ancient times, it represented the perfection of beauty in art and architecture – not for any particular reason we know of, but apparently from aesthetic considerations alone. Obviously, the golden ratio did not surface in ancient Greece as the limit of an inductively defined sequence! We don't know how or when the golden ratio was first specified, but the golden ratio appears in many constructions in ancient Greek geometry. The ancient Greek geometers considered the most beautiful of all rectangles to be one for which the ratio of its length to its width is $\frac{1+\sqrt{5}}{2}$; the straight line joining two nonadjacent vertices of a regular pentagon with sides of unit length is $\frac{1+\sqrt{5}}{2}$ (see figure on next page).



A study of ancient art and architecture shows that the golden ratio was a foundation of design. More recently, the golden ratio has been investigated in connection with music. There are many other situations in which the golden ratio plays an (explicit or implicit) role. For our purpose, it suffices to say that Example 19.27 shows that inductively defined sequences with *very simple* inductive formulas can have a very interesting limits indeed! However, we are left to wonder why the golden ratio is noticeable in so many diverse situations.

I do not want to leave the reader with a misimpression: When we define a sequence by a simple inductive formula and show that the limit of the sequence exists, we still may not be able to compute the exact limit of the sequence by the method we used in Example 19.27. For example, as mentioned earlier, the sequence in Example 19.21 can be defined inductively by letting $s_1 = \frac{1}{2}$ and defining $s_{n+1} = \frac{2n+1}{2n+2} s_n$; however, if we try to compute the exact value of $\lim_{n\to\infty} s_n$ using the method in Example 19.27, we get nothing:

$$\lim_{n\to\infty} s_{n+1} \overset{19.4}{=} \lim_{n\to\infty} \tfrac{2n+1}{2n+2} \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_n .$$

The Graphical Method

We look at Example 19.27 from another point of view, one that really serves to explain what is going on. Our comments about the specific example generalize and, thus, can be employed to aid the reader's intuition with respect to many inductively defined sequences.

Consider the function f given by $f(x) = 1 + \frac{1}{x}$ for all $x \neq 0$. The inductively defined sequence in Example 19.27 can be defined in terms of f: the inductive formula becomes $s_{n+1} = f(s_n)$. "Simple enough," you say, "but so what!" The value of this point of view is as follows:

Draw the graph of f; start at $x = 1$, move vertically to the graph, then move horizontally to the line $y = x$, then vertically to the graph, then horizontally to the line $y = x$, and so on. The first point we got on the graph was $f(1) = s_2$, the second point we got on the graph was $f(f(1)) = s_3$, and the n^{th} time we touch the graph in this procedure we get the the term s_{n+1}, which coincides with $f^n(1)$ where $f^n = f \circ f \circ \cdots \circ f$ (f appearing n times); f^n is called the n^{th} *iterate* *of* f. It is evident from your picture that the terms of the sequence are heading towards the positive fixed point of f; simple algebra shows the positive fixed point of f is $\frac{1+\sqrt{5}}{2}$. You can also see from your picture that successive terms of the sequence are alternately below and above the positive fixed point of f with the odd terms increasing towards the fixed point of f and the even terms decreasing towards the fixed point of f. The fact that the sequence converges to a fixed point of f is not an accident – see Exercise 19.28.

In practice, the graphical method we just described works for any inductively defined sequence provided that the formula for the inductive definition is functional in nature and not too complicated. Of course, the conclusions we arrive at using the method still need to be proved: the graphical method is limited to helping our intuition.

The exercises that follow are designed to give the reader experience with the graphical method. Although we do not call attention to the graphical method in most of the exercises, we intend that the reader use the method to make conjectures before doing anything else (with the exceptions of Exercises 19.28 and 19.36); then, of course, the reader must try to verify the conjectures. Conjectures followed by attempts to verify the conjectures is, after all, the way mathematics is done; I doubt that many readers could just jump in and work the exercises without first getting a feeling for what is going on, and the graphical method is well suited for building up intuition.

Our first exercise verifies a statement we made when we discussed the graphical method.

Exercise 19.28: Let $X \subset \mathsf{R}^1$, and let $f : X \to X$ be a continuous function. Define a sequence $\{s_n\}_{n=1}^{\infty}$ as follows: Let s_1 be any given point of X and, assuming we have defined s_n, let $s_{n+1} = f(s_n)$. Prove that if the sequence $\{s_n\}_{n=1}^{\infty}$ converges to a point $p \in X$, then p is a fixed point of f (i.e., $f(p) = p$).

Exercise 19.29: Use the graphical method to conjecture whether or not we could have started with any $s_1 > 0$ in Example 19.27 and obtained the golden mean $\frac{1+\sqrt{5}}{2}$ as the limit of $\{s_n\}_{n=1}^{\infty}$. What about when $s_1 < 0$?

Exercise 19.30: As a variation of Example 19.27, let m be a given real number, let $s_1 = 1$ and, assuming we have defined s_n, let $s_{n+1} = m - \frac{1}{s_n}$. Use the graphical method to conjecture about which real numbers m result in convergence of the sequence $\{s_n\}_{n=1}^{\infty}$ and to conjecture about the value of the limit (in terms of m) when the sequence converges.

(*Hint:* Use Exercise 19.28.)

Exercise 19.31: Let $s_1 = 0$ and, assuming we have defined s_n, let $s_{n+1} = 1 + \frac{s_n+1}{5}$. Determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges and, if the sequence converges, find its limit.

Exercise 19.32: Let $s_1 = 1$ and, assuming we have defined s_n, let $s_{n+1} = 1 + \frac{1}{1+s_n}$. Determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges and, if the sequence converges, find its limit.

Exercise 19.33: Let $s_1 = 1$ and, assuming we have defined s_n, let $s_{n+1} = \frac{s_n^2+3}{2s_n}$. Determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges and, if the sequence converges, find its limit.

Exercise 19.34: Let $s_1 = \frac{1}{8}$ and, assuming we have defined s_n, let $s_{n+1} = \frac{1}{2}s_n^2 + 1$. Determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges and, if so, find its limit.

Exercise 19.35: Let $s_1 = 1$ and, assuming we have defined s_n, let $s_{n+1} = s_n^2 - 1$. Determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ converges and, if the sequence converges, find its limit.

Exercise 19.36: We know from Exercise 19.25 that the series $\Sigma_{i=1}^{\infty}\frac{1}{i!}$ converges. Define the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of $\Sigma_{i=1}^{\infty}\frac{1}{i!}$inductively; then see if the method we used to compute the limit L near the end of Example 19.27 can be applied here to the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums to find the exact value of $\Sigma_{i=1}^{\infty}\frac{1}{i!}$.

The first term s_1 of an inductively defined sequence is called the *initial value* of the sequence. In general, inductively defined sequences are unstable with respect to their initial values. This means that with the same inductive formula, small changes in the initial value can produce significant changes in the properties of the resulting sequences – one sequence may converge while others that start as near it as we wish may diverge or may converge but to limits that are far away from the limit of the original convergent sequence. Our final exercise below illustrates these behaviors.

Exercise 19.37: We define sequences $\{s_{t,n}\}_{n=1}^{\infty}$ for any given real number t as follows: $s_{t,1} = t$ and, assuming we have defined $s_{t,n}$, let

$$s_{t,n+1} = \begin{cases} 2s_{t,n} & \text{, if } s_{t,n} \leq \frac{1}{2} \\ -\frac{1}{2}s_{t,n} + \frac{5}{4} & \text{, if } s_{t,n} > \frac{1}{2}. \end{cases}$$

Find a real number t_0 with the following properties: The sequence $\{s_{t_0,n}\}_{n=1}^{\infty}$ with inital value t_0 converges to a number L; there are points t as close to t_0 as we wish such that the sequences $\{s_{t,n}\}_{n=1}^{\infty}$ with initial values t diverge to $-\infty$; and there are points t as close to t_0 as we wish such that the sequences $\{s_{t,n}\}_{n=1}^{\infty}$ with initial values t converge to the same number $M \neq L$.

4. Arbitrary Closeness and Continuity via Sequences

We introduced the notions of arbitrary closeness and continuity in Chapter II. We show how these concepts can be reformulated in terms of sequences.

Theorem 19.38: Let $p \in \mathsf{R}^1$, and let $A \subset \mathsf{R}^1$ such that $A \neq \emptyset$. Then $p \sim A$ if and only if there is a sequence $\{s_n\}_{n=1}^{\infty}$ of points in A such that $\lim_{n\to\infty} s_n = p$.

Proof: Assume that $p \sim A$. Then, by Theorem 2.3,

$$(p - \tfrac{1}{n}, p + \tfrac{1}{n}) \cap A \neq \emptyset \quad \text{for each } n \in \mathsf{N}.$$

Hence, we can let $s_n \in (p - \frac{1}{n}, p + \frac{1}{n}) \cap A$ for each $n \in \mathsf{N}$, thereby obtaining a sequence $\{s_n\}_{n=1}^{\infty}$. Clearly, $s_n \in A$ for each $n \in \mathsf{N}$ and $\lim_{n\to\infty} s_n = p$ (by the Archimedean Property (Theorem 1.22)).

Conversely, assume that there is a sequence $\{s_n\}_{n=1}^{\infty}$ of points in A such that $\lim_{n\to\infty} s_n = p$. Let $S = \{s_n : n \in \mathsf{N}\}$. Then, by the definitions of convergence of sequences and arbitrary closeness, we have that $p \sim S$. Therefore, since $S \subset A$, we have by Exercise 2.10 that $p \sim A$. ¥

Our next theorem characterizes continuity of a function at a point in terms of sequences. One value of the theorem is that we can analyze some properties of continuous functions easier with sequences than with limits.

Theorem 19.39: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$. Then f is continuous at p if and only if whenever $\{s_n\}_{n=1}^{\infty}$ is a sequence of points in X converging to p, then the sequence $\{f(s_n)\}_{n=1}^{\infty}$ converges to $f(p)$.

Proof: Assume that f is continuous at p. Let $\{s_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} s_n = p$. Then, by Theorem 18.6, $\lim_{n\to\infty} f(s_n) = f(p)$.

Conversely, assume the sequence condition in our theorem. We show that f is continuous at p using the definition of continuity (below Exercise 2.22).

Let $A \subset X$ such that $p \sim A$. Then, by Theorem 19.38, there is a sequence $\{s_n\}_{n=1}^{\infty}$ of points in A such that $\lim_{n\to\infty} s_n = p$. Thus, by our assumption, $\lim_{n\to\infty} f(s_n) = f(p)$. Hence, letting $S = \{s_n : n \in \mathsf{N}\}$, we have that $f(p) \sim f(S)$ (by the definitions of convergence of sequences and arbitrary closeness). Thus, since $S \subset A$ and, hence, $f(S) \subset f(A)$, we have by Exercise 2.10 that $f(p) \sim f(A)$. This proves that f is continuous at p. ¥

For the remainder of the section, we discuss an aspect of the proofs of Theorem 19.38 and Theorem 19.39.

In the first part of the proof of Theorem 19.38, we used (without specifically saying so) a set-theoretic axiom called the Countable Axiom of Choice. This axiom says that there is a choice function for any countably infinite collection $\mathcal{C}$ of nonempty sets; that is, there is a function $\varphi : \mathcal{C} \to \cup\mathcal{C}$ such that $\varphi(C) \in C$ for all $C \in \mathcal{C}$. In the proof of Theorem 19.38, we implicitly applied the axiom to the collection $\mathcal{C} = \{(p - \frac{1}{n}, p + \frac{1}{n}) \cap A : n \in \mathsf{N}\}$.

In the proof of Theorem 19.39, we used the part of Theorem 19.38 whose proof used the Countable Axiom of Choice. Thus, in effect, we used the Countable Axiom of Choice in proving Theorem 19.39.

I see no way to avoid using of the Countable Axiom of Choice in the proofs of Theorem 19.38 or Theorem 19.39. This is not to say that the Countable Axiom of Choice is something to be avoided whenever possible! Rather, it suggests a question: Does Theorem 19.38 or Theorem 19.39 imply the Countable Axiom of Choice for (countable) collections of nonempty sets of real numbers? I note that Theorem 19.38 and Theorem 19.39 extend directly to all metric spaces; in the general setting of metric spaces, Theorem 19.38 implies Theorem 19.39 which, in turn, implies the Countable Axiom of Choice – see Norbert Brunner, *Sequential continuity*, Kyungpook Math. J. **22**(1982), 233.

Theorem 3.11 is closely related to Theorem 19.39. However, I did not use the Countable Axiom of Choice in the proof of Theorem 3.11; instead, I made use of the sets A_δ (it is insightful to prove that part of Theorem 3.11 again, this time using Theorem 19.38). Nevertheless, the proof of Theorem 19.39 used Theorem 19.38, and the proof of Theorem 19.38 seems to rest on *defining* a sequence with the designated properties; defining such a sequence, even in the real line, seems to require the Countable Axiom of Choice (but I am not sure).

Although we are now aware of the Countable Axiom of Choice, we will no longer mention the axiom or use it explicitly. Our failure to mention the axiom when it is used will not bother you, and the axiom's explicit use is more distracting than helpful (see comments about the general Axiom of Choice following the proof of Theorem 18.18).

Exercise 19.40: Let $A, B \subset \mathsf{R}^1$. If $p \sim A$ and $q \sim B$, then $p + q \sim A + B$ where $A + B = \{a + b : a \in A \text{ and } b \in B\}$.

The result in Exercise 19.40 can be interpreted as showing that addition is continuous; of course, since addition is a function from the plane R^2 to the real line R^1, a complete understanding of this is predicated on knowing the notion of arbitrary closeness for R^2. We define arbitrary closeness for any metric space near the end of section 4 in the next chapter.

Exercise 19.41: A comment in the proof of Theorem 2.11 suggested that a direct argument for $(A \cup B)^\sim \subset A^\sim \cup B^\sim$ can be done with other methods, and we gave a contrapositive argument for the containment. Now that Theorem 19.38 says how we can view arbitrary closeness in terms of sequences, a direct argument for $(A \cup B)^\sim \subset A^\sim \cup B^\sim$ is easy. Give the argument.

Chapter XX: Subsequences and Cauchy Sequences

We continue our study of the basic properties of sequences that we began in the preceding chapter.

Throughout the chapter (unless we say otherwise or context makes it obvious), the term *sequence* means a sequence of points in R^1 (i.e., a *numerical sequence*).

In section 1 we introduce the important notion of a subsequence. In section 2 we prove that every bounded sequence has a convergent monotonic subsequence. In section 3 we define the notion of a Cauchy sequence and use the theorem in section 2 to prove that every Cauchy sequence converges. This leads us, in section 4, to revisit the Completeness Axiom in section 1 of Chapter I and develop a notion of completeness for metric spaces in general.

1. Subsequences

We define what we mean by a subsequence of a sequence and discuss the notion. One value in considering subsequences is that even though a sequence may not converge, the sequence may have many convergent subsequences that are useful. The theorem in the next section gives an applicable condition under which a convergent subsequence of a sequence exists.

Definition: Let $\{s_n\}_{n=1}^{\infty}$ be a sequence. For any strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of natural numbers, the sequence $\{s_{n_i}\}_{i=1}^{\infty}$ is called a *subsequence of the sequence* $\{s_n\}_{n=1}^{\infty}$. In other words, a subsequence of a sequence s is the composition, $s \circ t$, of s with any strictly increasing sequence t of natural numbers.

Note that a subsequence of a subsequence of $\{s_n\}_{n=1}^{\infty}$ is still a subsequence of $\{s_n\}_{n=1}^{\infty}$. We denote subsequences of subsequences with triple subscripts, $\{s_{n_{i_k}}\}_{k=1}^{\infty}$.

In the definition of subsequence, the requirement that $\{n_i\}_{i=1}^{\infty}$ is strictly increasing is important to remember. Consider the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$; the sequence all of whose terms are 1, as well as the sequence whose terms are $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$, are not subsequences of the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ even though their terms are terms of $\{\frac{1}{n}\}_{n=1}^{\infty}$.

Exercise 20.1: Determine all the convergent subsequences of $\{(-1)^n\}_{n=1}^{\infty}$. Show that $\{(-1)^n\}_{n=1}^{\infty}$ has uncountably many subsequences but only countably many convergent subsequences. (Countable sets are discussed in section 1 of Chapter XV.)

Exercise 20.2: Prove the following (obvious) result carefully, referring diligently to each item used in Chapter I (the axioms in section 1 of Chapter I, assumptions about N in 1.18, and any theorem in Chapter I that must be used):

If $\{n_i\}_{i=1}^{\infty}$ is a strictly increasing sequence of natural numbers, then $n_i \geq i$ for all $i \in \mathsf{N}$; hence, for any real number N, there exists $k \in \mathsf{N}$ such that $n_i \geq N$ for all $i \geq k$.

It is evident from Exercise 20.2 that every subsequence of $\{\frac{1}{n}\}_{n=1}^{\infty}$ must converge to 0, the limit of $\{\frac{1}{n}\}_{n=1}^{\infty}$. This observation about subsequences of $\{\frac{1}{n}\}_{n=1}^{\infty}$ illustrates the general result in the next exercise. The result in the exercise is natural and is easy to prove, but would be false without requiring in the definition of subsequence that $\{n_i\}_{i=1}^{\infty}$ is strictly increasing.

Exercise 20.3: Every subsequence, $\{s_{n_i}\}_{i=1}^{\infty}$, of a convergent sequence $\{s_n\}_{n=1}^{\infty}$ converges, and $\lim_{i\to\infty} s_{n_i} = \lim_{n\to\infty} s_n$.

Exercise 20.4: It follows from Exercise 15.2 that the set Q of all rational numbers is countable. Thus, there is a one-to-one function f from N onto the set Q of all rational numbers. What real numbers are limits of subsequences of the sequence $\{f(n)\}_{n=1}^{\infty}$?

2. The Bolzano Weierstrass Theorem

We prove the Bolzano-Weierstrass Theorem, which says that every bounded sequence has a convergent monotonic subsequence. This is an existence theorem – it may be difficult to find a specific convergent subsequence. For example, try the following exercise:

Exercise 20.5: Find a convergent subsequence of the sequence $\{\sin(n)\}_{n=1}^{\infty}$.

The proof of the Bolzano-Weierstrass Theorem uses the result in Exercise 5.16, which says that every bounded infinite subset of R^1 has a limit point. Actually, the Bolzano-Weierstrass Theorem and the result in Exercise 5.16 are equivalent: Half of the equivalence will be shown in the proof of the Bolzano-Weierstrass Theorem; the other half of the equivalence is left for the reader to prove in Exercise 20.8.

Theorem 20.6 (Bolzano-Weierstrass Theorem): Every bounded sequence has a convergent monotonic subsequence.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence. Let $A = \{s_n : n \in \mathsf{N}\}$ (the set of values of $\{s_n\}_{n=1}^{\infty}$).

If A is a finite set, then there is a point $a \in A$ such that $s_n = a$ for infinitely many n. Hence, by a simple induction, we obtain a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of natural numbers such that $s_{n_i} = a$ for all $i \in \mathsf{N}$. Then the sequence $\{s_{n_i}\}_{i=1}^{\infty}$ is a monotonic subsequence of $\{s_n\}_{n=1}^{\infty}$ and, clearly, $\{s_{n_i}\}_{i=1}^{\infty}$ converges to a. This proves our theorem when A is a finite set. Therefore, we assume for the rest of the proof the A is an infinite set.

Thus, by Exercise 5.16, A has a limit point $p \in \mathsf{R}^1$. We prove that there is a monotonic subsequence of $\{s_n\}_{n=1}^{\infty}$ that converges to p.

For each $k \in \mathsf{N}$, let

$$L_k = A \cap (p - \tfrac{1}{k}, p), \quad R_k = A \cap (p, p + \tfrac{1}{k}).$$

Since p is a limit point of A, we have by Exercise 2.33 that $(p - \frac{1}{k}, p + \frac{1}{k}) \cap A$ is an infinite set for each $k \in \mathsf{N}$. Hence, for each $k \in \mathsf{N}$, at least one of the sets L_k or R_k is infinite. Thus, L_k is infinite for infinitely many k or R_k is infinite for infinitely many k. We assume by symmetry that L_k is infinite for infinitely many $k \in \mathsf{N}$. Then, since $L_{k+1} \subset L_k$ for all $k \in \mathsf{N}$, it follows immediately that L_k is infinite for all $k \in \mathsf{N}$. Note that p is a limit point of L_k for all $k \in \mathsf{N}$.

We now use induction to define a subsequence $\{s_{n_i}\}_{i=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ with the properties claimed by our theorem.

Let $s_{n_1} \in L_1$. Assume inductively that we have defined $s_{n_k} \in L_k$ for some $k \in \mathsf{N}$; note that since $s_{n_k} \in L_k$, $s_{n_k} < p$. Then, since L_{k+1} is an infinite set and p is a limit point of L_{k+1}, there is a point $s_{n_{k+1}} \in L_{k+1}$ such that $n_{k+1} > n_k$ and $s_{n_{k+1}} > s_{n_k}$. Therefore, by the Induction Principle (Theorem 1.20), we have defined s_{n_i} for each $i \in \mathsf{N}$. In other words, we have defined a sequence $\{s_{n_i}\}_{i=1}^{\infty}$.

From the constuction defining $\{s_{n_i}\}_{i=1}^{\infty}$, we see that $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence of natural numbers; hence, $\{s_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{s_n\}_{n=1}^{\infty}$. Furthermore, $s_{n_{i+1}} > s_{n_i}$ for each $i \in \mathsf{N}$, which shows that $\{s_{n_i}\}_{i=1}^{\infty}$ is increasing. Finally, since $s_{n_i} \in L_i$ for each $i \in \mathsf{N}$, we have that

$$p - \tfrac{1}{k_i} < s_{n_i} < p \quad \text{for each } i \in \mathsf{N};$$

therefore, $\{s_{n_i}\}_{i=1}^{\infty}$ converges to p (by the Squeeze Theorem for Sequences (Theorem 19.6) since $\{\frac{1}{k_i}\}_{i=1}^{\infty}$ converges to 0 by the Archimedean Property (Theorem 1.22)). ¥

Exercise 20.7: True or false: If $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence such that all convergent subsequences of $\{s_n\}_{n=1}^{\infty}$ converge to the same point, then the sequence converges.

Exercise 20.8: Show how to prove the result in Exercise 5.16 from the Bolzano - Weierstrass Theorem (Theorem 20.6). Thus, the two results are equivalent (since we proved the other direction in the proof of Theorem 20.6).

3. Cauchy Sequences

We characterize convergent sequences in terms of a condition commonly called the *Cauchy criterion for convergence* (Cauchy refers to the French mathematician Augustin Louis Cauchy, 1789 - 1857). The Cauchy criterion plays an important role in studying convergence of sequences and series, especially sequences and series of functions. Aside from that, the notion of a Cauchy sequence provides an appropriate generalization to the setting of metric spaces of the Completeness Axiom in section 1 of Chapter I, as we will see in the next section.

Definition: A *Cauchy sequence* is a sequence $\{s_n\}_{n=1}^{\infty}$ with the following property: For each $\epsilon > 0$, there exists a number N such that

$$|s_i - s_j| < \epsilon \quad \text{for all } i, j \geq N.$$

Exercise 20.9: Every convergent sequence is a Cauchy sequence.

We prove that every Cauchy sequence converges – the converse of the result in Exercise 20.9 above. The proof is based on the Bolzano-Weierstrass Theorem (Theorem 20.6): The first lemma below enables us to apply the Bolzano-Weierstrass Theorem, and the next lemma pinpoints the other ingredient in the proof.

Lemma 20.10: Every Cauchy sequence is bounded.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then, by definition (with $\epsilon = 1$), there is a natural number N such that

$$|s_i - s_j| < 1 \text{ for all } i, j \geq N.$$

In particular, $|s_i - s_N| < 1$ for all $i \geq N$. Hence, only finitely many terms can be at a distance more than 1 from s_N, namely, the terms $s_1, s_2, ..., s_{N-1}$. Thus, we see that $\{s_n\}_{n=1}^{\infty}$ is bounded as follows: For any $i \geq N$,

$$|s_i| = |s_i - s_N + s_N| \leq |s_i - s_N| + |s_N| < 1 + |s_N|;$$

therefore, letting $M = \max\{|s_1|, |s_2|, ..., |s_{N-1}|, 1+|s_N|\}$, it is clear that $|s_n| \leq M$ for all $n \in \mathsf{N}$.. ¥

We know from Exercise 20.3 that if $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence with limit L, then every subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to L. The converse is obvious (since a sequence is a subsequence of itself). Our next lemma shows that a stronger result in the converse direction is true for Cauchy sequences.

Lemma 20.11: If *some* subsequence of a Cauchy sequence converges, then the entire sequence converges (to the same limit as the subsequence).

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence such that some subsequence, say $\{s_{n_i}\}_{i=1}^{\infty}$, converges. Let $L = \lim_{i\to\infty} s_{n_i}$.

We now proceed to prove that $\lim_{n\to\infty} s_n = L$.

Let $\epsilon > 0$. Since $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there exists N such that

(1) $|s_i - s_j| < \frac{\epsilon}{2}$ for all $i, j \geq N$.

Since $L = \lim_{i\to\infty} s_{n_i}$, there exists $k \in \mathsf{N}$ such that

(2) $|s_{n_k} - L| < \frac{\epsilon}{2}$.

Now, for any $n \geq N$, we have

$$|s_n - L| = |s_n - s_{n_k} + s_{n_k} - L| \leq |s_n - s_{n_k}| + |s_{n_k} - L| \overset{(1),\,(2)}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, we have proved that $\lim_{n\to\infty} s_n = L$. ¥

The proof of our theorem is now merely a matter of applying the Bolzano-Weierstrass Theorem and the preceeding two lemmas:

Theorem 20.12: Every Cauchy sequence converges.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence. By Lemma 20.10, $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence. Hence, by the Bolzano-Weierstrass Theorem (Theorem

20.6), $\{s_n\}_{n=1}^{\infty}$ has a convergent subsequence. Therefore, by Lemma 20.11, the sequence $\{s_n\}_{n=1}^{\infty}$ converges. ¥

Corollary 20.13: A sequence converges if and only if it is a Cauchy sequence.

Proof: Combine Exercise 20.9 and Theorem 20.12. ¥

It is natural to wonder whether the condition defining a Cauchy sequence can be replaced by the following condition which involves only two successive terms at a time: For each $\epsilon > 0$, there exists a number N such that

$$(*)\ |s_{n+1} - s_n| < \epsilon \quad \text{for all } n \geq N.$$

The following example shows that this condition is not strong enough to imply that the sequence is a Cauchy sequence.

Example 20.14: Let $\{s_n\}_{n=1}^{\infty}$ be the sequence defined by $s_n = \Sigma_{i=1}^{n} \frac{1}{i}$. Since $|s_{n+1} - s_n| = \frac{1}{n+1}$, it follows from the Archimedean Property (Theorem 1.22) that $\{s_n\}_{n=1}^{\infty}$ satisfies condition (*) above. However, $\{s_n\}_{n=1}^{\infty}$ is not a Cauchy sequence, which we show two ways: First, for any $n \in \mathsf{N}$,

$$|s_{2n} - s_n| = \Sigma_{i=0}^{n-1} \frac{1}{2n-i} \geq \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2},$$

so $\{s_n\}_{n=1}^{\infty}$ is not a Cauchy sequence by the definition of Cauchy sequence; second, for any $n \in \mathsf{N}$,

$$s_{2^n} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\Sigma_{i=5}^{8} \tfrac{1}{i}\right) + \left(\Sigma_{i=9}^{16} \tfrac{1}{i}\right) + \cdots + \left(\Sigma_{i=2^{n-1}+1}^{2^n} \tfrac{1}{i}\right)$$
$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \tfrac{4}{8} + \tfrac{8}{16} + \cdots + \tfrac{2^{n-1}}{2^n} = 1 + n\left(\tfrac{1}{2}\right),$$

which shows that $\{s_n\}_{n=1}^{\infty}$ is not bounded, hence $\{s_n\}_{n=1}^{\infty}$ is not a Cauchy sequence by Lemma 20.10 (or Theorem 20.12).

Exercise 20.15: Let $\{s_n\}_{n=1}^{\infty}$ be the sequence defined by $s_n = \Sigma_{i=1}^{n} \frac{1}{i^2}$. Show using the definition of a Cauchy sequence that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy Sequence. (The limit of the sequence $\{s_n\}_{n=1}^{\infty}$ is $\frac{\pi^2}{6}$, but this is difficult to prove.)

(*Hint:* For any natural number $i > 1$, $\frac{1}{(i-1)i} = \frac{1}{i-1} - \frac{1}{i}$.)

Exercise 20.16: Fix any real number $p \geq 2$, and let $\{s_n\}_{n=1}^{\infty}$ be the sequence defined by $s_n = \Sigma_{i=1}^{n} \frac{1}{i^p}$. Show using Exercise 20.15 that the sequence $\{s_n\}_{n=1}^{\infty}$ converges. Be sure to include *all* details.

The sequence in Exercise 20.16 is the sequence of partial sums of an important type of series called a p-series. In the next chapter we show that a p-series converges if and only if $p > 1$ (Example 21.29). Exercise 20.16 shows that the method we will use to prove this is not really necessary when $p \geq 2$.

Exercise 20.17: Is the sequence $\{\frac{n!}{10^n}\}_{n=1}^{\infty}$ a Cauchy sequence?

Exercise 20.18: Is the sequence $\{\sin(n) - \cos(n)\}_{n=1}^{\infty}$ a Cauchy sequence?

4. Cauchy Sequences and the Completeness Axiom

We discuss how the notion of a Cauchy sequence leads to an appropriate notion of completeness for metric spaces.

A *metric space* is a set X together with a distance function d for X (the definition of a distance function is in Exercise 1.30). As is customary, we use the ordered pair notation (X, d) to denote a metric space X with distance function d.

The general study of analysis takes place in metric spaces. We already know that R^1 with $d(x, y) = |x - y|$ is a metric space (Exercise 1.30). We give a few more examples of metric spaces simply to indicate the variety of spaces that occur (we do not prove that d is a distance function for any of the examples):

Example 20.19: The following are metric spaces:

(1) R^n, the set of all n-tuples of real numbers, with

$$d((x_i)_{i=1}^n, (y_i)_{i=1}^n) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}.$$

(2) The set of all continuous functions on $[0, 1]$ with

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

(3) The set of all functions on $[0, 1]$ that have a continuous derivative with

$$d(f, g) = \sup_{x \in [0,1]} (|f(x) - g(x)| + |f'(x) - g'(x)|)$$

(4) The set of all continuous functions on $[0, 1]$ with

$$d(f, g) = \int_0^1 |f - g|.$$

(5) The set of all sequences of real numbers with

$$d((s_n)_{n=1}^\infty, (t_n)_{n=1}^\infty) = \Sigma_{n=1}^\infty \frac{1}{2^n} \frac{|s_n - t_n|}{1 + |s_n - t_n|}.$$

(6) The set of all bounded sequences of real numbers with

$$d((s_n)_{n=1}^\infty, (t_n)_{n=1}^\infty) = \sup_{n \in \mathsf{N}} |s_n - t_n|.$$

(7) Any nonempty set X with

$$d(x, y) = \begin{cases} 1 & \text{, if } x \neq y \\ 0 & \text{, if } x = y. \end{cases}$$

The Completeness Axiom in section 1 of Chapter I played an important role in most of the material in the previous chapters. In particular, the Completeness Axiom was the central ingredient in the proofs many results that, ostensibly, had nothing to do with the axiom; we mention only four such results: the existence of square roots (Theorem 1.25), the Intermediate Value Theorem (Theorem 5.2), the Maximum-Minimum Theorem (Theorem 5.13), and the derivative test for strictly increasing and strictly decreasing functions in Theorem 10.17. We discussed the relationship between Theorem 10.17 and the Completeness Axiom in detail after Exercise 10.18. Of course, the Completeness Axiom was also used (indirectly) when any of the results just mentioned, or other results like them, were used; for example, the Completeness Axiom was used indirectly in the proof that continuous functions are integrable (Theorem 12.33).

Therefore, since metric spaces are the setting for analysis, it is only reasonable that we should want a notion that we could use in general metric spaces to the same advantage that we used the Completeness Axiom in R^1. Of course, we can not use the Completeness Axiom itself since its statement is in terms of an order, which metric spaces almost never have. However, the result in the exercise below will suggest a notion of completeness for metric spaces. We continue our discussion after the exercise.

Exercise 20.20: The Completeness Axiom in section 1 of Chapter I is equivalent to the statement *every Cauchy sequence converges*.

(*Hint:* To prove that the Completeness Axiom implies the statement *every Cauchy sequence converges*, we merely need to recall various proofs in turn: the proof of Theorem 5.11, then Exercise 5.16, then the proof of Theorem 20.6, and, finally, the proof of Theorem 20.12. To prove the reverse implication, show that statement *every Cauchy sequence converges* implies the Nested Interval Property, and then apply Exercise 5.17).

Next, note that the definition of a Cauchy sequence – unlike the Completeness Axiom – has a direct generalization for any metric space:

Definition: Let (X, d) be a metric space. A sequence $\{s_n\}_{n=1}^{\infty}$ of points of X is called a *Cauchy sequence with respect to* d provided that for each $\epsilon > 0$, there is a number N such that

$$d(s_i, s_j) < \epsilon \text{ for all } i, j \geq N.$$

Also, note that the definition of convergence for sequences in section 8 of Chapter IV generalizes directly to metric spaces:

Definition: Let (X, d) be a metric space, and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points of X. We say that the *sequence* $\{s_n\}_{n=1}^{\infty}$ *converges with respect to* d *to a point* $p \in X$ provided that for each $\epsilon > 0$, there exists N such that $d(s_n, p) < \epsilon$ for all $n \geq N$. We call p the *limit of the sequence* $\{s_n\}_{n=1}^{\infty}$. We write $\lim_{n\to\infty} s_n = p$ or $\{s_n\}_{n=1}^{\infty} \to p$ to denote that a sequence $\{s_n\}_{n=1}^{\infty}$ converges to p.

Thus, we have natural definitions of Cauchy sequences and convergent sequences for any metric space. In addition, the exercise above says that we could have taken the statement *every Cauchy sequence of real numbers converges* to be the Completeness Axiom in section 1 of Chapter I. Therefore, the following definition seems reasonable as a definition for completeness in the general setting of metric spaces.

Definition: A metric space (X, d) is said to be *complete,* or *Cauchy complete*, provided that every Cauchy sequence with respect to d converges (to a point of X).

The question of whether or not our definition of a complete metric space is valuable can only be decided later, on the merits of the theory that evolves.

We have probably created a misimpression that we want to correct. We arrived at our definition of completeness for metric spaces solely on the basis of two facts: Cauchy completeness in R^1 is equivalent to the Completeness Axiom in Chapter I, and the definition of a Cauchy sequence in R^1 extends directly to any metric space. Actually, there are other properties which, by the same reasoning, could have become our definition of completeness in metric spaces. We will discuss one such property and indicate why the property would not be appropriate as a definition for completeness in metric spaces.

First, recall our definitions of arbitrary closeness, limit point and bounded set for subsets of R^1 (in sections 1 and 4 of Chapter II and section 2 of Chapter V); these notions have natural, straightforward analogues for metric spaces:

Definition: Let (X, d) be a metric space, let $p \in X$ and let A be a nonempty subset of X. We let

$$dist_d(p, A) = glb\,\{d(p, a) : a \in A\}.$$

- *p is arbitrarily close to A with respect to d*, written $p \sim_d A$ provided that $dist_d(p, A) = 0$.
- *p is a limit point of A with respect to d* provided that $p \sim_d A - \{p\}$ (which presupposes $A - \{p\} \neq \emptyset$). We let A^ℓ denote the set of all limit points of A.
- A of X is *bounded with respect to d* provided that (assuming $A \neq \emptyset$)

  $$\sup\,\{d(a_1, a_2) : a_1, a_2 \in A\} < \infty;$$

  also, by definition, the empty set is bounded.

In view of the definition above, the following property, which we originally considered for R^1 (in Exercise 5.16), makes sense in any metric space X:

(*) *Every bounded infinite subset of X has a limit point in X.*

Furthermore, the Completeness Axiom in section 1 of Chapter I is equivalent to (*) when $X = \mathsf{R}^1$ (see Exercises 5.16 and 5.17; it is easy to prove that (*) for R^1 implies the Nested Interval Property). Why then not use (*) as the definition of completeness for metric spaces in general?

We can not answer the question completely at this time. Nevertheless, we can say that for metric spaces in general, (*) implies Cauchy completeness but the converse implication is false (Exercises 20.21 and 20.22 below). The fact of the matter is that (*) is too strong to be useful in general – not many metric spaces satisfy (*) – while, on the other hand, Cauchy completeness works well (but, we must wait to find this out).

Exercise 20.21: In the preceding paragraph, we said that if a metric space has the property in (*), then the space is Cauchy complete. Prove this.

(*Hint:* Prove that Lemmas 20.10 and 20.11 hold in any metric space.)

Exercise 20.22: Prove that d in (2) of Example 20.19 is a distance function. In connection with the discussion at the end of the section, prove that the metric space in (2) is Cauchy complete but that the space contains an infinite set with no limit point in the space.

Exercise 20.23: Prove that d in (4) of Example 20.19 is a distance function. Is the metric space Cauchy complete?

Exercise 20.24: Prove that d in (7) of Example 20.19 is a distance function. Find a simple necessary and sufficient condition for a sequence in this metric space to be a Cauchy sequence (and prove your answer is correct). Is the metric space complete? What subsets of the space have limit points?

Our final two exercises are fundamental results in metric spaces that require completeness.

Exercise 20.25: Let (X, d_X) be a metric space, and let (Y, d_Y) be a Cauchy complete metric space. Let $A \subset X$, and let $f : A \to Y$ be a uniformly continuous function (which means that for each $\epsilon > 0$, there is a $\delta > 0$ such that if $a_1, a_2 \in A$ and $d_X(a_1, a_2) < \delta$, then $d_Y(f(a_1), f(a_2)) < \epsilon$; compare with the definition above Exercise 12.28).

Then there is a unique uniformly continuous function $g : A \cup A^\ell \to Y$ such that $g|A = f$. (The function g is called *the uniformly continuous extension of* f.)

In the light of Exercise 20.25, you might want to revisit Exercise 16.21: The exponential function $f(t) = a^t$ is uniformly continuous on any closed and bounded interval $[a, b]$ by Theorem 12.31 and, thus, its restriction to the rationals in $[a, b]$ is uniformly continuous.

Our next exercise concerns contraction maps. Let (X, d) be a metric space; a function $f : X \to X$ is called a *contraction map* provided that there exists $\lambda < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \lambda d(x, y)$.

Exercise 20.26: If (X, d) is a Cauchy complete metric space and $f : X \to X$ is a contraction map, then f has a unique fixed point (i.e., $f(p) = p$ for some unique point $p \in X$).

(*Hint:* Fix any point $x \in X$ and prove that the sequence $\{f^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence, where $f^n = f \circ f \circ \cdots \circ f$ with f appearing n times. Start by showing that for any n, $d(f^n(x), f^{n+1}(x)) \leq \lambda^n d(x, f(x))$; then use that the sequence of partial sums of the series $\Sigma_{i=1}^{\infty} \lambda^i$ is a Cauchy sequence by Theorem 15.4.)

The result in Exercise 20.26 has numerous applications in differential equations and dynamical systems.

Chapter XXI: Numerical Series

We introduced series very briefly in section 1 of Chapter XV. Recall that a series $\Sigma_{i=1}^{\infty} a_i$ is, by definition, nothing more than the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums $s_n = \Sigma_{i=1}^{n} a_i$. Thus, having developed a theory for sequences in the last two chapters, we can use our understanding of sequences to develop a theory for series. You may conclude there is nothing to do – simply apply our work on sequences to series. However, series are special types of sequences and, therefore, have an inherent structure that sequences in general do not have; we devote this chapter and the next chapter to uncovering that structure.

There are two main problems in the theory of series – determining when series converge and finding the exact sum of specific convergent series. Finding the exact value for the sum of a convergent series is almost always very difficult. One exception is convergent geometric series, whose sums we found in Theorem 15.4. We find the exact value for the sum of another series in the last section of this chapter (others are in later chapters); however, in the meantime, we focus on the problem of determining when series converge; solutions for this problem are called *convergence tests*.

Convergence tests involve conditions on the terms of series. We present a number of different kinds of convergence tests in this chapter and in the next chapter.

In section 1, we state (as exercises) a few basic results about series in general. In the next three sections, we prove and illustrate three types of convergence tests – i^{th} term tests (including the Alternating Series Test), comparison tests, and the Integral Test. We conclude the chapter by finding the exact sum of an important series; as a result, we prove that the base e of the natural logarithm function is irrational.

We make general comments about series and notation.

Even though we denote a series by $\Sigma_{i=1}^{\infty} a_i$ and think of a series as an infinite sum, it must be pointed out that we are never actually adding an infinite number of numbers together to obtain the sum of a series; instead, the sum of a series is a limit of finite sums (the partial sums).

The notation $\Sigma_{i=1}^{\infty} a_i$ has dual meanings – it represents a series as well as the sum of a series. The context will make it clear which we mean.

We sometimes write $\Sigma_{i=1}^{\infty} a_i < \infty$ as shorthand for saying that the series $\Sigma_{i=1}^{\infty} a_i$ converges.

We include as a series any sum of the form $\Sigma_{i=n}^{\infty} a_i$, where n is a given integer.

1. General Elementary Results

This section consists entirely of exercises accompanied by a few comments. Our purpose is to provide readers a chance to refresh their understanding of the definitions under the heading Series in section 1 of Chapter XV. The exercises are elementary in the sense that their verifications only use the most basic results about sequences in the previous two chapters.

Our first exercise gives the simple convergence relationship between a given series $\Sigma_{i=1}^{\infty} a_i$ and the series $\Sigma_{i=n}^{\infty} a_i$.

Exercise 21.1: A series $\Sigma_{i=1}^{\infty} a_i$ converges if and only if the series $\Sigma_{i=n}^{\infty} a_i$ converges for all $n \in \mathsf{N}$; furthermore, if either series converges, then $\Sigma_{i=1}^{\infty} a_i = \Sigma_{i=n}^{\infty} a_i + \Sigma_{i=1}^{n-1} a_i$ for all $n \in \mathsf{N}$ such that $n \geq 2$.

Exercise 21.2: If the series $\Sigma_{i=1}^{\infty} a_i$ converges then, for each $\epsilon > 0$, there is a number N such that $|\Sigma_{i=n}^{\infty} a_i| < \epsilon$ for all $n \geq N$.

The *termwise sum of two series*, $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$, is the series $\Sigma_{i=1}^{\infty}(a_i+b_i)$; the *termwise difference of the series* is the series $\Sigma_{i=1}^{\infty}(a_i - b_i)$ or the series $\Sigma_{i=1}^{\infty}(b_i - a_i)$.

Exercise 21.3: Let $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$ be two convergent series, say

$$\Sigma_{i=1}^{\infty} a_i = a \quad \text{and} \quad \Sigma_{i=1}^{\infty} b_i = b.$$

Then $\Sigma_{i=1}^{\infty}(a_i + b_i) = a + b$ and $\Sigma_{i=1}^{\infty}(a_i - b_i) = a - b$.

The *termwise product of two series*, $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$, is the series $\Sigma_{i=1}^{\infty} a_i b_i$. Having just considered the termwise sum and the termwise difference of two series, we should now consider the termwise product of two series. However, termwise products of series behave badly in two respects. First, the termwise product of two convergent series may diverge (you will be asked to supply an example in Exercise 21.17). Second, even when the termwise product of two convergent series converges, the termwise product may not converge to the product of the sums of the two series; for example, by Theorem 15.4, $\Sigma_{i=1}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{2}$ and $\Sigma_{i=1}^{\infty}\left(\frac{1}{3}\right)^i\left(\frac{1}{3}\right)^i = \Sigma_{i=1}^{\infty}\left(\frac{1}{9}\right)^i = \frac{1}{8}$. We do not discuss termwise products anymore at this time. Later, we consider termwise products in several exercises; our most definitive positive result is in the next chapter (Exercise 22.3). Actually, Cauchy products (introduced in section 6 of Chapter XXII) are more appropriate products for series than termwise products.

Our next exercise is the distributive law for constants over series.

Exercise 21.4: If $c \neq 0$, then a series $\Sigma_{i=1}^{\infty} a_i$ converges if and only if the series $\Sigma_{i=1}^{\infty} c a_i$ converges; if either series converges, then $\Sigma_{i=1}^{\infty} c a_i = c\Sigma_{i=1}^{\infty} a_i$.

Next, we have a result about grouping finitely many terms of a convergent series together infinitely often.

Exercise 21.5: Let $\Sigma_{i=1}^{\infty} a_i$ be a series. Let $\varphi : \mathsf{N} \to \mathsf{N}$ be a strictly increasing function such that $\varphi(1) = 1$. Let $\Sigma_{i=1}^{\infty} b_i$ be the series whose i^{th} term is the sum, in parentheses, of the terms $a_{\varphi(i)+1}, a_{\varphi(i)+2}, \cdots, a_{\varphi(i+1)}$ except when $i = 1$, as seen below:

$$\Sigma_{i=1}^{\infty} b_i = (a_{\varphi(1)} + a_2 + \cdots a_{\varphi(2)}) + (a_{\varphi(2)+1} + a_{\varphi(2)+2} + \cdots a_{\varphi(3)})$$
$$+ \cdots + (a_{\varphi(i)+1} + a_{\varphi(i)+2} + \cdots a_{\varphi(i+1)}) + \cdots .$$

If the series $\Sigma_{i=1}^{\infty} a_i$ converges, then the series $\Sigma_{i=1}^{\infty} b_i$ converges and $\Sigma_{i=1}^{\infty} b_i = \Sigma_{i=1}^{\infty} a_i$.

We note that for divergent series, the grouping in Exercise 21.5 may result in a convergent series. For example, the series $1+(-1)+1+(-1)+\cdots$ diverges, but the series $[1+(-1)]+[1+(-1)]+\cdots$ converges.

Next, we adapt the Cauchy criterion for sequences to series:

Exercise 21.6 (Cauchy Criterion for Convergence of Series): A series $\Sigma_{i=1}^{\infty} a_i$ converges if and only if for each $\epsilon > 0$, there is a number N such that

$$|\Sigma_{i=m}^{n} a_i| < \epsilon \quad \text{whenever } n \geq m \geq N.$$

Exercise 21.7: Determine whether the series $\Sigma_{i=1}^{\infty}\left(\cos(\frac{1}{i}) - \cos(\frac{1}{i+1})\right)$ converges.

2. i^{th} Term Tests

When trying to determine whether a series converges or diverges, the simplest thing to check first is whether the terms of the series converge to 0: If the terms do not converge to 0, then the series diverges; this is called the i^{th} Term Test. Thus, the i^{th} Term Test is a test for divergence, not convergence (the converse of the i^{th} Term Test is false, which we show after we prove the test).

We prove the i^{th} Term Test and its companion, the Alternating Series Test. The Alternating Series Test shows that the i^{th} Term Test *is* a test for convergence for alternating series that satisfy a simple condition (which we will show is necessary). The i^{th} term tests are useful because it is obviously easier to work with the individual terms of a series than with the partial sums of a series.

Theorem 21.8 (i^{th} Term Test): If $\Sigma_{i=1}^{\infty} a_i < \infty$, then $\lim_{i\to\infty} a_i = 0$.

Proof: Let $s_n = \Sigma_{i=1}^{n} a_i$ for each $n \in \mathsf{N}$. The key idea is to note that

$$a_i = s_i - s_{i-1} \quad \text{for all } i \geq 2.$$

Then, since $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} s_{n-1}$ exist and are equal,

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty}(s_i - s_{i-1}) \overset{19.3}{=} \lim_{i\to\infty} s_i - \lim_{i\to\infty} s_{i-1} = 0. \;\text{¥}$$

Exercise 21.9: Explain how the i^{th} Term Test follows immediately from the Cauchy Criterion for Convergence of Series (Exercise 21.6).

We can use the i^{th} Term Test to see that a series diverges: $\Sigma_{i=1}^{\infty}(-1)^n \frac{n}{n+3}$ diverges by the test. However, we can not use the i^{th} Term Test to show that a series converges: $\lim_{i\to\infty} \frac{1}{i} = 0$ (by Exercise 1.23), but the series $\Sigma_{i=1}^{\infty}\frac{1}{i}$ diverges since its sequence of partial sums is unbounded by Example 20.14.

Nevertheless, for certain types of series, the i^{th} Term Test completely determines whether the series converges. Our next theorem illustrates this. First, we give a definition.

Definition: An *alternating series* is a series for which the signs of consecutive terms alternate; in other words, an alternating series is a series that can be written in the form $\Sigma_{i=1}^{\infty}(-1)^i a_i$ or $\Sigma_{i=1}^{\infty}(-1)^{i+1} a_i$, where $a_i > 0$ for all $i \in \mathsf{N}$.

The test in our next theorem is one of the easiest tests for convergence to apply. We address the assumption that the terms are decreasing after we prove the theorem.

Theorem 21.10 (Alternating Series Test): Let $\Sigma_{i=1}^{\infty}(-1)^i a_i$ be an (alternating) series such that

$$a_i \geq a_{i+1} > 0 \text{ for all } i \in \mathsf{N}.$$

Then $\Sigma_{i=1}^{\infty}(-1)^i a_i$ converges if and only if $\lim_{i\to\infty} a_i = 0$.

Proof: If $\Sigma_{i=1}^{\infty}(-1)^i a_i$ converges, then $\lim_{i\to\infty}(-1)^i a_i = 0$ by the i^{th} Term Test (Theorem 21.8); therefore, $\lim_{i\to\infty} a_i = 0$.

We prove the other half of the equivalence. The idea behind the proof is to consider the sequence of even - numbered partial sums separately from the sequence of odd - numbered partial sums. We proceed as follows.

Assume that

(1) $\lim_{i\to\infty} a_i = 0$.

Let $s_n = \Sigma_{i=1}^{n}(-1)^i a_i$ for each $n \in \mathsf{N}$.

Note that we can write s_{2n} in the following two ways:

$$(*)\ s_{2n} = -(a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{2n-1} - a_{2n})$$

and

$$(**)\ s_{2n} = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2n-2} - a_{2n-1}) + a_{2n}.$$

By assumption in our theorem, $a_i - a_{i+1} > 0$ for all $i \in \mathsf{N}$. Thus, by (*), the sequence $\{s_{2n}\}_{n=1}^{\infty}$ is decreasing and, by (**), the sequence $\{s_{2n}\}_{n=1}^{\infty}$ is bounded below by $-a_1$ and, hence, is bounded (since the sequence is decreasing). Therefore, by the Bounded Monotonic Sequence Property (Theorem 19.20), we have that

(2) $\{s_{2n}\}_{n=1}^{\infty}$ converges.

Finally, we turn our attention to the sequence of odd - numbered partial sums. Clearly, $s_{2n+1} = s_{2n} - a_{2n+1}$ for each $n \in \mathsf{N}$; hence, using (1) and (2) to apply Theorem 19.3, we obtain

$$\lim_{n\to\infty} s_{2n+1} \overset{19.3}{=} \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} a_{2n+1} \overset{(1)}{=} \lim_{n\to\infty} s_{2n}.$$

(we used Exercise 20.3 in conjunction with (1) for the last equality). Therefore, by Exercise 19.18, $\lim_{n\to\infty} s_n$ exists. ¥

After the proof of the i^{th} Term Test, we noted that the series $\Sigma_{i=1}^{\infty}\frac{1}{i}$ diverges. Let us change this series by making consecutive terms alternate in sign, obtaining $\Sigma_{i=1}^{\infty}(-1)^i\frac{1}{i}$ or $\Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$; then we know from the Alternating Series Test that both of these series converge. The series $\Sigma_{i=1}^{\infty}\frac{1}{i}$ is called the *harmonic series*, and the series $\Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$ is called the *alternating harmonic series.*

It may seem in the Alternating Seres Test that we do not really need the terms a_i to be decreasing, but that it is enough to merely assume that $\lim_{i\to\infty} a_i = 0$ and that $a_i > 0$ for each i. However, the series in the following exercise shows that the assumption that the terms decrease to 0 is necessary:

Exercise 21.11: Consider the alternating series $\Sigma_{i=1}^{\infty}(-1)^i a_i$, where for each $i \in \mathsf{N}$,

$$a_{2i-1} = \tfrac{1}{2^i} \quad \text{and} \quad a_{2i} = \tfrac{1}{i+1}.$$

Prove that the series diverges even though $\lim_{n\to\infty} a_i = 0$.

It is natural to wonder if there is an analogue of the Alternating Seres Test for series whose terms change sign frequently (but not from each term to the next). The example in the next exercise shows that, for a simple case, there is no such analogue.

Exercise 21.12: Consider the series $\Sigma_{i=1}^{\infty}(-1)^{\sigma(i)}\frac{1}{i}$, where σ defined below makes the signs of the terms repeat in the pattern $+, +, -, +, +, -, \ldots$:

$$\sigma(i) = \begin{cases} 0 & \text{, if } i = 3j-2 \text{ or } 3j-1 \text{ for some } j \in \mathsf{N} \\ 1 & \text{, if } i = 3j \text{ for some } j \in \mathsf{N}. \end{cases}$$

Prove that the series diverges even though the series is "almost alternating" and satisfies the other assumptions in the Alternating Seres Test.

Our next exercise gives an estimate of the error between the n^{th} partial sum and the sum of an alternating series when the series satisfies the assumptions in the Alternating Series Test.

Exercise 21.13: Let $\Sigma_{i=1}^{\infty}(-1)^i a_i$ be a convergent (alternating) series such that $a_i \geq a_{i+1} > 0$ for all $i \in \mathsf{N}$. Then

$$\left|\Sigma_{i=1}^{\infty}(-1)^i a_i - \Sigma_{i=1}^{n}(-1)^i a_i\right| \leq a_{n+1} \quad \text{for all } n \in \mathsf{N}.$$

Exercise 21.14: Assume that $a_i \geq a_{i+1} > 0$ for all $i \in \mathsf{N}$. If $\Sigma_{j=1}^{\infty} a_{i_j} < \infty$ for some subsequence $\{a_{i_j}\}_{j=1}^{\infty}$ of $\{a_i\}_{i=1}^{\infty}$, then $\Sigma_{i=1}^{\infty}(-1)^i a_i < \infty$.

Exercise 21.15: Determine whether the series $\Sigma_{i=1}^{\infty}\cos(\frac{1}{i})$ converges.

Exercise 21.16: True or false: The series $\Sigma_{i=1}^{\infty}(\cos(x))^i\sin(x)$ converges for all $x \in \mathsf{R}^1$.

Exercise 21.17: Give an example of two convergent series, $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$, such that the series $\Sigma_{i=1}^{\infty} a_i b_i$ diverges.

3. Comparison Tests

We prove the Comparison Test and the Limit Comparison Test. The Comparison Test directly compares the terms of two series; the Limit Comparison Test is concerned with the limit of the ratios of the terms of two series.

Theorem 21.18 (Comparison Test): Assume that $0 \leq a_i \leq b_i$ for each $i \in \mathsf{N}$. If $\Sigma_{i=1}^{\infty} b_i < \infty$, then $\Sigma_{i=1}^{\infty} a_i < \infty$.

Proof: Let $s_n = \Sigma_{i=1}^{n} a_i$ and $t_n = \Sigma_{i=1}^{n} b_i$ for each $n \in \mathsf{N}$.

Since $a_i \geq 0$ for each $i \in \mathsf{N}$, it is clear that the sequence $\{s_n\}_{n=1}^{\infty}$ is increasing; also, the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded since, by our assumptions that $0 \leq a_i \leq b_i$ and $\Sigma_{i=1}^{\infty} b_i < \infty$, we have

$$0 \le s_n \le t_n \le \Sigma_{i=1}^{\infty} b_i \quad \text{for each } n \in \mathsf{N}.$$

Therefore, by the Bounded Monotonic Sequence Property (Theorem 19.20), $\{s_n\}_{n=1}^{\infty}$ converges. ¥

Exercise 21.19: Series of the form $\Sigma_{i=1}^{\infty} \frac{d_i}{10^i}$ are called *decimals* and are denoted by $.d_1 d_2 ... d_i ...$. Prove that every decimal converges (in common language, every decimal represents a real number).

(*Hint:* Make use of geometric series (Theorem 15.4).)

Theorem 21.20 (Limit Comparison Test): Let $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$ be two series whose terms are positive.

(1) If $0 < \lim_{i\to\infty} \frac{a_i}{b_i} < \infty$, then either both of the series $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$ converge or both of them diverge.

(2) If $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$ and $\Sigma_{i=1}^{\infty} b_i$ converges, then $\Sigma_{i=1}^{\infty} a_i$ converges.

(3) If $\lim_{i\to\infty} \frac{a_i}{b_i} = \infty$ and $\Sigma_{i=1}^{\infty} b_i$ diverges, then $\Sigma_{i=1}^{\infty} a_i$ diverges.

Proof: We prove each part in turn.

Proof of part (1)*:* Let $L = \lim_{i\to\infty} \frac{a_i}{b_i}$. Then, since $L > 0$ (by assumption), there is a number N such that

$$\left| \frac{a_i}{b_i} - L \right| < \frac{L}{2} \quad \text{for all } i \ge N.$$

Hence, $\frac{L}{2} < \frac{a_i}{b_i} < \frac{3L}{2}$ for all $i \ge N$. Thus, since $b_i > 0$ for all i (by assumption), we have

$$(*) \ \frac{L}{2} b_i < a_i < \frac{3L}{2} b_i \quad \text{for all } i \ge N.$$

Now, if $\Sigma_{i=1}^{\infty} b_i$ converges, then $\Sigma_{i=1}^{\infty} \frac{3L}{2} b_i$ converges (by Exercise 21.4) and, thus, $\Sigma_{i=N}^{\infty} \frac{3L}{2} b_i$ converges (by Exercise 21.1). Hence, $\Sigma_{i=N}^{\infty} a_i$ converges by (*) and the Comparison Test (Theorem 21.18). Therefore, $\Sigma_{i=1}^{\infty} a_i$ converges (by Exercise 21.1).

On the other hand, if $\Sigma_{i=1}^{\infty} b_i$ diverges, then $\Sigma_{i=1}^{\infty} \frac{L}{2} b_i$ diverges (by Exercise 21.4) and, thus, $\Sigma_{i=N}^{\infty} \frac{L}{2} b_i$ diverges (by Exercise 21.1). Hence, $\Sigma_{i=N}^{\infty} a_i$diverges by (*) and the (contrapositive of) the Comparison Test. Therefore, $\Sigma_{i=1}^{\infty} a_i$ diverges (by Exercise 21.1).

This proves part (1) of our theorem.

Proof of part (2)*:* Since $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$ (by assumption), there is a number N such that $\frac{a_i}{b_i} < 1$ for all $i \ge N$. Thus, since $a_i, b_i > 0$ for all i, we have

$$(**) \ 0 < a_i < b_i \quad \text{for all } i \ge N.$$

Since $\Sigma_{i=1}^{\infty} b_i$ converges (by assumption), $\Sigma_{i=N}^{\infty} b_i$ converges (by Exercise 21.1). Hence, $\Sigma_{i=N}^{\infty} a_i$ converges by (**) and the Comparison Test (Theorem 21.18). Therefore, $\Sigma_{i=1}^{\infty} a_i$ converges (by Exercise 21.1).

This proves part (2) of our theorem.

Proof of part (3)*:* Part (3) follows from part (2) by noting that since $\lim_{i\to\infty} \frac{a_i}{b_i} = \infty$ (by assumption), we have by Exercise 18.3 that

$$\lim_{i\to\infty} \frac{b_i}{a_i} = 0;$$

therefore, by part (2), if $\Sigma_{i=1}^{\infty} a_i$ converges, then $\Sigma_{i=1}^{\infty} b_i$ converges, which proves part (3) of our theorem. ¥

Obviously, the problem in applying the comparison tests is to find a suitable series to compare the given series with. There is no definitive answer for this problem, but we can suggest a working principle that is often effective when the i^{th} term of the original series is a quotient of algebraic functions of i (or even when transcendental functions that are algebraic in i are involved): *Try comparing with the series that is obtained from the original series by eliminating all expressions in the numerator and the denominator of the i^{th} term except those of the highest power.* We illustrate as follows:

Example 21.21: Consider the series $\Sigma_{i=1}^{\infty} \frac{i^2+8i}{i^{\frac{9}{2}}+4}$. According to the suggestion above, we take as the comparison series the series $\Sigma_{i=1}^{\infty} \frac{i^2}{i^{\frac{9}{2}}}$, which reduces to $\Sigma_{i=1}^{\infty} \frac{1}{i^{\frac{5}{2}}}$. Now, with the Limit Comparison Test in mind, we see that

$$\lim_{i\to\infty} \frac{\frac{i^2+8i}{i^{\frac{9}{2}}+4}}{\frac{1}{i^{\frac{5}{2}}}} = \lim_{i\to\infty} \frac{i^{\frac{9}{2}}+8i^{\frac{7}{2}}}{i^{\frac{9}{2}}+4} = \lim_{i\to\infty} \frac{1+\frac{8}{i}}{1+\frac{4}{i^{\frac{9}{2}}}} = 1;$$

furthermore, the series $\Sigma_{i=1}^{\infty} \frac{1}{i^{\frac{5}{2}}}$ converges (by Exercise 20.16). Therefore, by part (1) of the Limit Comparison Test, the series $\Sigma_{i=1}^{\infty} \frac{i^2+8i}{i^{\frac{9}{2}}+4}$ converges.

Exercise 21.22: Determine whether the series $\Sigma_{i=1}^{\infty} \frac{\ln(i)}{i^3+i}$ converges two ways: (1) using the Comparison Test; (2) using the Limit Comparison Test.

Exercise 21.23: Determine whether the series $\Sigma_{i=1}^{\infty} \frac{\ln(i)+i^2}{i^3+i}$ converges.

Exercise 21.24: Determine whether the series $\Sigma_{i=2}^{\infty} \frac{1}{\ln(i)}$ converges two ways: (1) using the Comparison Test; (2) using the Limit Comparison Test.

Exercise 21.25: Determine whether the series $\Sigma_{i=1}^{\infty} \sin(\frac{1}{i})$ converges.

Exercise 21.26: True or false: If $\Sigma_{i=1}^{\infty} a_i$ is a convergent series and $\{a_{i_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_i\}_{i=1}^{\infty}$, then the series $\Sigma_{k=1}^{\infty} a_{i_k}$ converges.

Exercise 21.27: True or false: If $a_i \geq 0$ for all $i \in \mathsf{N}$ and the series $\Sigma_{i=1}^{\infty} a_i$ converges, then the series $\Sigma_{i=1}^{\infty} a_i^2$ converges.

Exercise 21.28: True or false: If $\Sigma_{i=1}^{\infty} a_i$ is a convergent series whose terms are all positive, then the series $\Sigma_{i=1}^{\infty} \frac{\sqrt{a_i}}{i}$ converges.

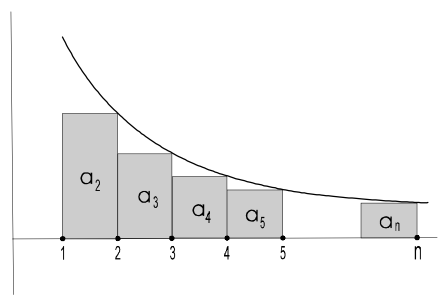
4. The Integral Test

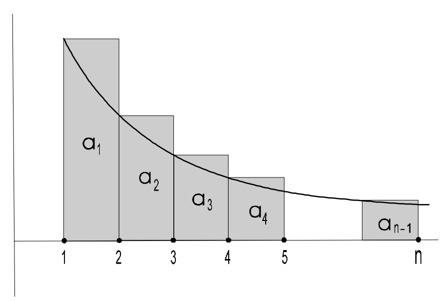
In Theorem 19.7 we initiated an important way to study convergence of sequences – by using continuous functions defined on the interval $[1, \infty)$. This led to using the derivative (specifically, l'Hôpital's rules) to study convergence of sequences. In this chapter and the two previous chapters, especially in exercises, we have seen how well this works for sequences as well as for series. We now come to the role that the *integral* plays in determining convergence or divergence of series.

We are going to use integrals of continuous functions defined on the interval $[0, \infty)$ to study convergence of series. The rudimentary idea that connects integrals to convergence of series is very simple: any term a_i of a series can be interpreted as the (signed) area of a rectangle of height a_i and width 1. We explain how to apply the idea.

Assume that $\Sigma_{i=1}^{\infty} a_i$ is a series whose terms are positive and that f is a continuous decreasing function on $[1, \infty)$ such that $f(i) = a_i$ for each i; then, for each natural number $n \geq 2$ (see figures below),

$$\Sigma_{i=2}^{n} a_i \leq \int_1^n f \leq \Sigma_{i=1}^{n-1} a_i;$$





it follows that $\lim_{n\to\infty} \int_1^n f < \infty$ if and only if the series $\Sigma_{i=1}^{\infty} a_i$ converges (we give a formal proof later). In this way, we have reduced the study of convergence of many series to the study of integrals.

Before we explicitly state and prove the theorem that we have indicated (namely, Theorem 21.31), we give an important example to illustrate what we have just discussed.

Series of the form $\Sigma_{i=1}^{\infty} \frac{1}{i^p}$, where p is a fixed real number, are called *p-series*. We know from Exercise 20.16 that p-series converge when $p \geq 2$. We now determine all the values of p for which p-series converge. We note that the exact values for the sums of many convergent p-series are not known.

Example 21.29: We show that the p-series $\Sigma_{i=1}^{\infty}\frac{1}{i^p}$ converges when $p > 1$ and that the series diverges when $p \leq 1$.

The case when $p > 0$ and $p \neq 1$ relate to the discussion above. We first dispense with the other cases: If $p < 0$, then $\lim_{i\to\infty}\frac{1}{i^p} = \lim_{i\to\infty} i^{-p} = \infty$ (since $-p > 0$), so the series $\Sigma_{i=1}^{\infty}\frac{1}{i^p}$ diverges by the i^{th} Term Test (Theorem 21.8). If $p = 0$, then the series obviously diverges. Finally, if $p = 1$, then the series $\Sigma_{i=1}^{\infty}\frac{1}{i^p}$ diverges by Example 20.14 and Exercise 20.9.

We assume from now on that $p > 0$ and that $p \neq 1$.

Let $f(x) = \frac{1}{x^p}$ for all $x \geq 1$. Then, since $f'(x) = -px^{-p-1}$ (by Theorem 16.31) and $-p < 0$, we see that $f'(x) < 0$ for all x; hence, by Theorem 10.17, f is strictly decreasing. Next note that f is continuous (by Theorem 16.31 and Theorem 6.14); hence, $f|[1,t]$ is continuous for any $t \geq 1$ (by Exercise 5.3). Thus, letting

$$g(x) = \frac{x^{-p+1}}{-p+1} \quad \text{for all } x \geq 1$$

and noting that $g'(x) = f(x)$ for all $x \geq 1$ (by Theorem 16.31), we have by the Fundamental Theorem of Calculus (Theorem 14.2) that

$$(^*) \; \int_1^t f = g(t) - g(1) = \frac{1}{1-p}\left(\frac{1}{t^{p-1}} - 1\right) \quad \text{for all } t > 1.$$

Now, if $p > 1$, then $\lim_{t\to\infty}\frac{1}{t^{p-1}} = 0$; hence, by (*),

$$\lim_{t\to\infty}\int_1^t f = \frac{1}{1-p}(-1) = \frac{1}{p-1} < \infty.$$

Therefore, based on the discussion preceding the example, we conclude that the series $\Sigma_{i=1}^{\infty}\frac{1}{i^p}$ converges when $p > 1$.

On the other hand, if $0 < p < 1$, then $\lim_{t\to\infty}\frac{1}{t^{p-1}} = \infty$; thus, by (*),

$$\lim_{t\to\infty}\int_1^t f = \infty.$$

Therefore, according to the discussion preceding the example, we conclude that the series $\Sigma_{i=1}^{\infty}\frac{1}{i^p}$ diverges when $0 < p < 1$. This completes the verifications for the example.

We formulate our discussion at the beginning of the section into a theorem. First, we introduce the following natural and convenient notation and terminology.

Definition: If $\int_a^t f$ exists for all $t \geq a$ and $\lim_{t\to\infty}\int_a^t f$ exists (i.e., is finite), then we denote the limit by $\int_a^{\infty} f$ and say *the integral* $\int_a^{\infty} f$ *converges.*

Exercise 21.30: If $\int_a^{\infty} f$ converges and $b \geq a$, then $\int_b^{\infty} f$ converges and

$$\int_a^{\infty} f = \int_a^b f + \int_b^{\infty} f.$$

Theorem 21.31 (Integral Test): Assume that $\Sigma_{i=1}^{\infty} a_i$ is a series whose terms are positive and that f is a continuous decreasing function on $[1,\infty)$ such that $f(i) = a_i$ for each i. Then the series $\Sigma_{i=1}^{\infty} a_i$ converges if and only if $\int_1^{\infty} f$ converges.

Proof: We indicated the essential ideas for the proof at the beginning of the section; we now fill in the details.

We note for repeated use (usually without saying so) that $\int_a^b f$ exists whenever $1 \le a \le b$ by Theorem 12.33 (and Exercise 5.3).

We first prove the following (which are the inequalities we stated when we referred to the figures above):

(1) $\Sigma_{i=2}^{n} a_i \le \int_1^n f \le \Sigma_{i=1}^{n-1} a_i$ for each $n \in \mathsf{N}$ such that $n \ge 2$.

Proof of (1)*:* Fix $n \in \mathsf{N}$ such that $n \ge 2$. Since f is decreasing and $f(i+1) = a_{i+1}$ for all $i \in \mathsf{N}$, we have that

$$a_{i+1} \le f(x) \quad \text{when } i \le x \le i+1.$$

Hence, by Exercise 13.15,

$$a_{i+1} \le \int_i^{i+1} f \quad \text{for each } i \in \mathsf{N}.$$

Therefore,

$$\Sigma_{i=2}^{n} a_i \le \Sigma_{i=1}^{n-1} \int_i^{i+1} f \overset{13.40}{=} \int_1^n f.$$

This proves the first inequality in (1). The proof of the second inequality in (1) is similar: Since f is decreasing and $f(i) = a_i$ for all $i \in \mathsf{N}$, we have that

$$f(x) \le a_i \quad \text{when } i \le x \le i+1.$$

Hence, by Exercise 13.15,

$$\int_i^{i+1} f \le a_i \quad \text{for each } i \in \mathsf{N}.$$

Therefore,

$$\int_1^n f \overset{13.40}{=} \Sigma_{i=1}^{n-1} \int_i^{i+1} f \le \Sigma_{i=1}^{n-1} a_i.$$

This proves the second inequality in (1). Therefore, we have proved (1).

Now, we use (1) to prove the theorem.

Recall that we are assuming in the theorem that $f(a_i) = a_i > 0$ for each i and that f is a continuous decreasing function on $[1,\infty)$. Thus, since any point of $[1,\infty)$ is less than some natural number (by Theorem 1.22), we see that

(2) $f(x) > 0$ for all $x \ge 1$.

Note from (1) that

(3) $\int_1^t f \le \int_1^n f$ when $n-1 \le t \le n$ and $n \in \mathsf{N}$ with $n \ge 2$.

Now, assume that $\int_1^\infty f$ converges. Then, by (3), $\int_1^t f \le \int_1^\infty f$ for all $t \ge 1$ (actually, the inequality is strict, but this is not important here). Thus, by the first inequality in (1), $\Sigma_{i=2}^n a_i \le \int_1^\infty f$ for all $n \in \mathsf{N}$ such that $n \ge 2$. Hence, adding a_1 to both sides,

$$\Sigma_{i=1}^n a_i \le a_1 + \int_1^\infty f \quad \text{for all } n \in \mathsf{N}. \quad .$$

Thus, since $0 < \Sigma_{i=1}^n a_i$ for all $n \in \mathsf{N}$, we have proved that the sequence of partial sums of the series $\Sigma_{i=1}^\infty a_i$ is bounded (by 0 and $a_1 + \int_1^\infty f$). Furthermore, the sequence of partial sums is increasing (since $a_i > 0$ for all i). Therefore, by the Bounded Monotonic Sequence Property (Theorem 19.20), the sequence of partial sums converges. In other words, the series $\Sigma_{i=1}^\infty a_i$ converges.

Conversely, assume that $\int_1^\infty f$ does not converge. With the definition of $\int_1^\infty f$ in mind, we recall from the beginning of the proof that $\int_1^t f$ exists for all $t \ge 1$; note that $\int_1^t f \ge 0$ for all $t \ge 1$ (by (2) and Exercise 12.25). Thus, since $\int_1^\infty f$ does not converge, we see that

$$\lim_{t\to\infty} \int_1^t f = \infty.$$

Hence, by (3), $\lim_{n\to\infty} \int_1^n f = \infty$. Thus, by the second inequality in (1),

$$\lim_{n\to\infty} \Sigma_{i=1}^{n-1} a_i = \infty.$$

Therefore, the series $\Sigma_{i=1}^\infty a_i$ diverges. ¥

In regard to the Integral Test, we should not be misled into believing that the sum of the series is the value of the integral. The second figure near the beginning of the section suggests that

$$\Sigma_{i=1}^\infty a_i > \int_1^\infty f.$$

To give a specific example, the series $\Sigma_{i=1}^\infty \frac{1}{i^2}$ converges (Example 21.29) and it is clear $\Sigma_{i=1}^\infty \frac{1}{i^2} > 1$; however, $\int_1^\infty \frac{1}{x^2} = 1$.

Exercise 21.32: In connection with the preceding discussion, assume (as in the Integral Test) that $\Sigma_{i=1}^\infty a_i$ is a convergent series whose terms are positive and that f is a continuous decreasing function on $[1, \infty)$ such that $f(i) = a_i$ for each i. Prove or give a counterexample: $\Sigma_{i=1}^\infty a_i > \int_1^\infty f$.

Exercise 21.33: We know from Example 21.29 that the series $\Sigma_{i=1}^\infty \frac{1}{i^{\frac{3}{2}}}$ converges. Find n such that the sum of the first n terms of the series approximates the sum of the series within an accuracy of .1 ? What about an accuracy within .01 ?

Exercise 21.34: Find all real numbers r such that the series $\Sigma_{i=2}^\infty \frac{1}{i(\ln(i))^r}$ converges.

Exercise 21.35: Determine whether the series $\Sigma_{i=1}^\infty \frac{i}{e^i}$ converges.

Exercise 21.36: Determine whether the series $\Sigma_{i=1}^{\infty}\frac{\sqrt{i^5+i^4+1}}{3i^4+i^2+2}$ converges.

Exercise 21.37: Determine whether the series $\Sigma_{i=1}^{\infty}\tan^{-1}(\frac{1}{i})$ converges in two ways: (1) using the Integral Test; (2) using the Limit Comparison Test (Theorem 21.20).

(*Hint:* For (1), use ideas that we used to integrate the natural logarithm function in section (5) of Chapter XVI, recalling the formula in Exercise 8.27.)

Exercise 21.38: Construct an example of a continuous function f defined on $[1,\infty)$ such that $\int_1^{\infty} f$ converges but the series $\Sigma_{i=1}^{\infty} f(i)$ diverges.

5. e as the Sum of a Series

At this point we postpone further development of convergence tests until the next chapter. We briefly turn to the other main aspect of the theory of series, that of finding the exact value of the sum of a convergent series. Specifically, we show that the sum of the series $\Sigma_{i=0}^{\infty}\frac{1}{i!}$ is the base e of the natural logarithm function. (Recall that $i! = i(i-1)(i-2)\cdots(2)(1)$ and $0! = 1$.)

So far, we know almost nothing about the number e. For example, Is e a rational number? We originally defined e, somewhat abstractly, in Theorem 16.23 as the number whose value under the natural logarithm function is 1. We represented e as the limit of a sequence in Exercise 16.27; we now represent e as the sum of a series. After we prove that $e = \Sigma_{i=0}^{\infty}\frac{1}{i!}$, we use this series representation of e prove that e is irrational.

We use the Binomial Theorem to prove that $e = \Sigma_{i=0}^{\infty}\frac{1}{i!}$. We prove the Binomial Theorem by a counting argument; for this purpose, we introduce the following combinatorial notions:

Definition: Let n and k be nonnegative integers with $k \leq n$.

- The following expression is called a *binomial coefficient*:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- A choice of k distinct objects without regard to the order in which the objects are chosen is called a *combination*. We let $C(n,k)$ denote the number of ways to choose k distinct objects from n distinct objects. By convention (or since there is only one way to choose no objects), $C(n,0) = 1$ and $C(0,0) = 1$.

- An arrangement of k distinct objects in a given order is called a *permutation*. We let $P(n,k)$ is the number of ways to choose *and* order k distinct objects from n distinct objects; as in the case of combinations, $P(n,0) = 1$ and $P(0,0) = 1$.

We prove two lemmas concerning the definitions we just gave.

Lemma 21.39: If $k \leq n$ are natural numbers, then

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1).$$

Proof: There are n choices for the first object and, for each first choice, there are $n-1$ choices for the second object; thus, $n(n-1)$ is the number of ways to pick and order the first two objects. Then, for each first and second choice, there are $n-2$ choices for the third object, thus $n(n-1)(n-2)$ ways to pick and order the first three objects; and so on, for a total of $n(n-1)(n-2)\cdots(n-k+1)$ ways to pick and order k distinct objects from n distinct objects. ¥

Lemma 21.40: Let $k \leq n$ are nonnegative integers, then $C(n,k) = \binom{n}{k}$.

Proof: If $k = 0$, then (by definitions) $C(n,0) = 1$ and $\binom{n}{k} = \frac{n!}{0!n!} = 1$. Thus, the lemma is proved when $k = 0$.

Assume that $k > 0$, and let S be an n-element set ($k \leq n$). If T be a k-element subset of S then the number of ways to order the k elements of T is, by definition, $P(k,k)$, which, by Lemma 21.39, is $k!$. Thus, the number of ways to choose and order all k-element subsets of S is $k!$ times the number of k-element subsets of S, which is $k!C(n,k)$. On the other hand, $P(n,k)$ is, by defintion, the number of ways to choose and order all k-element subsets of S. Hence,

$$k!C(n,k) = P(n,k).$$

Therefore,

$$C(n,k) = \frac{P(n,k)}{k!} \overset{21.39}{=} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \text{ ¥}$$

The Binomial Theorem follows easily from Lemma 21.40:

Theorem 21.41 (Binomial Theorem): For any $x, y \in \mathsf{R}^1 - \{0\}$ and any $n \in \mathsf{N}$,

$$(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof: Fix n. For reference, we note that

$$(*)\ (x+y)^n = \underbrace{(x+y)(x+y)(x+y)\cdots(x+y)}_{\text{n factors}}.$$

We consider the x terms in the n factors as comprising n objects that are distinct from one another by virtue of which factor they are in; the same for the y terms. Then, fixing k, we see that $x^{n-k}y^k$ in the summation in the theorem comes from choosing k $y's$ from k distinct factors in (*) and $n-k$ $x's$ from the other $n-k$ distinct factors in (*). Thus, since the number of ways to choose k $y's$ from the n distinct factors in (*) is $C(n,k)$ (by the definition of $C(n,k)$), the total number of terms in $(x+y)^n$ of the form $x^{n-k}y^k$ is $C(n,k)$ which, by Lemma 21.40, is $\binom{n}{k}$. ¥

We can now prove our series representation for e.

Theorem 21.42: $e = \Sigma_{i=0}^{\infty} \frac{1}{i!}$.

Proof: Note that for each integer $n \geq 0$,

$$\Sigma_{i=0}^{n} \frac{1}{i!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$$
$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \overset{15.4}{<} 3;$$

therefore, since the sequence $\{\Sigma_{i=0}^{n} \frac{1}{i!}\}_{n=1}^{\infty}$ is increasing, we have by Theorem 19.20 that

(1) $\Sigma_{i=0}^{\infty} \frac{1}{i!}$ converges.

We know from Exercise 16.27 that

(2) $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

For each $n \in \mathsf{N}$,

$$\left(1 + \frac{1}{n}\right)^n \overset{21.41}{=} \Sigma_{k=0}^{n} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k = \Sigma_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k}$$
$$= \Sigma_{k=0}^{n} \frac{n(n-1)\cdots(n-[k-1])}{n^k} \frac{1}{k!} \leq \Sigma_{k=0}^{n} \frac{1}{k!}.$$

Hence, by (1) and (2), $e \leq \Sigma_{k=0}^{\infty} \frac{1}{k!}$.

Thus, we have left to prove that $e \geq \Sigma_{i=0}^{\infty} \frac{1}{i!}$. Fix $n, m \in \mathsf{N}$ such that $n \geq m$. Then

$$\left(1 + \frac{1}{n}\right)^n \overset{21.41}{=} \Sigma_{k=0}^{n} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k \geq \Sigma_{k=0}^{m} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k$$
$$= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3$$
$$+ \cdots + \frac{n(n-1)\cdots(n-m+1)}{m!}\left(\frac{1}{n}\right)^m$$
$$= 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) + \cdots + \frac{1}{m!}\frac{n-1}{n}\cdots\frac{n-m+1}{n}.$$

Hence, holding m fixed, we obtain

$$e \overset{(2)}{=} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \quad \text{for each } m \in \mathsf{N}.$$

Therefore, using (1) to know that $\lim_{m\to\infty} \Sigma_{i=0}^{m} \frac{1}{i!}$ exists, we have

$$e \geq \lim_{m\to\infty} \Sigma_{i=0}^{m} \frac{1}{i!} = \Sigma_{i=0}^{\infty} \frac{1}{i!}. \text{ ¥}$$

The following corollary shows that the series $\Sigma_{i=0}^{\infty} \frac{1}{i!}$ converges very quickly to e. The corollary gives an estimate of the error between e and the n^{th} partial sum of the series $\Sigma_{i=0}^{\infty} \frac{1}{i!}$; the estimate will enable us to prove that e is irrational in Theorem 21.44.

Corollary 21.43: $0 < e - \Sigma_{i=0}^{n} \frac{1}{i!} < \frac{1}{n!n}$ for each $n \geq 1$.

Proof: Fix $n \geq 1$. Then, since $e = \Sigma_{i=0}^{\infty} \frac{1}{i!}$ by Theorem 21.42, we have (by Exercise 21.1) that

$$e - \Sigma_{i=0}^{n} \frac{1}{i!} = \Sigma_{i=n+1}^{\infty} \frac{1}{i!}.$$

Therefore,

$$\begin{aligned} e - \Sigma_{i=0}^{n} \tfrac{1}{i!} &= \tfrac{1}{(n+1)!} + \tfrac{1}{(n+2)!} + \tfrac{1}{(n+3)!} + \tfrac{1}{(n+4)!} + \cdots \\ &< \tfrac{1}{(n+1)!} \left(1 + \tfrac{1}{n+1} + \tfrac{1}{(n+1)^2} + \tfrac{1}{(n+1)^3} + \cdots\right) \\ &= \tfrac{1}{(n+1)!} \left(1 + \Sigma_{i=1}^{\infty} (\tfrac{1}{n+1})^i\right) \overset{15.4}{=} \tfrac{1}{(n+1)!} \left(1 + \tfrac{\frac{1}{n+1}}{1-\frac{1}{n+1}}\right) \\ &= \tfrac{1}{(n+1)!} \left(1 + \tfrac{1}{n}\right) = \tfrac{1}{(n+1)!} \tfrac{n+1}{n} = \tfrac{1}{n!n}. \end{aligned}$$ ¥

Theorem 21.44: e is irrational.

Proof: Suppose by way of contradiction that e is rational. Then $e = \frac{m}{n}$, where m and n are natural numbers. Hence, by Corollary 21.43,

$$0 < n! \left(\tfrac{m}{n} - \Sigma_{i=0}^{n} \tfrac{1}{i!}\right) < \tfrac{1}{n} < 1.$$

Thus, since $n! \left(\frac{m}{n} - \Sigma_{i=0}^{n} \frac{1}{i!}\right) = m(n-1)! - \Sigma_{i=0}^{n} \frac{n!}{i!}$ is obviously an integer and, hence, a natural number, we have a contradiction to the fact that there is no natural number strictly between 0 and 1 (Theorem 1.19). ¥

We obtain an infinite series representation for e^x for any real number x in Example 24.37.

Chapter XXII: Absolute Convergence

Absolute convergence is a notion of convergence of series that is particularly useful. We will see many examples of the use of absolute convergence when we study power series in Chapter XXIV and Chapter XXV.

We introduce the notion of absolute convergence in section 1. Then, as we did in the preceding chapter for convergence of series, we give tests for absolute convergence of series. We restrict ourselves to the two principal tests – the Ratio Test (in section 2) and the Root Test (in section 3). Then we show a direct relationship between the two tests (section 4). Next, we discuss a natural algebraic question about convergent series, namely, the question of what happens to convergence when we commute the terms of a series infinitely many times (section 5); although we could have asked this question a long time ago, we could not have answered it without the notion of absolute convergence. The final section concerns Cauchy products of series; we will see that Cauchy products behave in a more natural and predictable way than termwise products do.

One final comment: We will show that absolute convergence implies convergence (Theorem 22.1); thus, the Ratio Test and the Root Test in this chapter can be added to the tests for convergence in the previous chapter.

1. The Notion of Absolute Convergence

Consider the series $\Sigma_{i=1}^{\infty}\frac{\sin(i)}{i^2}$. It looks like the series converges, but can we tell from any of our previous tests? The series has infinitely many positive terms and infinitely many negative terms, and the signs of the terms occur somewhat irregularly in groups of 3 and 4 (e.g., although the first 21 terms change signs in groups of three, the next 4 terms are negative). Thus, we can not apply any of our previous tests to the series $\Sigma_{i=1}^{\infty}\frac{\sin(i)}{i^2}$; furthermore, recall from Exercise 21.12 that the Alternating Series Test does not adapt to nonalternating series even when the signs of their terms change in a *regular* pattern. On the other hand, if we disregard changes in sign by considering the series $\Sigma_{i=1}^{\infty}\frac{|\sin(i)|}{i^2}$, it is easy to see that the new series converges: $\frac{|\sin(i)|}{i^2} < \frac{1}{i^2}$ for each i and, thus, the series $\Sigma_{i=1}^{\infty}\frac{|\sin(i)|}{i^2}$ converges by the Comparison Test ($\Sigma_{i=1}^{\infty}\frac{1}{i^2}$ converges by Example 21.29).

But, what about the original series $\Sigma_{i=1}^{\infty}\frac{\sin(i)}{i^2}$? Believe it or not, we have stumbled into a way to easily show that the series converges: Simply note from the Triangle Inequality that

$$(*)\ \left|\Sigma_{i=m}^{n}\frac{\sin(i)}{i^2}\right| \leq \Sigma_{i=m}^{n}\left|\frac{\sin(i)}{i^2}\right| = \Sigma_{i=m}^{n}\frac{|\sin(i)|}{i^2} \quad \text{for all } n \geq m$$

and apply the Cauchy Criterion for Convergence of Series (Exercise 21.6) as follows: Let $\epsilon > 0$; since the series $\Sigma_{i=1}^{\infty}\frac{|\sin(i)|}{i^2}$ converges, the Cauchy criterion says there exists N such that $\Sigma_{i=m}^{n}\frac{|\sin(i)|}{i^2} < \epsilon$ when $n \geq m \geq N$; hence, by (*),

$$\left|\Sigma_{i=m}^{n}\frac{\sin(i)}{i^2}\right| < \epsilon \quad \text{when } n \geq m \geq N;$$

therefore, the series $\Sigma_{i=m}^{n}\frac{\sin(i)}{i^2}$ converges by the Cauchy criterion.

What we have done works in general. We state the general theorem after we introduce terminology that we use from now on.

Definition: A series $\Sigma_{i=1}^{\infty}a_i$ is said to be *absolutely convergent* (or to *converge absolutely*) provided that the series $\Sigma_{i=1}^{\infty}|a_i|$ converges.

Note that when all terms of a series are positive (or when all terms are negative), absolute convergence is the same as convergence. Thus, when it is not important to emphasize absolute convergence, we will often just say such series are convergent.

Theorem 22.1: If a series is absolutely convergent, then the series converges.

Proof: The proof merely consists of rewriting what we did above in the general setting of the theorem; we leave this for the reader to do. ¥

Exercise 22.2: Prove Theorem 22.1 as indicated. Also, give an example to show that the converse of Theorem 22.1 is false.

Thus, we have a new idea – using the series $\Sigma_{i=1}^{\infty}|a_i|$ to determine whether the series $\Sigma_{i=1}^{\infty}a_i$ converges. This is all well and good, but we need general tests for absolute convergence that are easy to apply. We give two such tests in the next two sections.

We know that the termwise product of two convergent series may diverge (Exercise 21.17). The following exercise provides a positive result for convergence of the termwise product. Although the result can be proved using only results in Chapter XXI (take this as a hint), the statement of the result could not be given in a concise way until we introduced the notion of absolute convergence.

Exercise 22.3: If the series $\Sigma_{i=1}^{\infty}a_i$ is absolutely convergent and the series $\Sigma_{i=1}^{\infty}b_i$ is any convergent series, then the termwise product $\Sigma_{i=1}^{\infty}a_ib_i$ is absolutely convergent (but does not necessarily converge to $(\Sigma_{i=1}^{\infty}a_i)(\Sigma_{i=1}^{\infty}b_i)$, as is illustrated following Exercise 21.3).

2. The Ratio Test

The Ratio Test and the Root Test are standard tests used to determine absolute convergence or divergence of series. Usually these tests are stated in terms of simple limits (of ratios and roots, respectively, of terms of the series). However, formulations of the tests in terms of upper and lower limits have much broader application (we give an example for the Root Test in the next section, Example 22.19) Thus, we present the tests in the more general form. This necessitates an introduction to upper and lower limits, which we give quickly in the first part of the section. Then we prove the Ratio Test.

Upper and Lower Limits

This part of the section consists of a definition and exercises directly related to the definition. The definition has three parts, but the definition is mainly terminology with no really new ideas involved. Hopefully the exercises will enable the reader to gain familiarity with the definition. We will use most of the results in the exercises at one time or another.

Definition: Let $\{s_i\}_{i=1}^{\infty}$ be a sequence and let

$$A = \{x \in \mathsf{R}^1 \cup \{\pm\infty\} : \text{some subsequence of } \{s_i\}_{i=1}^{\infty} \text{ converges to } x\}.$$

- The set A is called *the set of all subsequential limits of* $\{s_i\}_{i=1}^{\infty}$, and each point of A is called a *subsequential limit of* $\{s_i\}_{i=1}^{\infty}$. The set $A \cap \mathsf{R}^1$ is called *the set of all real subsequential limits of* $\{s_i\}_{i=1}^{\infty}$, and each point of $A \cap \mathsf{R}^1$ is called a *real subsequential limit of* $\{s_i\}_{i=1}^{\infty}$.
- The $\sup A$ is called the *upper limit* (or *limit superior*) *of the sequence* $\{s_i\}_{i=1}^{\infty}$ and is denoted by $\overline{\lim}_{i\to\infty} s_i$.
- The $\inf A$ is called the *lower limit* (or *limit inferior*) *of the sequence* $\{s_i\}_{i=1}^{\infty}$ and is denoted by $\underline{\lim}_{i\to\infty} s_i$.

Exercise 22.4: Let $\{s_i\}_{i=1}^{\infty}$ be a sequence. Then $\{s_i\}_{i=1}^{\infty}$ converges or $\{s_i\}_{i=1}^{\infty}$ diverges to ∞ or to $-\infty$ if and only if $\overline{\lim}_{i\to\infty} s_i = \underline{\lim}_{i\to\infty} s_i$, in which case the upper (and lower) limit of $\{s_i\}_{i=1}^{\infty}$ is the limit of $\{s_i\}_{i=1}^{\infty}$ or is ∞ or $-\infty$, respectively.

Exercise 22.5: The set of all real subsequential limits of a sequence $\{s_i\}_{i=1}^{\infty}$ is a closed set.
(*Hint:* Use Exercise 15.10.)

Exercise 22.6: For a sequence $\{s_i\}_{i=1}^{\infty}$, $\overline{\lim}_{i\to\infty} s_i$ is a subsequential limit of $\{s_i\}_{i=1}^{\infty}$.
(*Hint:* Use Exercise 22.5.)

Exercise 22.7: For a sequence $\{s_i\}_{i=1}^{\infty}$ and for any $p > \overline{\lim}_{i\to\infty} s_i$, there exists N such that $s_i < p$ for all $i \geq N$.

Exercise 22.8: If $\{s_i\}_{i=1}^{\infty}$ is a bounded sequence, then $\overline{\lim}_{i\to\infty} s_i$ is the only subsequential limit of $\{s_i\}_{i=1}^{\infty}$ that satisfies the condition (about p) in Exercise 22.7. Hence, $\overline{\lim}_{i\to\infty} s_i$ is the largest subsequential limit of $\{s_i\}_{i=1}^{\infty}$.

Exercise 22.9: The lower limit $\underline{\lim}_{i\to\infty} s_i$ satisfies results analogous to those in Exercises 22.6 - 22.8. Formulate the results for lower limits and prove them.

The Ratio Test

We are ready to state and prove the Ratio Test. After the proof, we give two examples to illustrate the test.

Theorem 22.10 (Ratio Test): Let $\Sigma_{i=1}^{\infty} a_i$ be a series none of whose terms is zero.

(1) If $\overline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| < 1$, then the series $\Sigma_{i=1}^{\infty} a_i$ is absolutely convergent.

(2) If $\underline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| > 1$, then the series $\Sigma_{i=1}^{\infty} a_i$ diverges.

(3) If $\underline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| \le 1 \le \overline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$, then there is no information about the convergence of $\Sigma_{i=1}^{\infty} a_i$.

Proof: Under the assumption in part (1), there is a number $r < 1$ and a natural number N such that $\left|\frac{a_{i+1}}{a_i}\right| < r$ for all $i \ge N$ (by Exercise 22.7). Hence, $|a_{i+1}| < |a_i|\, r$ for all $i \ge N$. Thus, for any given $i \in \mathsf{N}$,

$$(*)\ |a_{N+i}| < |a_{N+i-1}|\, r < (|a_{N+i-2}|\, r)r < \cdots < |a_N|\, r^i.$$

Since $0 \le r < 1$, the series $\Sigma_{i=1}^{\infty} |a_N|\, r^i$ is a convergent geometric series by Theorem 15.4. Thus, by (*) and the Comparison Test (Theorem 21.18), the series $\Sigma_{i=1}^{\infty} |a_{N+i}|$ converges. Therefore, the series $\Sigma_{i=1}^{\infty} |a_i|$ converges by Exercise 21.1. This proves part (1).of our theorem.

Under the assumption in part (2), there is a natural number M such that $\left|\frac{a_{i+1}}{a_i}\right| > 1$ for all $i \ge M$, which says that $|a_{i+1}| > |a_i|$ for all $i \ge M$. Hence, for any given $i \in \mathsf{N}$,

$$|a_{M+i}| > |a_{M+i-1}| > |a_{M+i-2}| > \cdots > |a_M|.$$

Thus, since $a_M \ne 0$, $\lim_{i\to\infty} a_i \ne 0$. Therefore, by the i^{th} Term Test (Theorem 21.8), the series $\Sigma_{i=1}^{\infty} a_i$ diverges. This proves part (2) of our theorem.

The proof of part (3) can be done by considering only the two series $\Sigma_{i=1}^{\infty} \frac{1}{i^2}$ and $\Sigma_{i=1}^{\infty} \frac{1}{i}$. However, we note that for any p-series $\Sigma_{i=1}^{\infty} \frac{1}{i^p}$,

$$\lim_{i\to\infty}\left|\frac{\frac{1}{(i+1)^p}}{\frac{1}{i^p}}\right| = \lim_{i\to\infty}\left(\frac{i}{i+1}\right)^p \overset{18.6}{=} 1^p = 1;$$

hence, $\underline{\lim}_{i\to\infty}\left|\frac{\frac{1}{(i+1)^p}}{\frac{1}{i^p}}\right| = 1 = \overline{\lim}_{i\to\infty}\left|\frac{\frac{1}{(i+1)^p}}{\frac{1}{i^p}}\right|$ (by Exercise 22.4); therefore, since a p-series converges when $p > 1$ and diverges when $p \le 1$ (by Example 21.29), we have proved part (3) of our theorem. ¥

The two series in the following exercise obviously diverge. Nevertheless, you are asked to see whether the Ratio Test shows that they diverge.

Exercise 22.11: What does the Ratio Test say about the series $\Sigma_{i=1}^{\infty} i$? What does the Ratio Test say about the series

$$\Sigma_{i=1}^{\infty} 2^{i-(-1)^i} = 2^2 + 2^1 + 2^4 + 2^3 + 2^6 + 2^5 + \cdots ?$$

We give two examples concerning the Ratio Test. In the first example, we illustrate how easy it is to apply the Ratio Test. In the second example, the Ratio Test fails; this example will be especially important when we get to the Root Test and the relationship between the Ratio Test and the Root Test.

Example 22.12: Consider the alternating series $\Sigma_{i=1}^{\infty} \frac{(-1)^{i+1} i!}{(1)(3)(5)\cdots(2i-1)}$. It is easy to apply the Ratio Test to see that the series converges: For each $i \in \mathsf{N}$, let a_i be the i^{th} term of the series. Then

$$\left|\frac{a_{i+1}}{a_i}\right| = \left|\frac{\frac{(-1)^{i+2}(i+1)!}{(1)(3)(5)\cdots(2i+1)}}{\frac{(-1)^{i+1}i!}{(1)(3)(5)\cdots(2i-1)}}\right| = \frac{(i+1)!}{i!}\frac{1}{2i+1} = \frac{i+1}{2i+1} \quad \text{for all } i \in \mathsf{N};$$

Hence, $\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \frac{1}{2}$. Thus, $\overline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \frac{1}{2} < 1$ (by Exercise 22.4). Therefore, by part (1) of the Ratio Test, the series $\Sigma_{i=1}^{\infty} \frac{(-1)^{i+1} i!}{(1)(3)(5)\cdots(2i-1)}$ converges (absolutely). The Alternating Series Test (Theorem 21.10) can also be used to show that the series converges, but some thought is required to see how (Exercise 22.14).

Example 22.13: Consider the series

$$\Sigma_{i=1}^{\infty} 2^{(-1)^i - i} = \frac{1}{2^2} + \frac{1}{2^1} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^5} + \cdots .$$

We show that the Ratio Test says nothing about the series. (You will be asked in Exercise 22.15 to prove that the series converges using a test in the previous chapter.)

For each $i \in \mathsf{N}$, let a_i be the i^{th} term of the series. Then, for all $i \in \mathsf{N}$,

$$\left|\frac{a_{i+1}}{a_i}\right| = \frac{2^{(-1)^{i+1}-(i+1)}}{2^{(-1)^i - i}} = 2^{(-1)^{i+1}-i-1-(-1)^i+i} = 2^{(-1)^{i+1}-(-1)^i-1}.$$

Thus,

$$\left|\frac{a_{i+1}}{a_i}\right| = \begin{cases} 2 & \text{, if } i \text{ is odd} \\ 2^{-3} & \text{, if } i \text{ is even.} \end{cases}$$

Hence, $\underline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| < 1 < \overline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$. Therefore, the Ratio Test says nothing about the series.

The first two exercises below concern the examples we just presented.

Exercise 22.14: Show that the series $\Sigma_{i=1}^{\infty} \frac{(-1)^{i+1} i!}{(1)(3)(5)\cdots(2i-1)}$ in Example 22.12 converges using the Alternating Series Test (Theorem 21.10).

Exercise 22.15: Show that the series $\Sigma_{i=1}^{\infty} 2^{(-1)^i - i}$ in Example 22.13 converges using a test in the preceding chapter (no fair using the Root Test, which we do not prove until the next section).

Exercise 22.16: Find all x such that the series $\Sigma_{i=2}^{\infty} \frac{x^i}{\ln(i)}$ converges.

3. The Root Test

We prove the Root Test and give two examples. The first example begins a comparison of the Root Test with the Ratio Test which we complete in section 4. The second example gives us a specific instance in which the Root Test applies but in which the simpler limit form of the test does not apply.

Theorem 22.17 (Root Test): Let $\Sigma_{i=1}^{\infty} a_i$ be a series.

(1) If $\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} < 1$, then the series $\Sigma_{i=1}^{\infty} a_i$ is absolutely convergent.

(2) If $\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} > 1$, then the series $\Sigma_{i=1}^{\infty} a_i$ diverges.

(3) If $\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} = 1$, then there is no information about the convergence of $\Sigma_{i=1}^{\infty} a_i$.

Proof: The proof is similar to the proof of the Ratio Test, but some modifications are needed (especially in the proof of parts (2) and (3)). We include the details.

Under the assumption in part (1), we can choose $r < 1$ and a natural number N such that $\sqrt[i]{|a_i|} < r$ for all $i \geq N$ (by Exercise 22.7). Hence,

$$(*)\ |a_i| < r^i \quad \text{for all } i \geq N.$$

Since $0 \leq r < 1$, the series $\Sigma_{i=1}^{\infty} r^i$ is a convergent geometric series by Theorem 15.4. Hence, the series $\Sigma_{i=N}^{\infty} r^i$ converges by Exercise 21.1. Thus, by (*) and the Comparison Test (Theorem 21.18), the series $\Sigma_{i=N}^{\infty} |a_i|$ converges. Therefore, the series $\Sigma_{i=1}^{\infty} |a_i|$ converges by Exercise 21.1. This proves part (1).of our theorem.

Next, we prove part (2). Recall from Exercise 22.6 that $\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|}$ is the limit of a subsequence,$\{\sqrt[i_k]{|a_{i_k}|}\}_{k=1}^{\infty}$, of the sequence $\{\sqrt[i]{|a_i|}\}_{i=1}^{\infty}$; hence, under the assumption in part (2), $\sqrt[i_k]{|a_{i_k}|} > 1$ for infinitely many k. Thus, $|a_{i_k}| > 1$ for (the same) infinitely many k. Hence, $\lim_{i\to\infty} a_i \neq 0$. Therefore, the series $\Sigma_{i=1}^{\infty} a_i$ diverges by the i^{th} Term Test (Theorem 21.8). This proves part (2) of our theorem.

We prove part (3) using p-series (as we did in the proof of part (3) of the Ratio Test). Fix p. We show that $\lim_{i\to\infty} \sqrt[i]{\frac{1}{i^p}} = 1$. We can show this directly, but we prefer to use encyclopedic recollection! The answer to Exercise 18.14 is that $\lim_{x\to\infty} x^{\frac{1}{x}} = 1$. Thus, since $(x^p)^{\frac{1}{x}} = (x^{\frac{1}{x}})^p$ (by part (4) of Theorem 16.30) and since the function $g(y) = y^p$ is continuous (by Theorem 16.31 and Theorem 6.14), we have by Theorem 18.6 that

$$\lim_{x\to\infty} (x^p)^{\frac{1}{x}} = 1.$$

Hence,

$$\lim_{i\to\infty} \sqrt[i]{\frac{1}{i^p}} = \lim_{i\to\infty} \frac{1}{(i^p)^{\frac{1}{i}}} \overset{4.19}{=} 1.$$

Thus, $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} = 1$ (by Exercise 22.4). Therefore, since a p-series converges when $p > 1$ and diverges when $p \leq 1$ (by Example 21.29), we have proved part (3) of our theorem. ¥

We return to the series $\frac{1}{2^2} + \frac{1}{2^1} + \frac{1}{2^4} + \frac{1}{2^3} + \cdots = \Sigma_{i=1}^{\infty} 2^{(-1)^i - i}$. We saw in Example 22.13 that the ratio says nothing about the series. It is easy to show that the series converges using more elementary tests than the Ratio Test and the Root Test (Exercise 22.15); nevertheless, we will use the series as an example for which the Ratio Test fails but the Root Test succeeds. We note that the reverse can not happen – when the Ratio Test succeeds, the Root Test succeeds, as we will prove in Theorem 22.23.

Example 22.18: For the reason mentioned above, we show that the Root Test can be used to prove that the series $\frac{1}{2^2} + \frac{1}{2^1} + \frac{1}{2^4} + \frac{1}{2^3} + \cdots = \Sigma_{i=1}^{\infty} 2^{(-1)^i - i}$ converges. Note that

$$\sqrt[i]{\left|2^{(-1)^i - i}\right|} = \left(\frac{2^{(-1)^i}}{2^i}\right)^{\frac{1}{i}} = \frac{1}{2}\left(2^{\frac{(-1)^i}{i}}\right) \text{ for each } i \in \mathsf{N}$$

and that

$$\lim_{i\to\infty} 2^{\frac{(-1)^i}{i}} \overset{16.23}{=} \lim_{i\to\infty} e^{\ln(2^{\frac{(-1)^i}{i}})} \overset{16.22}{=} \lim_{i\to\infty} e^{\frac{(-1)^i}{i}\ln(2)} \overset{18.6}{=} e^0 = 1.$$

Hence,

$$\lim_{i\to\infty} \sqrt[i]{\left|2^{(-1)^i - i}\right|} = \lim_{i\to\infty} \frac{1}{2}\left(2^{\frac{(-1)^i}{i}}\right) = \frac{1}{2}.$$

Thus, $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} = \frac{1}{2}$ (by Exercise 22.4). Therefore, by part (1) of the Root Test, the series $\Sigma_{i=1}^{\infty} 2^{(-1)^i - i}$ converges.

In many books, the Root Test and the Ratio Test are stated in terms of whether the limit $\lim_{i\to\infty}\sqrt[i]{|a_i|}$ is less than 1, greater than 1 (including ∞) or equal to 1. Obviously, these formulations of the tests follow from ours (by Exercise 22.4). Our next example shows that the Root Test we have given can be used when the Root Test in terms of the simple limit $\lim_{i\to\infty}\sqrt[i]{|a_i|}$ can not be used (because the limit does not exist as a number or ∞).

Example 22.19: We consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{2^i} + \frac{1}{3^i} + \cdots.$$

We show that the series has the properties mentioned above.

For each $i \in \mathsf{N}$, the i^{th} term, a_i, of the first series is given by

$$a_i = \begin{cases} \frac{1}{2^{\frac{i+1}{2}}} & \text{, if } i \text{ is odd} \\ \frac{1}{3^{\frac{i}{2}}} & \text{, if } i \text{ is even.} \end{cases}$$

Hence,

$$(*)\ \sqrt[i]{|a_i|} = \begin{cases} \left(\frac{1}{2^{\frac{i+1}{2}}}\right)^{\frac{1}{i}} = \left(\frac{1}{2}\right)^{\frac{i+1}{2i}} & \text{, if } i \text{ is odd} \\ \left(\frac{1}{3^{\frac{i}{2}}}\right)^{\frac{1}{i}} = \frac{1}{\sqrt{3}} & \text{, if } i \text{ is even.} \end{cases}$$

Since the exponential function $\left(\frac{1}{2}\right)^x$ is continuous and decreasing (by Corollary 17.20) and since the sequence $\{\frac{i+1}{2i}\}_{i=1}^{\infty}$ decreases to its limit $\frac{1}{2}$, we see that

$$\sup_{i\geq 1,\ i \text{ odd}} \left(\frac{1}{2}\right)^{\frac{i+1}{2i}} = \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Thus, by (*),

$$\sup_{i\geq 1} \sqrt[i]{|a_i|} = \frac{1}{\sqrt{2}}.$$

Hence, $\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} = \frac{1}{\sqrt{2}}$. Therefore, by part (1) of the Root Test, the series $\Sigma_{i=1}^{\infty} a_i$ converges. However, since

$$\lim_{i\to\infty,\ i \text{ odd}} \sqrt[i]{|a_i|} \overset{(*)}{=} \lim_{i\to\infty} \left(\frac{1}{2}\right)^{\frac{i+1}{2i}} \overset{16.23,16.22}{=} \lim_{i\to\infty} e^{\frac{i+1}{2i}\ln\left(\frac{1}{2}\right)}$$
$$= e^{\frac{1}{2}\ln\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2}}$$

and

$$\lim_{i\to\infty,\ i \text{ even}} \sqrt[i]{|a_i|} \overset{(*)}{=} \frac{1}{\sqrt{3}},$$

the limit $\lim_{i\to\infty} \sqrt[i]{|a_i|}$ does not exist.

You may think that the series in Example 22.19 converges simply because it is the termwise sum of two convergent geometric series. However, that only proves that the series $\Sigma_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{3^i}\right)$ converges. Nevertheless, it is instructive to work out a proof that the series in Example 22.19 converges using the series $\Sigma_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{3^i}\right)$.

Exercise 22.20: Prove that the series in Example 22.19 converges using the idea just mentioned.

Exercise 22.21: Does the Ratio Test (Theorem 22.10) show that the series in Example 22.19 converges?

4. Relationship between the Ratio Test and the Root Test

We know from experience that it is easier to compute ratios than to compute roots. Thus, we are inclined to try the Ratio Test before we try the Root Test unless the general form of the terms of the series suggests otherwise. For example, we would certainly try the Root Test first for the series $\Sigma_{i=1}^{\infty} \left(\frac{3-5i}{2+6i}\right)^i$; indeed, the Root Test easily shows that the series $\Sigma_{i=1}^{\infty} \left(\frac{3-5i}{2+6i}\right)^i$ converges, whereas trying the Ratio Test leads to an algebraic headache.

When it is not clear whether to try the Ratio Test or the Root Test, which should we try first? We saw in Example 22.18 that the Root Test can succeed when the Ratio Test fails. The reverse can not happen, as the theorem we will prove shows. The significance of the theorem is that in spite of our natural inclination to try the Ratio Test first, we should try the Root Test first (unless, of course, features of the terms of the series suggest trying the Ratio Test).

It is convenient to have the following lemma for the proof of our theorem.

Lemma 22.22: Let $\{s_i\}_{i=1}^{\infty}$ and $\{t_i\}_{i=1}^{\infty}$ be sequences such that $s_i \leq t_i$ for all $i \in \mathsf{N}$. If $\lim_{i\to\infty} t_i = p$ (including $p = \infty$), then $\overline{\lim}_{i\to\infty} s_i \leq p$.

Proof: Let $x = \lim_{k\to\infty} s_{i_k}$, where $\{s_{i_k}\}_{k=1}^{\infty}$ is a subsequence of $\{s_i\}_{i=1}^{\infty}$. Then, since $s_{i_k} \leq t_{i_k}$ for all k and since $\lim_{k\to\infty} t_{i_k} = p$ (by Exercise 20.3), we see easily that $x \leq p$. This proves that p is an upper bound for the set of all subsequential limits of $\{s_i\}_{i=1}^{\infty}$. Therefore, since $\overline{\lim}_{i\to\infty} s_i$ is a subsequential limit of $\{s_i\}_{i=1}^{\infty}$ (by Exercise 22.6), we have that $\overline{\lim}_{i\to\infty} s_i \leq p$. ¥

Theorem 22.23: If the Ratio Test shows absolute convergence or divergence, then so does the Root Test. In fact, for any sequence $\{a_i\}_{i=1}^{\infty}$ none of whose terms is zero,

$$\underline{\lim}_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| \leq \underline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} \leq \overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} \leq \overline{\lim}_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|.$$

Proof: The second inequality is obvious. We prove the third inequality, leaving the proof of the first inequality for the reader (the proof of the first inequality similar to our proof below; it uses the reader's results in Exercise 22.9 and the result for lower limits that is analogous to Lemma 22.22).

We assume for the proof that $\overline{\lim}_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| < \infty$ (otherwise, we are done). Then we can choose $p > \overline{\lim}_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. Hence, by Exercise 22.7, there is a natural number N such that $\left|\frac{a_{i+1}}{a_i}\right| < p$ for all $i \geq N$. Thus, $|a_{i+1}| < |a_i|\, p$ for all $i \geq N$. Therefore, for any given $i > N$,

$$|a_i| < |a_{i-1}|\, p < (|a_{i-2}|\, p)p < \cdots < |a_N|\, p^{i-N} = |a_N|\, p^{-N} p^i.$$

Hence, noting that $p \geq 0$ in order to know that $\sqrt[i]{|a_N|\, p^{-N}}$ exists, we have that

$$(*)\ \sqrt[i]{|a_i|} < \sqrt[i]{|a_N|\, p^{-N}}\, p \ \text{ for all } i \geq N.$$

Note that

$$\lim_{i\to\infty} \sqrt[i]{|a_N|\, p^{-N}} \overset{16.23,16.22}{=} \lim_{i\to\infty} e^{\frac{1}{i}\ln(|a_N|p^{-N})} = e^0 = 1.$$

Hence, $\lim_{i\to\infty} \sqrt[i]{|a_N|\, p^{-N}}\, p = p$ (by Theorem 19.4). Thus, (*) and Lemma 22.22,

$$\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|} \leq p.$$

Finally, having proved this last inequality for any $p > \overline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$, we have proved that

$$\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} \leq \overline{\lim}_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|,$$

which is the third inequality in our theorem.

It is easy to see that the inequalities in our theorem prove the first part of our theorem: part (1) of the Ratio Test implies part (1) of the Root Test by the third inequality in our theorem; part (2) of the Ratio Test implies part (2) of the Root Test by the first and second inequalities in our theorem. ¥

Exercise 22.24: Prove the first inequality in Theorem 22.23.

Exercise 22.25: Find all x such that the series $\Sigma_{i=1}^{\infty}\frac{(x-1)^i}{i2^i}$ converges.

Exercise 22.26: Find all x such that the series $\Sigma_{i=1}^{\infty}\frac{10^{ix}}{i!}$ converges.

Exercise 22.27: Is the series $\Sigma_{i=1}^{\infty}(-1)^i\frac{\sqrt{i+1}-\sqrt{i}}{i}$ absolutely convergent?

5. Rearrangements of Series

One general question in the back of our minds when studying series is the question of how much series behave like finite sums. When we started studying series we saw some results about series that resemble results about finite sums (the results in Exercises 21.3, 21.4 and 21.5). Since then we have not paid any attention to the similarity or difference between series and finite sums. We now focus on an aspect of comparing series with finite sums that is particularly interesting: When can we commute the terms of a convergent series infinitely many times and still, no matter how we commute the terms, have a convergent series? And, when we can do this, must the sums of the series be the same?

We provide complete answers to the questions by the end of the section (Theorem 22.35).

First, let us give an example of a convergent series such that for any given real number, we can commute the terms of the series so that the new series converges to the given real number. The procedure we use in the example will actually be more important than the example itself.

Example 22.28: We show that we can commute the terms of the alternating harmonic series $\Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$, which converges by Theorem 21.10, so the new series can be made to converge to any real number whatsoever.

We first make simple observations about the odd terms and the even terms of $\Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$; namely, $\Sigma_{i=1}^{\infty}\frac{1}{2i-1} = \infty$ and $\Sigma_{i=1}^{\infty}\frac{-1}{2i} = -\infty$. This is easily seen as follows: The series $\Sigma_{i=1}^{\infty}\frac{1}{2i}$ diverges (by Example 21.29 and Exercise 21.4) and, therefore, its sequence of partial sums, which is increasing, must not have an upper bound (by Theorem 19.20); thus, $\Sigma_{i=1}^{\infty}\frac{1}{2i} = \infty$; hence, $\Sigma_{i=1}^{\infty}\frac{-1}{2i} = -\infty$ and, since $\frac{1}{2i-1} > \frac{1}{2i}$ for each $i \in \mathsf{N}$, $\Sigma_{i=1}^{\infty}\frac{1}{2i-1} = \infty$ by the Comparison Test (Theorem 21.18).

Now, fix any number x. Since $\Sigma_{i=1}^{\infty} \frac{1}{2i-1} = \infty$, we can use the Well Ordering Principle (1.18) to let j_1 be the first natural number such that $\Sigma_{i=1}^{j_1} \frac{1}{2i-1} > x$. Similarly, since $\Sigma_{i=1}^{\infty} \frac{-1}{2i} = -\infty$, we can let k_1 be the first natural number such that

$$\Sigma_{i=1}^{j_1} \frac{1}{2i-1} + \Sigma_{i=1}^{k_1} \frac{-1}{2i} < x.$$

Next, let j_2 be the first natural number greater than j_1 such that

$$\Sigma_{i=1}^{j_2} \frac{1}{2i-1} + \Sigma_{i=1}^{k_1} \frac{-1}{2i} > x.$$

Then let k_2 be the first natural number greater than k_1 such that

$$\Sigma_{i=1}^{j_2} \frac{1}{2i-1} + \Sigma_{i=1}^{k_2} \frac{-1}{2i} < x.$$

Continuing the process as indicated (we omit the formal induction), we arrive at a series whose terms, in the order in which they occur, are

$$\frac{1}{1}, \ldots, \frac{1}{2j_1-1}, \frac{-1}{2}, \ldots, \frac{-1}{2k_1}, \frac{1}{2(j_1+1)-1}, \ldots, \frac{1}{2j_2-1}, \frac{-1}{2(k_1+1)}, \ldots, \frac{-1}{2k_2}, \ldots .$$

The terms of this new series are the terms of the alternating harmonic series, and each term of the alternating harmonic series appears only once as a term of the new series. Thus, we have obtained the new series solely by commuting the terms of the alternating harmonic series. Finally, since $\lim_{m\to\infty} \frac{1}{2j_m-1} = 0$ and $\lim_{m\to\infty} \frac{1}{2k_m} = 0$, it is easy to see from the construction that the sum of the new series is x.

Exercise 22.29: In addition to what we showed in Example 22.28, show that the terms of the alternating harmonic series $\Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$ can be commuted so that the sum of the resulting series is ∞.

When we commute the terms of a series, the new series we obtain is called a rearrangement of the original series. Let us note the precise definition:

Definition: A *rearrangement of a series* $\Sigma_{i=1}^{\infty} a_i$ is a series of the form $\Sigma_{i=1}^{\infty} a_{\varphi(i)}$, where φ is any one-to-one function from N onto N.

Rearrangements of Absolutely Convergent Series

It would seem that the procedure in Example 22.28 could be carried out more generally. For which series can we do this? That is, what general properties must a series have in order that for any given real number x, the procedure in Example 22.28 produces a rearrangement of the series such that the sum of the rearrangement is x? On reviewing the procedure in Example 22.28, we see that the sum of the positive terms of such a series should be ∞, the sum of the negative terms should be $-\infty$, and the limit of the positive terms, as well the limit of the negative terms, should be 0 (which was used at the end of Example 22.28 to show x was the sum of the rearrangement).

An absolutely convergent series $\Sigma_{i=1}^{\infty} a_i$ does not have the properties just mentioned: the sum of its positive terms, as well as the sum of its negative terms, is finite since the sums lie between $\Sigma_{i=1}^{\infty} - |a_i|$ and $\Sigma_{i=1}^{\infty} |a_i|$. However, this leaves open the possibility that for some absolutely convergent series $\Sigma_{i=1}^{\infty} a_i$, several real numbers between $\Sigma_{i=1}^{\infty} - |a_i|$ and $\Sigma_{i=1}^{\infty} |a_i|$ could be the sums of rearrangements of the series. We show this can not happen; we are then led to see if the procedure in Example 22.28 works for convergent series that are not absolutely convergent.

Theorem 22.30: If a series $\Sigma_{i=1}^{\infty} a_i$ is absolutely convergent, then every rearrangement of $\Sigma_{i=1}^{\infty} a_i$ converges to the sum $\Sigma_{i=1}^{\infty} a_i$.

Proof: Let $\Sigma_{i=1}^{\infty} a_{\varphi(i)}$ be a rearrangement of $\Sigma_{i=1}^{\infty} a_i$ (φ as in the definition of rearrangement). Let $\epsilon > 0$. Since the series $\Sigma_{i=1}^{\infty} |a_i|$ converges, we see from the Cauchy criterion in Exercise 21.6 that there is a natural number N such that

(1) $\Sigma_{i=m}^{n} |a_i| < \epsilon$ for all $n \geq m \geq N$.

Since φ maps N onto N, there is a natural number $K \geq N$ such that

(2) $\{1, 2, ..., N\} \subset \{\varphi(1), \varphi(2), ..., \varphi(K)\}$.

Now, let $\{s_n\}_{n=1}^{\infty}$ denote the sequence of partial sums of $\Sigma_{i=1}^{\infty} a_i$, and let $\{t_n\}_{n=1}^{\infty}$ denote the sequence of partial sums of $\Sigma_{i=1}^{\infty} a_{\varphi(i)}$. Fix $n \geq K$. Since $K \geq N$, each of the terms $a_1, a_2, ..., a_N$ appears exactly once in s_n; by (2) and the fact that φ is one-to-one, each of the terms $a_1, a_2, ..., a_N$ appears exactly once in t_n. Hence, the terms $a_1, a_2, ..., a_N$ will cancel one another in the difference $s_n - t_n$. After we cancel as just indicated, the terms left in $s_n - t_n$ are of the form a_i with $i > N$; denote the terms left by $b_1, b_2, ..., b_\ell$. Then,

$$|s_n - t_n| = \left|\Sigma_{i=1}^{\ell} b_i\right| \leq \Sigma_{i=1}^{\ell} |b_i| \overset{(1)}{<} \epsilon.$$

We have proved that for any $\epsilon > 0$, there exists K such that $|s_n - t_n| < \epsilon$ for all $n \geq K$. We also know from Theorem 22.1 that the series $\Sigma_{i=1}^{\infty} a_i$ converges, which means that $\lim_{n\to\infty} s_n = \Sigma_{i=1}^{\infty} a_i$. Therefore, it follows easily that $\lim_{n\to\infty} t_n = \Sigma_{i=1}^{\infty} a_i$. This proves that the rearrangement $\Sigma_{i=1}^{\infty} a_{\varphi(i)}$ converges to the sum of the series $\Sigma_{i=1}^{\infty} a_i$. ¥

Exercise 22.31: A series is absolutely convergent if and only if the series of its nonnegative terms converges and the series of its negative terms converges.

Rearrangements of Conditionally Convergent Series

In line with the discussion leading to Theorem 22.30, we now focus on convergent series that are not absolutely convergent. We show that the procedure in Example 22.28 generalizes to apply to such series. For conciseness, we note the following standard terminology.

Definition: A series that converges but is not absolutely convergent is said to be *conditionally convergent*.

The following lemma addresses the conditions about divergence to $\pm\infty$ of the positive and negative terms that we mentioned in the discussion preceding Theorem 22.30.

Lemma 22.32: If $\Sigma_{i=1}^{\infty} a_i$ is a conditionally convergent series, then the sum of the series of nonnegative terms of $\Sigma_{i=1}^{\infty} a_i$ is ∞ and the sum of the series of negative terms of $\Sigma_{i=1}^{\infty} a_i$ is $-\infty$.

Proof: For each $i \in \mathsf{N}$, let

$$p_i = \frac{|a_i| + a_i}{2} = \begin{cases} a_i & \text{, if } a_i \geq 0 \\ 0 & \text{, if } a_i \leq 0 \end{cases}, \quad q_i = \frac{|a_i| - a_i}{2} = \begin{cases} 0 & \text{, if } a_i \geq 0 \\ -a_i & \text{, if } a_i \leq 0 \end{cases}.$$

We first prove that

$$(*)\ \Sigma_{i=1}^{\infty} p_i = \infty \quad \text{and} \quad \Sigma_{i=1}^{\infty} q_i = \infty$$

and then we show that the lemma follows easily from (*).

Proof of (*)*:* If both the series $\Sigma_{i=1}^{\infty} p_i$ and $\Sigma_{i=1}^{\infty} q_i$ converged, then their termwise sum would converge (by Exercise 21.3); however, since their termwise sum satisfies

$$\Sigma_{i=1}^{\infty} (p_i + q_i) = \Sigma_{i=1}^{\infty} |a_i|,$$

their termwise sum diverges since the series $\Sigma_{i=1}^{\infty} a_i$ is not absolutely convergent. Hence, we have that

(1) $\Sigma_{i=1}^{\infty} p_i$ diverges or $\Sigma_{i=1}^{\infty} q_i$ diverges.

Now, note that for each $n \in \mathsf{N}$,

$$\Sigma_{i=1}^{n} a_i = \Sigma_{i=1}^{n} (p_i - q_i) = \Sigma_{i=1}^{n} p_i - \Sigma_{i=1}^{n} q_i;$$

thus, since the series $\Sigma_{i=1}^{\infty} a_i$ converges, the convergence of $\Sigma_{i=1}^{\infty} p_i$ implies the convergence of $\Sigma_{i=1}^{\infty} q_i$ (by Exercise 21.3) and vice versa. Hence, by (1), it must be that

(2) $\Sigma_{i=1}^{\infty} p_i$ diverges and $\Sigma_{i=1}^{\infty} q_i$ diverges.

Next, note that $p_i \geq 0$ for all $i \in \mathsf{N}$; hence, the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of the series $\Sigma_{i=1}^{\infty} p_i$ is increasing. Thus, by (2) and Theorem 19.20, the sequence $\{s_n\}_{n=1}^{\infty}$ has no upper bound. Therefore, $\Sigma_{i=1}^{\infty} p_i = \infty$. Similarly, $\Sigma_{i=1}^{\infty} q_i = \infty$. This proves (*).

Finally, we use (*) to complete the proof of our lemma.

By (*), there are infinitely many nonnegative terms of the series $\Sigma_{i=1}^{\infty} a_i$, say $a_{i_1}, a_{i_2}, ..., a_{i_n}, ...$ where the terms are indexed in the order in which they appear (meaning $i_1 < i_2 < \cdots < i_n < \cdots$). Then, for each $n \in \mathsf{N}$, the first i_n terms of the series $\Sigma_{i=1}^{\infty} p_i$ are the terms $a_{i_1}, a_{i_2}, ..., a_{i_n}$ together with (possibly) some terms that are 0. Hence,

$$\Sigma_{j=1}^{n} a_{i_j} = \Sigma_{j=1}^{i_n} p_j \quad \text{for each } n \in \mathsf{N}.$$

Therefore, by (*), $\Sigma_{j=1}^{\infty} a_{i_j} = \infty$.

Similarly, if $a_{k_1}, a_{k_2}, ..., a_{k_n}, ...$ are the negative terms (in order of appearance) of the series $\Sigma_{i=1}^{\infty} a_i$, then

$$\Sigma_{j=1}^{n} a_{k_j} = -\Sigma_{j=1}^{k_n} q_j \quad \text{for each } n \in \mathsf{N}.$$

Therefore, by (*), $\Sigma_{j=1}^{\infty} a_{k_j} = -\infty$. ¥

We now apply the procedure in Example 22.28 to prove the following theorem. Recall the definitions of $\overline{\lim}_{n\to\infty} s_n$ and $\underline{\lim}_{n\to\infty} s_n$ at the beginning of section 2 of this chapter.

Theorem 22.33: Any conditionally convergent series has a rearrangement that converges to any real number or whose sum is $\pm\infty$. Moreover, let $\Sigma_{i=1}^{\infty} a_i$ be a conditionally convergent series, and let $u \leq v$ be real numbers or $u = -\infty$ or $v = \infty$. Then there is a rearrangement $\Sigma_{i=1}^{\infty} a_{\varphi(i)}$ of $\Sigma_{i=1}^{\infty} a_i$ such that the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of $\Sigma_{i=1}^{\infty} a_{\varphi(i)}$ satisfies

$$\underline{\lim}_{n\to\infty} s_n = u \quad \text{and} \quad \overline{\lim}_{n\to\infty} s_n = v.$$

Proof: By Lemma 22.32, the series $\Sigma_{i=1}^{\infty} a_i$ has infinitely many nonnegative terms and infinitely many negative terms. Let $b_1, b_2, ..., b_i, ...$ denote the nonnegative terms of the series $\Sigma_{i=1}^{\infty} a_i$ listed in the order in which they appear, and let $c_1, c_2, ..., c_i, ...$ denote the negative terms of the series $\Sigma_{i=1}^{\infty} a_i$ listed in the order in which they appear.

We consider the case when u and v are finite.

By Lemma 22.32, $\Sigma_{i=1}^{\infty} b_i = \infty$. Hence, by the Well Ordering Principle (1.18), there is a first natural number j_1 such that $\Sigma_{i=1}^{j_1} b_i > v$. Then, since $\Sigma_{i=1}^{\infty} c_i = -\infty$ (by Lemma 22.32), there is a first natural number k_1 such that

$$\Sigma_{i=1}^{j_1} b_i + \Sigma_{i=1}^{k_1} c_i < u.$$

Next, there is a first natural number $j_2 > j_1$ such that (by Lemma 22.32)

$$\Sigma_{i=1}^{j_2} b_i + \Sigma_{i=1}^{k_1} c_i > v.$$

Then there is a first natural number $k_2 > k_1$ such that (by Lemma 22.32)

$$\Sigma_{i=1}^{j_2} b_i + \Sigma_{i=1}^{k_2} c_i < u.$$

Continuing the procedure as indicated (we omit the formal induction), we obtain a series whose terms, in the order in which they occur, are

$$b_1, ..., b_{j_1}, c_1, ..., c_{k_1}, b_{j_1+1}, ..., b_{j_2}, c_{k_1+1}, ..., c_{k_2}, ...$$

Note that this new series uses each term of the series $\Sigma_{i=1}^{\infty} a_i$ exactly once (and uses only the terms a_i); hence, the new series is a rearrangement of the original series $\Sigma_{i=1}^{\infty} a_i$.

We denote the new series by $\Sigma_{i=1}^{\infty} d_i$ and its sequence of partial sums by $\{s_n\}_{n=1}^{\infty}$.

For each $n \in \mathsf{N}$, we let

$$u_n = \Sigma_{i=1}^{k_n} d_i, \quad v_n = \Sigma_{i=1}^{j_n} d_i.$$

Since $k_1 < k_2 < \cdots < k_n < \cdots$ and $j_1 < j_2 < \cdots < j_n < \cdots$, we have that

(1) the sequences $\{u_n\}_{n=1}^{\infty}$and $\{v_n\}_{n=1}^{\infty}$ are subsequences of $\{s_n\}_{n=1}^{\infty}$.

Let us also note the following: Since the series $\Sigma_{i=1}^{\infty} a_i$ converges, $\lim_{i\to\infty} a_i = 0$ by the i^{th} Term Test (Theorem 21.8); thus, since the indices k_i and j_i are strictly increasing, we have that

(2) $\lim_{i\to\infty} c_{k_i} = 0$ and $\lim_{i\to\infty} b_{j_i} = 0$.

We now show that u and v are subsequential limits of $\{s_n\}_{n=1}^{\infty}$. It is clear from the construction that for each $n \in \mathsf{N}$,

$$|u - u_n| \leq |c_{k_n}| \quad \text{and} \quad |v - v_n| \leq |b_{j_n}|.$$

Hence, we see from (2) that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Thus, by (1), we have that

(3) u and v are subsequential limits of $\{s_n\}_{n=1}^{\infty}$.

Finally, we show that $\underline{\lim}_{n\to\infty} s_n = u$ and that $\overline{\lim}_{n\to\infty} s_n = v$. Fix $x < u$. Then, by (2), there exists N such that $c_{k_i} > \frac{x+u}{2}$ for all $i \geq N$. It then follows that no subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to x. Hence, $\underline{\lim}_{n\to\infty} s_n \geq u$. Therefore, by (3), $\underline{\lim}_{n\to\infty} s_n = u$. A similar argument shows that $\overline{\lim}_{n\to\infty} s_n = v$.

This completes the proof of our theorem when u and v are finite. The proof when one or both of u and v are infinite is left to the reader. ¥

Exercise 22.34: Prove Theorem 22.33 when u, v or both are infinite.

Main Theorem

We are now in a position to completely answer the questions we asked at the beginning of the section:

Theorem 22.35: For any series $\Sigma_{i=1}^{\infty} a_i$, the following three statements are equivalent to one another:

(1) Every rearrangement of the series $\Sigma_{i=1}^{\infty} a_i$ converges.

(2) Every rearrangement of the series $\Sigma_{i=1}^{\infty} a_i$ converges to the same sum.

(3) The series $\Sigma_{i=1}^{\infty} a_i$ is absolutely convergent.

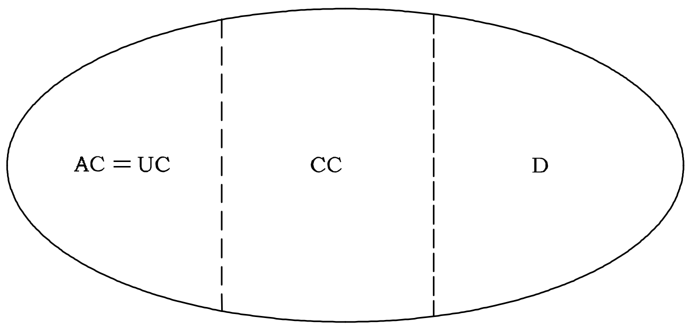
Proof: That (3) implies (2) is by Theorem 22.30. Obviously, (2) implies (1).

Finally, we prove that (1) implies (3). Assume that (1) holds. By Theorem 22.33, every conditionally convergent series has a nonconvergent rearrangement. Hence, by (1), the series $\Sigma_{i=1}^{\infty} a_i$ is not conditionally convergent. Thus, by the definition of conditionally convergent, the series $\Sigma_{i=1}^{\infty} a_i$ either diverges or is absolutely convergent. Now, note that the series $\Sigma_{i=1}^{\infty} a_i$ is a rearrangement of itself (by letting $\varphi(i) = i$ for all $i \in \mathsf{N}$); hence, by (1), the series $\Sigma_{i=1}^{\infty} a_i$

converges. Therefore, the series $\Sigma_{i=1}^{\infty} a_i$ is absolutely convergent. This proves (3). ¥

For completeness, we remark that series for which every rearrangement converges to the same sum are often said to be *unconditionally convergent*. Note that Theorem 22.35 says that the unconditionally convergent series are the same as the absolutely convergent series. Theorem 22.35 also shows that the phrase "to the same sum" is superfluous in the definition of unconditionally convergent.

In the figure below, we summarize convergence properties of series we have studied; the abbreviations AC, UC, CC and D, respectively, stand for absolutely convergent, unconditionally convergent, conditionally convergent and divergent.



Exercise 22.36: Give an example of a series that has only integers and $\pm\infty$ as sums of rearrangements and such that every integer, ∞ and $-\infty$ are sums of rearrangements of the series.

Exercise 22.37: Let S denote the sum of the alternating harmonic series $\Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$. The following series is a rearrangement of the alternating harmonic series:

$$\Sigma_{i=1}^{\infty} a_i = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots + \tfrac{1}{4i-3} + \tfrac{1}{4i-1} - \tfrac{1}{2i} + \cdots$$

Show that the sum of the series $\Sigma_{i=1}^{\infty} a_i$ is $\frac{3S}{2} \neq S$ by verifying the claims in (1) - (4) below:

(1) $S \neq 0$.

(2) $\Sigma_{i=1}^{\infty} b_i = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + ... = \frac{S}{2}$.

(3) The sequence of partial sums of the series $\Sigma_{i=1}^{\infty} a_i$ is a subsequence of the sequence of partial sums of the series $\Sigma_{i=1}^{\infty}\left((-1)^{i+1}\frac{1}{i} + b_i\right)$.

(4) $\Sigma_{i=1}^{\infty} a_i = \frac{3S}{2} \neq S$.

6. Cauchy Products

We discussed the termwise product of series following Exercise 21.3. There is another kind of product of series called the Cauchy product, which we define below.

The motivation for the Cauchy product of numerical series comes from power series. We will study power series in Chapters XXIV and XXV. For our purpose at this time, the term *power series* refers to a series of the form $\Sigma_{i=0}^{\infty} a_i x^i$, where a_i is a constant for each i and x is a variable. No knowledge of power series is needed for what we do here.

Let $\Sigma_{i=0}^{\infty} a_i$ and $\Sigma_{i=0}^{\infty} b_i$ be two numerical series, and consider the two power series $\Sigma_{i=0}^{\infty} a_i x^i$ and $\Sigma_{i=0}^{\infty} b_i x^i$. If we multiply the two power series together as though they were polynomials and collect terms, we obtain the power series

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 \\ + \cdots + (\Sigma_{i=0}^{n} a_i b_{n-i})x^n + \cdots;$$

this power series is called the *Cauchy product of the two power series* $\Sigma_{i=0}^{\infty} a_i x^i$ *and* $\Sigma_{i=0}^{\infty} b_i x^i$, and is denoted by $\left(\Sigma_{i=0}^{\infty} a_i x^i\right) \times \left(\Sigma_{i=0}^{\infty} b_i x^i\right)$. The *Cauchy product of the numerical series* $\Sigma_{i=0}^{\infty} a_i$ *and* $\Sigma_{i=0}^{\infty} b_i$ is the series we just defined but without the x's and is denoted by $(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i)$; in other words,

$$(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i) = \Sigma_{n=0}^{\infty} (\Sigma_{i=0}^{n} a_i b_{n-i}).$$

We see that Cauchy products, rather than termwise products, are the natural product for series, even for numerical series. This is because if $\Sigma_{i=0}^{\infty} a_i$ and $\Sigma_{i=0}^{\infty} b_i$ are numerical series, then the n^{th} partial sum of their Cauchy product is the finite sum obtained by multiplying the two polynomials $\Sigma_{i=0}^{n} a_i x^i$ and $\Sigma_{i=0}^{n} b_i x^i$ together.

We know that termwise product of two convergent series may diverge (Exercise 21.17). The same unfortunate behavior occurs with Cauchy products:

Exercise 22.38: The series $\Sigma_{i=0}^{\infty}(-1)^i \frac{1}{\sqrt{i+1}}$ converges, but its Cauchy product with itself diverges.

However, if at least one of two convergent series converges absolutely, then their Cauchy product converges and, moreover, it converges to the expected value:

Theorem 22.39: If $\Sigma_{i=0}^{\infty} a_i$ is absolutely convergent and $\Sigma_{i=0}^{\infty} b_i$ is convergent, then the Cauchy product $(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i)$ converges to $(\Sigma_{i=0}^{\infty} a_i)\,(\Sigma_{i=0}^{\infty} b_i)$; in other words, the Cauchy product converges to the product of the sums of the two series $\Sigma_{i=0}^{\infty} a_i$ and $\Sigma_{i=0}^{\infty} b_i$.

Proof: We first establish some notation for use in our computations: For each $n \geq 0$, let c_n be the n^{th} term of $(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i)$, so

$$c_n = \Sigma_{i=0}^{n} a_i b_{n-i},$$

and denote the n^{th} partial sums of the three series by

$$A_n = \Sigma_{i=0}^n a_i, \quad B_n = \Sigma_{i=0}^n b_i, \quad S_n = \Sigma_{i=0}^n c_i.$$

Also, let A, B, and S denote the sums of the three series,

$$A = \Sigma_{i=0}^\infty a_i, \quad B = \Sigma_{i=0}^\infty b_i, \quad S = \Sigma_{i=0}^\infty c_i.$$

Finally, for each $n \geq 0$, let

$$\beta_n = B_n - B.$$

Our theorem claims that $S = AB$. We prove this by showing that

$$\lim_{n\to\infty} S_n = AB.$$

Note that for each $n \geq 0$,

$$\begin{aligned} S_n &= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots + (\Sigma_{i=0}^n a_i b_{n-i}) \\ &= a_0B_n + a_1B_{n-1} + a_2B_{n-2} + \cdots + a_nB_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + a_2(B + \beta_{n-2}) + \cdots + a_n(B + \beta_0) \\ &= A_nB + \Sigma_{i=0}^n a_i\beta_{n-i}. \end{aligned}$$

Hence, letting $\gamma_n = \Sigma_{i=0}^n a_i\beta_{n-i}$ for each $n \geq 0$, we have proved that

$$S_n = A_nB + \gamma_n \quad \text{for each } n \geq 0.$$

Thus, since $\lim_{n\to\infty} A_nB = AB$ (by Theorem 19.4), our theorem will be proved once we prove

$$(*)\ \lim_{n\to\infty} \gamma_n = 0.$$

Proof of (*): Let $\epsilon > 0$. By assumption in our theorem, $\Sigma_{i=0}^\infty |a_i| < \infty$. We let

$$\alpha = \Sigma_{i=0}^\infty |a_i|.$$

Since $\alpha < \infty$, we see from Exercise 21.2 that there is a natural number N such that

(1) $\Sigma_{i=n}^\infty |a_i| < \epsilon$ for all $n > N$.

Since $\Sigma_{i=0}^\infty b_i$ converges and $\beta_n = B_n - B$, we have that $\lim_{n\to\infty} \beta_n = 0$; hence, we can assume that N in (1) is large enough so that

(2) $|\beta_n| < \epsilon$ for all $n > N$.

Next, note that for each $n > N$,

$$\gamma_n = \Sigma_{i=0}^n a_i\beta_{n-i} = \Sigma_{i=0}^{n-N-1} a_i\beta_{n-i} + \Sigma_{i=n-N}^n a_i\beta_{n-i};$$

hence, for each $n > N$,

$$\begin{aligned}|\gamma_n| &\leq \left|\Sigma_{i=0}^{n-N-1} a_i \beta_{n-i}\right| + \left|\Sigma_{i=n-N}^{n} a_i \beta_{n-i}\right| \\ &\leq \Sigma_{i=0}^{n-N-1} |a_i| \left|\beta_{n-i}\right| + \left|\Sigma_{i=n-N}^{n} a_i \beta_{n-i}\right| \\ &\overset{(2)}{\leq} \Sigma_{i=0}^{n-N-1} |a_i| \epsilon + \left|\Sigma_{i=n-N}^{n} a_i \beta_{n-i}\right| \leq \alpha\epsilon + \Sigma_{i=n-N}^{n} |a_i| \left|_i\beta_{n-i}\right|.\end{aligned}$$

Thus, letting $\beta = \max\{|\beta_0|, |\beta_1|, ..., |\beta_N|\}$, we have that

$$|\gamma_n| \leq \alpha\epsilon + \beta\Sigma_{i=n-N}^{n} |a_i| \quad \text{for all } n > N.$$

Hence, by (1),

$$|\gamma_n| \leq \alpha\epsilon + \beta\epsilon \quad \text{for all } n > 2N.$$

Therefore, since $\epsilon > 0$ was arbitrary, we have proved (*). ¥

We compare Theorem 22.39 with the result for termwise products in Exercise 22.3. The assumptions in both results are the same, but there are two notable differences in the conclusions which we note below.

First, the termwise product is absolutely convergent (Exercise 22.3), whereas the Cauchy product may not be absolutely convergent, as we show with the following simple example: Just let $\Sigma_{i=0}^{\infty} a_i$ be the absolutely convergent series with $a_0 = 1$ and $a_i = 0$ for all $i \geq 1$; then we see that the Cauchy product of $\Sigma_{i=0}^{\infty} a_i$ with the convergent series $\Sigma_{i=0}^{\infty}(-1)^i \frac{1}{i+1}$ is conditionally convergent.

Second, the Cauchy product converges to the expected value, namely, to the product of the sums of the two series (Theorem 22.39), whereas the termwise product may not converge to the expected value (an example is in the discussion following Exercise 21.3).

The second difference between Theorem 22.39 and the result in Exercise 22.3 seems more important than the first. Thus, we are led to say that Cauchy products behave better than termwise products.

Finally, we raise a natural question that we will answer later. Assume that two series, $\Sigma_{i=0}^{\infty} a_i$ and $\Sigma_{i=0}^{\infty} b_i$, converge and that their Cauchy product $(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i)$ also converges. Then the question arises as to whether the Cauchy product must converge to $(\Sigma_{i=0}^{\infty} a_i)(\Sigma_{i=0}^{\infty} b_i)$. The answer is *yes*. However, the question is really a question about the continuity of power series; thus, we are not prepared to verify the answer at this time. You will be asked to carry out the verifications at the end of section 3 of Chapter XXIV (Exercise 24.33).

Chapter XXIII: Sequences of Functions

We studied numerical sequences in Chapter XIX and Chapter XX. Numerical sequences are functions whose domains are the natural numbers. In this chapter we study sequences of functions whose domains vary from being arbitrary sets to being intervals. We apply what we learn to series of functions in the next two chapters.

We study two notions of convergence of functions – pointwise convergence and uniform convergence. In section 1 we present examples to show that the properties of continuity, differentiability and integrability are not preserved under pointwise convergence. In section 2 we introduce uniform convergence and prove the Cauchy criterion for uniform convergence. In sections 3, 4 and 5 we obtain results relating uniform convergence to continuity, differentiability and integrability.

1. Pointwise Convergence

The following definition expresses the most natural way to consider convergence for a sequence of functions:

Definition: Let X be a set, and let $f_n : X \to \mathsf{R}^1$ be a function for each $n \in \mathsf{N}$. We say that the sequence $\{f_n\}_{n=1}^{\infty}$ of functions *converges pointwise on* X provided that for each $x \in X$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges (to a real number).

Assume that $\{f_n\}_{n=1}^{\infty}$ converges pointwise on X, and let $f : X \to \mathsf{R}^1$ be defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for each } x \in X;$$

then we call f the *pointwise limit of* $\{f_n\}_{n=1}^{\infty}$, and we say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise to* f *on* X.

Pointwise convergence is the simplest type of convergence of functions imaginable; unfortunately, however, pointwise convergence is not strong enough to preserve continuity, differentiability or integrability. We show this in examples below.

Let us first show that the problem of the continuity of pointwise limits of continuous functions can be viewed as a problem about interchanging the order in which we take limits.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions converging pointwise on a subset X of R^1 to a function f. Let $p \in X$. By Corollary 3.13, f is continuous at any isolated point of X. Thus, to examine the continuity of f at p, we may as well assume p is a limit point of X. Then, by Corollary 3.13, f is continuous at p if and only if

$$\lim_{x\to p} f(x) = f(p);$$

thus, since $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$, the continuity of f at p is equivalent to having

$$\lim_{x\to p} \lim_{n\to\infty} f_n(x) = f(p).$$

On the other hand, solely by our assumptions that each f_n is continuous and that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f, we have that

$$\lim_{n\to\infty} \lim_{x\to p} f_n(x) = f(p).$$

Therefore, the continuity of f at p is equivalent to the following equality:

$$\lim_{x\to p} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to p} f_n(x).$$

In other words, the question of whether a pointwise limit of a sequence of continuous functions is continuous at p is the same as asking whether the order in which we take limits does not matter.

Similarly, questions about derivatives as well as integrals of pointwise limits are really questions about the order in which we take limits. Thus, we sometimes express examples and theorems in terms of the order in which limits are taken.

Example 23.1: We show that a pointwise limit of continuous functions may not be continuous.

For each $n \in \mathsf{N}$, define the continuous function $f_n : [0,1] \to \mathsf{R}^1$ by $f(x) = x^n$ for all $x \in [0,1]$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function f given by

$$f(x) = \begin{cases} 0 & \text{, if } x \neq 1 \\ 1 & \text{, if } x = 1 \end{cases}$$

and f is not continuous at $x = 1$.

Example 23.1 also shows that differentiability is not preserved by pointwise convergence since the limit function is not continuous (recall Theorem 6.14). Our next example is stronger.

Example 23.2: We give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of differentiable functions whose pointwise limit is continuous but not differentiable.

The sequence $\{f_n\}_{n=1}^{\infty}$ comes from knowing beforehand that we want the pointwise limit of the sequence to be the function $f(x) = |x|$; we can then obtain f_n by replacing f in the interval $[-\frac{1}{n}, \frac{1}{n}]$ with part of a parabola that has the correct values and the correct derivatives at the points $\pm\frac{1}{n}$. The formula is as follows: For each $n \in \mathsf{N}$, let

$$f_n(x) = \begin{cases} \frac{nx^2}{2} + \frac{1}{2n} & \text{, if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ |x| & \text{, if } |x| \geq \frac{1}{n}. \end{cases}$$

We show that the absolute value function f is the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$: If $x \neq 0$, then $f_n(x) = |x| = f(x)$ for all n sufficiently large, and $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \frac{1}{2n} = 0 = f(0)$.

It is easy to verify that each function f_n is differentiable at every point of R^1 (use Theorem 6.15 for the points $x = \pm\frac{1}{n}$).

Finally, f is continuous on R^1 (by Exercise 3.6) but f is not differentiable at $x = 0$ (by Exercise 6.17).

Example 23.3: We give another example concerning pointwise limits and derivatives; in contrast to Example 23.2, this time the pointwise limit is differentiable but its derivative is not the pointwise limit of the derivatives of the approximating sequence. Thus, the example shows that the order in which we take pointwise limits and derivatives matters when the functions and their pointwise limit are differentiable.

For each $n \in \mathsf{N}$, let

$$f_n(x) = x^n - x^{n+1} \quad \text{for all } x \in [0,1].$$

Note that $f_n(1) = 0$ for each n and that $\lim_{n\to\infty} x^n = 0$ for each x such that $0 \leq x < 1$ (by Lemma 15.3); thus, the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$ to the zero function f. Since $f_n'(x) = nx^{n-1} - (n+1)x^n$ for each n, it follows that the sequence $\{f_n'\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$ to the function g given by

$$g(x) = \begin{cases} 0 & , \text{if } 0 \leq x < 1 \\ -1 & , \text{if } x = 1. \end{cases}$$

Not only is $g \neq f'$, but, in fact, g is not the derivative of any function by Theorem 10.50.

Our final two examples concern pointwise limits and integrals.

Example 23.4: We show that a sequence of integrable functions may converge pointwise to a bounded function that is not integrable.

Let A be the set of all rational numbers in the interval $[0,1]$. Since A is countable, we can index the set A with the natural numbers, $A = \{r_i : i \in \mathsf{N}\}$. For each $n \in \mathsf{N}$, let $A_n = \{r_i : i \leq n\}$ and define the function $f_n : [0,1] \to \mathsf{R}^1$ by

$$f_n(x) = \begin{cases} 0 & , \text{if } x \in A_n \\ 1 & , \text{if } x \notin A_n. \end{cases}$$

It is easy to see that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$ to the function $f : [0,1] \to \mathsf{R}^1$ given by

$$f(x) = \begin{cases} 0 & , \text{if } x \in A \\ 1 & , \text{if } x \notin A. \end{cases}$$

Each function f_n is bounded and is continuous except at the finitely many points of A_n; hence, each function f_n is integrable over $[0,1]$ (by Exercise 12.34 or by Theorem 15.33; in fact, it is easy to see using upper and lower sums that $\int_0^1 f_n = 1$ for each n). However, the pointwise limit f is not integrable over $[0,1]$ since f is the function in Example 12.12 (or by Theorem 15.33).

Note that each function f_n in Example 23.4 is discontinuous at only finitely many points but that the pointwise limit of the functions is not continuous at any point. In contrast, we mention the following result: *The pointwise limit of a sequence of continuous functions defined on an interval must be continuous at some point of every open subinterval.* We do not prove this result since its proof involves an idea we have not discussed (namely, the Baire Category Theorem); a proof is in the book by Ralph P. Boas, Jr. entitled *A Primer of Real Functions*, published by the Mathematical Association of America (The Carus Mathematical Monographs, Number 13, 1960, pp. 99 - 102).

Example 23.5: In analogy to what we showed for derivatives in Example 23.3, we show that the order in which we take limits and integrals matters. Specifically, we construct a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions on $[0,1]$ that converges pointwise to the zero function such that $\int_0^1 f_n = 1$ for each n.

We describe the function f_n as follows: Let T_n be the isosceles triangle in the upper half plane whose base is the segment on the x-axis from $(\frac{1}{n+1}, 0)$ to $(\frac{1}{n}, 0)$ and whose area is equal to 1; the graph of f_n on the interval $[\frac{1}{n+1}, \frac{1}{n}]$ consists of the two nonhorizontal sides of the triangle T_n (a "tent"), and all other values of f_n are 0. In a formula, $f_n : [0,1] \to \mathsf{R}^1$ is defined as follows (the point $\frac{2n+1}{2n(n+1)}$ is the midpoint between $\frac{1}{n+1}$ and $\frac{1}{n}$):

$$f_n(x) = \begin{cases} 4n^2(n+1)^2x - 4n^2(n+1) & \text{, if } \frac{1}{n+1} \le x \le \frac{2n+1}{2n(n+1)} \\ -4n^2(n+1)^2x + 4n(n+1)^2 & \text{, if } \frac{2n+1}{2n(n+1)} \le x \le \frac{1}{n} \\ 0 & \text{, otherwise.} \end{cases}$$

For any $x > 0$, $f_n(x) = 0$ for all n such that $\frac{1}{n} < x$ (hence, for all but finitely many n); thus, since $f_n(0) = 0$ for all n, the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the zero function f. However, by the construction, $\int_0^1 f_n = 1$, the area of the triangle T_n. Therefore, $\lim_{n\to\infty} \int_0^1 f_n \ne \int_0^1 f$.

This completes our examples.

Exercise 23.6: Find the pointwise limit of the sequence of derivatives of the functions f_n in Example 23.1. Do the same for the functions f_n in Example 23.2.

Exercise 23.7: For each $n \in \mathsf{N}$, define $f_n : [0,\infty) \to \mathsf{R}^1$ by $f_n(x) = \left(\frac{x}{n}\right)^n$. Find the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$.

Exercise 23.8: For each $n \in \mathsf{N}$, let $f_n(x) = \frac{\sin(nx)}{n}$ for all $x \in \mathsf{R}^1$. Find the pointwise limit f of the sequence $\{f_n\}_{n=1}^{\infty}$ and determine whether the pointwise limit of the sequence $\{f_n'\}_{n=1}^{\infty}$ of derivatives is f'.

Exercise 23.9: For each $n \in \mathsf{N}$, define $f_n : [0,1] \to \mathsf{R}^1$

$$f_n(x) = nx(1-x^2)^n \quad \text{for all } x \in [0,1].$$

Find the pointwise limit f of the sequence $\{f_n\}_{n=1}^{\infty}$. Is $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$?

2. Uniform Convergence

To lead us to the definition of uniform convergence, let us state the definition of pointwise convergence more explicitely than we did in the previous section: A sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise on X to a function f if and only if for each $x \in X$ and each $\epsilon > 0$, there exists N depending on x as well as ϵ such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n \geq N.$$

Uniform convergence only requires N to depend on ϵ:

Definition: Let X be a set, and let $f_n : X \to \mathsf{R}^1$ ($n \in \mathsf{N}$) and $f : X \to \mathsf{R}^1$ be functions. We say that the sequence $\{f_n\}_{n=1}^{\infty}$ of functions *converges uniformly on X to f*, or that f is the *uniform limit of $\{f_n\}_{n=1}^{\infty}$ on X*, provided that for each $\epsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n \geq N \text{ and all } x \in X.$$

The following exercise will help familiarize you with the definition of uniform convergence:

Exercise 23.10: Let X be a nonempty set, and let $f_n : X \to \mathsf{R}^1$ ($n \in \mathsf{N}$) and $f : X \to \mathsf{R}^1$ be bounded functions. For each $n \in \mathsf{N}$, let

$$M_n = \sup_{x\in X} |f_n(x) - f(x)|.$$

Then $M_n < \infty$ for each $n \in \mathsf{N}$ and the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f if and only if $\lim_{n\to\infty} M_n = 0$.

Obviously, uniform convergence is stronger than pointwise convergence; more specifically, uniform convergence to a function f implies pointwise convergence to the same function f. We will see in subsequent sections that uniform convergence is strong enough to eliminate "defects" illustrated by examples in the preceding section.

We illustrate some ideas involved in proving uniform convergence as well as nonuniform convergence with the following example. (The reader will discover other techniques for proving uniform convergence and nonuniform convergence by working Exercises 23.14 and 23.15.)

Example 23.11: For each $n \in \mathsf{N}$, define $f_n : [0,\infty) \to \mathsf{R}^1$ by

$$f_n(x) = \frac{nx}{2n+x} \quad \text{for all } x \in [0,\infty).$$

We show that for any $b \geq 0$, the sequence $\{f_n|[0,b]\}_{n=1}^{\infty}$ converges uniformly on $[0,b]$ but that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0,\infty)$.

Since uniform convergence to a function f implies pointwise convergence to f, we first find the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$ For a given $x \in [0,\infty)$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{2n+x} = \lim_{n\to\infty} \frac{x}{2+\frac{x}{n}} \overset{19.2,\ 19.5}{=} \frac{x}{2}.$$

Hence, the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,\infty)$ to the function f given by $f(x) = \frac{x}{2}$ for all $x \in [0,\infty)$.

Now, fix $b \geq 0$. We show that the sequence $\{f_n|[0,b]\}_{n=1}^{\infty}$ converges uniformly on $[0,b]$ to $f|[0,b]$. The following calculations tell us how to proceed: For all $x \in [0,b]$,

$$(*)\ |f_n(x) - f(x)| = \left|\frac{nx}{2n+x} - \frac{x}{2}\right| = \left|\frac{-x^2}{4n+2x}\right| = \frac{x^2}{4n+2x} \leq \frac{x^2}{4n} \leq \frac{b^2}{4n}.$$

All we have left to do is to incorporate (*) into the definition of uniform convergence: Let $\epsilon > 0$. Then, by the Archimedean Property (Theorem 1.22), there is a natural number N such that $\frac{b^2}{4\epsilon} < N$. Hence, $\frac{b^2}{4n} < \epsilon$ for all $n \geq N$. Therefore,

$$|f_n(x) - f(x)| \overset{(*)}{\leq} \frac{b^2}{4n} < \epsilon \quad \text{for all } n \geq N \text{ and all } x \in [0,b].$$

This proves that the sequence $\{f_n|[0,b]\}_{n=1}^{\infty}$ converges uniformly on $[0,b]$ to f.

Finally, we show that $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0,\infty)$. Since a uniformly convergent sequence must converge uniformly to its pointwise limit, the only function to which the sequence $\{f_n\}_{n=1}^{\infty}$ could converge uniformly on $[0,\infty)$ is the function f. However, since

$$|f_n(n) - f(n)| = \left|\frac{n^2}{2n+n} - \frac{n}{2}\right| = \frac{n}{6} \quad \text{for all } n \in \mathsf{N},$$

the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0,\infty)$ to f (since, by the Archimedean Property, for any $\epsilon > 0$, there exists $j \in \mathsf{N}$ such that $6\epsilon < j$, so $|f_n(n) - f(n)| > \epsilon$ for all $n \geq j$).

The Cauchy criterion came up in connection with convergence of numerical sequences (section 3 of Chapter XX). We now use the criterion to characterize uniform convergence in a useful way.

Theorem 23.12 (Cauchy Criterion for Uniform Convergence): Let X be a set, and let $f_n : X \to \mathsf{R}^1$ be a function for each $n \in \mathsf{N}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to a function f if and only if for each $\epsilon > 0$, there exists N such that

$$|f_i(x) - f_j(x)| < \epsilon \quad \text{for all } i, j \geq N \text{ and all } x \in X.$$

Proof: Assume that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to a function f. Let $\epsilon > 0$. Then there exists N such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } n \geq N \text{ and all } x \in X.$$

Therefore, for all $i, j \geq N$ and all $x \in X$,

$$|f_i(x) - f_j(x)| \leq |f_i(x) - f(x)| + |f(x) - f_j(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, assume the condition in our theorem involving ϵ. Then, for each fixed point $x \in X$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence and, hence, converges by Theorem 20.12. For each $x \in X$, we denote the limit of the sequence $\{f_n(x)\}_{n=1}^{\infty}$ by $f(x)$. This defines a function $f : X \to \mathsf{R}^1$; in other words, f is the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$. In fact, as we now show, f is the uniform limit of the sequence $\{f_n\}_{n=1}^{\infty}$: Let $\epsilon > 0$; then, since we are assuming the condition in our theorem for ϵ, there exists N such that

$$(*)\ |f_i(x) - f_j(x)| < \frac{\epsilon}{2} \quad \text{for all } i, j \geq N \text{ and all } x \in X;$$

thus, for any given $n \geq N$, since $\lim_{j \to \infty} f_j(x) = f(x)$ for all $x \in X$,

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{j \to \infty} f_j(x)| \overset{(*)}{\leq} \frac{\epsilon}{2} < \epsilon \quad \text{for all } x \in X.$$

This proves that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f. ¥

Exercise 23.13: Determine which of the sequences $\{f_n\}_{n=1}^{\infty}$ in Examples 23.1 - 23.5 converge uniformly on their domains to their pointwise limits. A good explanation without all details suffices in most cases; however, give all details for the sequence in Example 23.2.

Exercise 23.14: For each $n \in \mathsf{N}$, define $f_n : [0, \infty) \to \mathsf{R}^1$ by

$$f_n(x) = \frac{x}{1 + nx^2} \quad \text{for all } x \in [0, \infty).$$

Determine whether the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[0, \infty)$.

Exercise 23.15: For each $n \in \mathsf{N}$, define $f_n : [0, \infty) \to \mathsf{R}^1$ by

$$f_n(x) = \frac{nx}{1 + n^2x^2} \quad \text{for all } x \in [0, \infty).$$

Determine whether the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[0, \infty)$.

Exercise 23.16: Let X be a set, and let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be sequences of functions from X to R^1 that converge uniformly on X to functions f and g, respectively. Then the sequences $\{f_n + g_n\}_{n=1}^{\infty}$ and $\{f_n - g_n\}_{n=1}^{\infty}$ converge uniformly on X to $f + g$ and $f - g$, respectively.

Does $\{f_n \cdot g_n\}_{n=1}^{\infty}$ converge uniformly on X to $f \cdot g$?

Exercise 23.17: Let X be a set, and let $f_n : X \to \mathsf{R}^1$ be a bounded function for each $n \in \mathsf{N}$. If the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to a function f, then f is bounded.

Exercise 23.18: This exercise shows that uniform convergence for bounded functions is really just plain old convergence in a certain metric space.

Let X be a nonempty set, and let $B(X) = \{f : X \to \mathsf{R}^1 : f \text{ is a bounded function}\}$. For each $f, g \in B(X)$, let

$$d(f,g) = \sup_{x\in X} |f(x) - g(x)|.$$

Prove that d is a distance function for $B(X)$. Then prove that convergence with respect to d is uniform convergence; that is, if $f_n \in B(X)$ for all $n \in \mathsf{N}$ and $f \in B(X)$, then the sequence $\{f_n\}_{n=1}^{\infty}$ converges with respect to d to f if and only if $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f (convergence in metric spaces is defined in section 4 of Chapter XX).

In addition, prove that the metric space $(B(X), d)$ is Cauchy complete (for which you will use Exercise 23.17).

Exercise 23.19: Let $X \subset \mathsf{R}^1$ let $p \in X$, and let $f_n : X \to \mathsf{R}^1$ and $f : X \to \mathsf{R}^1$ be functions. If the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of X such that $\lim_{n\to\infty} x_n = p$, then $\lim_{n\to\infty} f_n(x_n) = f(p)$. (See next exercise.)

Exercise 23.20: Show that uniform convergence can not be relaxed to pointwise convergence in Exercise 23.19 by considering the sequence $\{f_n\}_{n=1}^{\infty}$, where $f_n(x) = x^n$ for each $n \in \mathsf{N}$ and all $x \in [0,1]$.

Exercise 23.21: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from the closed and bounded interval $[a,b]$ to R^1 such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to a function f. Then, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \epsilon \quad \text{for all } n \in \mathsf{N} \text{ and all } x, y \in [a,b] \text{ such that } |x-y| < \delta.$$

(*Hint:* Make use of Theorem 12.31.)

3. Uniform Convergence and Continuity

In Theorem 23.22, we show that uniform convergence allows us to interchange the order in which we take limits. As a consequence, we obtain that the uniform limit of continuous functions is continuous (Corollary 23.23). We conclude by giving a sufficient condition for pointwise convergence to imply uniform convergence (Theorem 23.24).

Note that in Theorem 23.22 below, the point p is not necessarily a point of X; this feature of the theorem is important as we will see when we prove our result about derivatives in section 4.

Theorem 23.22: Let $X \subset \mathsf{R}^1$, let $p \in \mathsf{R}^1$ such that p is a limit point of X (we do not assume that $p \in X$), and let $f_n : X \to \mathsf{R}^1$ and $f : X \to \mathsf{R}^1$ be functions. Assume that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f and that

$$\lim_{x\to p} f_n(x) = q_n \quad \text{for each } n \in \mathsf{N}.$$

Then the limits $\lim_{x\to p} f(x)$ and $\lim_{n\to\infty} q_n$ exist and are equal; in other words,

$$\lim_{x\to p} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to p} f_n(x).$$

Proof: Let $\epsilon > 0$. By Theorem 23.12, there exists N such that

(1) $|f_i(x) - f_j(x)| < \frac{\epsilon}{3}$ for all $i, j \geq N$ and all $x \in X$.

Fix $i, j \geq N$. Then, since $\lim_{x \to p} f_n(x) = q_n$ for each $n \in \mathsf{N}$, the definition of limit gives us a $\delta_i > 0$ such that

$$|f_i(x) - q_i| < \frac{\epsilon}{3} \text{ for all } x \in X - \{p\} \text{ such that } |x - p| < \delta_i$$

and a $\delta_j > 0$ such that

$$|f_j(x) - q_j| < \frac{\epsilon}{3} \text{ for all } x \in X - \{p\} \text{ such that } |x - p| < \delta_j.$$

Hence, for all $x \in X - \{p\}$ such that $|x - p| < \min\{\delta_i, \delta_j\}$,

$$|q_i - q_j| \leq |q_i - f_i(x)| + |f_i(x) - f_j(x)| + |f_j(x) - q_j| < \frac{2\epsilon}{3} + |f_i(x) - f_j(x)|;$$

Thus, by (1), $|q_i - q_j| < \epsilon$. Hence, having proved this last inequality whenever $i, j \geq N$, we have proved that the sequence $\{q_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore, by Theorem 20.12, the sequence $\{q_n\}_{n=1}^{\infty}$ converges, say

(2) $\lim_{n \to \infty} q_n = q$.

We prove that $\lim_{x \to p} f(x) = q$, which will complete the proof of our theorem. The proof is guided by the following inequality:

(3) $|f(x) - q| \leq |f(x) - f_n(x)| + |f_n(x) - q_n| + |q_n - q|$
for all $x \in X$ and $n \in \mathsf{N}$.

We proceed as follows. By the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ to f and by (2), we can choose a single $k \in \mathsf{N}$ such that

(4) $|f(x) - f_k(x)| < \frac{\epsilon}{3}$ for all $x \in X$, and $|q_k - q| < \frac{\epsilon}{3}$.

Furthermore, for this choice of k, since $\lim_{x \to p} f_k(x) = q_k$, there exists $\delta > 0$ such that

(5) $|f_k(x) - q_k| < \frac{\epsilon}{3}$ for all $x \in X - \{p\}$ such that $|x - p| < \delta$.

Then, for all $x \in X - \{p\}$ such that $|x - p| < \delta$, we see from (3), (4) and (5) that

$$|f(x) - q| < \epsilon.$$

This proves that $\lim_{x \to p} f(x) = q$. ¥

Example 23.1 shows that pointwise convergence may not preserve continuity. On the other hand, uniform convergence does preserve continuity:

Corollary 23.23: The uniform limit of continuous functions is continuous. In fact, let $X \subset \mathsf{R}^1$, let $p \in X$, and let f_n be a function from X to R^1 such that f_n is continuous at p for each $n = 1, 2, ...$; if $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to a function f, then f is continuous at p.

Proof: The corollary follows immediately from Theorem 23.22 and Corollary 3.13. ¥

We can see from Example 23.5 that a sequence of continuous functions can converge pointwise to a continuous function and, yet, the convergence may not be uniform. Thus, under the assumption that f is the pointwise limit of $\{f_n\}_{n=1}^{\infty}$, the converse of Corollary 23.23 is false. However, with additional conditions, the converse of Corollary 23.23 is true. We present one such theorem (the assumptions in the theorem about X are each necessary – see Exercise 23.25).

Theorem 23.24: Let X be a closed and bounded subset of R^1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from X to R^1 such that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a continuous function f on X. If

$$f_n(x) \geq f_{n+1}(x) \quad \text{for all } n \in \mathsf{N} \text{ and } x \in X,$$

then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f.

Proof: For each $n \in \mathsf{N}$, let $g_n = f_n - f$. We prove that $\{g_n\}_{n=1}^{\infty}$ converges uniformly on X to the zero function g (from which our theorem follows, as we will see).

Note the following four facts (the first of which is by Corollary 4.4; the other three follow immediately from the assumptions in our theorem):

(1) g_n is continuous on X for each $n \in \mathsf{N}$;

(2) $g_n(x) \geq 0$ for all $n \in \mathsf{N}$ and $x \in X$;

(3) $\{g_n\}_{n=1}^{\infty}$ converges pointwise on X to the zero function g;

(4) $g_n(x) \geq g_{n+1}(x)$ for all $n \in \mathsf{N}$ and $x \in X$.

Let $\epsilon > 0$. For each $x \in X$, there is by (2) and (3) a natural number n_x such that $0 \leq g_{n_x}(x) < \epsilon$. Thus, since g_{n_x} is continuous at x (by (1)), we see from Exercise 3.14 that there is an open interval J_x such that $x \in J_x$ and

$$0 \leq g_{n_x}(y) < \epsilon \quad \text{for all } y \in J_x \cap X.$$

Therefore, by (4), we have that

(5) $0 \leq g_n(y) < \epsilon$ for all $n \geq n_x$ and all $y \in J_x \cap X$.

We have shown that for each $x \in X$, there are a natural number n_x and an open interval J_x such that $x \in J_x$ and such that (5) holds. The collection $\mathcal{C} = \{J_x : x \in X\}$ is then an open cover of X; hence, by Exercise 15.13, $\mathcal{C}$ has a finite subcover $\mathcal{S}$, say

$$\mathcal{S} = \{J_{x_1}, J_{x_2}, ..., J_{x_k}\},\ k < \infty.$$

In other words,

(6) $X \subset \cup_{i=1}^{k} J_{x_i}$

and (5) holds for each J_{x_i}, which we state explicitly:

(7) $0 \leq g_n(y) < \epsilon$ for all $n \geq n_{x_i}$ and all $y \in J_{x_i} \cap X$.

Now, let

$$N = \max\{n_{x_1}, n_{x_2}, ..., n_{x_k}\}.$$

Fix $n \geq N$ and $x \in X$. Then, by (6), $x \in J_{x_m}$ for some m; thus, since $n \geq n_{x_m}$, we have by (7) that $0 \leq g_n(x) < \epsilon$. This proves that

$$0 \leq g_n(x) < \epsilon \quad \text{for all } n \geq N \text{ and all } x \in X,$$

which proves that

(8) $\{g_n\}_{n=1}^{\infty}$ converges uniformly on X to the zero function g.

Finally, note that $g_n + f = f_n$ for each $n \in \mathsf{N}$ and $g + f = f$. Therefore, by (8) and Exercise 23.16, $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to f. ¥

Exercise 23.25: Show that Theorem 23.24 fails when X is bounded but not closed by considering the functions $f_n : (0, 1] \to \mathsf{R}^1$ defined by

$$f_n(x) = \frac{1}{nx+1} \quad \text{for each } n \in \mathsf{N} \text{ and all } x \in X.$$

Also, show that Theorem 23.24 fails when X is closed but not bounded.

Exercise 23.26: Prove that Theorem 23.24 remains true when we replace the assumption $f_n(x) \geq f_{n+1}(x)$ with the assumption $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathsf{N}$ and $x \in X$ (and retain the other assumptions).

4. Uniform Convergence and Differentiability

Although uniform convergence preserves continuity, uniform convergence does not preserve differentiability; this is shown by Example 23.2. Moreover, a sequence $\{f_n\}_{n=1}^{\infty}$ of differentiable functions may converge uniformly to a differentiable function and, yet, the sequence $\{f'_n\}_{n=1}^{\infty}$ of derivatives may not even converge pointwise to any function (see Exercise 23.8).

If we assume that both of the sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f'_n\}_{n=1}^{\infty}$ converge uniformly on an interval, then the limit of $\{f'_n\}_{n=1}^{\infty}$ is the derivative of the limit of $\{f_n\}_{n=1}^{\infty}$. The following theorem is more general:

Theorem 23.27: Let $f_n : [a, b] \to \mathsf{R}^1$ be differentiable for each $n \in \mathsf{N}$ such that for some point $p \in [a, b]$, the sequence $\{f_n(p)\}_{n=1}^{\infty}$ converges. If the sequence $\{f'_n\}_{n=1}^{\infty}$ of derivatives converges uniformly on $[a, b]$ to a function g, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to a function f and $f' = g$, thus

$$f'(x) = \lim_{n\to\infty} f'_n(x) \quad \text{for all } x \in [a, b].$$

Proof: Let $\epsilon > 0$. Since $\{f_n(p)\}_{n=1}^{\infty}$ converges, $\{f_n(p)\}_{n=1}^{\infty}$ is a Cauchy sequence (by Exercise 20.9); hence, there exists N_1 such that

(1) $|f_i(p) - f_j(p)| < \frac{\epsilon}{2}$ for all $i, j \geq N_1$.

Since $\{f'_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$, Theorem 23.12 shows there exists N_2 such that

(2) $\left|f'_i(x) - f'_j(x)\right| < \frac{\epsilon}{2(b-a)}$ for all $i, j \geq N_2$ and all $x \in [a, b]$.

Fix $i, j \in \mathsf{N}$ such that $i, j \geq N_2$, and fix $y, z \in [a, b]$ such that $y \neq z$. Note that $f_i - f_j$ is differentiable (by Theorem 7.3) and, hence, continuous (by Theorem 6.14); therefore, by the Mean Value Theorem (Theorem 10.2), there is a point c such that

$$\frac{f_i(z)-f_j(z)-(f_i(y)-f_j(y))}{z-y} = (f_i - f_j)'(c) \overset{7.3}{=} f_i'(c) - f_j'(c).$$

Hence,

$$|f_i(z) - f_j(z) - f_i(y) + f_j(y)| = \left|f_i'(c) - f_j'(c)\right| |z - y|.$$

Thus, by (2), we have proved the following:

(3) $|f_i(z) - f_j(z) - f_i(y) + f_j(y)| < \frac{\epsilon}{2(b-a)} |z - y|$ for all $i, j \geq N_2$ and all $y, z \in [a, b]$ such that $y \neq z$;

Since $\frac{\epsilon}{2(b-a)} |z - y| \leq \frac{\epsilon}{2}$ for all $y, z \in [a, b]$, we have by (3) that

(4) $|f_i(z) - f_j(z) - f_i(y) + f_j(y)| < \frac{\epsilon}{2}$ for all $i, j \geq N_2$ and all $y, z \in [a, b]$.

Next, let

$$N = \max\{N_1, N_2\}.$$

Note from the Triangle Inequality that for any $i, j \in \mathsf{N}$ and $x \in [a, b]$,

$$|f_i(x) - f_j(x)| \leq |f_i(x) - f_j(x) - f_i(p) + f_j(p)| + |f_i(p) - f_j(p)|;$$

hence, by (1) and (4),

$$|f_i(x) - f_j(x)| < \epsilon \quad \text{for all } i, j \geq N \text{ and all } x \in [a, b].$$

Therefore, by Theorem 23.12, we have proved that

(5) $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to a function f.

This proves the first part of our theorem.

It remains to prove that $f' = g$, where g is as in our theorem.

Fix a point $q \in [a, b]$. Let $X = [a, b] - \{q\}$. Define functions $\varphi_n, \varphi : X \to \mathsf{R}^1$ as follows:

$$\varphi_n(x) = \frac{f_n(x)-f_n(q)}{x-q} \text{ for each } n \in \mathsf{N}, \quad \varphi(x) = \frac{f(x)-f(q)}{x-q}.$$

By (5), $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[a, b]$ to f; Hence, it follows using Theorems 19.3 and 19.4 that

(6) $\{\varphi_n\}_{n=1}^{\infty}$ converges pointwise on X to $\varphi|X$.

Note from Exercise 6.10 that

(7) $\lim_{x \to q} \varphi_n(x) = f_n'(q)$ for each $n \in \mathsf{N}$.

We prove that the convergence in (6) is actually uniform. For all $i, j \in \mathsf{N}$ and all $x \in X$,

$$\left|\varphi_i(x)-\varphi_j(x)\right| = \left|\frac{f_i(x)-f_i(q)}{x-q} - \frac{f_j(x)-f_j(q)}{x-q}\right| = \frac{|f_i(x)-f_j(x)-f_i(q)+f_j(q)|}{|x-q|};$$

hence, by (3),

$$\left|\varphi_i(x)-\varphi_j(x)\right| < \frac{\epsilon}{2(b-a)} \quad \text{for all } i, j \geq N \text{ and all } x \in X.$$

Thus, by Theorem 23.12,

(8) $\{\varphi_n\}_{n=1}^{\infty}$ converges uniformly on X to $\varphi|X$.

With Theorem 23.22 in mind, we note the following: q is a limit point of X; $\{\varphi_n\}_{n=1}^{\infty}$ converges uniformly on X to $\varphi|X$ by (8); and, by (7),

$$\lim_{x\to q} \varphi_n(x) = f_n'(q) \text{ for each } n \in \mathsf{N}.$$

Hence, by Theorem 23.22, the limits $\lim_{x\to q} \varphi(x)$ and $\lim_{n\to\infty} f_n'(q)$ exist and

$$\lim_{x\to q} \varphi(x) = \lim_{n\to\infty} f_n'(q);$$

furthermore, by the definition of φ and Exercise 6.10,

$$\lim_{x\to q} \varphi(x) = f'(q).$$

Therefore, f is differentiable at q and $f'(q) = \lim_{n\to\infty} f_n'(q) = g(q)$ (by the definition of g in our theorem). ¥

Exercise 23.28: Let $C^1([a,b]) = \{f : [a,b] \to \mathsf{R}^1 :$ the first derivative of f exists and is continuous on $[a,b]\}$. For each $f, g \in C^1([a,b])$, let (see Example 20.19 (3))

$$d(f,g) = \sup_{x\in[a,b]} (|f(x)-g(x)| + |f'(x)-g'(x)|)$$

Then d is a distance function for $C^1([a,b])$ and the metric space $(C^1([a,b]), d)$ is Cauchy complete.

5. Uniform Convergence and Integrability

We know from Example 23.4 and Example 23.5 that integrals do not behave well with respect to pointwise limits. We show that uniform convergence preserves integrability and that the integral of the uniform limit is the limit of the integrals.

Theorem 23.29: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions that are integrable over $[a,b]$. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to a function f, then f is integrable over $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof: We use the notation U_P, L_P, $M_i(f)$, $m_i(f)$ and Δx_i, which we introduced at the beginning of section 2 of Chapter XII.

We first prove that f is integrable over $[a, b]$. Let $\epsilon > 0$. Then, by the definition of uniform convergence, there is a natural number k such that

(1) $|f_k(x) - f(x)| < \frac{\epsilon}{3(b-a)}$ for all $x \in [a, b]$.

Since f_k is integrable over $[a, b]$, we have by Theorem 12.15 that there is a partition $P = \{x_0, x_1, ..., x_m\}$ of $[a, b]$ such that

(2) $U_P(f_k) - L_P(f_k) < \frac{\epsilon}{3}$.

By (1), $f(x) < f_k(x) + \frac{\epsilon}{3(b-a)}$ for all $x \in [a, b]$; hence, considering $\frac{\epsilon}{3(b-a)}$ as the function on $[a, b]$ that is constantly $\frac{\epsilon}{3(b-a)}$, we have that

(3) $U_P(f) \leq U_P(f_k + \frac{\epsilon}{3(b-a)}) \overset{13.2}{\leq} U_P(f_k) + U_P(\frac{\epsilon}{3(b-a)}) = U_P(f_k) + \frac{\epsilon}{3}$.

By (1), $f(x) > f_k(x) - \frac{\epsilon}{3(b-a)}$ for all $x \in [a, b]$; hence, as in the proof of (3), we have that

(4) $L_P(f) \geq L_P(f_k - \frac{\epsilon}{3(b-a)}) \overset{13.2}{\geq} L_P(f_k) + L_P(\frac{-\epsilon}{3(b-a)}) = L_P(f_k) - \frac{\epsilon}{3}$.

Now, by (3) and (4), $U_P(f) - L_P(f) \leq U_P(f_k) - L_P(f_k) + \frac{2\epsilon}{3}$; hence, by (2),

$$U_P(f) - L_P(f) < \epsilon.$$

Therefore, by Theorem 12.15, we have proved that f is integrable over $[a, b]$.

Finally, we prove that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$. Let $\epsilon > 0$. Then, by the definition of uniform convergence, there is a natural number N such that

(5) $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $n \geq N$ and all $x \in [a, b]$.

Let $n \geq N$; then, since f and f_n are integrable over $[a, b]$,

$$\left|\int_a^b f - \int_a^b f_n\right| \overset{13.12}{=} \left|\int_a^b (f - f_n)\right| \overset{13.17}{\leq} \int_a^b |f - f_n| \overset{(5),\ 13.14}{\leq} \int_a^b \frac{\epsilon}{b-a} \overset{12.13}{=} \epsilon.$$

Therefore, we have proved that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. ¥

Exercise 23.30: For each $n \in \mathsf{N}$, define $f_n : [0, 1] \to \mathsf{R}^1$ by

$$f_n(x) = \begin{cases} n - n^2 x & \text{, if } 0 \leq x < \frac{1}{n} \\ 0 & \text{, if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

Find the pointwise limit f of $\{f_n\}_{n=1}^{\infty}$, and show that $\int_0^1 f \neq \lim_{n\to\infty} \int_a^b f_n$. (This example is simpler than Example 23.5, but it is not as strong as Example 23.5 since the function f here is not continuous.)

Exercise 23.31: Use Theorem 23.29 to prove the following weaker (but useful) version of Theorem 23.27: Assume that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[a,b]$ to f, that the sequence $\{f_n'\}_{n=1}^{\infty}$ of derivatives converges uniformly on $[a,b]$ to g, and that f_n' is continuous for each n. Then f is differentiable and $f' = g$.

(*Hint:* Use the Fundamental Theorem of Calculus (Theorem 14.2).)

Exercise 23.32: Use Exercise 23.17 and Theorem 15.33 to give another proof that the uniform limit of integrable functions on $[a,b]$ is integrable over $[a,b]$ (the first conclusion of Theorem 23.29).

Exercise 23.33: Give an example to show that Theorem 23.29 does not extend to integrals over unbounded intervals (as defined above Exercise 21.30); more precisely, give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of functions defined on $[1,\infty)$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1,\infty)$ to a function f, $\int_1^{\infty} f_n < \infty$ for each $n \in \mathsf{N}$, $\int_1^{\infty} f < \infty$ and, yet,

$$\int_1^{\infty} f \neq \lim_{n\to\infty} \int_1^{\infty} f_n.$$

Exercise 23.34: Let $X = \{f : [0,1] \to \mathsf{R}^1 : f \text{ is continuous}\}$ with the distance function d given by

$$d(f,g) = \int_0^1 |f-g| \quad \text{for all } f,g \in X$$

(see Exercise 20.23). True or False: If the sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions converges on $[0,1]$ with respect to d to a continuous function f, then $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$ to f. (Convergence with respect to a metric is the second definition below Exercise 20.20.)

Chapter XXIV: Power Series

In section 1 of Chapter XV we defined a numerical series to be its sequence of partial sums. We now define a *series* $\Sigma_{i=1}^{\infty} f_i$ *of functions* $f_i : X \to \mathsf{R}^1$ to be the sequence $\{\Sigma_{i=1}^{n} f_i\}_{n=1}^{\infty}$ of its partial sums.

Thus, our study of numerical series and of sequences of functions in the previous three chapters sets the stage for examining series of functions.

In section 1 we present three general results for series of functions that follow directly from previous results about sequences of functions. Throughout the rest of the chapter we focus on power series. Proofs of most of our results on power series use one or another of the theorems in section 1 or use consequences of the theorems in section 1.

A *power series* is a series that can be written in the form $a_0 + \Sigma_{i=1}^{\infty} a_i (x-c)^i$. It is convenient and customary to write the general form for a power series as $\Sigma_{i=0}^{\infty} a_i (x-c)^i$, where we understand that the first term $a_0(x-c)^0$ is a_0 when $x = c$ (i.e., $0^0 = 1$ here, Chapter XVIII notwithstanding!). Note that if $n \geq 0$ and $k_i \geq 0$ are integers and $k_i < k_{i+1}$ for each i, then the series $\Sigma_{i=n}^{\infty} a_i (x-c)^{k_i}$ is a power series (since it can be written in the required form by using zeros for missing terms).

We show that power series always converge on an interval (section 2). We then study properties of power series on their intevals of convergence – uniform convergence and continuity (section 3), differentiation and integration (section 4).

We continue our study of power series in the next chapter, where we investigate the Taylor series for a function.

1. General Results for Series of Functions

We know from sections 1 and 2 of Chapter XXIII what it means for a sequence of functions to converge pointwise or uniformly; thus, we know what it means for a series $\Sigma_{i=1}^{\infty} f_i$ of functions converges pointwise or uniformly. Nevertheless, we state the definition for completeness and to emphasize what the limit (or sum) of a convergent series of functions is understood to be.

Definition: Let X be a set, and let $f_i : X \to \mathsf{R}^1$ be a function for each $i \in \mathsf{N}$. We say that *the series* $\Sigma_{i=1}^{\infty} f_i$ *converges pointwise* (*uniformly*) *on* X provided that the sequence $\{\Sigma_{i=1}^{n} f_i\}_{n=1}^{\infty}$ of partial sums converges pointwise (uniformly, respectively) on X to the function f given by $f(x) = \Sigma_{i=1}^{\infty} f_i(x)$ for each $x \in X$. We call f the *limit* or *sum of the series* $\Sigma_{i=1}^{\infty} f_i$.

We prove three fundamental theorems about the sum of a series of functions. The first theorem concerns continuity of the sum, the second theorem concerns differentiability of the sum, and the third theorem concerns integrability of the sum. We apply the theorems to power series in sections 3 and 4.

Our first theorem is due to Karl Weierstrass (1815 - 1897):

Theorem 24.1 (Weierstrass M Test): Let X be a set, and let $\Sigma_{i=1}^{\infty} f_i$ be a series of functions $f_i : X \to \mathsf{R}^1$ such that for each i, $|f_i(x)| \leq M_i$ for all $x \in X$. If $\Sigma_{i=1}^{\infty} M_i < \infty$, then the series $\Sigma_{i=1}^{\infty} f_i$ converges uniformly on X. Hence, if each f_i is continuous on X, then $\Sigma_{i=1}^{\infty} f_i$ is continuous on X.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\Sigma_{i=1}^{\infty} f_i$.

Let $\epsilon > 0$. Then, since $\Sigma_{i=1}^{\infty} M_i < \infty$, we have by Exercise 21.6 that there exists N such that

$$\Sigma_{i=m}^{n} M_i < \epsilon \quad \text{for all } n \geq m \geq N.$$

Thus, for all $n \geq m \geq N$ and all $x \in X$, we have (assuming $m > 1$)

$$|s_n(x) - s_{m-1}(x)| = |\Sigma_{i=m}^{n} f_i(x)| \leq \Sigma_{i=m}^{n} |f_i(x)| \leq \Sigma_{i=m}^{n} M_i < \epsilon.$$

Hence, by applying Theorem 23.12 to the sequence $\{s_n\}_{n=1}^{\infty}$, we obtain that $\{s_n\}_{n=1}^{\infty}$ converges uniformly on X. This proves that the series $\Sigma_{i=1}^{\infty} f_i$ converges uniformly on X (recall definition at the beginning of the section).

To prove the last part of our theorem, assume that each f_i is continuous on X. Then, by Corollary 4.6, each partial sum $s_n = \Sigma_{i=1}^{n} f_i$ is continuous on X. Also, as we already proved, $\{s_n\}_{n=1}^{\infty}$ converges uniformly on X to $\Sigma_{i=1}^{\infty} f_i$. Therefore, $\Sigma_{i=1}^{\infty} f_i$ is continuous on X by Corollary 23.23. ¥

Our next theorem says that under certain conditions, we can find the derivative of the sum of a series of differentiable functions by differentiating the series term by term (just as we do to find the derivative of a polynomial).

Theorem 24.2: Let $\Sigma_{i=1}^{\infty} f_i$ be a series of differentiable functions $f_i : [a, b] \to \mathsf{R}^1$ such that the series $\Sigma_{i=1}^{\infty} f_i(p)$ converges for some point $p \in [a, b]$. If the series $\Sigma_{i=1}^{\infty} f_i'$ converges uniformly on $[a, b]$, then the series $\Sigma_{i=1}^{\infty} f_i$ converges uniformly on $[a, b]$ and

$$(\Sigma_{i=1}^{\infty} f_i)' = \Sigma_{i=1}^{\infty} f_i'.$$

Proof: For each $n \in \mathsf{N}$, let $s_n = \Sigma_{i=1}^{n} f_i$. By assumption, the sequence $\{s_n(p)\}_{n=1}^{\infty}$ converges; also, since

$$s_n' \overset{7.2}{=} \Sigma_{i=1}^{n} f_i' \quad \text{for each } n \in \mathsf{N},$$

the sequence $\{s_n'\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to the function $\Sigma_{i=1}^{\infty} f_i'$. Therefore, by Theorem 23.27, the sequence $\{s_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to the function $\Sigma_{i=1}^{\infty} f_i$ and

$$(\Sigma_{i=1}^{\infty} f_i)' = \Sigma_{i=1}^{\infty} f_i'. \text{ ¥}$$

Finally, we prove a theorem concerning when we can integrate the sum of a series of integrable functions term by term.

Theorem 24.3: Let $\Sigma_{i=1}^{\infty} f_i$ be a series of functions $f_i : [a,b] \to \mathsf{R}^1$ such that f_i is integrable over $[a,b]$ for each i. If the series $\Sigma_{i=1}^{\infty} f_i$ converges uniformly on $[a,b]$, then $\Sigma_{i=1}^{\infty} f_i$ is integrable over $[a,b]$ and

$$\int_a^b \Sigma_{i=1}^{\infty} f_i = \Sigma_{i=1}^{\infty} \int_a^b f_i.$$

Proof: For each $n \in \mathsf{N}$, let $s_n = \Sigma_{i=1}^{n} f_i$. By assumption, the sequence $\{s_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to the function $\Sigma_{i=1}^{\infty} f_i$. Therefore, by Theorem 23.29, the function $\Sigma_{i=1}^{\infty} f_i$ is integrable over $[a,b]$ and

$$\int_a^b \Sigma_{i=1}^{\infty} f_i \overset{23.29}{=} \lim_{n\to\infty} \int_a^b s_n = \lim_{n\to\infty} \int_a^b \Sigma_{i=1}^{n} f_i$$

$$\overset{13.4}{=} \lim_{n\to\infty} \Sigma_{i=1}^{n} \int_a^b f_i = \Sigma_{i=1}^{\infty} \int_a^b f_i. \text{ ¥}$$

Exercise 24.4: If X is a set and a series $\Sigma_{i=1}^{\infty} f_i$ of functions converges uniformly on X, then the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly on X to the zero function.

Exercise 24.5: The series $\Sigma_{i=1}^{\infty} \frac{\sin(ix)}{2^i}$ converges uniformly on R^1.

Exercise 24.6: The series $\Sigma_{i=1}^{\infty} \frac{(-1)^i}{i+x^2}$ converges uniformly on R^1.
(*Hint:* Use Exercise 21.13.)

Exercise 24.7: Consider a "double sequence" $\{a_{i,j}\}_{i,j=1}^{\infty}$. If

$$\Sigma_{j=1}^{\infty} |a_{i,j}| = b_i < \infty \text{ for each } i \text{ and } \Sigma_{i=1}^{\infty} b_i < \infty,$$

then the order of summations over i and j separately can be interchanged; that is,

$$\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} a_{i,j} = \Sigma_{j=1}^{\infty} \Sigma_{i=1}^{\infty} a_{i,j}.$$

(*Hint:* Let $X = \{\frac{1}{n} : n \in \mathsf{N}\} \cup \{0\}$. For each $i \in \mathsf{N}$, define $f_i : X \to \mathsf{R}^1$ by $f_i(\frac{1}{n}) = \Sigma_{j=1}^{n} a_{i,j}$ and $f_i(0) = \Sigma_{j=1}^{\infty} a_{i,j}$ (why is $f_i(0) < \infty$?). Let $g_n = \Sigma_{i=1}^{n} f_i$ and let $g = \Sigma_{i=1}^{\infty} f_i$. Proceed as indicated in the four steps below:

1. Use Theorem 24.1 to show that g is continuous at 0, hence $\lim_{n\to\infty} g(\frac{1}{n}) = \Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} a_{i,j}$.
2. $\lim_{k\to\infty} g_n(\frac{1}{k}) = \lim_{k\to\infty} \Sigma_{i=1}^{n} \Sigma_{j=1}^{k} a_{i,j} = \lim_{k\to\infty} \Sigma_{j=1}^{k} \Sigma_{i=1}^{n} a_{i,j}$.
3. Explain why Theorem 23.22 can be applied to show $\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} a_{i,j} = \lim_{n\to\infty} \Sigma_{j=1}^{\infty} \Sigma_{i=1}^{n} a_{i,j}$.
4. Complete the proof.)

2. The Interval of Convergence of Power Series

The definition of a power series is in the introduction to this chapter.

It is somewhat surprising that the set of points at which a power series converges is always an interval. We prove this in Theorem 24.10, which shows that a power series is actually absolutely convergent at each point of the interval except possibly at the end points. Then we give examples that illustrate all possible intervals that can be the set of points at which a power series converges.

You should keep in mind throughout our discussion of power series that absolute convergence implies convergence (by Theorem 22.1).

We use upper limits of sequences, which we discussed at the beginning of section 2 of Chapter XXII.

Lemma 24.8: If $\{s_i\}_{i=1}^{\infty}$ is a sequence of nonnegative numbers and $b > 0$, then

$$\overline{\lim}_{i\to\infty} bs_i = b\,\overline{\lim}_{i\to\infty} s_i$$

where, if $\overline{\lim}_{i\to\infty} s_i = \infty$, we interpret $b\infty$ to be ∞.

Proof: Assume first that $\overline{\lim}_{i\to\infty} s_i = \infty$. Then, by Exercise 22.6, there is a subsequence $\{s_{i_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} s_{i_j} = \infty$. Thus, since $b > 0$, $\lim_{j\to\infty} bs_{i_j} = \infty$. Hence, $\overline{\lim}_{i\to\infty} bs_i = \infty$. Therefore, $\overline{\lim}_{i\to\infty} bs_i = b\,\overline{\lim}_{i\to\infty} s_i$. This proves our lemma when $\overline{\lim}_{i\to\infty} s_i = \infty$.

Therefore, we assume from now on that

(1) $\overline{\lim}_{i\to\infty} s_i < \infty$.

Let

$$A = \{x \in \mathsf{R}^1 : \text{some subsequence of } \{s_i\}_{i=1}^{\infty} \text{ converges to } x\}$$

and let

$$B = \{x \in \mathsf{R}^1 : \text{some subsequence of } \{bs_i\}_{i=1}^{\infty} \text{ converges to } x\}.$$

By the definition of upper limit, we have that

(2) $\overline{\lim}_{i\to\infty} s_i = \sup A$ and $\overline{\lim}_{i\to\infty} bs_i = \sup B$.

Since $b \neq 0$, we see from Theorem 19.4 that $x \in A$ if and only if $bx \in B$. In other words,

(3) $bA = B$, where $bA = \{bx : x \in A\}$.

By (1) and Exercise 22.6, $\left(\overline{\lim}_{i\to\infty} s_i\right) \in A$. Thus, A is nonempty and, since $s_i \geq 0$ for all i, A is also bounded. Therefore, since $b > 0$, we can apply Lemma 13.7 to obtain that

(4) $\sup bA = b \sup A$.

Finally,

$$\overline{\lim}_{i\to\infty} bs_i \overset{(2)}{=} \sup B \overset{(2)}{=} \sup bA \overset{(4)}{=} b \sup A \overset{(2)}{=} b\,\overline{\lim}_{i\to\infty} s_i. \text{ ¥}$$

Exercise 24.9: Prove that Lemma 24.8 remains true for any sequence $\{s_i\}_{i=1}^{\infty}$. Is it necessary to assume in Lemma 24.8 that $b > 0$?

We now prove our main theorem.

Theorem 24.10: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series, and let

$$\sigma = \overline{\lim}_{i\to\infty} \sqrt[i]{|a_i|}.$$

(1) If $0 < \sigma < \infty$, then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is absolutely convergent (hence, convergent) at all x such that $|x-c| < \frac{1}{\sigma}$, and $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ diverges at all x such that $|x-c| > \frac{1}{\sigma}$.

(2) If $\sigma = 0$, then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is absolutely convergent (hence, convergent) at all real numbers x.

(3) If $\sigma = \infty$, then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ diverges at all real numbers $x \neq c$.

Proof: We will use the Root Test (Theorem 22.17). For this purpose, note that

$$\overline{\lim}_{i\to\infty} \sqrt[i]{|a_i(x-c)^i|} = \overline{\lim}_{i\to\infty} (\sqrt[i]{|a_i|}\,|x-c|) \overset{24.8}{=} \sigma\,|x-c|$$

($\sigma\,|x-c| = \infty$ when $\sigma = \infty$ by the convention adopted in Lemma 24.8). Therefore, by the Root Test, $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is absolutely convergent at all x such that $\sigma\,|x-c| < 1$ and $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ diverges at all x such that $\sigma\,|x-c| > 1$. Our theorem now follows easily. ¥

Theorem 24.10 shows that the set of all the points x at whch a power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges is an interval centered at c. This leads to the following terminology:

Definition: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series.

- The *interval of convergence of* $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is the set of all real numbers x for which $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges.

- The *radius of convergence of* $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is the radius of the interval of convergence of $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ (as measured from the center c of the interval of convergence). We will see in forthcoming examples that the radius of convergence may be 0, any positive number, or ∞.

- The *interior of the interval I of convergence of* $\Sigma_{i=0}^{\infty} a_i(x-c)^i$, denoted by $int(I)$, is the interval of convergence without its end points (more generally, the interior of a subset X of R^1 is the largest open set contained in X). Thus, if r is the radius of convergence of $\Sigma_{i=0}^{\infty} a_i(x-c)^i$, then the interior of the interval of convergence of $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is $(c-r, c+r)$ when $0 < r < \infty$, R^1 when $r = \infty$, and $\emptyset$ when $r = 0$.

Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a given power series for which we have found that σ in Theorem 24.10 satisfies the condition in part (1). Then we know that $(c - \frac{1}{\sigma}, c + \frac{1}{\sigma})$ is the interior of the interval I of convergence; however, we can not conclude that $I = (c - \frac{1}{\sigma}, c + \frac{1}{\sigma})$. The reason is that Theorem 24.10 says nothing about the convergence of the series at the end points $c \pm \frac{1}{\sigma}$. In fact, the series may or may not converge at one or another of the end points $c \pm \frac{1}{\sigma}$; thus, we must be careful to check the end points in order to find the entire interval of convergence (see examples below).

We remark that the definition of the interval of convergence of a power series is not specifically tied in a computational way to Theorem 24.10; indeed, it is sometimes preferable to find the interval of convergence using the Ratio Test (see Example 24.17) or other methods.

The statement of Theorem 24.10 is somewhat technical. It will be convenient to have the following less specific formulation of parts (1) and (2) of the theorem:

Corollary 24.11: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series with interval of convergence I and radius of convergence r, where $0 < r \leq \infty$. Then the series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is absolutely convergent at all x such that $|x-c| < r$ and, if $r < \infty$, the series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ diverges when $|x-c| > r$.

Examples of Intervals of Convergence

The six examples that follow illustrate all the types of intervals of convergence that a power series can have. Regarding the types of intervals not in the examples, see the comment following Example 24.17; also, see Exercise 24.18.

Example 24.12: We give an example of a power series whose interval of convergence is a bounded open interval.

Consider the power series $\Sigma_{i=1}^{\infty} x^i$. By Theorem 24.10 (note that $\sigma = 1$ here), $\Sigma_{i=1}^{\infty} x^i$ is absolutely convergent when $|x| < 1$ and diverges when $|x| > 1$. Clearly, $\Sigma_{i=1}^{\infty} x^i$ also diverges when $x = \pm 1$. Hence, the interval of convergence of $\Sigma_{i=0}^{\infty} x^i$ is the open interval $(-1, 1)$, and the radius of convergence is 1. We note that what we have shown also follows from Theorem 15.4. In fact, Theorem 15.4 shows that the series $\Sigma_{i=1}^{\infty} x^i$ represents the function f given by $f(x) = \frac{x}{1-x}$ on the interval $(-1, 1)$; thus, it is interesting to observe that even though f is naturally defined at $x = -1$, the series $\Sigma_{i=1}^{\infty} x^i$ does not represent f on $[-1, 1)$.

Our next two examples involve the two types of bounded half-open intervals.

Example 24.13: We give an example of a power series whose interval of convergence is of the form $[a, b)$.

Consider the power series $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$. By Example 19.8, $\lim_{i\to\infty} i^{\frac{1}{i}} = 1$; hence, by Theorem 19.5, $\lim_{i\to\infty} \sqrt[i]{\frac{1}{i}} = 1$. Thus, by Exercise 22.4,

$$\overline{\lim}_{i\to\infty} \sqrt[i]{\tfrac{1}{i}} = 1.$$

Therefore, by Theorem 24.10, $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ is absolutely convergent when $|x-1| < 1$ and diverges when $|x-1| > 1$. Hence, to determine the interval of convergence, we have to see what happens at the points $x = 0$ and $x = 2$. When $x = 0$, our series becomes the alternating harmonic series $\Sigma_{i=1}^{\infty}(-1)^i \frac{1}{i}$, which converges (by Theorem 21.10); on the other hand, when $x = 2$, our series becomes the harmonic series $\Sigma_{i=1}^{\infty} \frac{1}{i}$, which diverges (by Example 21.29). Therefore, the interval of convergence of $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ is the half-open interval $[0, 2)$.

Example 24.14: We give an example of a power series whose interval of convergence is of the form $(a, b]$.

This example is easily done by changing the series in example 24.13 to the series $\Sigma_{i=1}^{\infty} \frac{(-1)^i}{i}(x-1)^i$. Then, as in the previous example, the series converges on $(0, 2)$, but the convergence - divergence behavior at the end points is opposite to that in the preceding example. Therefore, the interval of convergence is $(0, 2]$.

Example 24.15: We give an example of a power series whose interval of convergence is a bounded closed interval with different end points.

Consider the series $\Sigma_{i=1}^{\infty} \frac{1}{i^2} x^i$. By Example 19.8, $\lim_{i\to\infty} i^{\frac{1}{i}} = 1$; hence, by Theorem 19.4, $\lim_{i\to\infty} i^{\frac{2}{i}} = 1$. Thus,

$$\lim_{i\to\infty} \sqrt[i]{\frac{1}{i^2}} = \lim_{i\to\infty} \frac{1}{i^{\frac{2}{i}}} \overset{19.5}{=} 1;$$

therefore, by Exercise 22.4,

$$\overline{\lim}_{i\to\infty} \sqrt[i]{\frac{1}{i^2}} = 1.$$

Hence, by Theorem 24.10, $\Sigma_{i=1}^{\infty} \frac{1}{i^2} x^i$ is absolutely convergent when $|x| < 1$ and diverges when $|x| > 1$. Furthermore, the series $\Sigma_{i=1}^{\infty} \frac{1}{i^2} x^i$ is absolutely convergent when $x = \pm 1$ since the series $\Sigma_{i=1}^{\infty} \frac{1}{i^2}$ converges by Example 21.29 and, hence, the series $\Sigma_{i=1}^{\infty} \left| \frac{1}{i^2}(-1)^i \right|$ converges. Therefore, the interval of convergence of $\Sigma_{i=1}^{\infty} \frac{1}{i^2} x^i$ is the closed interval $[-1, 1]$. Unlike the previous two examples, the interval of convergence for this example is the same as the interval on which the series is absolutely convergent.

Example 24.16: We give an example of a power series whose interval of convergence is a single point (i.e., the closed interval $[c, c]$).

The series $\Sigma_{i=1}^{\infty} i^i x^i$ only converges when $x = 0$ by part (3) of Theorem 24.10 since $\overline{\lim}_{i\to\infty} \sqrt[i]{i^i} = \overline{\lim}_{i\to\infty} i = \infty$. Therefore, the interval of convergence of $\Sigma_{i=1}^{\infty} i^i x^i$ is the closed interval $[0, 0]$ and the radius of convergence is 0.

Example 24.17: We give an example of a power series whose interval of convergence is $(-\infty, \infty)$.

Of course, there is the trivial example, namely, $\Sigma_{i=0}^{\infty} 0 x^i$. But, let us give a more meaningful and important example:

Consider the series $\Sigma_{i=0}^{\infty} \frac{1}{i!} x^i$. In order to apply Theorem 24.10, we would need to compute $\overline{\lim}_{i\to\infty} \sqrt[i]{\frac{1}{i!}}$. It is much easier to use the Ratio Test (Theorem 22.10): For any given x,

$$\overline{\lim}_{i\to\infty} \left| \frac{\frac{x^{i+1}}{(i+1)!}}{\frac{x^i}{i!}} \right| = \overline{\lim}_{i\to\infty} \left| \frac{x}{i+1} \right| = 0;$$

hence, by part (1) of the Ratio Test, the series $\Sigma_{i=0}^{\infty} \frac{1}{i!} x^i$ is absolutely convergent at any given x. Therefore, the interval of convergence of $\Sigma_{i=0}^{\infty} \frac{1}{i!} x^i$ is $(-\infty, \infty)$ and the radius of convergence is ∞. (You may suspect from Theorem 21.42 that $\Sigma_{i=0}^{\infty} \frac{1}{i!} x^i = e^x$ for all $x \in \mathsf{R}^1$; this is true, as we show in Example 24.37).

The only types of intervals we have not considered in the examples above are intervals of the forms (a, ∞), $(-\infty, a)$, $[a, \infty)$ and $(-\infty, a]$. None of these types of intervals can be intervals of convergence by Theorem 24.10. Also, note the following exercise:

Exercise 24.18: Our purpose in Examples 24.12 - 24.16 was to illustrate all types of bounded intervals of convergence that a power series can have. In the examples, the five types of bounded intervals had specific end points. Show that the specific nature of the end points did not matter; that is, for each of the five types of bounded intervals with arbitrary end points a and b, construct a power series whose interval of convergence is that interval.

Exercise 24.19: Find the interval I of convergence of the power series $\Sigma_{i=1}^{\infty} \frac{(-1)^i}{\sqrt[3]{i}} x^i$.

Exercise 24.20: Find the interval I of convergence of the power series $\Sigma_{i=1}^{\infty} \frac{1+(-1)^i}{3(i!)} x^i$.

Exercise 24.21: Find the interval I of convergence of the power series $\Sigma_{i=1}^{\infty} \sqrt{i}(5x+2)^i$.

3. Uniform Convergence of Power Series

A power series may not converge uniformly on the entire interior of its interval of convergence; examples illustrating this are in Exercises 24.24 and 24.25. However, we prove that a power series converges uniformly on any closed and bounded interval in the interior of its interval of convergence (Theorem 24.22). This theorem is of central importance to our development of the calculus of power series in the next section.

Even though uniform convergence may not be present, we prove that a power series is always continuous on its entire interval of convergence (Corollary 24.30). This result is a consequence of Abel's Theorem on uniform convergence (Theorem 24.29).

We conclude with an application of the continuity of power series to Cauchy products of numerical series (Exercise 24.33); we briefly discussed the application at the end of section 6 of Chapter XXII.

Theorem 24.22: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series with interval of convergence I. Then, for any $\alpha > 0$ such that $[c-\alpha, c+\alpha] \subset int(I)$, the series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on $[c-\alpha, c+\alpha]$.

Proof: Let $r \leq \infty$ denote the radius of convergence of $\Sigma_{i=0}^{\infty} a_i(x-c)^i$. Let $J = [c-\alpha, c+\alpha]$ and let $p = c+\alpha$. We make two observations in order to apply the Weierstrass M Test (Theorem 24.1), which will prove our theorem.

First, since $|p-c| = \alpha < r$, $\Sigma_{i=0}^{\infty} \left|a_i(p-c)^i\right|$ converges by Corollary 24.11; thus, since $p-c = \alpha$,

$$\Sigma_{i=0}^{\infty} \left|a_i \alpha^i\right| < \infty.$$

Second, for all $x \in J$ and all i,

$$\left|a_i(x-c)^i\right| = |a_i|\,|x-c|^i \leq |a_i|\,\alpha^i = \left|a_i\alpha^i\right|.$$

Therefore, we can now apply Theorem 24.1 to see that $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on J. ¥

We note the following corollary (which is a special case of our main result about continuity in Corollary 24.30).

Corollary 24.23: Assume that $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is a power series with interval of convergence I. Define $f : I \to \mathsf{R}^1$ by

$$f(x) = \Sigma_{i=0}^{\infty} a_i(x-c)^i \quad \text{for all } x \in I.$$

Then f is continuous on $int(I)$.

Proof: Let $p \in int(I)$. Then there is a closed interval $[c-\alpha, c+\alpha] \subset int(I)$ such that $p \in (c-\alpha, c+\alpha)$. Therefore, f is continuous at p by Theorem 24.22 and by Corollary 23.23 applied to the partial sums $\Sigma_{i=0}^{n} a_i(x-c)^i$ (which are continuous by Theorem 4.16). ¥

In the next section we strengthen Corollary 24.23 by proving that the function f is differentiable on $int(I)$ (Theorem 24.35).

The following three exercises below illustrate the necessity of considering closed and bounded subintervals of $int(I)$ in Theorem 24.22.

Exercise 24.24: We know from Example 24.13 that the interval of convergence of $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ is the half-open interval $[0,2)$. Thus, by Theorem 24.22, the series $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ is converges uniformly on any closed interval $[\epsilon, 2-\epsilon]$, $\epsilon > 0$. However, prove that the series $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ does not converge uniformly on $[0,2)$. (Compare with Exercise 24.27.)

Exercise 24.25: The interval of convergence of the power series $\Sigma_{i=0}^{\infty} \frac{1}{i!}x^i$ is R^1 by Example 24.17. Prove that $\Sigma_{i=0}^{\infty} \frac{1}{i!}x^i$ does not converge uniformly on R^1.

(*Hint:* Find a lower bound for $\Sigma_{i=0}^{\infty} \frac{1}{i!}x^i - \Sigma_{i=0}^{n} \frac{1}{i!}x^i$ in terms of n and x for all $x > 0$.)

Exercise 24.26: The interval of convergence of the power series $\Sigma_{i=1}^{\infty} x^i$ is the open interval $(-1,1)$ by Theorem 15.4. Prove that $\Sigma_{i=1}^{\infty} x^i$ does not converge uniformly on $(-1,1)$.

(*Hint:* From the proof of Theorem 15.4, $\Sigma_{i=1}^{n} x^i = \frac{x-x^{n+1}}{1-x}$ when $-1 < x < 1$.)

The power series in Exercise 24.24 is conditionally convergent at the end point of its interval I of convergence and not uniformly convergent on I. In contrast, the following exercise shows that absolute convergence at an end point implies uniform convergence on the entire interval of convergence.

Exercise 24.27: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series whose interval of convergence I includes an end point η. If $\Sigma_{i=0}^{\infty} a_i(\eta-c)^i$ is absolutely convergent, then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is uniformly convergent on I.

Exercise 24.28: Assume that the power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges on the open interval $I = (c-\alpha, c+\alpha)$, where $\alpha > 0$, and that $a_i \neq 0$ for at least one i. Then there exists $\epsilon > 0$ such that $\Sigma_{i=0}^{\infty} a_i(x-c)^i \neq 0$ for all $x \neq c$ such that $|x-c| < \epsilon$. (An interesting application is in Exercise 25.28.)

Abel's Theorem

We know from Corollary 24.23 that a power series is continuous on the interior of its interval of convergence. We show that a power series is continuous on its interval of convergence even when the interval of convergence includes an end point (Corollary 24.30). The following theorem, due to Niels Abel (1802-1829), shows even more:

Theorem 24.29 (Abel's Theorem): Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series with interval of convergence I and radius of convergence $r < \infty$. If $c+r \in I$ (i.e., $\Sigma_{i=0}^{\infty} a_i r^i$ converges), then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on $[c, c+r]$; if $c-r \in I$ (i.e., $\Sigma_{i=0}^{\infty} a_i(-r)^i$ converges), then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on $[c-r, c]$.

Proof: We only prove the first part of the theorem (the proof of the second part is similar).

Assume that $c+r \in I$; that is, $\Sigma_{i=0}^{\infty} a_i r^i$ converges. Let

$$A_j = \Sigma_{i=j}^{\infty} a_i r^i \quad \text{for all } j,$$

and note from Exercise 21.1 that $A_j < \infty$ and that $A_j - A_{j+1} = a_j r^j$ for all j. Thus, since we can assume that $r > 0$, we have that

(1) $a_j = \frac{A_j - A_{j+1}}{r^j}$ for all j.

Now, let $\epsilon > 0$. Then, since $\Sigma_{i=0}^{\infty} a_i r^i$ converges, there exists N such that (by Exercise 21.2)

(2) $|A_j| = \left|\Sigma_{i=j}^{\infty} a_i r^i\right| < \frac{\epsilon}{2}$ for all $j \geq N$.

Fix x such that $c \leq x < c+r$ and fix $n \geq N$. Then

$$\Sigma_{i=n}^{\infty} a_i(x-c)^i \overset{(1)}{=} \Sigma_{i=n}^{\infty} \frac{A_i - A_{i+1}}{r^i}(x-c)^i$$

$$= \Sigma_{i=n}^{\infty} \left(\frac{A_i}{r^i}(x-c)^i - \frac{A_{i+1}}{r^i}(x-c)^i\right)$$

$$= \frac{A_n}{r^n}(x-c)^n + \Sigma_{i=n}^{\infty} \left(-\frac{A_{i+1}}{r^i}(x-c)^i + \frac{A_{i+1}}{r^{i+1}}(x-c)^{i+1}\right)$$

$$= \frac{A_n}{r^n}(x-c)^n + \Sigma_{i=n}^{\infty} \frac{A_{i+1}}{r^{i+1}}[-r(x-c)^i + (x-c)^{i+1}]$$

$$= \frac{A_n}{r^n}(x-c)^n + (x-c)^n \Sigma_{i=n}^{\infty} \frac{A_{i+1}}{r^{i+1}}[-r(x-c)^{i-n} + (x-c)^{i-n+1}]$$

$$= \frac{A_n}{r^n}(x-c)^n + (x-c)^n \Sigma_{i=n}^{\infty} \frac{A_{i+1}}{r^{i+1}}(x-c)^{i-n}[-r + (x-c)]$$

$$= \frac{A_n}{r^n}(x-c)^n + (x-c)^n[-r + (x-c)] \Sigma_{i=n}^{\infty} \frac{A_{i+1}}{r^{i+1}}(x-c)^{i-n}.$$

Thus, since $x - c \geq 0$ and $-r + (x-c) < 0$ and since $r > 0$,

$$\left|\Sigma_{i=n}^{\infty} a_i(x-c)^i\right|$$
$$\leq \frac{|A_n|}{r^n}(x-c)^n + (x-c)^n[r-(x-c)]\Sigma_{i=n}^{\infty}|A_{i+1}|\frac{(x-c)^{i-n}}{r^{i+1}}.$$

Hence, by (2),

$$\left|\Sigma_{i=n}^{\infty} a_i(x-c)^i\right| < \frac{\epsilon}{2r^n}(x-c)^n + (x-c)^n[r-(x-c)]\frac{\epsilon}{2}\Sigma_{i=n}^{\infty}\frac{(x-c)^{i-n}}{r^{i+1}}.$$

Thus, since $\frac{(x-c)^{i-n}}{r^{i+1}} = \left(\frac{x-c}{r}\right)^{i-n}\frac{1}{r^{n+1}}$,

$$\left|\Sigma_{i=n}^{\infty} a_i(x-c)^i\right| < \frac{\epsilon}{2r^n}(x-c)^n + (x-c)^n[r-(x-c)]\frac{\epsilon}{2r^{n+1}}\Sigma_{i=n}^{\infty}\left(\frac{x-c}{r}\right)^{i-n};$$

also, since $0 \leq \frac{x-c}{r} < 1$,

$$\Sigma_{i=n}^{\infty}\left(\frac{x-c}{r}\right)^{i-n} \overset{15.4}{=} 1 + \frac{\frac{x-c}{r}}{1-\frac{x-c}{r}} = \frac{r}{r-(x-c)}.$$

Hence,

$$\left|\Sigma_{i=n}^{\infty} a_i(x-c)^i\right| < \frac{\epsilon}{2r^n}(x-c)^n + (x-c)^n[r-(x-c)]\frac{\epsilon}{2r^{n+1}}\frac{r}{r-(x-c)}$$
$$= \frac{\epsilon}{2r^n}(x-c)^n + (x-c)^n\frac{\epsilon}{2r^n} = \frac{\epsilon}{r^n}(x-c)^n = \epsilon\left(\frac{x-c}{r}\right)^n.$$

Thus, since $0 \leq \frac{x-c}{r} < 1$,

$$\left|\Sigma_{i=n}^{\infty} a_i(x-c)^i\right| < \epsilon.$$

We have proved that for any $\epsilon > 0$, there exists N (depending only on the fact that $\Sigma_{i=0}^{\infty} a_i r^i$ converges) such that

(3) $\left|\Sigma_{i=n}^{\infty} a_i(x-c)^i\right| < \epsilon$ for all $n \geq N$ and all $x \in [c, c+r)$.

The inequality in (3) also holds when $x = c+r$ by (2) with $j = n$. Therefore, we have proved that $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on $[c, c+r]$. ¥

Corollary 24.30: Every power series is continuous on its entire interval of convergence.

Proof: Let I denote the interval of convergence of a power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$. The case when I is an open interval (including when $I = \mathsf{R}^1$) is taken care of by Corollary 24.23. So, assume that I is not an open interval. Then I is a bounded half-open interval or a bounded closed interval (recall the comment above Exercise 24.18); hence, the corollary follows from Abel's Theorem and Corollary 23.23 (applied to the partial sums $\Sigma_{i=0}^{n} a_i(x-c)^i$ on one and/or the other intervals $[c, c+r]$ and $[c-r, c]$.) ¥

Exercise 24.31: If $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series with interval of convergence I, then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on any closed and bounded subinterval of I. Hence, if I itself is a closed interval, then $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on I.

We summarize some main aspects of what we have shown. Let I denote the interval of convergence of some power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$. The power series always converges uniformly on any closed and bounded subinterval of I (Exercise 24.31). Assume that I contains an end point η, say $\eta < c$. Then, whereas Abel's Theorem shows that mere conditional convergence at η is strong enough to guarantee uniform convergence on $[\eta, c]$, the example in Exercise 24.24 shows that conditional convergence at η is *not* strong enough to guarantee uniform convergence on all of I. On the other hand, the result in Exercise 24.27 shows that absolute convergence at η guarantees uniform convergence of the power series on all of I.

Exercise 24.32: Look carefully at the proof of Abel's Theorem; state the theorem in a slightly more general way so that the proof of the more general result is essentially the same as the proof of Abel's Theorem and so that the more general result immediately implies Theorem 24.22. (Thus, we could have written the section more concisely!)

We conclude with an application of Abel's Theorem to numerical series. The application concerns Cauchy products, which we introduced in section 6 of Chapter XXII.

First, recall that when the termwise product of two convergent numerical series converges, it is not necessarily the case that the termwise product converges to the product of the sums of the two series (a simple example is in the discussion following Exercise 21.3). This does not happen for Cauchy products:

Exercise 24.33: If $\Sigma_{i=0}^{\infty} a_i$, $\Sigma_{i=0}^{\infty} b_i$ and $(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i)$ each converge, then

$$(\Sigma_{i=0}^{\infty} a_i) \times (\Sigma_{i=0}^{\infty} b_i) = (\Sigma_{i=0}^{\infty} a_i)\,(\Sigma_{i=0}^{\infty} b_i).$$

(*Hint:* Show that the three power series $\Sigma_{i=0}^{\infty} a_i x^i$, $\Sigma_{i=0}^{\infty} b_i x^i$ and $\left(\Sigma_{i=0}^{\infty} a_i x^i\right) \times \left(\Sigma_{i=0}^{\infty} b_i x^i\right)$ converge on the interval $[0,1]$. Recall Theorem 22.39.)

4. Calculus of Power Series

We show that derivatives and integrals of power series can be computed the same way that they are computed for polynimials – term by term. We show that term-by-term differentiation is valid at each point of the interior of the interval of convergence (Theorem 24.35) and that term-by-term integration is valid over any closed subinterval of the interior of the interval of convergence (Theorem 24.41). We conclude with an application to numerical series; namely, our theorem on integration of power series and other results enable us to find the value of the series $\Sigma_{i=1}^{\infty}(-1)^i \frac{1}{i}$ (Exercise 24.43).

Differentiating Power Series

We require the following lemma for proving our theorem about differentiating power series.

Lemma 24.34: If the power series $\Sigma_{i=0}^{\infty} a_i (x-c)^i$ has radius of convergence $r \leq \infty$, then the power series $\Sigma_{i=1}^{\infty} i a_i (x-c)^{i-1}$ also has radius of convergence r.

Proof: By Theorem 24.10, it suffices to show that

$$(*)\ \overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} = \overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}.$$

Proof of (*)*:* By Exercise 22.7, $\overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}$ is the largest subsequential limit of $\{\sqrt[i]{|ia_i|}\}_{i=1}^{\infty}$. Also, note that since $\sqrt[i]{i} \geq 1$ for all i, $\sqrt[i]{|a_i|} \leq \sqrt[i]{|ia_i|}$ for all i. Thus, since $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|}$ is a subsequential limit of $\{\sqrt[i]{|a_i|}\}_{i=1}^{\infty}$ (by Exercise 22.6), we have that

(1) $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} \leq \overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}$.

Now, assume by way of contradiction that

$$\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} < \overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}.$$

Then there is a point p such that

(2) $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} < p < \overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}$.

By Exercise 22.6, there is a subsequence $\{i_j a_{i_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty}\sqrt[i_j]{\left|i_j a_{i_j}\right|} = \overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}.$$

Hence, by (2), there exists N such that

(3) $\sqrt[i_j]{\left|i_j a_{i_j}\right|} > p$ for all $j \geq N$.

Since $\lim_{i\to\infty}\sqrt[i]{i} = 1$ (by Example 19.8), $\lim_{j\to\infty}\sqrt[i_j]{i_j} = 1$ (by Exercise 20.3); hence, we see from (3) that there exists M such that

$$\sqrt[i_j]{\left|a_{i_j}\right|} > p \text{ for all } j \geq M.$$

It follows that $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} > p$; however, this contradicts (2). Hence, we have proved that

(4) $\overline{\lim}_{i\to\infty}\sqrt[i]{|a_i|} \geq \overline{\lim}_{i\to\infty}\sqrt[i]{|ia_i|}$.

By (1) and (4), we have proved (*). ¥

Theorem 24.35: Let $\Sigma_{i=0}^{\infty}a_i(x-c)^i$ be a power series with interval of convergence I. Define $f : I \to \mathsf{R}^1$ by

$$f(x) = \Sigma_{i=0}^{\infty}a_i(x-c)^i \text{ for all } x \in I.$$

Then f is differentiable at all points in $int(I)$ and

$$f'(x) = \Sigma_{i=1}^{\infty}ia_i(x-c)^{i-1} \text{ for all } x \in int(I).$$

Proof: Fix $\alpha > 0$ such that $[c-\alpha, c+\alpha] \subset int(I)$ (we assume that $int(I) \neq \emptyset$ since, otherwise, the theorem is true vacuously). We need only prove our theorem for all $x \in [c-\alpha, c+\alpha]$.

Let J be the interval of convergence of $\Sigma_{i=1}^{\infty}ia_i(x-c)^{i-1}$. By Lemma 24.34, $int(J) = int(I)$. Hence, $[c-\alpha, c+\alpha] \subset int(J)$. Thus, by Theorem 24.22, $\Sigma_{i=1}^{\infty}ia_i(x-c)^{i-1}$ converges uniformly on $[c-\alpha, c+\alpha]$. Therefore, by Theorem 24.2,

$$\left(\Sigma_{i=0}^{\infty} a_i (x-c)^i\right)' = \Sigma_{i=1}^{\infty} i a_i (x-c)^{i-1} \quad \text{for all } x \in [c-\alpha, c+\alpha]. \ ¥$$

One point about Theorem 24.35 should be clarified: Even though the radii of convergence of the power series for f and f' are the same (by the theorem), the intervals of convergence of f and f' may be different. This is because the interval of convergence of f' may omit an end point of the interval of convergence of f. We give an example in the following exercise:

Exercise 24.36: By Example 24.13, the interval of convergence of the power series $\Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ is $[0,2)$. Define f on $[0,2)$ by $f(x) = \Sigma_{i=1}^{\infty} \frac{1}{i}(x-1)^i$ for all $x \in [0,2)$. Find the interval of convergence of f'. Is f differentiable at $x = 0$?

At the end of Example 24.17, we said that $\Sigma_{i=0}^{\infty} \frac{1}{i!} x^i = e^x$ for all $x \in \mathsf{R}^1$; we are now in a position to prove this.

Example 24.37: As an application of Theorem 24.35, we show that the series $\Sigma_{i=0}^{\infty} \frac{1}{i!} x^i$ converges to e^x for all $x \in \mathsf{R}^1$.

Let $f(x) = \Sigma_{i=0}^{\infty} \frac{1}{i!} x^i$. We showed in Example 24.17 that f is defined on all of R^1. Hence, by Theorem 24.35, we see that for any given point $x \in \mathsf{R}^1$,

$$f'(x) = \Sigma_{i=1}^{\infty} \tfrac{1}{i!}(i) x^{i-1} = \Sigma_{i=1}^{\infty} \tfrac{1}{(i-1)!} x^{i-1} = \Sigma_{i=0}^{\infty} \tfrac{1}{i!} x^i = f(x).$$

Thus, by Exercise 16.25, there is a constant c such that $f(x) = ce^x$ for all $x \in \mathsf{R}^1$. Therefore, since $f(0) = 1$ (remember, $0^0 = 1$), $c = 1$ and $f(x) = e^x$.

We strengthen Theorem 24.35 in the corollary below. First, we introduce some terminology and notation.

Definition: Let f be a function that is differentiable at a point p. We define the n^{th} *derivative of f at p*, written $f^{(n)}(p)$, by induction as follows: $f^{(1)}(p) = f'(p)$ and, assuming that $f^{(k)}(p)$ exists and that $f^{(k)}$ is differentiable at p for a given $k \in \mathsf{N}$, $f^{(k+1)}(p) = (f^{(k)})'(p)$. (Do not confuse the notation $f^{(n)}$ with the notation f^n for the n^{th} iterate of f used in the latter part of section 3 of Chapter XIX.) In addition, it is convenient to adopt the notational convention that $f^{(0)} = f$.

We say that a function f is *infinitely differentiable at a point p* provided that $f^{(n)}(p)$ exists for all $n \in \mathsf{N}$; we say that f is *infinitely differentiable on a set X* provided that f is infinitely differentiable at each point of X.

Corollary 24.38: Let $\Sigma_{i=0}^{\infty} a_i (x-c)^i$ be a power series with interval of convergence I. Then the function $f : I \to \mathsf{R}^1$ defined by

$$f(x) = \Sigma_{i=0}^{\infty} a_i (x-c)^i \quad \text{for all } x \in I$$

is infinitely differentiable on $int(I)$ and, for any $n \in \mathsf{N}$,

$$f^{(n)}(x) = \Sigma_{i=n}^{\infty} (i)(i-1)\cdots(i-n+1) a_i (x-c)^{i-n} \quad \text{for all } x \in int(I).$$

Hence, $f^{(n)}(c) = n! a_n$ for each $n \geq 0$ (recall that $f^{(0)} = f$).

Proof: The first part of the corollary follows by a straightforward induction using Theorem 24.35. The formula $f^{(n)}(c) = n!a_n$ comes from setting $x = c$ in the formula for $f^{(n)}(x)$ in the first part and observing that all terms in the summation are zero except the first term, which is $n!a_n$ (recall our conventional agreement at the beginning of the chapter that $0^0 = 1$ for power series). ¥

The formula $f^{(n)}(c) = n!a_n$ in Corollary 24.38 is more interesting than it may appear to be at first glance: Assume that a function f can be represented by a power series on an open interval I centered at c, say $f(x) = \Sigma_{i=0}^{\infty} a_i(x-c)^i$ for all $x \in I$; then the formula $f^{(n)}(c) = n!a_n$ shows that the coefficients a_i of the power series representation of f are uniquely and completely determined by $f(c)$ and the values of the derivatives of f at the single point c.

A word of caution: A function f that is infinitely differentiable on an open interval I centered at c may not be represented by the power series $\Sigma_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$. An example is in the next chapter (Example 25.1); in fact, we devote the next chapter to investigating when such a representation is valid.

Exercise 24.39: Define $f : (-1, 1) \to \mathsf{R}^1$ by $f(x) = \frac{x}{1-x}$. Find a simple formula for $f^{(n)}(0)$ for each $n = 1, 2, \ldots$.

(*Hint:* Use Theorem 15.4.)

Integrating Power Series

We use the result in the following exercise in the proof of our theorem about integrability; the proof of the result is similar to the proof Lemma 24.34, and you are asked to fill in the details.

Exercise 24.40: If the power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ has radius of convergence $r \leq \infty$, then the power series $\Sigma_{i=0}^{\infty} \frac{a_i}{i+1}(x-c)^{i+1}$ also has radius of convergence r.

Theorem 24.41: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series with interval of convergence I. Then, for any $a, b \in int(I)$ such that $a \leq b$, $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is integrable over $[a, b]$ (with respect to x) and

$$\int_a^b \Sigma_{i=0}^{\infty} a_i(x-c)^i = \Sigma_{i=0}^{\infty} \int_a^b a_i(x-c)^i$$
$$= \Sigma_{i=0}^{\infty} \tfrac{a_i}{i+1}(b-c)^{i+1} - \Sigma_{i=0}^{\infty} \tfrac{a_i}{i+1}(a-c)^{i+1}.$$

Proof: Fix $a, b \in int(I)$ such that $a \leq b$. Then, by Exercise 24.31, $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges uniformly on $[a, b]$. Also, each term $a_i(x-c)^i$ of the series is integrable over $[a, b]$ by Theorem 12.33. Hence, by Theorem 24.3, $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is integrable over $[a, b]$ and

$$\int_a^b \Sigma_{i=0}^{\infty} a_i(x-c)^i = \Sigma_{i=0}^{\infty} \int_a^b a_i(x-c)^i.$$

Thus, since

$$\int_a^b a_i(x-c)^i \overset{14.2}{=} \tfrac{a_i}{i+1}(b-c)^{i+1} - \tfrac{a_i}{i+1}(a-c)^{i+1} \quad \text{for each } i,$$

we have that

$$\int_a^b \Sigma_{i=0}^{\infty} a_i(x-c)^i = \Sigma_{i=0}^{\infty}\left(\frac{a_i}{i+1}(b-c)^{i+1} - \frac{a_i}{i+1}(a-c)^{i+1}\right);$$

furthermore, since $a, b \in int(I)$, we see from Lemma 24.40 that each of the series $\Sigma_{i=0}^{\infty}\frac{a_i}{i+1}(b-c)^{i+1}$ and $\Sigma_{i=0}^{\infty}\frac{a_i}{i+1}(a-c)^{i+1}$ converges. Therefore, by Exercise 21.3,

$$\int_a^b \Sigma_{i=0}^{\infty} a_i(x-c)^i = \Sigma_{i=0}^{\infty}\frac{a_i}{i+1}(b-c)^{i+1} - \Sigma_{i=0}^{\infty}\frac{a_i}{i+1}(a-c)^{i+1}.$$ ¥

Exercise 24.42: True or false: If the interval of convergence of the power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ is the interval $[a, b]$, then

$$\int_a^b \Sigma_{i=0}^{\infty} a_i(x-c)^i = \Sigma_{i=0}^{\infty}\frac{a_i}{i+1}(b-c)^{i+1} - \Sigma_{i=0}^{\infty}\frac{a_i}{i+1}(a-c)^{i+1}.$$

Exercise 24.43: Prove that $\Sigma_{i=1}^{\infty}(-1)^i\frac{1}{i} = \ln(\frac{1}{2})$ as follows:

First, show that $\frac{1}{1-t} = \Sigma_{i=1}^{\infty} t^{i-1}$ when $-1 < t < 1$ by using Theorem 15.4; then use Theorem 14.2 and Theorem 24.41 to show that $\ln(\frac{1}{1-x}) = \Sigma_{i=1}^{\infty}\frac{x^i}{i}$ when $-1 < x < 1$; finally, apply Corollary 24.30. Other results are also needed; show all details.

Chapter XXV: Taylor Series

We say that a real-valued function f defined on an interval I centered at c is *represented on* I *by a power series at* c provided that

$$f(x) = \Sigma_{i=0}^{\infty} a_i (x-c)^i \quad \text{for all } x \in I.$$

We are concerned with the question of what functions f can be represented by a power series. In other words, we shift our emphasis from starting with a power series and obtaining a function on the interval of convergence, as in the preceding chapter, to starting with a function and trying to obtain a power series that represents the function.

Corollary 24.38 shows that our investigation is restricted to special types of functions and to power series that have a particular form: If a function f is represented on an interval I by a power series $\Sigma_{i=0}^{\infty} a_i (x-c)^i$, then f is infinitely differentiable on $int(I)$ and the coefficients a_i are unique, namely

$$f(x) = \Sigma_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!} (x-c)^i \quad \text{for all } x \in I.$$

For any function f that is infinitely differentiable at a point c, we call the power series $\Sigma_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!} (x-c)^i$ the *Taylor series of* f *at* c (whether or not the series represents f on an interval centered at c). Taylor series are named after Brook Taylor (1685-1731). The Taylor series $\Sigma_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!} x^i$ of f at 0 is called the *Maclaurin series of* f, named after Colin Maclaurin (1698-1746).

For example, the function e^x is represented on R^1 by its Maclaurin series (Example 24.38).

It is natural to conjecture that any function that is infinitely differentiable on an open interval I centered at c is represented on I by its Taylor series at c, or at least that it is represented by its Taylor series at c on some small subinterval of I centered at c. However, this is not the case as we will show in Example 25.1.

Thus, we are led to try to find conditions under which an infinitely differentiable function is represented by its Taylor series at c on an open interval centered at c. We give such conditions in terms of the remainder that we define in section 2. We write the remainder in two ways (sections 3 and 4); we give applications of the two forms of the remainder to the evaluation of integrals and to the approximation of values of functions and integrals (mostly in exercises).

In the final section, we study properties that functions which are represented by their Taylor series have in common with polynomials. Among other results, we prove the following theorem: *If a function is represented by its Taylor series at* c *on some open interval* I *centered at* c*, then the function is represented by its Taylor series at any point* b *of* I *on the largest subinterval of* I *centered at* b (Theorem 25.27). This result sounds straightforward, but it requires a clever proof with new ideas. In addition to the new ideas involved in the proof of the theorem, the proof of the theorem is partly based on the Lagrange form of the remainder which we introduced in section 3; thus, we change our emphasis from computational applications of the remainder in sections 3 and 4 to a theoretical application.

We remark that in addition to the inherent scholarly reasons for studying Taylor series, there is a practical reason as well. Many calculators and computers use Taylor polynomials to provide provide approximate evaluations for transcendental functions (the trigonometric functions, the logarithmic functions, the exponential functions, the hyperbolic functions, and so on). In order to evaluate the limitations of answers provided by technology, it is important to understand the method that is used by technology (see the comment in Exercise 25.17).

1. Infinitely Differentiable Functions Not Represented by Taylor Series

As mentioned in the introduction, the natural conjecture – an infinitely differentiable function on an open interval I centered at c is represented by its Taylor series at c – is false. The key to constructing a counterexample to the conjecture comes from Exercise 24.28: Suppose that we were able to find an infinitely differentiable function g on $[0, \infty)$ such that $g^{(n)}(0) = 0$ for all $n \geq 0$ and such that $g(x) > 0$ for all $x > 0$; then, extending g to h by letting $h(x) = g(-x)$ for all $x < 0$, we see that h is infinitely differentiable but that, by Exercise 24.28, h is not represented on any open interval $(-a, a)$ by its Maclaurin series. However, finding a function g with the required properties is not as easy as we may imagine. One such function is defined by $g(x) = e^{\frac{-1}{x^2}}$ when $x > 0$ and $g(0) = 0$ (it is not easy to show that $g^{(n)}(0) = 0$ for all $n \geq 0$; we do so in Example 25.1).

We give three examples of functions f that are infinitely differentiable on R^1 but that are not represented on R^1 by their Maclaurin series. In the first example, the Maclaurin series converges on R^1 but does not converge to f except at $x = 0$. In the second example, f is represented by its Maclaurin series on an open interval but not on all of R^1. In the third example, the Maclaurin series for f diverges at all $x \neq 0$ (this example is in Exercise 25.3).

Example 25.1: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0. \end{cases}$$

Then f is infinitely differentiable on R^1 but the Maclaurin series of f is constantly zero.

By various previous results, f is infinitely differentiable at each $x \neq 0$ (use Corollary 16.24, Theorem 7.13 and the Chain Rule (Theorem 7.23)). Thus, we must show that f is infinitely differentiable at $x = 0$ and that the Maclaurin series of f is constantly zero; specifically, we must show that $f^{(i)}(0)$ exists and is 0 for all integers $i \geq 0$. We first prove (1) and (2) below.

(1) For each $i \in \mathsf{N}$ and each $x \neq 0$, $f^{(i)}(x) = e^{\frac{-1}{x^2}} P_i(\frac{1}{x})$,
where $P_i(t)$ is a polynomial in t.

Proof of (1)*:* Fix $x \neq 0$. Using Corollary 16.24, the Chain Rule (Theorem 7.23) and Lemma 7.5, we see that

$$(*)\ f^{(1)}(x) = e^{\frac{-1}{x^2}} \left(\frac{2}{x^3}\right);$$

hence, letting $P_1(t) = 2t^3$ for all $t \in \mathsf{R}^1$, we have that

$$f^{(1)}(x) = e^{\frac{-1}{x^2}} P_1(\tfrac{1}{x}).$$

Now, assume inductively that for some $i \in \mathsf{N}$, $f^{(i)}(x) = e^{\frac{-1}{x^2}} P_i(\frac{1}{x})$, where $P_i(t)$ is a polynomial in t. Then

$$\begin{aligned} f^{(i+1)}(x) &\overset{7.4}{=} e^{\frac{-1}{x^2}} \left(P_i(\tfrac{1}{x})\right)' + P_i(\tfrac{1}{x}) \left(e^{\frac{-1}{x^2}}\right)' \\ &\overset{7.23}{=} e^{\frac{-1}{x^2}} P_i'(\tfrac{1}{x}) \left(\tfrac{-1}{x^2}\right) + P_i(\tfrac{1}{x}) \left(e^{\frac{-1}{x^2}}\right)' \\ &\overset{(*)}{=} e^{\frac{-1}{x^2}} P_i'(\tfrac{1}{x}) \left(\tfrac{-1}{x^2}\right) + P_i(\tfrac{1}{x}) e^{\frac{-1}{x^2}} \left(\tfrac{2}{x^3}\right) \\ &= e^{\frac{-1}{x^2}} \left[P_i'(\tfrac{1}{x}) \left(\tfrac{-1}{x^2}\right) + P_i(\tfrac{1}{x}) \left(\tfrac{2}{x^3}\right)\right]; \end{aligned}$$

hence, letting $P_{i+1}(t) = -t^2 P_i'(t) + 2t^3 P_i(t)$ for all $t \in \mathsf{R}^1$, we have that

$$f^{(i+1)}(x) = e^{\frac{-1}{x^2}} P_{i+1}(\tfrac{1}{x}).$$

Therefore, by the Induction Principle (Theorem 1.20), we have proved (1).

(2) For each integer $k \geq 0$, $\lim_{x \to 0} \frac{e^{\frac{-1}{x^2}}}{x^k} = 0$.

Proof of (2)*:* On making the substitution $u = \frac{1}{x}$, we see using Exercise 18.4 that (2) becomes

$$(\#)\ \lim_{u \to \infty} \frac{u^k}{e^{u^2}} = 0 \quad \text{for each integer } k \geq 0.$$

Clearly, (#) holds when $k = 0$. Assuming inductively that (#) holds for some integer $k \geq 0$, we see that (#) holds for $k+1$ as follows (the final equality uses Theorem 18.5 for the case of Theorem 4.9):

$$\lim_{u \to \infty} \frac{u^{k+1}}{e^{u^2}} \overset{18.9}{=} \lim_{u \to \infty} \frac{(k+1)u^k}{2ue^{u^2}} = \lim_{u \to \infty} \frac{k+1}{2u} \frac{u^k}{e^{u^2}} \overset{18.5}{=} 0.$$

Therefore, (2) holds by the Induction Principle (Theorem 1.20).

Finally, we show that $f^{(i)}(0) = 0$ for all integers $i \geq 0$. By definition, $f^{(0)}(0) = f(0) = 0$. Assume inductively that $f^{(i)}(0) = 0$ for some $i \geq 0$. Then, letting P_i be as in (1), say

$$P_i(t) = \Sigma_{n=0}^{m} a_n t^n \quad \text{for all } t \in \mathsf{R}^1,$$

we have

$$f^{(i+1)}(0) \overset{6.10}{=} \lim_{x\to 0} \frac{f^{(i)}(x)-f^{(i)}(0)}{x-0} \overset{(1)}{=} \lim_{x\to 0} \frac{e^{\frac{-1}{x^2}} P_i(\frac{1}{x})}{x}$$
$$= \lim_{x\to 0} \left(e^{\frac{-1}{x^2}} \Sigma_{n=0}^{m} a_n \left(\frac{1}{x}\right)^{n+1} \right) = \lim_{x\to 0} \left(\Sigma_{n=0}^{m} e^{\frac{-1}{x^2}} a_n \left(\frac{1}{x}\right)^{n+1} \right)$$

thus, since $\lim_{x\to 0} e^{\frac{-1}{x^2}} a_n \left(\frac{1}{x}\right)^{n+1} \overset{(2)}{=} 0$ for each n, we have by Theorem 4.5 that

$$f^{(n+1)}(0) = 0.$$

Therefore, by the Induction Principle, we have proved that $f^{(i)}(0) = 0$ for all i.

Example 25.2: We give an example of an infinitely differentiable function f on R^1 such that f is represented by its Maclaurin series on an open interval centered at 0 but such that f is not represented by its Maclaurin series on all of R^1.

Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by letting $f(x) = \frac{1}{1+x^2}$ for all $x \in \mathsf{R}^1$. It is easy to check that f is infinitely differentiable on R^1. We find the Maclaurin series for f as follows: By Theorem 15.4 (and Exercise 21.1)

$$\frac{1}{1-r} = \Sigma_{i=0}^{\infty} r^i \quad \text{when } -1 < r < 1;$$

hence, substituting $-x^2$ for r, we obtain that

$$(*)\ f(x) = \Sigma_{i=0}^{\infty}(-x^2)^i = \Sigma_{i=0}^{\infty}(-1)^i x^{2i} \quad \text{when } -1 < x < 1.$$

Thus, the series $\Sigma_{i=0}^{\infty}(-1)^i x^{2i}$ is the Maclaurin series for f for all $x \in \mathsf{R}^1$. Therefore, the Maclaurin series for f only converges when $-1 < x < 1$ (by (*) and Theorem 15.4).

Exercise 25.3: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \Sigma_{i=0}^{\infty} e^{-i} \cos(i^2 x) \quad \text{for all } x \in \mathsf{R}^1.$$

Prove that f is, indeed, defined at each $x \in \mathsf{R}^1$ (i.e., $\Sigma_{i=0}^{\infty} e^{-i}\cos(i^2 x) < \infty$ for all $x \in \mathsf{R}^1$) and that f is infinitely differentiable at each $x \in \mathsf{R}^1$ using Theorems 24.1 and 24.2. Prove that the Maclaurin series for f diverges for each $x \neq 0$ as follows:

Since the odd terms of the Maclaurin series are all 0, we need only consider the even terms. Fix $x \in \mathsf{R}^1$ such that $x \neq 0$. For any given $k \in \mathsf{N}$, verify that the absolute value of the $(2k)^{\text{th}}$ term of the Maclaurin series satisfies

$$\left| \frac{f^{(2k)}(0)}{(2k)!} x^{2k} \right| = \frac{x^{2k}}{(2k)!} \Sigma_{i=0}^{\infty} e^{-i} (i^2)^{2k} > \frac{x^{2k}}{(2k)!} e^{-2k} \left((2k)^2\right)^{2k}$$
$$= \frac{1}{(2k)!} \left(\frac{x(2k)^2}{e} \right)^{2k} > \left(\frac{x(2k)^2}{2ke} \right)^{2k} = \left(\frac{2kx}{e}\right)^{2k},$$

and apply the i^{th} Term Test (Theorem 21.8) to see that the Maclaurin series for f diverges at x.

2. Taylor Polynomials and the Remainder Term

We define and discuss Taylor polynomials and remainder terms in general.

Let I be an interval centered at c, and let $f : I \to \mathsf{R}^1$ be a function that is at least n times differentiable at c. The polynomial $\Sigma_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i$ is called the n^{th} *- degree Taylor polynomial of* f *at* c. For each $x \in I$, we let

$$R_n(x;c) = f(x) - \Sigma_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i;$$

$R_n(x;c)$ is called the n^{th} *remainder term at* x. (The notation and terminology do not reflect the dependence of the remainder term on f, but it will always be clear from context what function f we are considering.)

We briefly discuss the relationship between Taylor polynomials and linear approximation. Recall that one motivation for the derivative at the beginning of Chapter VI was the idea of a tangent line; this led to a discussion of linear approximation in section 3 of Chapter VI. The equation for linear approximation is the equation of the tangent line to the graph of a differentiable function f at the point $(c, f(c))$,

$$y = f(c) + f'(c)(x-c);$$

clearly, $f(c) + f'(c)(x-c)$ is the Taylor polynomial of f at c of degree 1. Thus, Taylor polynomials are a natural extension of linear approximation.

Moreover, if T is a Taylor polynomial of f at c of degree $n > 1$, then T almost always approximates f on a (small) open interval at c more accurately than linear approximation does. This is because the first n derivatives of T at c are the same as the first n derivatives of f at c − in particular, even when $n = 2$, T takes into account the shape (concavity) of the curve $y = f(x)$, whereas linear approximation does not take into account the shape of the curve. On the other hand, with respect to numerical computations, there is a disadvantage in using Taylor polynomials of high degree rather than linear approximation: it is obviously more complicated to compute approximate values for $f(x)$ using Taylor polynomials of degree greater than 1 than it is using linear approximations.

We complete this section with a simple theorem.

Theorem 25.4: Assume that f is defined on an interval I centered at c and that f is infinitely differentiable at c. Then f is represented on I by its Taylor series at c if and only if

$$\lim_{n\to\infty} R_n(x;c) = 0 \quad \text{for each } x \in I.$$

Proof: The theorem follows from the definition of convergence of a series since the n^{th} degree Taylor polynomial of f at c is simply the n^{th} partial sum of the Taylor series of f at c. ¥

Even though Theorem 25.4 is very obvious, the theorem points establishes a method for determining when a function is represented by its Taylor series. We will find two ways to evaluate remainder terms (Theorems 25.6 and 25.18), and we will apply the results in various ways.

Exercise 25.5: Prove that any polynomial is represented by its Taylor series on R^1 at any point c.

Find the Taylor series for the polynomial $f(x) = x^4 - 3x^2 + 2x + 5$ at $c = 2$.

3. The Lagrange Form of the Remainder

In Theorem 25.6, we give a formula for $R_n(x;c)$ in terms of the $(n+1)^{\text{st}}$ derivative of the function f. The formula is called the *Lagrange form of the remainder*. Joseph-Louis Lagrange (1736-1813) was the first person to state the Mean Value Theorem; in fact, when $n = 0$, the Lagrange form of the remainder reduces to

$$f(x) - f(c) = f'(z)(x - c),$$

which is the formula in the Mean-Value Theorem (Theorem 10.2). This suggests proving Theorem 25.6 by using the Mean Value Theorem, which we will do (we actually use Rolle's Theorem).

The Lagrange form of $R_n(x;c)$ is similar to the $(n+1)^{\text{st}}$ term of the Taylor series for f at c; however, note that $f^{(n+1)}$ is evaluated at a point $z \neq c$ in the theorem.

Theorem 25.6 (Lagrange Remainder Theorem): Let I be an open interval centered at c, and let $f : I \to \mathsf{R}^1$ be a function that is $n+1$ times differentiable on I for a given fixed n. Then, for each point $x \in I$, there is a point z strictly between x and c (unless $x = c$, in which case $z = c$) such that

$$R_n(x;c) = \tfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Proof: Fix $x \in I$. Since the theorem is obvious if $x = c$ (recall that $0^0 = 1$ for series), we assume that $x \neq c$. Then we can define a function $g : I \to \mathsf{R}^1$ as follows:

$$g(t) = f(x) - \Sigma_{i=0}^{n} \tfrac{f^{(i)}(t)}{i!}(x-t)^i - R_n(x;c)\tfrac{(x-t)^{n+1}}{(x-c)^{n+1}} \quad \text{for all } t \in I.$$

We see that $g(x) = 0$ and $g(c) = 0$ since

$$\begin{aligned} g(x) &= f(x) - \Sigma_{i=0}^{n} \tfrac{f^{(i)}(x)}{i!}(x-x)^i - R_n(x;c)\tfrac{(x-x)^{n+1}}{(x-c)^{n+1}} \\ &= f(x) - f^{(0)}(x) = 0 \end{aligned}$$

and

$$\begin{aligned} g(c) &= f(x) - \Sigma_{i=0}^{n} \tfrac{f^{(i)}(c)}{i!}(x-c)^i - R_n(x;c)\tfrac{(x-c)^{n+1}}{(x-c)^{n+1}} \\ &= R_n(x;c) - R_n(x;c) = 0. \end{aligned}$$

Thus, since g is differentiable on I (because f is $n+1$ times differentiable on I), Rolle's Theorem (Lemma 10.1) shows that there is a point z strictly between x

and c such that $g'(z) = 0$. On the other hand, differentiating g at z directly by using formulas in Chapter VII, we obtain (remember, x is fixed)

$$g'(z) = 0 - \left(\Sigma_{i=1}^{n} \frac{f^{(i)}(z)}{i!} i(x-z)^{i-1}(-1) + \Sigma_{i=0}^{n} \frac{f^{(i+1)}(z)}{i!}(x-z)^{i}\right)$$
$$- \frac{R_n(x,c)}{(x-c)^{n+1}}(n+1)(x-z)^{n}(-1)$$
$$= \Sigma_{i=1}^{n} \frac{f^{(i)}(z)}{(i-1)!}(x-z)^{i-1} - \Sigma_{i=0}^{n} \frac{f^{(i+1)}(z)}{i!}(x-z)^{i}$$
$$+(n+1)\frac{R_n(x,c)}{(x-c)^{n+1}}(x-z)^{n}$$
$$= -\frac{f^{(n+1)}(z)}{n!}(x-z)^{n} + (n+1)\frac{R_n(x,c)}{(x-c)^{n+1}}(x-z)^{n}.$$

Thus, since $g'(z) = 0$,

$$-\frac{f^{(n+1)}(z)}{n!}(x-z)^{n} + (n+1)\frac{R_n(x,c)}{(x-c)^{n+1}}(x-z)^{n} = 0,$$

which, since $z \neq x$, gives us that

$$R_n(x;c) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}. \text{ ¥}$$

Let X be a set, and let $\mathcal{F}$ be a family of real-valued functions defined on X. We say that the family $\mathcal{F}$ is *uniformly bounded on* X provided that there is a number M such that

$$|f(x)| < M \quad \text{for all } f \in \mathcal{C} \text{ and all } x \in X.$$

Corollary 25.7: Let I be an open interval centered at c, and let $f : I \to \mathsf{R}^1$ be a function that is infinitely differentiable on I. If the family of all derivatives of f is uniformly bounded on I, then f is represented on I by its Taylor series at c.

Proof: By the definition of uniformly bounded (above), there is a number $M > 0$ such that

(1) $\left|f^{(n+1)}(z)\right| < M$ for all integers $n \geq 0$ and all $z \in I$.

Now, fix a point $x \in I$. Let $\epsilon > 0$. Since the series $\Sigma_{n=0}^{\infty} \frac{(x-c)^{n+1}}{(n+1)!}$ converges by the Ratio Test (Theorem 22.10), we have that $\lim_{n\to\infty} \frac{(x-c)^{n+1}}{(n+1)!} = 0$ by the i^{th} Term Test (Theorem 21.8). Hence, there exists N such that

$$\left|\frac{(x-c)^{n+1}}{(n+1)!}\right| < \frac{\epsilon}{M} \text{ for all } n \geq N.$$

Thus, by (1) and Theorem 25.6,

$$|R_n(x;c)| < \epsilon \quad \text{for all } n \geq N.$$

We have proved that $\lim_{n\to\infty} R_n(x;c) = 0$ for each $x \in I$. Therefore, by Theorem 25.4, f is represented on I by its Taylor series at c. ¥

We discuss the function $f(x) = \sin(x)$ in the next two examples.

Example 25.8: The function $f(x) = \sin(x)$ is represented on R^1 by its Maclaurin series. This follows immediately from Corollary 25.7 since the derivatives of $\sin(x)$ are $\pm\cos(x)$ and $\pm\sin(x)$, which are uniformly bounded on R^1. In addition, we can easily compute the Maclaurin series for $\sin(x)$ as follows:

$$f^{(0)}(0) = 0,\ \ f^{(1)}(0) = 1,\ \ f^{(2)}(0) = 0,\ \ f^{(3)}(0) = -1,\ \ f^{(4)}(0) = 0$$

at which point the pattern begins to repeat since $f^{(4)}(x) = \sin(x)$. Therefore, for all $x \in \mathsf{R}^1$,

$$\sin(x) = 1x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots = \Sigma_{i=0}^{\infty} \tfrac{(-1)^i}{(2i+1)!}x^{2i+1}.$$

Example 25.9: The first three nonzero terms of the Maclaurin series for $\sin(x)$ in the preceding example approximate $\sin(1)$ as $\frac{101}{120}$. Thus, since the coefficient of x^6 in the Maclaurin series for $\sin(x)$ is 0, we can use $R_6(1;0)$ to estimate the error between $\sin(1)$ and $\frac{101}{120}$. Hence,

$$\left|\sin(1) - \tfrac{101}{120}\right| = |R_6(1;0)| \overset{25.6}{=} \left|\tfrac{-\cos(z)}{7!}(1-0)^7\right| \leq \tfrac{1}{7!} = \tfrac{1}{5040}.$$

Corollary 25.7 is somewhat limited since even very simple functions whose families of derivatives are not uniformly bounded are represented on an open interval by their Taylor series at the center point c; for example, the function $f(x) = x^2$ is obviously represented on R^1 by its Maclaurin series, but $f^{(1)}$ is not bounded. Nevertheless, we can easily extend Corollary 25.7 to a situation where the derivatives of f are need not be uniformly bounded on the entire given interval I:

Corollary 25.10: Let I be an open interval centered at c, and let $f : I \to \mathsf{R}^1$ be a function that is infinitely differentiable on I. Let $I_1 \subset I_2 \subset I_3 \subset \cdots$ be open intervals centered at c such that $I = \cup_{n=1}^{\infty} I_n$. If the family of all derivatives of f is uniformly bounded on I_n for each n, then f is represented on I by its Taylor series at c.

Proof: For each $n = 1, 2, ...,$ f is represented on I_n by its Taylor series at c by Corollary 25.7. ¥

The first three exercises below are concerned with Corollary 25.10.

Exercise 25.11: In Example 24.38, we showed that the function $f(x) = e^x$ is represented on R^1 by its Maclaurin series. Explain why this fact follows immediately from Corollary 25.10.

Exercise 25.12: Show how to work Exercise 25.11 using the Lagrange Remainder Theorem (Theorem 25.6). Compare your work with what you did to work Exercise 25.11.

Exercise 25.13: Can Corollary 25.10 be used to show that the function $f(x) = \frac{x}{1-x}$ is represented on $(-1, 1)$ by its Maclaurin series?

(*Hint:* See Exercise 24.39.)

Exercise 25.14: Let $f(x) = \cos(x)$ for all $x \in \mathsf{R}^1$. Prove that f is represented on R^1 by its Maclaurin series and find the Maclaurin series series for f.

Exercise 25.15: Note that we can not use the Fundamental Theorem of Calculus (Theorem 14.2) to evaluate $\int_0^1 \sin(x^2)$ since we do not know a specific function whose derivative is $\sin(x^2)$. Evaluate (write) $\int_0^1 \sin(x^2)$ in the form of a numerical series. Estimate the answer to within three decimal places of accuracy using Exercise 21.13.

Exercise 25.16: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{, if } x \neq 0 \\ 1 & \text{, if } x = 0. \end{cases}$$

This is the function f in Exercise 18.30, where we asked if you thought the function is infinitely differentiable at $x = 0$. We can now answer this question easily: Use Example 25.8 to find the Maclaurin series for f and to conclude that f is represented on R^1 by its Maclaurin series; therefore, f is infinitely differentiable by Theorem 24.35.

Find a concise formula for $f^{(n)}(0)$.

Exercise 25.17: Let f be the function in Exercise 25.16. Use the first three (nonzero) terms of the Maclaurin series for f to show that $\int_{-1}^1 f$ is approximately $\frac{1703}{900}$. (As in Exercise 25.15, we have no way to evaluate this integral exactly except as an infinite series. Most hand calculators will not give an answer for the integral since the integral is entered as $\int_{-1}^1 \frac{\sin(x)}{x}$ and there is a division by zero that the calculator can not handle.)

4. The Integral Form of the Remainder

In Theorem 25.18, we give a formula for $R_n(x;c)$ in terms of an integral of the $(n+1)^{\text{st}}$ derivative of the function f (and other terms). The formula is called the *integral form of the remainder*. We give an example of when the integral form of the remainder can be applied to show that a function can be represented by its Taylor series but the Lagrange form of the remainder can not be applied for that purpose (Example 25.19). Just as in the case of the Lagrange form of the remainder, the integral form for the remainder is useful for estimating the error between the value of a function and the values of its Taylor polynomials.

Theorem 25.18 (Integral Remainder Theorem): Let I be an open interval centered at c, and let $f : I \to \mathsf{R}^1$ be a function that is $n+1$ times differentiable on I. Then, for each point $x \in I$,

$$R_n(x;c) = \int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt.$$

Proof: Fix $x \in I$. Since the lemma is obvious when $x = c$, we assume that $x \neq c$. Define a function $g : I \to \mathsf{R}^1$ as follows (g is the first two expressions for the function g in the proof of Theorem 25.6):

$$g(t) = f(x) - \Sigma_{i=0}^{n} \frac{f^{(i)}(t)}{i!}(x-t)^i \quad \text{for all } t \in I.$$

We observe the following (for the first equality, recall from the introduction to Chapter XXIV that $0^0 = 1$ for power series):

(1) $g(x) = 0$ and $g(c) = R_n(x;c)$.

Since f is $n+1$ times differentiable on I, g is differentiable on I. Hence, as in the proof of Theorem 25.6,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Hence, by part (2) of the Fundamental Theorem of Calculus (Theorem 14.2),

$$\int_c^x -\frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt = g(x) - g(c).$$

Thus, by (1),

$$\int_c^x -\frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt = -R_n(x;c).$$

Therefore, since

$$\int_c^x -\frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt \overset{13.11}{=} -\int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt,$$

we have proved our theorem. ¥

We can apply the Integral Remainder Theorem in situations when we can not apply the Lagrange Remainder Theorem or Corollary 25.10. We illustrate with the following example:

Example 25.19: Let $f(x) = \ln(x)$ for all $x \in (0,2)$. We can not use the Lagrange Remainder Theorem or Corollary 25.10 to show that f is represented on $(0,2)$ by its Taylor series at 1. The reason is that

$$(*)\ f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n} \quad \text{for all } n \geq 1 \text{ and all } x \in (0,2)$$

and, thus, the family of derivatives of f is not uniformly bounded on any open interval at at 1.

However, we can use the Integral Remainder Theorem to show that f is represented on $(0,2)$ by its Taylor series at $1:$ Fix $x \in (0,2)$; then

$$R_n(x;c) \overset{25.18}{=} \int_1^x \frac{(-1)^{n+2}\frac{n!}{x^{n+1}}}{n!}(x-t)^n\, dt \overset{13.11}{=} \frac{(-1)^{n+2}}{x^{n+1}} \int_1^x (x-t)^n\, dt$$
$$\overset{14.2}{=} \frac{(-1)^{n+2}}{x^{n+1}} \left(\frac{(x-1)^{n+1}}{n+1}\right) = \frac{(-1)^{n+2}}{n+1} \left(\frac{x-1}{x}\right)^{n+1}$$

and, thus, $\lim_{n\to\infty} R_n(x;c) = 0$ by Lemma 15.3. Therefore, by Theorem 25.4, f is represented on $(0,2)$ by its Taylor series at 1. Thus, as is worth noting, we see from (*) that $\ln(x) = \Sigma_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}(x-1)^i$ for all $x \in (0,2)$.

Exercise 25.20: Evaluate $\int_0^1 (1-t)^4 e^t$ using Theorem 25.18.

5. Analytic Functions

A function that is represented by its Taylor series at c on some open interval centered at c is said to be *analytic at* c. A function that is analytic at each point of an open interval I is said to be *analytic on* I.

It is natural to think of analytic functions as being the infinite analogue of polynomials. This leads us to try to find properties that analytic functions have in common with polynomials. We obtain three basic properties of analytic functions that are analogues of properties of polynomials (Theorem 25.21, Theorem 25.27 and Exercise 25.28).

One elementary property of polynomials is that two polynomials with the same value and the same derivatives at a given point are actually the same polynomial. Analytic functions have the same property:

Theorem 25.21: Let I be an open interval, and let $f, g : I \to \mathsf{R}^1$ be functions that are analytic on I. If there is a point $p \in I$ such that

$$f^{(n)}(p) = g^{(n)}(p) \quad \text{for all } n \geq 0,$$

then $f = g$.

Proof: Let

$$A = \{x \in I : f^{(n)}(x) = g^{(n)}(x) \text{ for all } n \geq 0\}.$$

We prove that A is an open set. Let $a \in A$. Since f and g are analytic at a, there exists $\epsilon > 0$ such that

(1) $f(x) = \Sigma_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ and $g(x) = \Sigma_{i=0}^{\infty} \frac{g^{(i)}(a)}{i!}(x-a)^i$
for all $x \in J_\epsilon(a) = (a - \epsilon, a + \epsilon)$.

Since $a \in A$, $f^{(i)}(a) = g^{(i)}(a)$ for all $i \geq 0$. Hence, by (1), $g(x) = f(x)$ for all $x \in J_\epsilon(a)$. Thus, clearly, for each $i \geq 0$, $g^{(i)}(x) = f^{(i)}(x)$ for all $x \in J_\epsilon(a)$. Hence, $J_\epsilon(a) \subset A$. This proves that each point of A lies in an open interval contained in A. Therefore, A is an open set by Theorem 15.6.

Next, we prove that A contains all its limit points in I. Let $q \in I$ such that q is a limit point of A. Then, by Exercise 2.29, $q \sim A$. Hence, by Theorem 19.38, there is a sequence $\{a_i\}_{i=1}^{\infty}$ of points in A such that

(2) $\lim_{i\to\infty} a_i = q$.

Since $a_i \in A$ for each i, we have that

(3) $f^{(n)}(a_i) = g^{(n)}(a_i)$ for all n and for all i.

Since f and g are infinitely differentiable at q (by Corollary 24.38), $f^{(n)}$ and $g^{(n)}$ are continuous at q for each n (by Theorem 6.14). Thus, by (2) and Theorem 19.39,

$$\lim_{i\to\infty} f^{(n)}(a_i) = f^{(n)}(q) \quad \text{and} \quad \lim_{i\to\infty} g^{(n)}(a_i) = g^{(n)}(q) \quad \text{for each } n.$$

Therefore, by (3),

$$f^{(n)}(q) = g^{(n)}(q) \quad \text{for each } n.$$

This proves that $q \in A$. Therefore, we have proved that A contains all its limit points in I.

Next, we prove that $A = I$ (the argument we are about to give is not necessary for those readers who are familiar with connected sets). Suppose by way of contradiction that $A \neq I$. Then there is a point $t \in I$ such that $t \notin A$. We assume without loss of generality that $p < t$, and we let

$$B = \{x \in A : x < t\}.$$

Then, since $p \in A$, we have that $B \neq \emptyset$. Hence, $lub\, B = \ell$ exists. If $\ell \in A$, then, since A is an open set, it follows easily that ℓ is not the *least* upper bound of A, a contradiction. Hence, $\ell \notin A$. Then it follows easily using Theorem 19.39 that ℓ is a limit point of A; thus, since A contains all its limit points in I, we again have that $\ell \in A$, a contradiction. Therefore, we have proved that $A = I$.

Finally, since $A = I$, $f^{(n)}(x) = g^{(n)}(x)$ for all $n \geq 0$ and all $x \in I$. Therefore, since $f^{(0)} = f$ and $g^{(0)} = g$, we have in particular that $f = g$. ¥

Another property of polynomials is that they are represented on R^1 by their Taylor series at every point of R^1 (Exercise 25.5). We prove the analogue of this for analytic functions in Theorem 25.27.

The proof of Theorem 25.27 is somewhat intricate, and we break the proof down into several lemmas. We prove three lemmas about power series and a lemma about Taylor series. The first two lemmas together give a characterization for lower bounds of the radius of convergence of a power series (Corollary 25.24). We state the first two lemmas separately since the characterization we obtain is not specific enough in one direction to be able to be used directly in the proof of Theorem 25.27.

Lemma 25.22: Assume that the power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ has radius of convergence at least r, where $r > 0$. Then, for each number α such that $0 < \alpha < r$, there exists $\beta > 0$ such that

$$|a_i| \leq \tfrac{\beta}{\alpha^i} \quad \text{for all } i.$$

Proof: Fix α such that $0 < \alpha < r$. Let $I = (c - r, c + r)$, and note that I is contained in the interval of convergence of $\Sigma_{i=0}^{\infty} a_i(x-c)^i$. Thus, since $c + \alpha \in I$, the series $\Sigma_{i=0}^{\infty} a_i\alpha^i$ converges. Hence, by the i^{th} Term Test (Theorem 21.8), $\lim_{i\to\infty} a_i\alpha^i = 0$. Thus, there exists $\beta > 0$ such that $\beta \geq \left|a_i\alpha^i\right|$ for all i. Therefore, since $\alpha > 0$,

$$|a_i| \leq \tfrac{\beta}{\alpha^i} \quad \text{for all } i. \text{ ¥}$$

Lemma 25.23: Let $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ be a power series. Assume that $\alpha, \beta > 0$ satisfy

$$|a_i| \leq \frac{\beta}{\alpha^i} \quad \text{for all } i.$$

Then, for all x such that $|x-c| < \alpha$, the series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges (absolutely).

Proof: Fix x such that $|x-c| < \alpha$. Then

$$\Sigma_{i=0}^{\infty} \left|a_i(x-c)^i\right| = \Sigma_{i=0}^{\infty} |a_i| \, |x-c|^i \leq \Sigma_{i=0}^{\infty} \frac{\beta}{\alpha^i} \, |x-c|^i = \Sigma_{i=0}^{\infty} \beta \left|\frac{x-c}{\alpha}\right|^i.$$

Thus, since $\left|\frac{x-c}{\alpha}\right| < 1$, the series $\Sigma_{i=0}^{\infty} \beta \left|\frac{x-c}{\alpha}\right|^i$ converges by Theorem 15.4 (and Exercise 21.1). Hence, by the Comparison Test (Theorem 21.18), the series $\Sigma_{i=0}^{\infty} \left|a_i(x-c)^i\right|$ converges. Therefore, by Theorem 22.1, we have that the series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ converges. ¥

The preceding two lemmas give the following characterization:

Corollary 25.24: Let $r > 0$. A power series $\Sigma_{i=0}^{\infty} a_i(x-c)^i$ has radius of convergence at least r if and only if for each number α such that $0 < \alpha < r$, there exists $\beta > 0$ such that

$$|a_i| \leq \frac{\beta}{\alpha^i} \quad \text{for all } i.$$

Our next lemma concerns the sum of a specific power series. We use the lemma in the proof of Lemma 25.26.

Lemma 25.25: For any given integer $n \geq 0$ and any x such that $-1 < x < 1$,

$$\Sigma_{i=n}^{\infty} \frac{i!}{(i-n)!} x^{i-n} = \frac{n!}{(1-x)^{n+1}}.$$

Proof: Define $f : (-1,1) \to \mathsf{R}^1$ by

$$f(x) = \frac{1}{1-x} \quad \text{for all } x \in (-1,1).$$

An easy induction using the way we differentiate quotients (Theorem 7.6) shows that

(1) $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ for all $n \geq 0$ and all $x \in (-1,1)$.

By Theorem 15.4, $\Sigma_{i=1}^{\infty} x^i = \frac{x}{1-x}$ for all $x \in (-1,1)$; hence, by Exercise 21.1,

$$f(x) = \Sigma_{i=0}^{\infty} x^i \quad \text{for all } x \in (-1,1).$$

Thus, by Corollary 24.38,

(2) $f^{(n)}(x) = \Sigma_{i=n}^{\infty} \frac{i!}{(i-n)!} x^{i-n}$.

Our lemma follows from (1) and (2). ¥

It is convenient to have the following technical lemma since we use it twice in the proof of Theorem 25.27.

Lemma 25.26: Assume that a function f is represented by its Taylor series at c on an open interval $I = (c-\eta, c+\eta)$. Let $p \in I$. Then, for any number α such that $|p-c| < \alpha < \eta$, there is a number $\beta > 0$ such that

$$\left|f^{(n)}(p)\right| \leq \frac{n!\beta\alpha}{(\alpha-|p-c|)^{n+1}} \quad \text{for all } n \geq 0.$$

Proof: Let $a_i = \frac{f^{(i)}(c)}{i!}$ for each integer $i \geq 0$. Then, by assumption,

$$f(x) = \Sigma_{i=0}^{\infty} a_i (x-c)^i \quad \text{for all } x \in I.$$

Hence, by Corollary 24.38, we have that

(1) $f^{(n)}(p) = \Sigma_{i=n}^{\infty} \frac{i!}{(i-n)!} a_i (p-c)^{i-n}$ for all $n \geq 0$.

Fix α such that $|p-c| < \alpha < \eta$. Then, since the series $\Sigma_{i=0}^{\infty} a_i (x-c)^i$ converges for each $x \in I$, we have by Lemma 25.22 that there exists $\beta > 0$ such that

(2) $|a_i| \leq \frac{\beta}{\alpha^i}$ for all i.

Now, for any integer $n \geq 0$,

$$\left|f^{(n)}(p)\right| \overset{(1)}{\leq} \Sigma_{i=n}^{\infty} \frac{i!}{(i-n)!} |a_i| \, |p-c|^{i-n} \overset{(2)}{\leq} \Sigma_{i=n}^{\infty} \frac{i!}{(i-n)!} \frac{\beta}{\alpha^i} |p-c|^{i-n}$$

$$= \frac{\beta}{\alpha^n} \Sigma_{i=n}^{\infty} \frac{i!}{(i-n)!} \left(\frac{|p-c|}{\alpha}\right)^{i-n} \overset{25.25}{=} \frac{\beta}{\alpha^n} \frac{n!}{\left(1-\frac{|p-c|}{\alpha}\right)^{n+1}}$$

$$= \frac{n!\beta\alpha}{(\alpha-|p-c|)^{n+1}}. \text{ ¥}$$

We are now ready to prove our main theorem. The proof makes use of the Lagrange Remainder Theorem (Theorem 25.6).

Theorem 25.27: Assume that a function f is represented by its Taylor series at c on an open interval $I = (c-\eta, c+\eta)$ centered at c. Then f is analytic on I. Moreover, let $b \in I$, and let J denote the largest subinterval of I centered at b; in other words,

$$J = (b-\tau, b+\tau), \quad \text{where } \tau = \eta - |b-c|;$$

then $f|J$ is represented by its Taylor series at b on the entire open interval J.

Proof: Let b and J be as in the theorem.

By Lemma 25.26, we have that for any α such that $|b-c| < \alpha < \eta$, there exists $\beta > 0$ such that

$$\left|f^{(n)}(b)\right| \leq \frac{\beta\alpha(n!)}{(\alpha-|b-c|)^{n+1}} \quad \text{for all } n \geq 0;$$

thus, letting $\gamma = \frac{\beta\alpha}{\alpha-|b-c|}$ and $\lambda = \alpha - |b-c|$, we have

$$\left|\frac{f^{(n)}(b)}{n!}\right| \leq \frac{\gamma}{\lambda^n} \quad \text{for all } n \geq 0.$$

Hence, by Lemma 25.23, the series $\Sigma_{n=0}^{\infty}\frac{f^{(n)}(b)}{n!}(x-b)^n$ converges for all x such that $|x-b|<\lambda$. Furthermore, we could have chosen α as close to η as we wished, which means that we can assume that λ is as close to $\eta-|b-c|=\tau$ as we wish; therefore, since $J=(b-\tau,b+\tau)$, we have shown that

(1) $\Sigma_{n=0}^{\infty}\frac{f^{(n)}(b)}{n!}(x-b)^n$ converges for all $x\in J$.

We define a function $g:J\to \mathsf{R}^1$ using (1) by

$$g(x)=\Sigma_{i=0}^{\infty}\frac{f^{(i)}(b)}{i!}(x-b)^i \quad \text{for all } x\in J.$$

We will show that $g(x)=f(x)$ for all $x\in J$, which will complete the proof of our theorem.

We first prove that $g=f$ in an open subinterval J_ϵ of J about b. Let

$$J_\epsilon=(b-\epsilon,b+\epsilon), \quad \text{where } \epsilon=\tfrac{1}{4}(\eta-|b-c|).$$

Fix a point $q\in J_\epsilon$. Since f is represented by its Taylor series at c on $I=(c-\eta,c+\eta)$, f is infinitely differentiable on J_ϵ (by Corollary 24.38). Hence, we can apply the Lagrange Remainder Theorem (Theorem 25.6) to see that for each $n\geq 0$, there is a point z_n strictly between q and b (unless $q=b$, in which case $z_n=b$) such that

$$f(x)-\Sigma_{i=0}^{n}\frac{f^{(i)}(b)}{i!}(q-b)^i=\frac{f^{(n+1)}(z_n)}{(n+1)!}(q-b)^{n+1}.$$

Therefore, to show that $g(q)=f(q)$, it suffices to show that

$$(\#)\ \lim_{n\to\infty}\frac{f^{(n+1)}(z_n)}{(n+1)!}(q-b)^{n+1}=0.$$

Proof of (#): We assume throughout the proof of (#) that $b>c$ (the proof when $b<c$ is analogous, and the theorem we are proving is obvious when $b=c$).

We first consider the case when $q<b$. Then $q<z_n<b$ for all n. Fix α such that $|b-c|<\alpha<\eta$. Then, since $|z_n-c|<\alpha<\eta$, we see form Lemma 25.26 that there exists $\beta>0$ such that

$$\left|f^{(n+1)}(z_n)\right|\leq\frac{(n+1)!\beta\alpha}{(\alpha-|z_n-c|)^{n+2}}.$$

Thus, since $\alpha-|b-c|<\alpha-|z_n-c|$ for all n, we have that

$$\left|f^{(n+1)}(z_n)\right|<\frac{(n+1)!\beta\alpha}{(\alpha-|b-c|)^{n+2}}.$$

Hence, we have that

$$\text{(i)}\ \left|\frac{f^{(n+1)}(z_n)}{(n+1)!}(q-b)^{n+1}\right|<\frac{\beta\alpha}{\alpha-|b-c|}\left(\frac{|q-b|}{\alpha-|b-c|}\right)^{n+1}.$$

Now, note that $|b-c|+\epsilon<\eta$; thus, we can assume that $\alpha>|b-c|+\epsilon$. Then $\frac{|q-b|}{\alpha-|b-c|}<1$. Therefore, it follows from (i) and Lemma 15.3 that

$$\lim_{n\to\infty}\left|\frac{f^{(n+1)}(z_n)}{(n+1)!}(q-b)^{n+1}\right| = 0.$$

This proves (#) when $q < b$.

To finish the proof of (#), assume that $q > b$. Then $b < z_n < q$ for all n. Fix α such that $|q-c| < \alpha < \eta$. Then, since $|z_n - c| < \alpha < \eta$, we see form Lemma 25.26 that there exists $\beta > 0$ such that

$$\left|f^{(n+1)}(z_n)\right| \leq \frac{(n+1)!\beta\alpha}{(\alpha-|z_n-c|)^{n+2}}.$$

Thus, since $\alpha - |q-c| < \alpha - |z_n - c|$ for all n, we have that

$$\left|f^{(n+1)}(z_n)\right| < \frac{(n+1)!\beta\alpha}{(\alpha-|q-c|)^{n+2}}.$$

Hence, we have that

$$\text{(ii)}\ \left|\frac{f^{(n+1)}(z_n)}{(n+1)!}(q-b)^{n+1}\right| < \frac{\beta\alpha}{\alpha-|q-c|}\left(\frac{|q-b|}{\alpha-|q-c|}\right)^{n+1}.$$

Note that

$$|b-c| + 2\epsilon = |b-c| + \tfrac{1}{2}(\eta - |b-c|) = \tfrac{1}{2}(\eta + |b-c|) < \eta;$$

Hence, we can assume that $\alpha > |b-c| + 2\epsilon$. Then $\frac{|q-b|}{\alpha-|q-c|} < 1$. Therefore, we see from (ii) and Lemma 15.3 that

$$\lim_{n\to\infty}\left|\frac{f^{(n+1)}(z_n)}{(n+1)!}(q-b)^{n+1}\right| = 0.$$

This completes the proof of (#).

Since we proved (#) for any point $q \in J_\epsilon$, we have proved that $g|J_\epsilon = f|J_\epsilon$. Hence, by the way we defined g, we have proved that f is analytic at b. Therefore, since b was any point of I, we have proved that f is analytic on I. This proves the first part of the theorem.

We prove the last part of the theorem using the first part and Theorem 25.21. Since $g = f$ on J_ϵ, $g^{(n)}(b) = f^{(n)}(b)$ for all $n \geq 0$. Thus, the definition of g may be written as

$$g(x) = \Sigma_{i=0}^{\infty}\frac{g^{(i)}(b)}{i!}(x-b)^i \quad \text{for all } x \in J.$$

In other words, g is represented by its Taylor series at b on J. Hence, by the first part of the theorem, g is analytic on J. Also, by the first part of the theorem, f is analytic on J. Thus, since $g^{(n)}(b) = f^{(n)}(b)$ for all $n \geq 0$, we have by Theorem 25.21 that $g = f$ on J. Therefore, in view of the way we originally defined g, we have that $f|J$ is represented by its Taylor series at b on the entire open interval J. ¥

Recall that a nonconstant polynomial has only finitely many roots (Exercise 9.21). This is false for analytic maps: the sine function is analytic (Example

25.8) but $\sin(n\pi) = 0$ for all $n = \pm1, \pm2, ...$ (also, see Exercise 25.29). However, nonconstant analytic functions have the next best property – their roots are isolated – which you are asked to prove:

Exercise 25.28: If f is a nonconstant analytic function on an open interval I, then $\{x \in I : f(x) = 0\}$ does not have a limit point in I. Thus, f has only finitely many roots in any closed and bounded subinterval of I.

(*Hint:* Make use of Exercise 24.28.)

Exercise 25.29: In relation to the preceding exercise, give an example of an analytic function on a bounded open interval I that has infinitely many roots in I.

Exercise 25.30: If f is analytic on an open interval I, then there is a largest open interval J containing I for which there is an analytic function g on J such that $g|I = f$. (The function g is called the *analytic continuation of* f.)

(*Hint:* By Theorem 25.21, if φ and ψ are analytic functions on an open interval $K \supset I$ such that $\varphi|I = f$ and $\psi|I = f$, then $\varphi = \psi$.)

Exercise 25.31: Let f and g be analytic at a point c. Quote results in previous chapters that show that $f+g$, $f-g$ and the product $f \cdot g$ is analytic at c. (The result for $\frac{f}{g}$, $g(c) \neq 0$, is also true, but not by merely quoting previous results.)

Final Comments

We conclude with two suggestions for the further study of analytic functions.

First, we have only covered very few of the many important basic results about analytic functions of a real variable. A treatment of such results is in, for example, the book by Steven G. Krantz and Harold R. Parks entitled *A Primer of Real Analytic Functions*, published by Birkhäuser (Boston, Basel, Berlin), 1992. This book also contains important results about analytic functions of several variables.

Second, analytic functions of a complex variable is a fascinating and profound subject with numerous applications to other areas of mathematics and to the physical sciences. The complex plane is a natural setting for analytic functions since, for example, the "pathology" exhibited by the examples in section 1 does not exist for functions of a complex variable – namely, if a function f on the complex plane C is differentiable (meaning that for each point $z_0 \in \mathsf{C}$, $\lim_{z\to(0,0)} \frac{f(z)-f(z_0)}{z-z_0}$ exists), then f is represented locally by a power series. (For this reason, differentiable functions of a complex variable are called *analytic* rather than differentiable often before the result we just stated is proved.) You are well prepared to begin your study of complex function theory.