neighborhood of 0 in $[-\frac{1}{4}, 1)$, but $(-\frac{1}{4}, \frac{1}{2})$ is not an ϵ-neighborhood of 0 in $[-\frac{1}{4}, 1)$; for $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\}$, $\{0, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, ..., \frac{1}{n}, ...\}$ is a neighborhood of 0 in X (the $\frac{1}{9}$-neighborhood of 0 in X), but $\{0, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, ..., \frac{1}{9+2n}, ...\}$ is not a neighborhood of 0 in X.

Exercise 9.2: Let $X \subset \mathsf{R}^1$, and let $p \in X$. The intersection of finitely many neighborhoods of p in X is a neighborhood of p in X.

Exercise 9.3: If I is an open interval and $p \in I$, then any neighborhood U of p in I contains an open interval J such that $p \in J$; hence, J is an open neighborhood of p in I and both J and U are open neighborhoods of p in R^1.

Exercise 9.4: Let $X \subset \mathsf{R}^1$, and let $p \in X$. When is $\{p\}$ a neighborhood of p in X?

2. Local and Global Maxima and Minima

We localize the notions of maximum value and minimum value of a function (defined in section 2 of Chapter V). In order to avoid ambiguity, from now on we call the maximum value and the minimum value of a function the global maximum value and the global minimum value of the function.

Definition: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$.

- f has a *local maximum at* p provided that there is a neighborhood U of p in X such that $f(p) \geq f(x)$ for all $x \in U$.
- f has a *local minimum at* p provided that there is a neighborhood U of p in X such that $f(p) \leq f(x)$ for all $x \in U$.
- f has a *global* (or *absolute*) *maximum at* p provided that $f(p) \geq f(x)$ for all $x \in X$, in which case we call $f(p)$ the *global maximum value of* f.
- f has a *global* (or *absolute*) *minimum at* p provided that $f(p) \leq f(x)$ for all $x \in X$, in which case we call $f(p)$ the *global minimum value of* f.
- Local maxima and local minima are called *local extrema*; global maxima and global minima are called *global* (or *absolute*) *extrema*.

We give an example to illustrate the concepts we just introduced.

Example 9.5: Define f on $[0, 3]$ as follows:

$$f(x) = \begin{cases} 3x & \text{, if } 0 \leq x \leq 1 \\ -x + 4 & \text{, if } 1 \leq x \leq 2 \\ 2x - 2 & \text{, if } 2 \leq x \leq 3. \end{cases}$$

Then f has local minima at $x = 0$ and 2, local maxima at $x = 1$ and 3, and global extrema at $x = 0$ and 3.

Next, we give an example for which we have more questions than answers. Our purpose is to motivate the value of the theorem we are about to prove; we return to the example after we prove the theorem.

Example 9.6: Define $f : [0,4] \to \mathsf{R}^1$ by $f(x) = x(x-2)(x-4)$. Note that $f(x) = 0$ when $x = 0, 2$ and 4; also, from the signs of the terms, we see that $f(x) > 0$ when $0 < x < 2$ and that $f(x) < 0$ when $2 < x < 4$. It now follows from the Maximum-Minimum Theorem (Theorem 5.13) that f has a global maximum value at some point of $[0,2]$ and a global minimum value at some point of $[2,4]$. Also, f has a local minimum at $x = 0$ and a local maximum at $x = 4$; obviously, f does not have global extrema at $x = 0, 4$. The questions are: At what points are the global extrema attained? What are the values of the global extrema? Are there any local extrema occurring at points in the open interval $(0,4)$ that are not global extrema and, if so, at what points do they occur? We answer the questions in Example 9.10.

The theorem below gives an important relation between local extrema and derivatives. The relation is only true in the direction stated; for example, $f(x) = x^3$ (all $x \in \mathsf{R}^1$) has derivative zero at $p = 0$ and yet has no local extrema.

Theorem 9.7: Let I be an open interval, and let $f : I \to \mathsf{R}^1$ be a function that is differentiable at a point $p \in I$. If f has a local extremum at p, then $f'(p) = 0$.

Proof: Assume that f has a local maximum at p. Then there is a neighborhood U of p in I such that $f(p) \geq f(x)$ for all $x \in U$. By Exercise 9.3, there is an open interval $(s,t) \subset U$ such that $p \in (s,t)$. Note that $(s,t) \subset I$ (since $U \subset I$); hence, we have that

(1) $f(p) \geq f(x)$ for all $x \in (s,t)$.

The proof now proceeds by analyzing the sign of $\frac{f(x)-f(p)}{x-p}$ when $s < x < p$ and when $p < x < t$: By (1), $f(x) - f(p) \leq 0$ for all $x \in (s,t)$; hence,

(2) $\frac{f(x)-f(p)}{x-p} \geq 0$ if $s < x < p$ and $\frac{f(x)-f(p)}{x-p} \leq 0$ if $p < x < t$.

Now, since f is differentiable at p, we know from Theorem 6.15 that

$$f'_-(p) = f'(p) = f'_+(p).$$

Furthermore, by (2), $f'_-(p) \geq 0$ and $f'_+(p) \leq 0$. Therefore, $f'(p) = 0$.

This proves the theorem when f has a local maximum at p. We leave the case when f has a local minimum at p as an exercise (below). ¥

Exercise 9.8: Prove Theorem 9.7 for the case when f has a local minimum at p.

Exercise 9.9: Give an example to show that the analogue of Theorem 9.7 for closed intervals is false.

Theorem 9.7 gives us a way to determine where a differentiable function on an interval may have local or global extrema. Sometimes, we can even determine the types of extrema the function has. We illustrate with two examples. The first example is a continuation of Example 9.6.

Example 9.10: Define $f : [0,4] \to \mathsf{R}^1$ by $f(x) = x(x-2)(x-4)$, the function in Example 9.6. We apply Theorem 9.7 to answer the questions we asked in Example 9.6.

To find the derivative of f, it is convenient to write f in unfactored form (to avoid using the product theorem for derivatives twice): $f(x) = x^3 - 6x^2 + 8x$ and thus, by Theorem 7.12,

$$f'(x) = 3x^2 - 12x + 8.$$

Hence, $f'(x) = 0$ when $x = 2 \pm \frac{2}{3}\sqrt{3}$. Moreover, we knew in Example 9.6 that f has its global maximum value at some point of $(0,2)$ and its global minimum value at some point of $(2,4)$. Therefore, by Theorem 9.7, we can now conclude that f must have its global maximum at $x = 2 - \frac{2}{3}\sqrt{3}$, its global minimum at $x = 2 + \frac{2}{3}\sqrt{3}$, and the global extrema do not occur at any other point; the global maximum value is $f(2 - \frac{2}{3}\sqrt{3}) = \frac{16}{9}\sqrt{3}$ and the global minimum value is $f(2 - \frac{2}{3}\sqrt{3}) = -\frac{16}{9}\sqrt{3}$. Finally, by Theorem 9.7, f has no local extrema at points in the open interval $(0,4)$ that are not global extrema. Thus, taking into account the end points, the local extrema occur at $x = 0, 2 \pm \frac{2}{3}\sqrt{3}, 4$ and the global extrema occur at $x = 2 \pm \frac{2}{3}\sqrt{3}$.

Example 9.11: Define $f : [-2,3] \to \mathsf{R}^1$ by $f(x) = x^3 - 3x^2 + 2$. Then, by Theorem 7.12,

$$f'(x) = 3x^2 - 6x.$$

Hence, $f'(x) = 0$ when $x = 0$ or 2. Thus, by Theorem 9.7, the only possible points at which f could have local extrema are $0, 2$ and the end points -2 and 3 (end point extrema are not taken care of by Theorem 9.7). Now, we see whether extrema occur at these points and, if so, what types of extrema they are. We list the values of f at the four points $-2, 0, 2$ and 3:

$$f(-2) = -18,\ f(0) = 2,\ f(2) = -2,\ f(3) = 2.$$

Therefore, $f(-2) = -18$ is the global minimum of f and $f(0) = f(3) = 2$ is the global maximum of f. What about $f(2) = -2$? This appears be a local minimum for f since the function f seems to go down to -2 on $[0,2]$ and then up to 2 on $[2,3]$; but, can we be sure that f has a local minimum at 2? Yes – we can be sure by using Theorem 9.7 together with the Maximum - Minimum Theorem (Theorem 5.13). We argue as follows: By the Maximum - Minimum Theorem, f has a minimum value m on $[0,3]$; since $f(2) = -2$, m does not occur at the end points of $[0,3]$; thus, by Theorem 9.7 applied to the open interval $(0,3)$, m occurs when $x \in (0,3)$ and $f'(x) = 0$; therefore, $x = 2$ is the only possibility and, hence, $f(2) = m$. This proves that $f(2) = -2$ is a local minimum for f.

The argument in Example 9.11 to show f has a local minimum at $x = 2$ is somewhat tedious. Later, we will have a simple test at our disposal which will enable us to avoid such arguments (Theorem 10.19).

We clarify one point so as not to be misled by the examples above: A differentiable function on a closed interval *need not* have local extrema at end points of the interval even if the derivative of the function is zero at an end point. You are asked to find an example:

Exercise 9.12: Give an example of a differentiable function f on $[0, 1]$ such that $f'(0) = 0$ and, yet, 0 is not a local extremum of f. A picture of the function (rather than a formula) is sufficient, even preferred!

Exercise 9.13: Let $f(x) = x^3 + x^2 - 6x$. Find all points where f has local maxima and local minima; determine what kind of extremum occurs at each such point. Are there any global extrema?

3. Critical Points

In this section we bring into sharper focus the main ideas in the theorem and examples in the preceding section. We conclude with general comments.

We have seen that three types of points play the crucial role in finding and classifying extrema of a function on an interval: Points at which the derivative of the function is zero, end points of the interval (if there are any), and points at which the function is not differentiable (Example 9.5). We give a name to the types of these points that involve derivatives:

Definition: Let I be an interval, and let $f : I \to \mathsf{R}^1$ be a function. A point $p \in I$ that is not an end point of I is called a *critical point of* f provided that $f'(p) = 0$ or f is not differentiable at p.

We can now summarize what we have shown in the examples and the theorem in section 2 in a concise way:

Corollary 9.14: Let I be an interval, and let $f : I \to \mathsf{R}^1$ be a function. Then the local and global extrema (if they exist) must be attained at critical points of f or at an end point of I.

Proof: Assume that f has a local extremum at a point $p \in I$. Assume further that f is differentiable at p and that p is not an end point of I (remember: functions can be differentiable at end points according to our definition of derivative). Then, by Theorem 9.7 (applied to I without its end points), $f'(p) = 0$; therefore, p is a critical point of f. ¥

We comment in general about the ideas and, especially, the direction initiated in this chapter.

We have shifted our emphasis from finding global extrema to finding local extrema. At the same time, we have stressed the importance of finding global extrema. Why don't we just narrow down on finding global extrema and leave the problem of finding local extrema for later or omit it completely? The answer is simple: Finding local extrema *is* narrowing down on finding global extrema, as we have illustrated in examples, and it is easier to find local extrema first than it is to find global extrema directly (by virtue of Theorem 9.7).

Using local extrema to find global extrema is a special case of a general mathematical procedure – approximation. You have seen approximation at work when rounding off decimals, finding areas (if you had some contact with integral calculus or the work of the ancient Greeks), and in the section on linear approximation (section 3 of Chapter VI); in fact, the very definitions of limits and derivatives are based on approximation. We are now approximating global extrema by finding local extrema; as we have seen, this leads to finding the global extrema. "Necessity is the mother of invention," and, in this case, local extrema were born out of the desire to find global extrema.

We note that local extrema are important in connection with many aspects of mathematics and science. To mention only a few, local extrema are used in the physical sciences, in optimization, in dynamical systems, in economics, and in analyzing statistical data. That being said, we must add that local extrema are themselves interesting and that is enough reason to study them.

Exercise 9.15: Define $f : [-1, 1] \to \mathsf{R}^1$ by $f(x) = x^{\frac{4}{5}} + 3$. Find all points where f has local maxima and local minima; determine what kind of extremum occurs at each such point.

Exercise 9.16: Let f be defined on R^1 by $f(x) = 5x^{\frac{2}{3}} + x^{\frac{5}{3}} + 1$. Find all points where f has local maxima and local minima; determine what kind of extremum occurs at each such point.

Exercise 9.17: Prove the assertion in the introduction to the chapter that of all the rectangles having a given perimeter, the one with the largest area is the one that is most symmetric (the square).

Exercise 9.18: Find the point on the circle $x^2 + y^2 = 1$ that is closest to $(2, 0)$. (You know the answer, but use the methods in this chapter.)

Exercise 9.19: Assume that $f : \mathsf{R}^1 \to \mathsf{R}^1$ is differentiable and that $f'(x) \neq 0$ for all x. Then f is one-to-one.

Exercise 9.20: Give examples of polynomials of degree 3 that have no critical point, only one critical point, and two critical points.

Exercise 9.21: A polynomial of degree $n > 0$ has at most n roots. (A *root of a function* is a point at which the function has value 0.)

Exercise 9.22: Give an example of a nonconstant function $f : \mathsf{R}^1 \to \mathsf{R}^1$ such that every real number is a critical point of f and such that $f'_+(x)$ exists for every $x \in \mathsf{R}^1$.

Chapter X: The Mean Value Theorem and Consequences

We prove the Mean Value Theorem in section 1. Then, section by section, we derive different types of important results from the theorem. We emphasize curve sketching in conjunction with the results in sections 3 and 4.

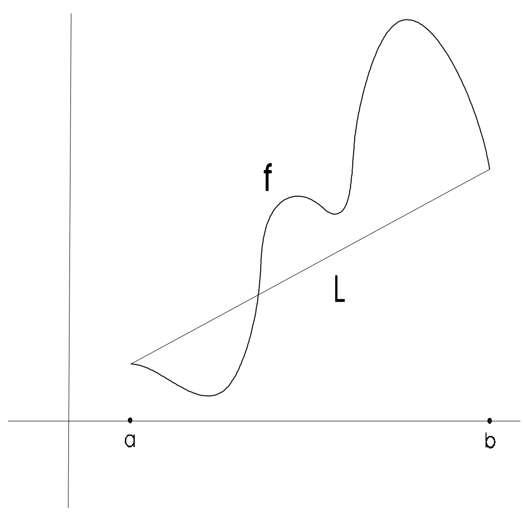
1. The Mean Value Theorem

If you travel 100 miles in 2 hours, it is obvious that at some point during the trip your velocity must be 50 miles per hour (your average velocity). In general terms, let f be a differentiable function that gives the distance $f(t)$ an object has traveled as a function of time t; then it is intuitively evident that the average velocity of the object over a time interval $[a, b]$ must be its instantaneous velocity at some time t_0 between a and b:

$$f'(t_0) = \frac{f(b)-f(a)}{b-a}.$$

This is the substance of the Mean Value Theorem, but certainly not the proof! Let us indicate the geometrical idea behind the proof.

In the figure below, the slope of the line segment L joining $(a, f(a))$ and $(b, f(b))$ is the average velocity of the object. Imagine that we continuously move L up (or down) parallel to itself. We eventually arrive at the last time the moving line segments touch the graph of f; at that moment, the line segment is tangent to the graph of f at a point $(t_0, f(t_0))$, which says $f'(t_0) = \frac{f(b)-f(a)}{b-a}$.



The discussion we just presented is not a proof; for example, how do we know there is a last time the moving line segments touch the graph of f ? Nevertheless, the discussion is an intuitively plausible argument that provides insight into why the Mean Value Theorem is true.

We proceed to the precise statement and proof of the Mean Value Theorem. We first prove a special case of the theorem from which the theorem follows. The special case is due to Michel Rolle (1652 - 1719), who eventually became a vocal opponent of calculus, calling it a "collection of ingenious fallacies."

Lemma 10.1 (Rolle's Theorem): Assume that f is continuous on $[a, b]$, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there is a point $p \in (a, b)$ such that $f'(p) = 0$.

Proof: If f is a constant function, then $f'(x) = 0$ for all x and, thus, the lemma is true. Hence, we assume for the purpose of proof that f is not a constant function. Then there is a point $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Hence, either $f(x_0) < 0$ or $f(x_0) > 0$.

Assume first that $f(x_0) < 0$. By the Maximum - Minimum Theorem (Theorem 5.13), f attains its global minimum value at a point p. Since $f(x_0) < 0$, clearly $f(p) < 0$; hence, $p \in (a, b)$. In particular, then, f is differentiable at p. Therefore, by Theorem 9.7, $f'(p) = 0$.

The case when $f(x_0) > 0$ is handled similarly by taking p to be a point at which f attains its global maximum value (or, perhaps you have a simpler proof based on past experience?). ¥

Theorem 10.2 (Mean Value Theorem): Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $p \in (a, b)$ such that

$$f'(p) = \tfrac{f(b)-f(a)}{b-a}.$$

Proof: In functional notation, the equation of the line going through the two points $(a, f(a))$ and $(b, f(b))$ is

$$g(x) = \tfrac{f(b)-f(a)}{b-a}(x-a) + f(a).$$

Define $h : [a, b] \to \mathsf{R}^1$ by letting $h = f - g$. (For geometric insight into what we do next, locate the local extrema of h in the figure on the preceding page.)

We see that h satisfies the assumptions of Lemma 10.1: h is continuous on $[a, b]$ by Corollary 4.4, h is differentiable on (a, b) by Theorem 7.3, and $h(a) = h(b) = 0$ by the formulas for g and h. Hence, by Lemma 10.1, there is a point $p \in (a, b)$ such that $h'(p) = 0$. Therefore,

$$0 = h'(p) \overset{7.3}{=} f'(p) - g'(p) \overset{6.2}{=} f'(p) - \tfrac{f(b)-f(a)}{b-a},$$

which gives that $f'(p) = \tfrac{f(b)-f(a)}{b-a}$. ¥

Exercise 10.3: Define $f : [-2, 2] \to \mathsf{R}^1$ by $f(x) = x^3 - 3x + 3$. Find all numbers p in $[-2, 2]$ that satisfy the conclusion of the Mean Value Theorem.

Exercise 10.4: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is differentiable and $f'(x) \neq 1$ for all $x \in \mathsf{R}^1$, then there is at most one point $p \in \mathsf{R}^1$ such that $f(p) = p$.

Exercise 10.5: Let I be an open interval, and let $p \in I$. Assume that f is continuous on I and differentiable on $I - \{p\}$ and that $\lim_{x \to p} f'(x)$ exists. Then f is differentiable at p.

Exercise 10.6: Assume that f and g are continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $p \in (a, b)$ such that

$$f'(p)[g(b) - g(a)] = g'(p)[f(b) - f(a)].$$

2. Functions with Equal Derivatives

All constant functions (on an interval) have derivative zero. We prove that there are no other functions with derivative zero. Perhaps you think this is obvious. But then you may also think it is obvious that the only function whose derivative is itself is the function $f(x) = 0$; however, this is false! Furthermore, the prime example showing it is false is not just a curiosity – it is the exponential function $f(x) = e^x$, which has numerous applications in probabilty theory, economics and the physical sciences. See Corollary 16.24; in Exercise 16.25 we determine all functions f such that $f' = f$.

Once we prove that constant functions are the only functions whose derivative is zero, it follows easily that any two functions on an interval that have the same derivative must differ by a constant; stated more insightfully, the graphs of the functions are vertical translations of one another. This result is so important that it is often referred to as the fundamental theorem of differential calculus. When we study the integral, we will see that the fundamental theorem of differential calculus is crucial to evaluating integrals – it is the important ingredient in proving the second part of the Fundamental Theorem of Calculus (Theorem 14.2).

Theorem 10.7: If f is continuous on $[a, b]$ and $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function.

Proof: Let $x \in [a, b]$ such that $x \neq a$. Note that f is continuous on the interval $[a, x]$ (by Exercise 5.3). Hence, we can apply the Mean Value Theorem (Theorem 10.2) to f on the interval $[a, x]$, thereby obtaining a point $p \in (a, x)$ such that

$$f'(p) = \tfrac{f(x) - f(a)}{x - a}.$$

Thus, since $f'(p) = 0$ (by assumption in the theorem), we see that $f(x) = f(a)$. This proves that $f(x) = f(a)$ for all $x \in [a, b]$. ¥

Theorem 10.8: If f and g are continuous on $[a, b]$ and $f'(x) = g'(x)$ for all $x \in (a, b)$, then f and g differ by a constant; in other words, there is a constant C such that $f(x) - g(x) = C$ for all $x \in [a, b]$.

Proof: Define $h : [a, b] \to \mathsf{R}^1$ by letting $h = f - g$. Then h is continuous on $[a, b]$ (by Corollary 4.4) and

$$h'(x) \overset{7.3}{=} f'(x) - g'(x) = 0, \text{ all } x \in (a, b).$$

Therefore, by Theorem 10.7, h is a constant function. ¥

We note that Theorem 10.7 and Theorem 10.8 are really the same theorem: Theorem 10.7 follows immediately from Theorem 10.8 by taking g in Theorem 10.8 to be the constant function $g(x) = 0$.

We close by noting that Theorem 10.8 holds when the functions are defined on any interval:

Theorem 10.9: Let I be any interval, and let E denote the set of end points of I (E may be empty). If $f, g : I \to \mathsf{R}^1$ are continuous on I and if $f'(x) = g'(x)$ for all $x \in I - E$, then f and g differ by a constant.

Proof: Recall from the proof of Theorem 8.4 that any interval is the countable union of an "increasing sequence" of closed and bounded intervals. Using this fact and Theorem 10.8, our theorem follows (we leave the details for the first exercise below). ¥

Exercise 10.10: Do the details for the proof of Theorem 10.9.

Exercise 10.11: Let $f(x) = x^5 - 3x^2 + 2$. Find all functions whose derivatives are f.

Exercise 10.12: Let $f(x) = (2x + 4)^8$. Find all functions whose derivatives are f.

Exercise 10.13: Let $f(x) = x\sqrt{x^2 + 7}$. Find all functions whose derivatives are f.

Exercise 10.14: Let $f(x) = \frac{1}{x^2}$. Find all functions whose derivatives are f. (Be careful – there may be more than you think!)

Exercise 10.15: Let $f(x) = |x - 1|$. Find all functions whose derivatives are f.

Exercise 10.16: Let f be the function given by

$$f(x) = \begin{cases} x + 2 & \text{, if } x < 0 \\ x & \text{, if } x \geq 0. \end{cases}$$

Is there a function $g : \mathsf{R}^1 \to \mathsf{R}^1$ such that $g' = f$?

3. Derivative Test for Local Extrema

Recall how hard we had to work in Example 9.11 to determine whether the function f had a local maximum or a local minimum at $x = 2$. We now provide a simple general test that will enable us to classify local extrema easily.

The test for classifying local extrema is based on the sign of the derivative. The following theorem shows what the sign of the derivative of a function says about the function. After we prove the theorem and discuss it, we give the test for classifying local extrema (Theorem 10.19).

Theorem 10.17: Assume that f is continuous on $[a, b]$ and differentiable on (a, b).

(1) If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

(2) If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.

Proof: Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. By the Mean Value Theorem (Theorem 10.2), there is a point $p \in (x_1, x_2)$ such that

$$f'(p) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Thus, under the assumption in part (1), $f(x_2) > f(x_1)$ and, under the assumption in part (2), $f(x_2) < f(x_1)$. ¥

Exercise 10.18: Give examples to show that the converses of parts (1) and (2) are false. However, prove the following partial converses to parts (1) and (2): If f is increasing (decreasing) on $[a, b]$, then $f'(x) \geq 0$ ($f'(x) \leq 0$, respectively) for all $x \in [a, b]$.

Theorem 10.17 is intuitively obvious: If all the tangent lines to the graph of a differentiable function have positive slopes, hence are strictly increasing, then surely the function is strictly increasing. However persuasive this argument may seem, it is still not a proof; it is no more a proof than saying, as a "proof" for Theorem 10.7, that if every tangent line to the graph of f is horizontal, then surely the function f is constant. The proof we gave for Theorem 10.17 is certainly short, deceptively short because the proof rests on so many previous results: The proof of Theorem 10.17 used the Mean Value Theorem, whose proof used Rolle's Theorem, whose proof depended essentially on the Maximum - Minimum Theorem; the proof of the Maximum - Minimum Theorem was by no means trivial and depended indispensably on the Completeness Axiom. Thus, in the final analysis, the underlying reason Theorem 10.17 is true is the Completeness Axiom. We conclude that Theorem 10.17 is not as obvious as it would seem to be or as trivial as its brief proof would suggest.

It is worthwhile to consider Theorem 10.17 and Theorem 10.7 together: The theorems show that the sign of a derivative on an interval has a lot to say about the nature of a function.

One final comment about Theorem 10.17: A differentiable function on an open interval can have a positive derivative at a particular point but not be strictly increasing in any neighborhood of the point. See Exercise 10.53.

We are ready to prove the derivative test for classifying local extrema.

Theorem 10.19 (First Derivative Test for Local Extrema): Let I be an open interval, and let $p \in I$. Assume that f is continuous on I and differentiable at each point of I except possibly at p. Let $[s, t] \subset I$ such that $p \in (s, t)$.

(1) If $f'(x) > 0$ for all $x \in (s, p)$ and $f'(x) < 0$ for all $x \in (p, t)$, then f has a local maximum at p.

(2) If $f'(x) < 0$ for all $x \in (s, p)$ and $f'(x) > 0$ for all $x \in (p, t)$, then f has a local minimum at p.

(3) If $f'(x) > 0$ for all $x \in (s,p) \cup (p,t)$, or if $f'(x) < 0$ for all $x \in (s,p) \cup (p,t)$, then f does not have a local extremum at p.

Proof: Assume the conditions in part (1). Then, by Theorem 10.17, f is strictly increasing on $(s,p]$ and f is strictly decreasing on $[p,t)$. Hence, $f(x) < f(p)$ when $s < x < p$ and $f(p) > f(x)$ when $p < x < t$. Thus, $f(p) \geq f(x)$ for all $x \in (s,t)$. Therefore, f has a local maximum at p. This proves part (1).

The proof of part (2) is similar.

We prove part (3) for the case when $f'(x) > 0$ for all $x \in (s,p) \cup (p,t)$. In this case, we have by Theorem 10.17 that f is strictly increasing on $(s,p]$ and on $[p,t)$. It follows easily that f is strictly increasing on (s,t). Thus,

$$f(y) < f(p) < f(z) \text{ whenever } s < y < p < z < t.$$

Therefore, we see that f does not have a local extremum at p (we leave the details to the reader).

The proof of part (3) for the case when $f'(x) < 0$ for all $x \in (s,p) \cup (p,t)$ is similar. ¥

The converse of part (1) of Theorem 10.19 is false (Exercise 10.26).

We illustrate how well the First Derivative Test for Local Extrema works:

Example 10.20: Let $f(x) = 2x^5 - 5x^4 - 10x^3$ for all $x \in \mathsf{R}^1$. We find all points at which f has local and global extrema and determine which extrema are local (or global) minima and which are local (or global) maxima. We also determine the maximal intervals on which f is strictly increasing or strictly decreasing. Finally, we sketch the graph of f using the information we have obtained (however, the sketch is incomplete, as we will see).

By the formula for differentiating polynomials (Theorem 7.12),

$$f'(x) = 10x^4 - 20x^3 - 30x^2.$$

To find where $f'(x) = 0$ (in order to apply Theorem 9.7), we factor $f'(x)$:

$$f'(x) = 10x^2(x^2 - 2x - 3) = 10x^2(x-3)(x+1).$$

Hence, by Theorem 9.7, the only possible points at which f has local extrema are $x = -1, 0, 3$.

The critical step for using Theorem 10.19 is to find the sign of f' on small intervals about the points $x = -1, 0, 3$. How small do we need the intervals to be? The answer comes from noting that f' is continuous: Hence, we can apply the Intermediate Value Theorem (Theorem 5.2) to f' to know that f' can not have opposite signs at two points without being 0 somewhere between the two points; thus, we only need to check the signs of f' at one point of each of the open intervals determined by the points $x = -1, 0, 3$. We can do this readily by inspecting the factored form of f'; we obtain the table below:

interval $\rightarrow$	$(-\infty, -1)$	$(-1, 0)$	$(0, 3)$	$(3, \infty)$
$sign f'(x) \rightarrow$	$+$	$-$	$-$	$+$

From the table and from Theorem 10.19, f has a local maximum at $x = -1$, a local minimum at $x = 3$, and no local extremum at $x = 0$. Furthermore, from the table and Theorem 10.17, the maximal intervals on which f is strictly increasing are $(-\infty, -1]$ and $[3, \infty)$, and the maximal interval on which f is strictly decreasing is $[-1, 3]$.

Next, we see that f has no global extrema: f has no global maximum since its only local maximum is $f(-1) = 3$ and $f(4) = 128$; f has no global minimum since its only local minimum is $f(3) = -189$ and $f(-3) = -621$. Actually, we can see that f has no global extrema without these types of numerical computations: Simply note that

$$f(x) = x^5(2 - \tfrac{5}{x} - \tfrac{10}{x^2}) \text{ for } x \neq 0,$$

which easily shows that f is neither bounded above nor bounded below.

Finally, using the information available, we obtain a picture of the graph of f (Figure 10.20 below). However, something is wrong: $f'(0) = 0$, so the x-axis should be tangent to the graph of f at the origin. In correcting this flaw, we must change the shape of the graph at some point to the left of the origin; we must also change the shape of the graph at some point to the right of the origin in order to avoid having a cusp at $(3, f(3))$. At the present time, it is not at all obvious where these changes should be made; moreover, for all we know, there may be many such changes, perhaps even at points $x < -1$ or at points $x > 3$. If this makes you wonder whether you really know how to graph $y = x^2$, then that is good!

We return to the problem of what is wrong with the graph of f in the next section. There we develop general ideas that solve the problem and that can be applied to other graphs. We arrive at a correct graph of the function f in Example 10.34.

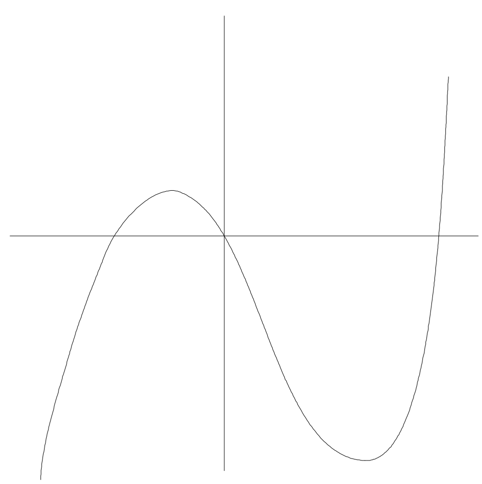


Figure 10.20

Exercise 10.21: Define $f : [0,6] \to \mathsf{R}^1$ by $f(x) = 4x^3 - 36x^2 + 77x$. Find all points at which f has local and global extrema, determine which extrema are local (or global) minima and which are local (or global) maxima, and determine the maximal intervals on which f is strictly increasing or strictly decreasing. Sketch the graph of f and discuss possible flaws in your graph as per the discussion of the graph we gave for Example 10.19. (We briefly discussed the function f after the proof of the Maximum - Minimum Theorem (Theorem 5.13)).

Exercise 10.22: Define $f : [-2,2] \to \mathsf{R}^1$ by $f(x) = x^4 - 2x^2 + 1$. Repeat Exercise 10.21 for this function.

Exercise 10.23: Define $f : [0,2] \to \mathsf{R}^1$ by $f(x) = \frac{x}{x^2+1}$. Repeat Exercise 10.21 for this function.

Exercise 10.24: In Figure 10.24 below, we have drawn a picture of the graph of the derivative of a function f. Determine all points at which f has local and global extrema, determine which extrema are local (or global) minima and which are local (or global) maxima, and determine the maximal intervals on which f is strictly increasing or strictly decreasing. Sketch the graph of f assuming that $f(0) = 0$.

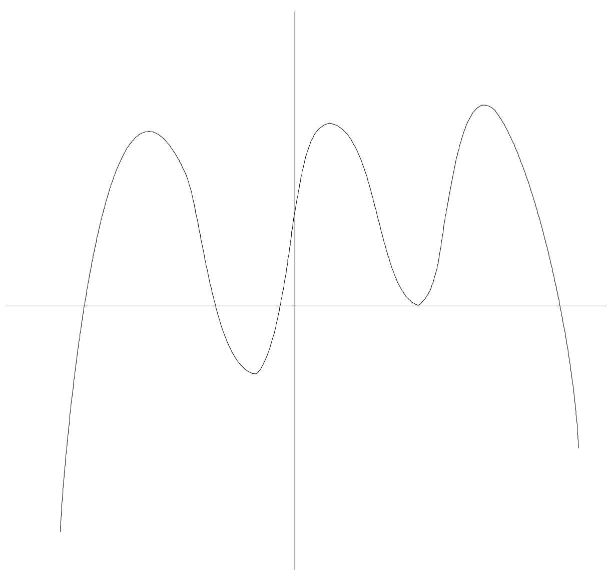


Figure 10.24

Exercise 10.25: Let $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ be differentiable functions such that $f'(x) < g'(x)$ for all $x \in \mathsf{R}^1$. Then there is at most one point p such that $f(p) = g(p)$.

Exercise 10.26: Draw a picture of the graph of a differentiable function on an open interval such that the function has a unique global maximum at a point p for which part (1) Theorem 10.19 does not apply.

4. Concavity

This section follows up on the discussion above Figure 10.20: We introduce concepts that describe the flaws in the preliminary graph in Figure 10.20 and that we can use to refine our graphing techniques in general. Specifically, we define the notions of concavity and inflection point, and we obtain results that connect the notions to derivatives. At the end of the section, we sketch the graph for Example 10.20 (this time correctly!).

Let I be an interval, let $a, b \in I$ such that $a \neq b$, and let $f : I \to \mathsf{R}^1$ be a function. The *chord joining* $(a, f(a))$ *and* $(b, f(b))$ is the line segment in the plane with end points $(a, f(a))$ and $(b, f(b))$.

Definition: Let I be an interval, and let $f : I \to \mathsf{R}^1$ be a function.

- We say that f is *concave up on* I provided that for any two different points $a, b \in I$, the chord joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f on (a, b); in other words,

$$f(x) < \tfrac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \text{when } a < x < b.$$

- We say that f is *concave down on* I provided that for any two different points $a, b \in I$, the chord joining $(a, f(a))$ and $(b, f(b))$ lies below the graph of f on (a, b); in other words,

$$f(x) > \tfrac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \text{when } a < x < b.$$

For example, $f(x) = x^3$ is concave up on $[0, \infty)$ and concave down on $(-\infty, 0]$. On the other hand, a linear function $f(x) = mx + b$ is not concave up or down on any interval.

In Theorem 10.29, we characterize the two types of concavity for a differentiable function on an interval in terms of the derivative of the function.

Lemma 10.27: Let I be an interval, let $f : I \to \mathsf{R}^1$ be a function, and let $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$. For each $i \neq j$, let $C_{i,j}$ denote the the chord joining $(x_i, f(x_i))$ and $(x_j, f(x_j))$.

(1) If f is concave up on I, then

$$\text{slope of } C_{1,2} < \text{slope of } C_{1,3} < \text{slope of } C_{2,3}.$$

(2) If f is concave down on I, then

$$\text{slope of } C_{1,2} > \text{slope of } C_{1,3} > \text{slope of } C_{2,3}.$$

Proof: We prove part (1); we leave the proof of part (2) to the reader (Exercise 10.28).

Assume that f is concave up on I. Let y_2 denote the second coordinate of the point on $C_{1,3}$ with first coordinate x_2. Since f is concave up on I, $f(x_2) < y_2$; hence, $f(x_2) - f(x_1) < y_2 - f(x_1)$. Thus,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} < \frac{y_2-f(x_1)}{x_2-x_1}.$$

Therefore, since the slope of $C_{1,2} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ and the slope of $C_{1,3} = \frac{y_2-f(x_1)}{x_2-x_1}$, we have proved that the slope of $C_{1,2} <$ slope of $C_{1,3}$.

The proof of the second inequality in part (1) is similar to what we have done: We rewrite $f(x_2) < y_2$ as $-y_2 < -f(x_2)$; then $f(x_3)-y_2 < f(x_3)-f(x_2)$. Thus,

$$\frac{f(x_3)-y_2}{x_3-x_2} < \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

Therefore, since the slope of $C_{1,3} = \frac{f(x_3)-y_2}{x_3-x_2}$ and the slope of $C_{2,3} = \frac{f(x_3)-f(x_2)}{x_3-x_2}$, we have proved that the slope of $C_{1,3} <$ slope of $C_{2,3}$.

This completes the proof of part (1) of the lemma. ¥

Exercise 10.28: Formulate and prove a simple theorem that can be applied to prove part (2) of Lemma 10.27 directly from part (1).

Theorem 10.29: Assume that f is differentiable on an open interval I.

(1) f is concave up on I if and only if f' is strictly increasing on I.

(2) f is concave down on I if and only if f' is strictly decreasing on I.

Proof: We prove part (1), leaving the proof of part (2) to the reader (Exercise 10.30).

Assume that f is concave up on I. Fix points $a, b \in I$ such that $a < b$. We show that $f'(a) < f'(b)$.

Fix points c and d such that $a < c < d < b$. Then, using part (1) of Lemma 10.27 twice, we see that

(i) $\frac{f(c)-f(a)}{c-a} < \frac{f(d)-f(a)}{d-a} < \frac{f(c)-f(d)}{c-d} < \frac{f(c)-f(b)}{c-b} < \frac{f(d)-f(b)}{d-b}$.

Then, using (1) of Lemma 10.27 for the first and third inequalities below,

$$\frac{f(x)-f(a)}{x-a} < \frac{f(c)-f(a)}{c-a} \overset{\text{(i)}}{<} \frac{f(d)-f(b)}{d-b} < \frac{f(y)-f(b)}{y-b}, \text{ if } a < x < c \text{ and } d < y < b.$$

Thus, since $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $f'(b) = \lim_{y\to b} \frac{f(y)-f(b)}{y-b}$ (by Exercise 6.10), we have that $f'(a) < f'(b)$. This proves that f' is strictly increasing on I.

Conversely, assume that f' is strictly increasing on I. Fix points $s, t \in I$ such that $s < t$. Fix a point x such that $s < x < t$. Note that f is continuous on $[s, t]$ by Theorem 6.14 (and Exercise 5.3). Therefore, we can apply the Mean Value Theorem (Theorem 10.2) to obtain points $p \in (s, x)$ and $q \in (x, t)$ such that

$$f'(p) = \frac{f(x)-f(s)}{x-s}, \quad f'(q) = \frac{f(t)-f(x)}{t-x}.$$

Thus, since f' is strictly increasing on (s, t) and $p < q$, we have that

(ii) $\frac{f(x)-f(s)}{x-s} < \frac{f(t)-f(x)}{t-x}$.

We now show that $f(x) < \frac{f(t)-f(s)}{t-s}(x-s) + f(s)$, which proves that f is concave up. By (ii),

$$[f(x) - f(s)](t - x) < [f(t) - f(x)](x - s);$$

hence,

$$f(x)(t - s) < f(t)(x - s) + f(s)(t - x);$$

thus,

(iii) $f(x) < f(t)\frac{x-s}{t-s} + f(s)\frac{t-x}{t-s}$.

Finally, subtracting and adding $f(s)\frac{x-s}{t-s}$ to the right - hand side of (iii), we have

$$\begin{aligned} f(x) &< \tfrac{f(t)-f(s)}{t-s}(x - s) + f(s)\tfrac{x-s}{t-s} + f(s)\tfrac{t-x}{t-s} \\ &= \tfrac{f(t)-f(s)}{t-s}(x - s) + f(s)\tfrac{t-s}{t-s} = \tfrac{f(t)-f(s)}{t-s}(x - s) + f(s). \end{aligned}$$ ¥

Exercise 10.30: Show that part (2) of Theorem 10.29 follows easily from part (1) using the theorem you discovered in Exercise 10.28.

In order to easily apply Theorem 10.29 to determine concavity, we need a simple test to determine whether a derivative f' is strictly increasing or strictly decreasing. Theorem 10.17 provides such a test when f' is differentiable; the corollary below states the test precisely.

We denote the derivative of f' by f''; f'' is called the *second derivative of* f. A function f that has a second derivative is said to be *twice differentiable.*

Corollary 10.31: Assume that f is twice differentiable on an open interval I.

(1) If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

(2) If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

Proof: The corollary follows directly from Theorems 10.17 and 10.29. ¥

We make two observations about Corollary 10.31. First, Corollary 10.31 does not apply to all differentiable functions since a function can be differentiable and yet not be twice differentiable. Second, the converses of parts (1) and (2) of Corollary 10.31 are false; for example, $f(x) = x^4$ shows the converse of part (1) is false.

Exercise 10.32: Verify the statements in the preceding paragraph (include an example to show that the converse of part (2) of Corollary 10.31 is false).

We will be concerned with points at which the concavity of a function changes. For example, we say that the function $f(x) = x^3$ changes concavity at $x = 0$ because f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. We give the following precise, general definition for change in concavity:

Definition. Let f be a function defined on an open interval I, and let $p \in I$. We say that f *changes concavity at the point* p provided that for some interval $(s, t) \subset I$, $f|(s, p)$ is concave one way (up or down) and $f|(p, t)$ is concave the other way (down or up, respectively).

When a function f changes concavity at a point p, we say that f *has an inflection point at* p, in which case we call $(p, f(p))$ an *inflection point of* f.[5]

The following theorem, in conjunction with Corollary 10.31, can enable us to determine the inflection points of a twice differentiable function; we will illustrate this in Example 10.34. We note that the theorem is analogous to Theorem 9.7 for local extrema.

Theorem 10.33: Assume that f is twice differentiable on an open interval I. If f has an inflection point at p, then $f''(p) = 0$.

Proof: Assume that f has an inflection point at p. Then there is an interval $[s, t] \subset I$ such that $f|(s, p)$ is concave one way and $f|(p, t)$ is concave the other way. Assume that $f|(s, p)$ is concave up and $f|(p, t)$ is concave down. Then, by Theorem 10.29, we have that

(*) f' is strictly increasing on (s, p) and f' is strictly decreasing on (p, t).

Since f' is differentiable and $[s, t] \subset I$, f' is continuous on $[s, t]$ by Theorem 6.14 (and Exercise 5.3). Hence, by the Maximum - Minimum Theorem (Theorem 5.13), the restricted function $f'|[s, t]$ has attains its maximum value at some point q of $[s, t]$. We see from (*) that $q = p$. Thus, f' has a local maximum at p. Therefore, by Theorem 9.7, $f''(p) = 0$.

The proof when $f|(s, p)$ is concave down and $f|(p, t)$ is concave up is similar and is omitted. ¥

Let's see how all this works. We return to Example 10.20:

Example 10.34: Let $f(x) = 2x^5 - 5x^4 - 10x^3$ for all $x \in \mathsf{R}^1$. In Example 10.20 we showed that f has a local maximum at $x = -1$, a local minimum at $x = 3$, and no global extrema. We noted some problems with the graph of f as depicted in Figure 10.20. We are now prepared to address the problems.

We use Theorem 10.33 to find all the *possible* inflection points of f:

$$f''(x) = 40x^3 - 60x^2 - 60x = 20x(2x^2 - 3x - 3);$$

thus, the points x at which $f''(x) = 0$ are $x = 0, \frac{3}{4} \pm \frac{\sqrt{33}}{4}$; hence, by Theorem 10.33, the only possible points at which f could have inflection points are $x = 0, \frac{3}{4} \pm \frac{\sqrt{33}}{4}$. Next, we use Corollary 10.31 to see which of the points $0, \frac{3}{4} \pm \frac{\sqrt{33}}{4}$ *is* a point at which f has an inflection point.

Note that f'' is continuous; thus, to apply Corollary 10.31, we only need to check the signs of f'' at one point of each of the open intervals determined by the points $x = 0, \frac{3}{4} \pm \frac{\sqrt{33}}{4}$ (we are using the Intermediate Value Theorem (Theorem 5.2)). Without using specific values for x, but merely inspecting the expression $f''(x) = 20x(2x^2 - 3x - 3)$ for any x very negative, for any $x < 0$ and very near 0, for any $x > 0$ and very near 0, and for any x very large (positive), we arrive at the following table:

[5] Notice in the definition of inflection point that an inflection point of f is a point $(p, f(p))$ on the graph of f, not the point p; the distinction emphasizes the fact that the graph of f is where the geometry inherent in the notion of inflection point is visible.

interval $\rightarrow$	$(-\infty, \frac{3}{4} - \frac{\sqrt{33}}{4})$	$(\frac{3}{4} - \frac{\sqrt{33}}{4}, 0)$	$(0, \frac{3}{4} + \frac{\sqrt{33}}{4})$	$(\frac{3}{4} + \frac{\sqrt{33}}{4}, \infty)$
$sign f''(x) \rightarrow$	$-$	$+$	$-$	$+$

Hence, by Corollary 10.31, f changes concavity at each of the points $x = 0, \frac{3}{4} \pm \frac{\sqrt{33}}{4}$. Therefore, f has an inflection point at each of these points and at no other point (by Theorem 10.33).

We also know from the table and Corollary 10.31 that f is concave up on $(\frac{3}{4} - \frac{\sqrt{33}}{4}, 0) \cup (\frac{3}{4} + \frac{\sqrt{33}}{4}, \infty)$ and that f is concave down on $(-\infty, \frac{3}{4} - \frac{\sqrt{33}}{4}) \cup (0, \frac{3}{4} + \frac{\sqrt{33}}{4})$.

Taking into account inflection points and concavity, we correct the graph of f that we drew in Figure 10.20:

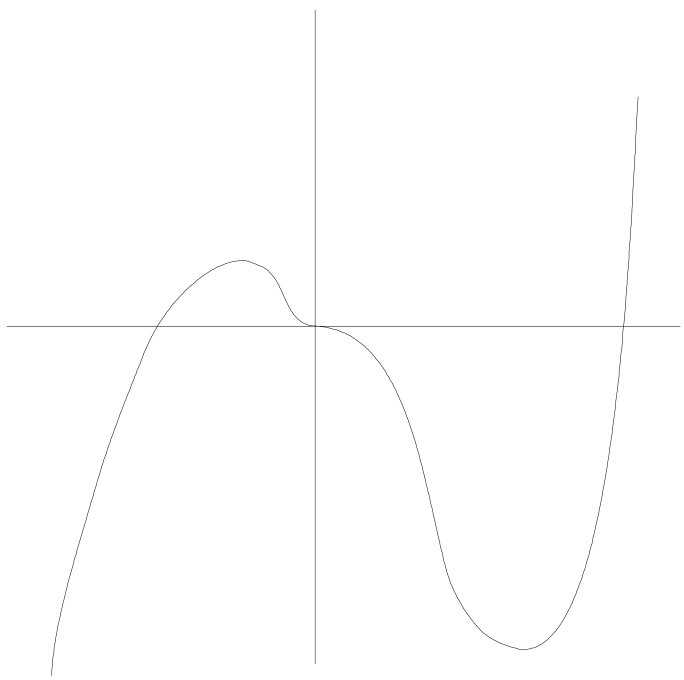


Figure 10.34

We conclude with an interesting theorem about polynomials. What really makes the theorem interesting is that the theorem is not true for differentiable functions in general, as you will be asked to show in Exercise 10.45. The proof of the theorem uses several previous results and must be done carefully (you will gain appreciation for the proof if you keep your solution to Exercise 10.26 in mind as you read the proof).

Theorem 10.35: A nonconstant polynomial has an inflection point at some point between any two points at which the polynomial has local extrema.

Proof: Assume the f is a polynomial with local extrema at p and q with $p < q$. Note that f has degree ≥ 3 (since nonconstant polynomials of degree ≤ 2 do not have two local extrema).

By Theorem 9.7, $f'(p) = 0$ and $f'(q) = 0$. Note that f' is a polynomial of degree ≥ 2 (by Theorem 7.12); hence, by Exercise 9.21, we can assume p and q were chosen so that $f'(x) \neq 0$ for all $x \in (p, q)$. Therefore, by the Intermediate Value Theorem (Theorem 5.2), $f'(x) > 0$ for all $x \in (p, q)$ or $f'(x) < 0$ for all $x \in (p, q)$. We assume for the proof that $f'(x) > 0$ for all $x \in (p, q)$ (the proof for the other case is similar).

By the Maximum - Minimum Theorem (Theorem 5.13), f' attains a maximum value on $[p, q]$ at a point r. Since $f'(p) = 0 = f'(q)$ and $f'(x) > 0$ for all $x \in (p, q)$, it is clear that $r \in (p, q)$. Hence, by Theorem 9.7, $f''(r) = 0$.

Now, since f'' is a polynomial of degree ≥ 1 (by Theorem 7.12), we see from Exercise 9.21 that there is a subinterval $[s, t]$ of (p, q) such that $r \in (s, t)$ and

$$f''(x) \neq 0 \text{ for all } x \in (s, r) \cup (r, t).$$

Thus, since f'' is continuous (because f'' is a polynomial), we have by the Intermediate Value Theorem that f'' does not change sign on (s, r) and f'' does not change sign on (r, t). Therefore, since f' has a local maximum at r, we see from part (3) of Theorem 10.19 that the sign of f'' on (s, r) is opposite to the sign of f'' on (r, t). Therefore, by Corollary 10.31, f changes concavity at r; in other words, f has an inflection point at r. ¥

In some exercises, we will ask you to find maximal intervals on which a function is concave up or down. The following theorem should be kept in mind when finding such maximal intervals:

Theorem 10.36: If f is continuous on $[a, b]$ and concave up (down) on (a, b), then f is concave up (down, respectively) on $[a, b]$.

Proof: Left as the first exercise below. ¥

Exercise 10.37: Prove Theorem 10.36.

Exercise 10.38: Define $f : [0, 6] \to \mathsf{R}^1$ by $f(x) = 4x^3 - 36x^2 + 77x$. Continue the analysis of f begun in Exercise 10.21 by finding all points at which f has an inflection point and determining the maximal intervals on which f is concave up or down. Sketch the graph of f eliminating flaws that may have occurred in your sketch for Exercise 10.21

Exercise 10.39: Define $f : [-2, 2] \to \mathsf{R}^1$ by $f(x) = x^4 - 2x^2 + 1$. This is the function in Exercise 10.22. Repeat Exercise 10.38 for this function.

Exercise 10.40: Define $f : [0, 2] \to \mathsf{R}^1$ by $f(x) = \frac{x}{x^2+1}$. This is the function in Exercise 10.23. Repeat Exercise 10.38 for this function.

Exercise 10.41: Sketch the graph of $f(x) = 8x^5 - 5x^4 - 20x^3 + 1$ identifying all local extrema, inflection points, and concavity.

Exercise 10.42: Repeat Exercise 10.41 for $f(x) = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}$.

Exercise 10.43: Repeat Exercise 10.41 for $f(x) = |x|(x+1)$.

Exercise 10.44: In Figure 10.44 below, we have drawn a picture of the graph of the second derivative of a function f. Assuming that $f(0) = 0$ and that $f'(0) = 0$, sketch the graph of f indicating the points at which all local extrema occur, the points at which f has inflection points, and the maximal intervals on which f is concave up or down.

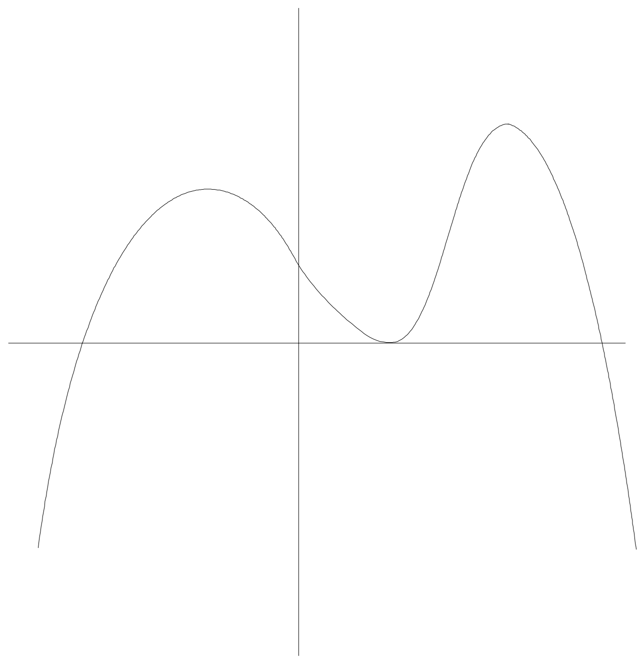


Figure 10.44

Exercise 10.45: Theorem 10.35 is not necessarily true for differentiable functions that are not polynomials. Show this by giving an example of a differentiable function on R^1 that has local extrema at different points but that has no inflection point.

Exercise 10.46: Any polynomial f of degree 3 has exactly one inflection point. Furthermore, if f crosses the x-axis at three distinct points a, b, c (i.e., has three distinct roots), then the inflection point of f occurs at the average $x = \frac{a+b+c}{3}$ of the roots.

Exercise 10.47: Let I be an open interval, and let $f : I \to \mathsf{R}^1$ be differentiable on I. Then f is concave up (down) on I if and only if for each $x \in I$, the graph of f lies above (below, respectively) the tangent line to the graph of f at $(x, f(x))$ except for the point $(x, f(x))$ itself.