into the function g. After we prove the theorem, we discuss the assumptions in the theorem.

Theorem 4.29 (Substitution Theorem): Let $X, Y, Z \subset \mathsf{R}^1$, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Let $p \in \mathsf{R}^1$ such that p is a limit point of X. If $\lim_{x\to p} f(x) = L$ and if g is continuous at L, then

$$\lim_{x\to p}(g \circ f)(x) = g(L).$$

Proof: We use Theorem 3.11. Let $A \subset X$ such that $p \sim A - \{p\}$. Then, since $\lim_{x\to p} f(x) = L$, we have by Theorem 3.11 that

$$L \sim f(A - \{p\}).$$

Thus, since g is continuous at L, the definition of continuity (above Example 2.23) gives us that

$$g(L) \sim g[f(A - \{p\})].$$

Hence, we have proved that for any $A \subset X$ such that $p \sim A - \{p\}$,

$$g(L) \sim (g \circ f)(A - \{p\}).$$

Therefore, by Theorem 3.11, $\lim_{x\to p}(g \circ f)(x) = g(L)$. ¥

Theorem 4.28 follows immediately from Theorem 4.29 using the characterization of continuity in Corollary 3.13. Nevertheless, we presented Theorem 4.28 first since it is less technical than Theorem 4.29 and since it is obviously the origin for Theorem 4.29.

The analogue of Theorem 4.29 for limits as x approaches infinity is in Theorem 18.6. It may enhance your understanding of Theorem 4.29 to read the proof of Theorem 18.6 now and adapt the proof to give an "epsilon - delta proof" of Theorem 4.29.

There is a natural question to ask about Theorem 4.29. It is the question of whether the analogous theorem is true when we interchange the assumptions about f and g; that is, assume f is continuous at p and $\lim_{y\to f(p)} g(y) = L$, and then conclude that $\lim_{x\to p}(g \circ f)(x) = L$. Of course, the assumption that $\lim_{y\to f(p)} g(y) = L$, makes no sense unless $f(p)$ is a limit point of Y (recall the definition of limit at the beginning of Chapter III). So, let's add the assumption that $f(p)$ is a limit point of Y to our other assumptions here. Now, what can go wrong? We see the problem when we try to prove the result:

Let $A \subset X$ such that $p \sim A - \{p\}$. Then, by our assumption that f is continuous at p,

$$f(p) \sim f(A - \{p\}).$$

Now, according to Theorem 3.11, we must prove $L \sim g(f(A - \{p\}))$ in order to know that $\lim_{x\to p}(g \circ f)(x) = L$. The definition of arbitrary closeness (section 1 of Chapter II) says that $L \sim g(f(A - \{p\}))$ means

$$dist(L, g(f(A - \{p\})) = 0.$$

Could $dist(L, g(f(A - \{p\})) > 0$? In other words, could $g(f(A - \{p\}))$ be bounded away from L? The simplest thing that could happen that would make $g(f(A - \{p\}))$ be bounded away from L is the following: $f(A - \{p\})$ is a single point q at which g jumps and for which $g(q) \neq L$. This suggests considering f to be the constant function $f(x) = q$ and letting g be a function that jumps at q but for which $g(q) \neq L = \lim_{y \to q} g(y)$. You should now be prepared to write down a counterexample to the analogue of Theorem 4.29 that we have tried to prove:

Exercise 4.30: Using the preceding discussion as a guide, find functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ such that f is continuous at $p = 0$, $\lim_{y \to f(p)} g(y) = L$, but $\lim_{x \to p} (g \circ f)(x) \neq L$.

Exercise 4.31: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is a function such that $\lim_{x \to p} f(x)$ exists for some point $p \in \mathsf{R}^1$, then $\lim_{x \to p} |f(x)| = |\lim_{x \to p} f(x)|$.

Exercise 4.32: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is a function such that $\lim_{x \to p} f(x)$ exists for some point $p \geq 0$, then $\lim_{x \to p} \sqrt{f(x)} = \sqrt{\lim_{x \to p} f(x)}$.

Exercise 4.33: For any two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$, define the *maximum function of f and g*, written $f \bigvee g$, and *the minimum function of f and g*, written $f \bigwedge g$, as follows: For each $x \in \mathsf{R}^1$,

$$(f \bigvee g)(x) = \max\{f(x), g(x)\}, \quad (f \bigwedge g)(x) = \min\{f(x), g(x)\}.$$

Prove that if f and g are continuous at p, then $f \bigvee g$ and $f \bigwedge g$ are continuous at p.

(*Hint:* What is $\frac{x+y}{2} + \frac{|x-y|}{2}$ for real numbers x and y?)

7. The Squeeze Theorem

The name Squeeze Theorem is very descriptive of what the theorem says: Consider three functions f, g and h defined on X into R^1 satisfying the conditions (where p is a limit point of X)

$$g(x) \leq f(x) \leq h(x), \text{ all } x \in X - \{p\}, \qquad \lim_{x \to p} g(x) = \lim_{x \to p} h(x);$$

the conditions suggest that the function f is squeezed by the functions g and h as x approaches p; the Squeeze Theorem says that f is then forced to have the same limit as x approaches p that g and h have.

The Squeeze Theorem is one of those theorems that is easy to prove but that states an important and useful point of view. The idea of squeezing to obtain a limit will come up in other contexts (e.g., the definition of the integral in Chapter XII).

Theorem 4.34 (Squeeze Theorem): Let $X \subset \mathsf{R}^1$, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. Assume that $f, g, h : X \to \mathsf{R}^1$ are functions such that

$$g(x) \leq f(x) \leq h(x), \text{ all } x \in X - \{p\}, \qquad \lim_{x \to p} g(x) = \lim_{x \to p} h(x) = L.$$

Then $\lim_{x \to p} f(x) = L$.

Proof: We prove the theorem using only the definition of limit.

Let $\epsilon > 0$. Since $\lim_{x\to p} g(x) = L$ and $\lim_{x\to p} h(x) = L$, there exist $\delta_1, \delta_2 > 0$ such that

$$|g(x) - L| < \epsilon \text{ for all } x \in X - \{p\} \text{ such that } |x - p| < \delta_1$$

and

$$|h(x) - L| < \epsilon \text{ for all } x \in X - \{p\} \text{ such that } |x - p| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for all $x \in X - \{p\}$ such that $|x - p| < \delta$,

$$L - \epsilon < g(x), h(x) < L + \epsilon$$

Thus, since $g(x) \leq f(x) \leq h(x)$, we must have that $L - \epsilon < f(x) < L + \epsilon$ for all $x \in X - \{p\}$ such that $|x - p| < \delta$. This proves that $\lim_{x\to p} f(x) = L$. ¥

Exercise 4.35: Find $\lim_{x\to 0} x \sin(\frac{1}{x})$.

Exercise 4.36: Find $\lim_{x\to 0} \sqrt{x^3 + x + 1} \sin(\frac{1}{x})$.

8. Limits of Sequences

We briefly introduce sequences and limits of sequences for use later. We will present an in-depth study of the general theory of sequences beginning with Chapter XIX. In the meantime, we will not use sequences often, and the material we present here is all we need.

In this section, we will see that the standard definition of limits for sequences can be considered to be a special case of the notion of limits for functions as defined in section 1 of Chapter III. As a consequence, almost all theorems about limits of functions that we have proved are automatically true for limits of sequences (Theorem 4.38).

A *sequence* is simply a function defined on the set $\mathsf{N} = \{1, 2, ...\}$. The range of a sequence can consist of any types of objects (for example, a sequence of statements, a sequence of vectors, a sequence of numbers and statements together, and so on). If s is a sequence, we denote the value of s at n by s_n (i.e., $s_n = s(n)$). We often denote a sequence s by writing $\{s_n\}_{n=1}^{\infty}$.

We want to focus on sequences whose values are real numbers. Such sequences are sometimes called *numerical sequences.* We prefer to just use the term sequence rather than numerical sequence; thus, unless we say otherwise or it is evident from context, we assume that the values of a sequence are real numbers.

We say that a *sequence* $\{s_n\}_{n=1}^{\infty}$ *coverges to a point* p provided that for each $\epsilon > 0$, there exists N such that $|s_n - p| < \epsilon$ for all $n \geq N$; we call p the *limit of the sequence* $\{s_n\}_{n=1}^{\infty}$. We write $\lim_{n\to\infty} s_n = p$ or $\{s_n\}_{n=1}^{\infty} \to p$ to denote that a sequence $\{s_n\}_{n=1}^{\infty}$ converges to p.

A sequence that converges is called a *convergent sequence*; a sequence that does not converge is called a *divergent sequence.*

For example, the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0 by the second part of Exercise 1.23 with $n_i = n$ for each i. However, we remind the reader that the fact that $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0 (as well as Exercise 1.23) depends essentially on the Archimedean Property – see the discussion leading to Lemma 1.21.

On the other hand, the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges (can you see why?).

We reformulate the definition of limits of sequences in terms of limits of functions as defined at the beginning of Chapter III. At first glance, the two notions of limit seem incompatible: limits of sequences are concerned with unbounded domain values, whereas limits of functions in Chapter III are involved only with bounded domain values. Nevertheless, the definition of $\lim_{x\to p} f(x)$ in Chapter III does *not* require p to be a point of the domain X of f; this is the underlying reason that we are able to formulate limits of sequences in terms of limits of functions as follows:

Exercise 4.37: Let $\{s_n\}_{n=1}^{\infty}$ be a sequence, and let $f : \{\frac{1}{n} : n \in \mathsf{N}\} \to R^1$ be the function defined by $f(\frac{1}{n}) = s_n$ for each $n \in \mathsf{N}$. Then $\{s_n\}_{n=1}^{\infty} \to p$ if and only if $\lim_{\frac{1}{n}\to 0} f(\frac{1}{n}) = p$.

Theorem 4.38: Results about limits in Chapter III and in this chapter apply to limits of sequences as well (except for Theorem 3.16).

Proof: The theorem is immediate from Exercise 4.37. ¥

We remark the fundamental notions of arbitrary closeness and continuity, which we introduced in Chapter II, can each be reformulated in terms of sequences. The reformulations are postponed until we begin a systematic study of sequences in Chapter XIX. The relevant results are Theorem 19.38 and Theorem 19.39; you are well prepared to read the results and their proofs now: the only background you need, aside from what we have already covered, is the analogue of Theorem 4.29 for limits as x approaches infinity (Theorem 18.6).

Chapter V: Two Properties of Continuous Functions

Of all the properties of continuous functions that I know, two properties stand out as being the most important. Both properties concern continuous functions on intervals: Let I be an interval, and let $f : I \to \mathsf{R}^1$ be a continuous function; then any number between two values of f is a value of f; if I is a closed and bounded interval, then f has a (unique) largest value, and a (unique) smallest value. We verify the two properties and give applications (in the exercises). We will see many more applications in subsequent chapters.

1. The Intermediate Value Theorem

A continuous function defined on a subset of R^1 may have two values such that no number between those two values is a value of the function. For example, let $X = \{0, 1\}$ and define f by letting $f(0) = 0$ and $f(1) = 1$. In fact, this kind of behavior can always happen when X is not an interval:

Exercise 5.1: Let X be any nonempty subset of R^1 such that X is not an interval. Then there is a continuous function $f : X \to \mathsf{R}^1$ such that for some two values, $f(a)$ and $f(b)$, no number between $f(a)$ and $f(b)$ is a value of f.

However, the behavior illustrated above can not occur when the domain of a continuous function is an interval:

Theorem 5.2 (Intermediate Value Theorem): Let I be an interval, and let $f : I \to \mathsf{R}^1$ be continuous. Then every number that lies between two values of f is a value of f. In other words, $f(I)$ is an interval.

Proof: Assume that $x_0, x_1 \in I$, with $x_0 < x_1$, and that $c \in \mathsf{R}^1$ such that

(1) $f(x_0) < c < f(x_1)$.

We prove that c is a value of f. (We will also need to prove that c is a value of f when $f(x_1) < c < f(x_0)$; we do this after we prove the theorem under the assumption in (1).)

We begin with a simple observation: Since $x_0, x_1 \in I$ and I is an interval, it is clear that

(2) $[x_0, x_1] \subset I$.

The following set A is central to the proof:

$$A = \{x \in [x_0, x_1] : f(x) < c\}.$$

By (1), $x_0 \in A$; hence, $A \neq \emptyset$. Also, x_1 is an upper bound for A by the way A is defined. Therefore, there is a least upper bound ℓ for A by the Completeness Axiom (section 1 of Chapter I). Since $x_0 \in A$ and x_1 is an upper bound for A, it is clear that

(3) $\ell \in [x_0, x_1]$.

We prove that $f(\ell) = c$ (note that f is defined at ℓ by (2) and (3)). We prove that $f(\ell) = c$ by proving (4) and (5) below.

(4) $f(\ell) \geq c$.

Proof of (4)*:* Suppose by way of contradiction that $f(\ell) < c$. Then, since f is continuous at ℓ, there is an open interval J such that $\ell \in J$ and $f(J) \subset (-\infty, c)$ (by Exercise 3.14). Since $f(\ell) < c$, we see from (1) that $\ell \neq x_1$; hence, by (3), $\ell < x_1$. Thus, there exists $p \in J$ such that $\ell < p < x_1$. Hence, by (3), $p \in [x_0, x_1]$. Therefore, by (2), $p \in I$, so f is defined at p. Thus, since $p \in J$, $f(p) \in (-\infty, c)$. Hence, since $p \in [x_0, x_1]$, we have that $p \in A$. Therefore, since $\ell < p$, we have a contradiction to ℓ being an upper bound for A. This proves (4).

(5) $f(\ell) \leq c$.

Proof of (5)*:* Suppose by way of contradiction that $f(\ell) > c$. Then, since f is continuous at ℓ, there is an open interval J' such that $\ell \in J'$ and $f(J') \subset (c, \infty)$ (by Exercise 3.14). Since $f(\ell) > c$, we see from (1) that $\ell \neq x_0$; hence, by (3), $\ell > x_0$. Thus, since ℓ is the *least* upper bound of A, there is a point $a \in A \cap J'$. Since $a \in A$, $f(a) < c$; on the other hand, since $a \in J'$, $f(a) > c$. This establishes a contradiction. Therefore, we have proved (5).

By (4) and (5), $f(\ell) = c$. This proves the theorem under the assumption in (1).

To complete the proof of the theorem, we must consider the case when (1) is changed to $f(x_1) < c < f(x_0)$ (and, as before, $x_0 < x_1$). We can prove the theorem for this case by going through the proof we have done and making the necessary adjustments; however, we prefer to sieze this opportunity to introduce a standard technique – a trick.

Observe that we have proved the theorem for *any* continuous function $g : I \to \mathsf{R}^1$ and for *any* point $b \in \mathsf{R}^1$ such that $g(x_0) < b < g(x_1)$, where $x_0, x_1 \in I$ and $x_0 < x_1$, We can use this to prove the theorem for the given function f when $f(x_1) < c < f(x_0)$, as follows:

Assume that $x_0, x_1 \in I$, with $x_0 < x_1$, and that $c \in \mathsf{R}^1$ such that the reverse inequalities in (1) hold, namely,

$$f(x_1) < c < f(x_0).$$

Consider the function $g = -f$. Then g is continuous by Corollary 4.10 and

$$g(x_0) < -c < g(x_1).$$

Hence, by the observation in the preceding paragraph, there is a point $p \in I$ such that $g(p) = -c$. Clearly, then, $f(p) = c$. ¥

Remember the "trick" employed in the last part of the proof of the Intermediate Value Theorem and use the trick to your advantage in the future.

The Intermediate Value Theorem is an existence theorem – it tells us that certain values of the function exist, but it does not tell us *where* in the domain a particular value of the function is attained. In addition, the proof is not constructive – the proof does not locate where a particular value is attained. For example, see Exercise 5.5.

We will see many applications of the Intermediate Value Theorem. We mention one application that is particularly important: We use the Intermediate Value Theorem to characterize one - to - one continuous functions on intervals as being either strictly increasing or strictly decreasing (Theorem 8.4); this leads to a proof of the Inverse Function Theorem (Theorem 8.7).

If X is any subset of R^1 and $f : X \to \mathsf{R}^1$ is continuous, then the Intermediate Value Theorem can be applied to intervals contained in X. We see why this is so from the first exercise below.

Exercise 5.3: The restriction of a continuous function is continuous (the restriction of a function is defined in the second paragraph of section 5 of Chapter III); in fact, if $f : X \to \mathsf{R}^1$ is continuous at a point $p \in X$ and if $X' \subset X$ such that $p \in X'$, then $f|X'$ is continuous at p.

Exercise 5.4: Use Theorem 5.2 to give a very short (and elegant) proof that every positive real number has a positive square root (which we proved earlier in Theorem 1.25).

Exercise 5.5: Let $f(x) = \frac{35}{\sqrt{2}x^{14}+5x^{10}+9x^8+3x^4+7}$. Show that $f(x) = \frac{10}{7}$ for some $x \in [0,1]$.

Exercise 5.6: Prove that if f is a polynomial of odd degree, then f has a root (i.e., $f(x) = 0$ for some $x \in \mathsf{R}^1$).

Give an example of a polynomial of even degree that does not have a root.

Exercise 5.7: If $f : [0,1] \to [0,1]$ is continuous, then $f(p) = p$ for some point p. (For any function f, a point p such that $f(p) = p$ is called a *fixed point of the function* f.)

Exercise 5.8: Assume that $f : [0,1] \to \mathsf{R}^1$ is continuous, $f(0) \leq 0$ and $f(1) \geq 1$. Then the equation $f(x) = x^2$ has a solution.

Exercise 5.9: You leave your home at 8 P.M. and walk to a friend's home, arriving at 8 : 30 P.M. You stay overnight, and the next evening you leave your friend's home at 8 P.M. and arrive home at 8 : 30 P.M., retracing exactly the same route as the evening before. At some time between 8 P.M. and 8 : 30 P.M., you are at exactly the same place on the route both evenings. Why?

Exercise 5.10: No interval is the union of two or more (including infinitely many) mutually disjoint open intervals.

(*Hint:* Assume to the contrary, and find a continuous function that contradicts the Intermediate Value Theorem (Theorem 5.2).)

2. The Maximum - Minimum Theorem

A continuous function defined on an interval may not have a largest or smallest value (e.g., $f(x) = x$ for $0 < x < 1$). On the other hand, when the interval is closed and bounded, any continuous function on the interval has *both* a largest value and a smallest value; this result is called the Maximum - Minimum Theorem. The theorem fails for bounded intervals that are not closed (by the example above) and for closed intervals that are not bounded (by restricting $f(x) = x$ to the closed interval $[1, \infty)$).

We devote this section to proving the Maximum - Minimum Theorem, which is Theorem 5.13.

We introduce terminology that we use throughout the section (and later).

Definition: We define bounded set, bounded function, maximum value and minimum value (extreme values).

- A subset X of R^1 is a *bounded set* provided that there exists $M > 0$ such that $X \subset (-M, M)$.
- A function $f : X \to \mathsf{R}^1$ is a *bounded function*, or is *bounded on* X, provided that $f(X)$ is a bounded set.
- The *maximum value*, or *largest value*, of a function $f : X \to \mathsf{R}^1$ is the value $f(p)$, if it exists, such that $f(p) \geq f(x)$ for all $x \in X$.
- The *minimum value*, or *smallest value*, of a function $f : X \to \mathsf{R}^1$ is the value $f(p)$, if it exists, such that $f(p) \leq f(x)$ for all $x \in X$.
- Taken together, the maximum value and the minimum value of a function (if they exist) are called the *extreme values* of the function.

We first prove a general theorem commonly called the Nested Interval Property. The Nested Interval Property seems to have nothing to do with the subject at hand; however, the property is the basis for the proof the Maximum - Minimum Theorem. The proof of the Nested Interval Property only uses the Completeness Axiom; thus, we could have presented the result and its proof back in Chapter I. We waited until now in order to give an immediate application of the result to a situation that is far removed from the result itself. Thus, the use of the Nested Interval Property to prove the Maximum - Minimum Theorem illustrates the many diverse and unexpected applications of the Nested Interval Property.

Theorem 5.11 (Nested Interval Property): Let $I_1, I_2, ..., I_n, ...$ be closed and bounded intervals such that $I_n \supset I_{n+1}$ for each n. Then

$$\cap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof: For each $n = 1, 2, \ldots$, let $I_n = [a_n, b_n]$. Since $I_n \supset I_{n+1}$ for each n, we see easily that

(1) $a_i \leq b_j$ for all $i, j = 1, 2, ...$.

Let $A = \{a_n : n = 1, 2, ...\}$. By (1), b_1 is an upper bound for A. Therefore, since $A \neq \emptyset$, A has a least upper bound ℓ (by the Completeness Axiom).

We show that $\ell \in \cap_{n=1}^{\infty} I_n$, which proves the theorem.

Since ℓ is an upper bound for A, $a_n \leq \ell$ for each n. Hence, to show that $\ell \in \cap_{n=1}^{\infty} I_n$, we are left to show that $\ell \leq b_n$ for each n. But this is easy to show: If it were true that $\ell > b_j$ for some j, then, since b_j is an upper bound for A by (1), ℓ would not be the *least* upper bound for A. ¥

The Nested Interval Property is simply another way of stating the Completeness Axiom; you will be asked to prove this (Exercise 5.17).

Next, we prove a preliminary lemma. The lemma will be subsumed by our main theorem; nevertheless, the lemma is a convenient way to break the proof of our main theorem into two parts.

Lemma 5.12: If $f : [a, b] \to \mathsf{R}^1$ is continuous, then f is bounded.

Proof: Let

$$A = \{x \in [a, b] : f([a, x]) \text{ is bounded}\}.$$

We prove the lemma by proving that $b \in A$.

Since $a \in A$, $A \neq \emptyset$; also, b is an upper bound for A. Therefore, by the Completeness Axiom, there is a least upper bound ℓ for A.

We prove that $b \in A$ by proving that $\ell \in A$ and then proving that $\ell = b$.

Note that $\ell \in [a, b]$ (since $a \in A$ and b is an upper bound for A). Hence, f is defined at ℓ and, therefore, f is continuous at ℓ. Thus, by Exercise 3.14, there is an open interval $J = (s, t)$ in R^1 such that

(1) $\ell \in J = (s, t)$ and $f(J) \subset (f(\ell) - 1, f(\ell) + 1)$.

We now prove that $\ell \in A$ (draw a picture as the proof progresses). Since $a \in A$, we assume for the proof that $\ell \neq a$; hence, $\ell \in (a, b]$. Thus, since $\ell \in J = (s, t)$ (by (1)) and since ℓ is the least upper bound for A, there is a point $x \in A$ such that $s < x \leq \ell$. Since $x \in A$, $f([a, x])$ is bounded. Also, since $[x, \ell] \subset J$ (by (1)), $f([x, \ell])$ is bounded (by (1)). Thus, since the union of two bounded sets is bounded and since $f([a, x]) \cup f([x, \ell]) = f([a, \ell])$, we have that $f([a, \ell])$ is bounded. Therefore, since $\ell \in [a, b]$, we have proved that $\ell \in A$.

Finally, we show that $\ell = b$. Suppose by way of contradiction that $\ell \neq b$. Then, since $\ell \in [a, b]$, $\ell < b$. Thus, since $\ell \in J$ (by (1)), there is a point $z \in [a, b] \cap J$ such that $z > \ell$. Since $\ell, z \in J$, we see that $[\ell, z] \subset J$; hence, by (1), $f([\ell, z])$ is bounded. Also, since $\ell \in A$ (proved above), $f([a, \ell])$ is bounded. Hence, $f([a, \ell]) \cup f([\ell, z])$ is bounded. Thus, $f([a, z])$ is bounded. Therefore, since $z \in [a, b]$, we have proved that $z \in A$. Thus, since $z > \ell$, we have a contradition to the fact that ℓ is an upper bound for A. Therefore, $\ell = b$.

We have proved that $\ell \in A$ and that $\ell = b$. Hence, $b \in A$. Therefore, from the definition of A, we see that f is bounded. ¥

We are ready to prove our main theorem. The proof illustrates an important technique - the bisection procedure - that has numerous applications in calculus as well as in other areas such as continuum theory, dynamics and chaos.

Theorem 5.13 (Maximum - Minimum Theorem): If $f : [a,b] \to \mathsf{R}^1$ is continuous, then f has a (unique) maximum value and a (unique) minimum value. Thus, $f([a,b])$ is a closed and bounded interval.

Proof: Since the theorem is trivially true when $a = b$, we assume for the proof that $a < b$. We prove that f has a maximum value; the proof that f has a minimum value is left for the reader (Exercise 5.14).

By Lemma 5.12, $f([a,b])$ is bounded. Therefore (since $f([a,b])$ is nonempty), we have by the Completeness Axiom that $f([a,b])$ has a least upper bound ℓ.

We prove that ℓ is a value of f; obviously, then, ℓ is the maximum value of f.

We inductively define closed and bounded intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ by bisecting, as follows: Let $I_1 = [a,b]$, and note that $\ell = lub\, f(I_1)$. Assume inductively that we have defined a closed and bounded interval $I_n = [a_n, b_n]$ for some $n \geq 1$ such that $\ell = lub\, f(I_n)$. Let m denote the midpoint of I_n (i.e., $m = \frac{a_n + b_n}{2}$). Then, since $\ell = lub\, f(I_n)$, we see easily that

$$\text{(i) } \ell = lub\, f([a_n, m]) \quad \text{or} \quad \text{(ii) } \ell = lub\, f([m, b_n]).$$

Define I_{n+1} to be $[a_n, m]$ if (i) holds and define I_{n+1} to be $[m, b_n]$ if (ii) holds and (i) does not hold. Then, by the Induction Principle (Theorem 1.20), we have defined I_n for each $n = 1, 2, \ldots$.

The intervals I_n have the following three important properties, each of which follows easily from the way we defined the intervals:

(1) For each n, $I_n \supset I_{n+1}$ and I_n is a closed and bounded interval;

(2) the length of I_n is $\frac{1}{2^{n-1}}(b-a)$ for each n;

(3) $\ell = lub\, f(I_n)$ for each n.

By (1) and the Nested Interval Property (Theorem 5.11), there is a point $p \in \bigcap_{n=1}^{\infty} I_n$.

We prove that $f(p) = \ell$. Suppose, as will lead to a contradiction, that $f(p) \neq \ell$. Then, since ℓ is an upper bound for $f([a,b])$, $f(p) < \ell$. Hence,

$$f(p) \in (-\infty, \tfrac{f(p)+\ell}{2}).$$

Thus, since f is continuous at p, we see from Exercise 3.14 that there is an open interval J such that $p \in J$ and

$$f(J) \subset (-\infty, \tfrac{f(p)+\ell}{2}).$$

Since J is an open interval containing p and since $p \in I_n$ for all n, it follows from (2) that $I_k \subset J$ for some k. Therefore,

$$f(I_k) \subset (-\infty, \tfrac{f(p)+\ell}{2}).$$

Hence, $\frac{f(p)+\ell}{2}$ is an upper bound for $f(I_k)$. Thus, since $\frac{f(p)+\ell}{2} < \ell$, we have a contradiction to (3). Therefore, $f(p) = \ell$.

We have proved that ℓ is the maximum value of f. Finally, assuming we have proved that f has a minimum value m (Exercise 5.14), we see that $f([a,b])$ is a closed and bounded interval: For by Theorem 5.2, $f([a,b]) = [m,\ell]$. ¥

The Maximum - Minimum Theorem and its proof have the same inherent characteristics as the Intermediate Value Theorem and its proof: The Maximum - Minimum Theorem is an existence theorem, and its proof does not tell us what the extreme values of f are or where in the domain $[a,b]$ they are attained. Let us illustrate. Consider the function f defined on the interval $[0,6]$ by

$$f(x) = 4x^3 - 36x^2 + 77x;$$

we know from the Maximum - Minimum Theorem that f has extreme values on $[0,6]$; however, we do not know (at this time) what the extreme values are or at which points in $[0,6]$ they are attained. See if you can find them, even with a hand calculator (but, *no* calculus allowed!).

Finding extreme values and where they are attained is a very important problem; differential calculus is designed to provide solutions to the problem. We will return to the problem of finding extreme values in our study of derivatives. We focus on the problem in Chapters IX and X (you will be asked to analyze the example above in Exercises 10.21 and 10.38).

Exercise 5.14: Finish the proof of Theorem 5.13 by proving that f has a minimum value.

Exercise 5.15: There are two positive real numbers such that the sum of their squares is $\sqrt{3}$ and such that their product is as large as possible.

Exercise 5.16: Use the bisection procedure in the proof of Theorem 5.13 to prove that any bounded infinite subset of R^1 has a limit point in R^1.

Exercise 5.17: Prove that the Nested Interval Property (Theorem 5.11) is equivalent to the Completeness Axiom. (The proof of Theorem 5.11 shows that the Completeness Axiom implies the Nested Interval Property).

Exercise 5.18: In addition to the assumptions for the Nested Interval Property (Theorem 5.11), assume that the length of the interval I_n is less than $\frac{1}{n}$. Then $\cap_{n=1}^{\infty} I_n$ contains exactly one point.

Exercise 5.19: True or false: If $f : (a,b] \to \mathsf{R}^1$ is continuous, then f has a maximum value or f has a minimum value.

Exercise 5.20: If X is an unbounded subset of R^1, then there is a continuous function $f : X \to \mathsf{R}^1$ such that f is unbounded.

Exercise 5.21: Is there a continuous function on R^1 that attains each of its values exactly twice?

Exercise 5.22: Is there a continuous function on R^1 that attains each of its values exactly three times?

Chapter VI: Introduction to the Derivative

The derivative of a function at a point is a general notion with numerous interpretations. The two most prominent interpretations are the instantaneous rate of change of a function at a point and the slope of the tangent line to the graph of a function at a point. In section 1 we first discuss the physical and geometric ideas that lead to the definition of the derivative; then we present the formal definition of the derivative and illustrate the definition in connection with tangent lines. In section 2 we relate differentiability to continuity. In the last section, we discuss linear approximation.

1. Definition of the Derivative

The definition of the derivative of a function took almost two millennia to be developed and rigorously understood. The notion comes from classical geometry.

For centuries, geometers were concerned with finding tangents to surfaces. Apollonius (262 - 190 B.C.) constructed tangents to conic sections. R. Descartes (1596 - 1650) used tangents to circles to find tangents to curves by "fitting" a circle to a point on the curve and declaring the tangent to the circle at the point to be the tangent to the curve at the point. P. Fermat (1601 - 1665) found tangents to curves using the so - called difference quotient we use today.[4]

But tangents to curves are not just a curiosity for geometers: Tangents to curves can describe physical action. We mention two examples. The first example concerns the tangent line itself, and the second example concerns the slope of the tangent line.

A tangent line to a curve can be interpreted as the path along which an object would naturally move were the object not constrained. You can see this experimentally: Attach a small weighty object to one end of a piece of string, twirl the object while holding the other end of the string, and then let go – the object goes in the direction tangent to its originally circular path at the point where it was released.

The slope of the tangent line to a curve at a point can be thought of as representing the velocity of a particle at the point. But wait! We all know what velocity is – $\frac{\text{distance}}{\text{time}}$ – but what is *velocity at a point* – $\frac{0}{0}$? Not hardly! And this is where the notion of limit steps in: Let $d(t)$ denote the distance a particle has moved from its initial position at time $t = 0$ to its position at time $t > 0$; assume that the particle is at a point p at time $t = t_0$. Then the velocity $v(p)$ of the particle at the point p should be the limit of the velocities over times $t \neq 0$ as $t \to t_0$ (if the limit exists):

$$v(p) = \lim_{t \to 0} \frac{d(t_0 + t) - d(t_0)}{t} .$$

[4] Fermat used difference quotients without the notion of limit, which came later. The modern day definition of limit is due to K. Weierstrass (1815 - 1897), but the essence of the notion is traceable back to I. Newton (1642 - 1727), who thought in terms of "ultimate ratios" of infinitesimal increments (without a definition).

We call $v(p)$ the *instantaneous velocity of the particle at* $x = p$, and we call $\frac{d(t_0+t)-d(t_0)}{t}$ the *average velocity of the particle over time* t ($t \neq 0$). Thus, the instantaneous velocity at p is the limit of the average velocities.

Now, note that for each time $t \neq 0$, the expression $\frac{d(t_0+t)-d(t_0)}{t}$ is the slope of the secant line joining the points $(t_0, d(t_0))$ and $(t_0 + t, d(t_0 + t))$ on the graph of the function d. The limit of these slopes, $\lim_{t\to 0} \frac{d(t_0+t)-d(t_0)}{t}$, should be considered to be the slope of the tangent line to the graph of the function d at $(t_0, d(t_0))$.

We conclude that we have two interpretations for $\lim_{t\to 0} \frac{d(t_0+t)-d(t_0)}{t}$: the instantaneous velocity of a particle at time t_0, and the slope of the tangent line to the graph of the function d at $(t_0, d(t_0))$.

The derivative of a function (which we define below) is merely a general formulation of what we have described. Indeed, there is no reason whatsoever to restrict what we have done to the setting of moving particles or to slopes of tangent lines. Nevertheless, the preceding discussion gives us an intuitive understanding, a point of reference if you like, for the definition of derivative.

Definition: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$ such that p is a limit point of X. We say that f *is differentiable at* p provided that the limit

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$$

exists, in which case we call the limit the *derivative of* f *at* p, denoted by $f'(p)$.

We say that f is *differentiable on* X (or just *differentiable* when the domain X is evident) provided that f is differentiable at each point of X.

If f is differentiable on X, then the *derivative of* f (*on* I) is the function f' that assigns the value $f'(x)$ to each point $x \in X$.

With the discussion above in mind, we sometimes use descriptive terminology: We call $f'(p)$ the *slope of the tangent line to the graph of* f *at* $(p, f(p))$; when we consider f to be the distance an object has traveled with respect to a variable x (usually time), we call $f'(p)$ the *instantaneous velocity of the object at* $x = p$. The descriptive phrases are no longer just intuitive ideas – as of now, they are defined to be the derivative of f at p.

We see that, in general, $f'(p)$ can be thought of as the *instantaneous rate of change of* f *at* p. Rate of change can refer to a number of physical quantities that change, for example, with time: The size of a population, the financial return on an investment, the amount of rainfall, the amount of a product produced in a chemical reaction or in a business, etc. The study of derivatives is, therefore, the study of many ideas at the same time. This illustrates a major aspect of the beauty of mathematics – the ability of mathematics to unify a number of seemingly different ideas.

The definition of the derivative at a point presupposes that the point is in the domain of the function and that the point is a limit point of the domain. Thus, when we assume that a function is differentiable at a point, we do not explicitly mention the conditions about the point.

Before proceeding, we make two comments about the defintion of derivative.

Our first comment concerns terminology. Assume that $f : [a,b] \to \mathsf{R}^1$ is a differentiable function. Then, according to our terminology, $f'(a)$ and $f'(b)$ are derivatives. This terminology contrasts with the usual terminology: in other books, $f'(a)$ and $f'(b)$ are called one - sided derivatives (i.e., derivatives from the right or from the left, respectively). The reason we prefer our terminology goes back to our comments about limits in the last section of Chapter III (above Exercise 3.17). We will consider one - sided derivatives when there is a good reason to do so; for example, one situation in which one - sided derivatives come up naturally is in the next section.

Our second comment is a clarification concerning the limit in the definition of derivative. In order that $\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$ make sense, 0 must be a limit point of $\{h \in \mathsf{R}^1 : p+h \in X\}$ (as required in the definition of limit in section 1, Chapter III); the reader should check that this is so:

Exercise 6.1: If $X \subset \mathsf{R}^1$ and p is a limit point of X, then 0 is a limit point of $\{h \in \mathsf{R}^1 : p+h \in X\}$.

We conclude the section with three examples. The examples illustrate various aspects of the definition of the derivative in relation to tangent lines.

We have defined the slope of the tangent line to the graph of a function f at $(p, f(p))$ to be $f'(p)$; thus, we had better make sure that when the graph of f itself is a line, then $f'(p)$ is the slope of the line (in the sense of precalculus):

Example 6.2: Let $f(x) = mx + b$, the slope - intercept form of a line with slope m. We show that $f'(p) = m$ for any point $p \in \mathsf{R}^1$:

$$f'(p) = \lim_{h\to 0} \frac{f(p+h)-f(p)}{h} = \lim_{h\to 0} \frac{m(p+h)+b-(mp+b)}{h} = \lim_{h\to 0} \frac{mh}{h}$$
$$= \lim_{h\to 0} m = m.$$

Our next example illustrates how to find the equation of the tangent line to the graph of a function at a point of the graph.

Example 6.3: Let $f(x) = x^2$ (all $x \in \mathsf{R}^1$). We find the equation of the tangent line to the graph of f at the point $(3, 9)$. We first compute the derivative of f : For any given point x,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h}$$
$$= \lim_{h\to 0} 2x + h \overset{4.16}{=} 2x + 0 = 2x.$$

Thus, $f'(3) = 6$. Therefore, the equation of the tangent line to the graph of f at the point $(3, 9)$ is $y - 9 = 6(x - 3)$, or $y = 6x - 9$.

In geometry we are accustomed to a tangent line being on "one side" of the curve and only touching the curve at the point of tangency. The following example shows that tangent lines as we have defined them do not always behave that way:

Example 6.4: Let $f(x) = x^3$ (all $x \in \mathsf{R}^1$). Then, for any given point x,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 \stackrel{4.16}{=} 3x^2.$$

Hence, $f'(0) = 0$, so the equation of the tangent line to the graph of f at $(0,0)$ is $y = 0$; therefore, since $x^3 < 0$ when $x < 0$ and $x^3 > 0$ when $x > 0$, the tangent line crosses the graph of f – the tangent line is not even locally on one side of the graph of f. Next, note that since $f'(1) = 3$, the equation of the tangent line to the graph of f at $(1,1)$ is $y = 3x - 2$; therefore, the tangent line intersects the graph of f at the two points $(1,1)$ and $(-2,-8)$.

Exercise 6.5: Find the points at which the function $f(x) = \frac{x}{x+1}$ is differentiable and find its derivative at those points.

Exercise 6.6: Find the points at which the function $f(x) = \sqrt{x}$ is differentiable and find its derivative at those points.

Exercise 6.7: Find the points at which the function $f(x) = \sqrt{x^2+1}$ is differentiable and find its derivative at those points.

Exercise 6.8: Let $f(x) = \frac{1}{\sqrt{x}}$. Find the equation of the tangent line to the graph of f at the point $(4, \frac{1}{2})$.

Exercise 6.9: Assume that f is a function defined on an open interval I and that f is differentiable at some point $p \in I$ with $f'(p) \neq 0$. Then there exists $\delta > 0$ such that for all $x \in I$ with $x \neq p$ and $|x-p| < \delta$, $f(x) \neq f(p)$.

Exercise 6.10: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$ such that p is a limit point of X. Then f is differentiable at p if and only if $\lim_{x\to p} \frac{f(x)-f(p)}{x-p}$ exists, in which case $\lim_{x\to p} \frac{f(x)-f(p)}{x-p} = f'(p)$.

Exercise 6.11: Assume that f is a function defined on an open interval I and that f is differentiable at some point $p \in I$. Find

$$\lim_{h\to 0} \frac{f(p+h)-f(p-h)}{h}.$$

Exercise 6.12: Assume that f is a function defined on an open interval I and that f is differentiable at some point $p \in I$. Find

$$\lim_{h\to 0} \frac{f(p+2h)-f(p)}{h}.$$

Exercise 6.13: Let f be a function defined on an open interval I such that f is differentiable at some point $p \in I$. Let φ be a function defined on an open interval J about 0 such that φ is continuous at 0, $\varphi(0) = 0$, and $\varphi(x) \neq 0$ when $x \neq 0$. Then

$$\lim_{h\to 0} \frac{f(p+\varphi(h))-f(p)}{\varphi(h)} = f'(p).$$

(*Hint:* The formula

$$g(y) = \begin{cases} \frac{f(p+y)-f(p)}{y} & \text{, if } y \neq 0 \\ f'(p) & \text{, if } y = 0. \end{cases}$$

defines a function g on some open interval about 0. Make use of g.)

2. Differentiability and Continuity

In the forthcoming discussion, we assume that the functions are defined on intervals (or at least on sets that have no isolated points).

Continuity and differentiability can be thought of as smoothness conditions on the graph of a function: Continuity says the graph of a continuous function is smooth enough so that the function does not jump; differentiability says the graph of a differentiable function is smooth enough to have a (unique) tangent line at each point. The natural question is Are the two smoothness conditions related? In other words, does differentiability imply continuity, does continuity imply differentiability, or does neither imply the other?

It seems reasonable to expect that differentiability implies continuity: if the graph of a function is smooth enough to have a (unique) tangent line at each point of its graph, then the graph of the function should be smooth enough to keep the function from jumping. However, our experience with Example 6.4 shows that tangent lines do not always work the way we expect, so we had better proceed with caution.

We want to try to show that if $\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$ exists, then $\lim_{x\to p} f(x) = f(p)$ (recall Theorem 3.12). To have a chance to prove this, we need to write x in terms of h (or h in terms of x); we have essentially already done this in Exercise 6.10, which says that

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} = \lim_{x\to p} \frac{f(x)-f(p)}{x-p}.$$

Now, can you see what to do next? Think about it before reading the proof of the theorem below.

Theorem 6.14: Let $X \subset \mathsf{R}^1$, and let $f : X \to \mathsf{R}^1$ be a function. If f is differentiable at p, then f is continuous at p.

Proof: We want to prove $\lim_{x\to p} f(x) = f(p)$ (recall Theorem 3.12). The key to answering the question in the preceding discussion is the observation that writing $\lim_{x\to p} f(x) = f(p)$ is the same as writing $\lim_{x\to p}(f(x) - f(p)) = 0$.

To see why what we just said works, recall that $\lim_{x\to p} \frac{f(x)-f(p)}{x-p} = f'(p)$ (by Exercise 6.10) and that $\lim_{x\to p}(x - p) = 0$ (by Theorem 4.16); then

$$\lim_{x\to p}(f(x) - f(p)) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p}(x - p) \overset{4.9}{=} f'(p) \cdot 0 = 0.$$

Therefore, $\lim_{x\to p} f(x) = f(p)$. ¥

Now, having proved that differentiability implies continuity, we address the question of whether the converse is true: Does continuity imply differentiability? If you worked Exercise 6.6, you already know the answer is *no*: The function $f(x) = \sqrt{x}$ is continuous at every point of $[0, \infty)$, but the function is not differentiable at $p = 0$. Perhaps you think this example is not very satisfactory because 0 is an end point of the interval on which f is defined – indeed, unusual things can happen at end points. But we can extend the function f so that the resulting function is continuous on the entire real line and not differentiable at 0: Simply let

$$g(x) = \begin{cases} \sqrt{x} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0. \end{cases}$$

"OK," you say, "the example with g is fine, but maybe we should extend the notion of tangent line to include $x = 0$ as a tangent line to the graph of $f(x) = \sqrt{x}$ at the origin."

"Yes, we can do that," I reply, "but that's a subject for another time."

An example showing something is false is one thing; a general principle showing *why* it is false is quite another. What is a general underlying principle that would lead easily to many continuous functions that are not differentiable?

The key to answering the question is to carefully examine why g in the discussion above is not differentiable; if you do so, you will arrive naturally at the notion of one-sided derivatives, defined below, and the simple theorem that follows.

Definition. Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$ such that p is a limit point of $X \cap (-\infty, p]$. We say f is *differentiable from the left at* p, written $f'_-(p)$, provided that the function $f|X \cap (-\infty, p]$ is differentiable at p, in which case $f'_-(p)$ is the derivative of $f|X \cap (-\infty, p]$ at p. We call $f'_-(p)$ *the left-hand derivative of* f *at* p (when it exists).

Similarly, assuming that p is a limit point of $X \cap [p, \infty)$, we say f is *differentiable from the right at* p, written $f'_+(p)$, provided that the function $f|X \cap [p, \infty)$ is differentiable at p, in which case $f'_+(p)$ is the derivative of $f|X \cap [p, \infty)$ at p. We call $f'_+(p)$ *the right-hand derivative of* f *at* p (when it exists).

Theorem 6.15: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$ such that p is a two-sided limit point of X. Then f is differentiable at p if and only if $f'_-(p) = f'_+(p)$, in which case

$$f'_-(p) = f'(p) = f'_+(p).$$

Proof: The theorem follows immediately from the theorem on one-sided limits (Theorem 3.16). ¥

We can now easily construct many functions that are continuous on R^1 but that are not differentiable at certain points. All we need to do is cut and paste: Start with two functions, f and g, that are continuous on R^1, that agree at a point p, but that have different derivatives at p (e.g., two functions whose graphs are straight lines with different slopes); then cut the domains at p and paste the restricted functions together, thereby forming the new function h given by

$$h(x) = \begin{cases} f(x) & \text{, if } x \leq p \\ g(x) & \text{, if } x > p. \end{cases}$$

The effect is that h has a "corner" in its graph at p which keeps h from being differentiable at p (but the "corner" does not keep h from being continuous). We give a specific example.

Example 6.16: Let

$$h(x) = \begin{cases} x & , \text{if } x \leq 0 \\ 3x & , \text{if } x > 0. \end{cases}$$

Then h is continuous (by Theorem 4.16 and Exercise 5.3 when $x \neq 0$, and by Theorem 3.12 and Theorem 3.16 when $x = 0$), and h is not differentiable at $x = 0$ by Theorem 6.15.

One-sided derivatives can be used to show that lines that really look like tangent lines are not tangent lines in the sense that we have defined them. Consider the function $f : \mathsf{R}^1 \to \mathsf{R}^1$ given by

$$f(x) = \begin{cases} x^2 & , \text{if } x \leq 0 \\ \sqrt{x} & , \text{if } x > 0. \end{cases}$$

Draw a picture of the graph of f and it will look like the x-axis is tangent to the graph of f at the origin. However, using one-sided derivatives (Theorem 6.15), we see that f is not differentiable at $x = 0$; thus, the x-axis is not tangent to the graph of f at the origin. Note that the x-axis *is* tangent to the graph of $f|(-\infty, 0]$ at the origin but that the x-axis is *not* tangent to the graph of $f|[0, \infty)$ at the origin.

We summarize the section in terms of our discussion at the beginning of the section: The smoothness that differentiability imposes on the graph of a function is stronger than the smoothness that continuity imposes on the graph.

Exercise 6.17: The function f given by $f(x) = |x|$ is continuous at every point of R^1, but f is not differentiable at 0.

Exercise 6.18: Let

$$f(x) = \begin{cases} x & , \text{if } x \leq 0 \\ x^2 + x & , \text{if } x > 0. \end{cases}$$

Is there a tangent line to the graph of f at the point $(0, 0)$?

Exercise 6.19: Are there constants a and b such that the function f given by

$$f(x) = \begin{cases} x^2 + 5 & , \text{if } x \leq 1 \\ ax + b & , \text{if } x > 1 \end{cases}$$

is differentiable?

Exercise 6.20: Are there constants a and b such that the function f given by

$$f(x) = \begin{cases} ax + 2 & , \text{if } x \leq b \\ x^3 + 3 & , \text{if } x > b \end{cases}$$

is differentiable?

Exercise 6.21: Give an example of a function that is continuous on R^1 and that is not differentiable at any integer.

Exercise 6.22: Give an example of a function that is continuous on R^1 and that is not differentiable at any of the points 1, $\frac{1}{2}$, $\frac{1}{3}$, ...

Exercise 6.23: Give an example of a function that is continuous on $[0, \infty)$ and that is not differentiable at 0, 1, $\frac{1}{2}$, $\frac{1}{3}$, ...

Exercise 6.24: Let

$$f(x) = \begin{cases} x\,|x| & \text{, if } x \text{ is rational} \\ 0 & \text{, if } x \text{ is irrational.} \end{cases}$$

Determine all points x at which f is differentiable.

Exercise 6.25: True or false: If $f : \mathsf{R}^1 \to \mathsf{R}^1$ is a function such that $f'_-(0)$ and $f'_+(0)$ both exist, then f is continuous at 0.

Exercise 6.26: Find all differentiable functions $f : \mathsf{R}^1 \to \mathsf{R}^1$ such that $f \circ f = f$.

3. Linear Approximation

We refer to a function whose graph is a straight line as a *linear function* (not to be confused with linear functions in the setting of linear algebra).

We can immediately determine the exact value of a given linear function at any particular point; for example, if $f(x) = 2x + 7$, then $f(65) = 137$. On the other hand, it is more difficult, sometimes impossible, to determine exact values of other types of functions – for example, $\sqrt{65}$ and $\sin(65)$.

Consider the tangent line to graph of a differentiable function f at a point $(p, f(p))$; the tangent line is a fairly good approximation to the graph of f near the point $(p, f(p))$. Thus, tangent lines should provide a way to find fairly close approximate values for functions such as $\sqrt{x}$ and $\sin(x)$.

The general procedure of using tangent lines to find approximate values is called *linear approximation*. We describe the procedure as follows: We are given (perhaps implicitly) a differentiable function f and a point x_0 at which we want to approximate $f(x_0)$. We first find a point p close to x_0 for which we know the value $f(p)$. Next, we determine the equation of the tangent line to the graph of f at $(p, f(p))$. Finally, we use the formula for the tangent line to obtain the desired approximation.

We illustrate:

Example 6.27: We approximate $\sqrt{65}$ using linear approximation. We are implicitly given the function $f(x) = \sqrt{x}$. The function f is differentiable at every $x > 0$ and the derivative is $f'(x) = \frac{1}{2\sqrt{x}}$ (which you know if you worked Exercise 6.6). We know $\sqrt{64} = 8$ and $\sqrt{81} = 9$, so we choose $p = 64$ since 64 is closer to 65 than 81 is (we could choose 65.61, which we find by computing $(8.1)^2$, but our point here is to avoid such tedious computations). The equation of the tangent line to the graph of f at $(64, 8)$ is

$$y = \frac{1}{2\sqrt{64}}x + 4 = \frac{1}{16}x + 4.$$

We write the equation of the tangent line using function notation, $y(x) = \frac{1}{16}x+4$, to show explicitly that we consider y to be a function of x. Then, finally,

$$y(65) = \frac{65}{16} + 4 = \frac{129}{16},$$

which is our linear approximation to $\sqrt{65}$.

We could expand the approximation in Example 6.27 into its decimal equivalent, 8.0625. If we wanted accuracy only to three decimal places to the right of the decimal point, would we round up or round down? We can numerically determine the answer:

$$(8.062)^2 = 64.996 \quad \text{and} \quad (8.063)^2 = 65.012,$$

thus we would round down to 8.062. However, if you know what the graph of $f(x) = \sqrt{x}$ looks like, then you know that all tangent lines to the graph of f lie above the graph of f (except where they touch the graph); therefore, rounding down is a good bet when we use linear approximation to estimate the square root of any positive number.

Exercise 6.28: Approximate $(5.137)^3$ using linear approximation.

Assume you only want accuracy to two decimal places to the right of the decimal point; would you round your answer up or would you round your answer down? Explain why without finding the decimal representing $(5.137)^3$,

Exercise 6.29: Approximate $\sin(31°)$ using linear approximation. (Use that the derivative of $\sin(x)$ is $\cos(x)$ when x is radian measure, which we will prove in Theorem 8.20; $1° = \frac{\pi}{180}$ radians.)