$$(p-\epsilon_1, p+\epsilon_1) \cap A = \emptyset \quad \text{and} \quad (p-\epsilon_2, p+\epsilon_2) \cap B = \emptyset.$$

Hence, letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we see that $\epsilon > 0$ and that

$$(p-\epsilon, p+\epsilon) \cap (A \cup B) = \emptyset.$$

Therefore, by Theorem 2.5, $p \notin (A \cup B)^{\sim}$. ¥

Exercise 2.12: Concerning a comment in the proof of Theorem 2.11, find the flaw in the following direct argument for $(A \cup B)^{\sim} \subset A^{\sim} \cup B^{\sim}$:

As in the proof of Theorem 2.11, we can assume that $A \neq \emptyset$ and $B \neq \emptyset$. Now, let $p \in (A \cup B)^{\sim}$. Then, by Theorem 2.5, $(p-\epsilon, p+\epsilon) \cap (A \cup B) \neq \emptyset$ for each $\epsilon > 0$. Hence,

$$[(p-\epsilon, p+\epsilon) \cap A] \cup [(p-\epsilon, p+\epsilon) \cap B] \neq \emptyset \text{ for each } \epsilon > 0.$$

Thus, $(p-\epsilon, p+\epsilon) \cap A \neq \emptyset$ or $(p-\epsilon, p+\epsilon) \cap B \neq \emptyset$ for each $\epsilon > 0$. Hence, by Theorem 2.5, $p \in A^{\sim}$ or $p \in B^{\sim}$. Therefore, $p \in A^{\sim} \cup B^{\sim}$.

Exercise 2.13: If $A_1, A_2, ..., A_n$ are finitely many subsets of R^1, then

$$(\cup_{i=1}^n A_i)^{\sim} = \cup_{i=1}^n A_i^{\sim}.$$

Exercise 2.14: Would the result in Exercise 2.13 remain true for infinitely many subsets of R^1 ? In other words, if $\{A_i : i \in \mathcal{I}\}$ is an infinite collection of subsets of R^1, then is it true that

$$(\cup\{A_i : i \in \mathcal{I}\})^{\sim} = \cup\{A_i^{\sim} : i \in \mathcal{I}\})\,?$$

Theorem 2.15: For any $A \subset \mathsf{R}^1$, $(A^{\sim})^{\sim} = A^{\sim}$.

Proof: By Theorem 2.7, $A \subset A^{\sim}$. Therefore, by Exercise 2.10, $A^{\sim} \subset (A^{\sim})^{\sim}$.

To prove the reverse containment, first note that $(A^{\sim})^{\sim} \subset A^{\sim}$ if $A = \emptyset$ (since $\emptyset^{\sim} = \emptyset$); hence, we assume for the proof that $A \neq \emptyset$. Thus, $A^{\sim} \neq \emptyset$ by Theorem 2.7. Now, let $p \in (A^{\sim})^{\sim}$. Let I be an open interval such that $p \in I$. Then, since $A^{\sim} \neq \emptyset$ and $p \in (A^{\sim})^{\sim}$, $I \cap A^{\sim} \neq \emptyset$ by Theorem 2.3. Hence, there exists a point $q \in I \cap A^{\sim}$. Thus, since $A \neq \emptyset$, $I \cap A \neq \emptyset$ by Theorem 2.3. We have proved that $I \cap A \neq \emptyset$ for any open interval I such that $p \in I$. Therefore, again by Theorem 2.3, $p \in A^{\sim}$. ¥

Exercise 2.16: If $A_1, A_2, ..., A_n$ are finitely many subsets of R^1, then

$$(\cap_{i=1}^n A_i)^{\sim} \subset \cap_{i=1}^n A_i^{\sim}.$$

Exercise 2.17: Give an example of two subsets A and B of R^1 such that $(A \cap B)^{\sim} \neq A^{\sim} \cap B^{\sim}$.

Exercise 2.18: Would the result in Exercise 2.16 remain true for infinitely many subsets of R^1 ?

3. The Definition of Continuity

You have surely had some experience with the idea of a continuous function; based on your experience, you know the intuitive meaning of continuity – a continuous function is a function that does not jump. Did you ever stop and try to figure out what it really means to say that a function does not jump? Let us examine this idea.

In general, a question asked in a negative way is harder to deal with than the corresponding question posed in the positive way. So, we ask *What does it mean for a function to jump*? If a function jumps, it seems reasonable that it must jump at some point in its domain. Thus, we ask *What does it mean for a function to jump at a point p in the domain of the function*? Certainly, everyone has an instinctive feeling – some mental picture – for what this means. Let us consider an example that everyone will agree is stereotypical of the (intuitive) idea of a function jumping at p:

Example 2.19: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} 0 & \text{, if } x \leq 0 \\ 1 & \text{, if } x > 0. \end{cases}$$

The function f jumps at $p = 0$. Surely you agree. But what is the underlying reason you agree? The reason is that if you look at positive numbers that are as close as you like to 0, but not equal to 0, their values under f are one unit away from $f(0)$.

Let us look at another example, one that is more complicated than the previous one.

Example 2.20: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} 0 & \text{, if } x \text{ is rational} \\ 1 & \text{, if } x \text{ is irrational.} \end{cases}$$

The function seems to jump at every point p. Why? If p is irrational, then you know from Theorem 1.26 that there rationals as close to p as you like, and the value of f at each rational is one unit away from $f(p)$. If p is rational, then (by the natural analogue of Theorem 1.26 for irrationals) there irrationals as close to p as you like, and the value of f at each irrational is one unit away from $f(p)$.

The two examples shed light on what it means for a function to jump at p. One need only observe the common thread in the two examples: The function f in each example jumps at p because there is a set A such that p is arbitrarily close to A but $f(p)$ is *not* arbitrarily close to $f(A)$. Notice that we say the condition holds for *some* set A, not for every set. Indeed, there are some sets A in the examples such that p is arbitrarily close to A *and* $f(p)$ *is arbitrarily close to* $f(A)$. You can see this by taking A to be any set containing p in both examples or, as is more illustrative, by taking $A = (-\infty, 0)$ in Example 2.19 and by taking A to be the set of all rationals except p when p is rational in Example 2.20.

Our discussion suggests the following definition:

Definition. Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$, and let $p \in X$. We say that f *jumps at* p provided that there is a subset, A, of X such that $p \sim A$ but $f(p) \not\sim f(A)$.

Exercise 2.21: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} \frac{1}{x} & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0. \end{cases}$$

Then f jumps at 0.

Exercise 2.22: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0. \end{cases}$$

Then f jumps at 0.

We do not claim that our definition for a function to jump at a point is "correct" – that can only be ascertained by checking numerous examples to see if the definition fits our intuition, and by seeing if the definition leads to appropriate theoretical developments. At this point, we accept the definition and use it to define continuity which, after all, is why we wanted the definition in the first place.

Definition. Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$, and let $p \in X$. We say that f is *continuous at* p provided that f does not jump at p. In other words, f is *continuous at* p provided that whenever $A \subset X$ and $p \sim A$, then $f(p) \sim f(A)$.

We say that f is *continuous on* X (or just *continuous* when the domain X is clear) provided that f does not jump at any point of X.

A simple kind of function that we know from past experience is continuous is a function whose graph is a straight line. We show this kind of function is continuous in the sense of the definition above. Thus, the example lends credibility to our definition of continuity.

Example 2.23: Fix $m, b \in \mathsf{R}^1$, and let $f : \mathsf{R}^1 \to \mathsf{R}^1$ be given by

$$f(x) = mx + b, \quad \text{all } x \in \mathsf{R}^1.$$

The function f is continuous.

To prove this, let $p \in \mathsf{R}^1$ and let $A \subset \mathsf{R}^1$ such that $p \sim A$.

If $m = 0$, then $f(p) = b$ and $f(A) = \{b\}$; thus, since $b \sim \{b\}$ by Theorem 2.7, we have that $f(p) \sim f(A)$. This proves that f is continuous at p when $m = 0$.

Next, assume that $m > 0$. We show that $f(p) \sim f(A)$ by using Theorem 2.3. For this purpose, let $I = (a, c)$ be an open interval such that $f(p) \in I$. Let J be the open interval defined by

$$J = (\tfrac{a-b}{m}, \tfrac{c-b}{m}).$$

We see that $p \in J$ as follows: Since $f(p) \in I$, $a < mp + b < c$; thus, since $m > 0$, $\frac{a-b}{m} < p < \frac{c-b}{m}$, so $p \in J$. Therefore, since $p \sim A$, we have by Theorem 2.3 that there is a point $x \in J \cap A$. Since $x \in J$, $\frac{a-b}{m} < x < \frac{c-b}{m}$; thus, since $m > 0$,

$$a < mx + b < c;$$

hence, $f(x) \in I$. Also, since $x \in A$, $f(x) \in f(A)$. Hence, $f(x) \in I \cap f(A)$. This proves that any open interval containing $f(p)$ has a nonempty intersection with $f(A)$. Thus, by Theorem 2.3, $f(p) \sim f(A)$. Therefore, we have proved that f is continuous at p when $m > 0$.

Finally, assume that $m < 0$. Then the proof that f is continuous at p is similar to the proof when $m > 0$: Let I be as before, redefine J to be the interval $(\frac{c-b}{m}, \frac{a-b}{m})$, and make the obvious changes necessitated by the assumption that $m < 0$.

We prove in the next chapter that our definition of continuity is correct in the sense that it is equivalent to the definition of continuity that you have (probably) seen in your study of calculus. So, why did we define continuity as we did? The answer is purely philosophical: *We adhere to the principle that a definition should convey as much as possible the fundamental idea behind the notion being defined.*

Exercise 2.24: Define $f : \mathsf{R}^1 \to \mathsf{R}^1$ by letting $f(x) = x^2$. Then f is continuous.

Exercise 2.25: Define $f : \mathsf{R}^1 - \{0\} \to \mathsf{R}^1$ by letting $f(x) = \frac{1}{x}$. Then f is continuous.

Exercise 2.26: Define $f : [0,1] \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} x & \text{, if } x = \frac{1}{n} \text{ for some } n \in \mathsf{N} \\ 0 & \text{, otherwise.} \end{cases}$$

At what points p is f continuous?

Exercise 2.27: Any function $f : \mathsf{N} \to \mathsf{R}^1$ is continuous.

4. Limit Points and Isolated Points

Limit points and isolated points of sets will be important in our discussion of limits in the next chapter.

Definition. Let $X \subset \mathsf{R}^1$. A point $p \in \mathsf{R}^1$ is called a *limit point of* X provided that $p \sim X - \{p\}$[2]. A point of X that is not a limit point of X is called an *isolated point of* X.

We let X^ℓ denote the set of all limit points of X.

[2] $\mathsf{X} - \{\mathsf{p}\}$ denotes all the points of X except p (if $\mathsf{p} \notin \mathsf{X}$, obviously $\mathsf{X} - \{\mathsf{p}\} = \mathsf{X}$). More generally, for any two sets A and B, $\mathsf{A} - \mathsf{B} = \{\mathsf{x} \in \mathsf{A} : \mathsf{x} \notin \mathsf{B}\}$; the set $\mathsf{A} - \mathsf{B}$ is called the *complement of* B *in* A.

Exercise 2.28: What are the limit points of $\{15\}$? What are the limit points of the interval $(0, 1)$? What are the limit points of Q? What are the limit points of $X = \{\frac{1}{n} : n \in \mathsf{N}\}$?

Exercise 2.29: For any $A \subset \mathsf{R}^1$, $A^{\sim} = A \cup A^{\ell}$.

Exercise 2.30: If $A, B \subset \mathsf{R}^1$ such that $A \subset B$, then $A^{\ell} \subset B^{\ell}$.

Exercise 2.31: For any $A, B \subset \mathsf{R}^1$, $(A \cup B)^{\ell} = A^{\ell} \cup B^{\ell}$.

Exercise 2.32: For any $A \subset \mathsf{R}^1$, $(A^{\ell})^{\ell} \subset A^{\ell}$. Must $(A^{\ell})^{\ell} = A^{\ell}$?

Exercise 2.33: Let $A \subset \mathsf{R}^1$, and let $p \in \mathsf{R}^1$. Then p is a limit point of A if and only if for each $\epsilon > 0$, the open interval $(p - \epsilon, p + \epsilon)$ contains infinitely many points of A.

Exercise 2.34: Let $X \subset \mathsf{R}^1$ and let $A \subset X$. If p is an isolated point of X, then $p \sim A$ if and only if $p \in A$.

We conclude with a simple theorem that shows that functions are always continuous at any isolated point of their domain. In other words, continuity is only in question at points of the domain that are limit points of the domain.

Theorem 2.35: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let p be an isolated point of X. Then f is continuous at p.

Proof: We prove that f is continuous at p by showing that f satisfies the definition of continuity at p (which is below Exercise 2.22).

Let $A \subset X$ such that $p \sim A$. Then, by Exercise 2.34, $p \in A$. Hence, $f(p) \in f(A)$. Thus, by Theorem 2.7, $f(p) \sim f(A)$. Therefore, we have proved that f is continuous at p. ¥

Chapter III: The Notion of Limit

We define and discuss the notion of limit of a function, commonly denoted in calculus by $\lim_{x\to p} f(x)$. In section 2, we reformulate the notion of limit completely in terms of arbitrary closeness. In section 3, we use the result in section 2 to show that our definition of continuity in the preceding chapter is equivalent to the definition of continuity as presented in calculus. In section 4, we present our rationale for introducing continuity before limits (which is contrary to common practice). In the final section, we discuss one - sided limits.

1. The Definition of Limit

You encountered limits in calculus. We state the definition for $\lim_{x\to p} f(x)$ as it is presented in calculus but in a slightly more general way – we replace the assumption in the calculus definition that f is defined at all points $x \neq p$ in an open interval about p with the less restrictive assumption that p is a limit point of the domain of f. (See the comments at the end of section 5.)

Definition. Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. We say that L *is the limit of* f *as* x *approaches* p, written $\lim_{x\to p} f(x) = L$, provided that for any given number $\epsilon > 0$, there is a number $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$, we have

$$|f(x) - L| < \epsilon.$$

The definition is complicated. Let us interpret the definition informally as a game: You give me any error $\epsilon > 0$, meaning that you will allow the values of f to deviate from L but only by less than ϵ; I win the game if for any such allowed error, I can find a δ - neighborhood of p such that the values of the function f on the neighborhood with p removed are within the prescribed error ϵ from L.

It is important in the definition of limit that we did not require p to be a point of X. Indeed, many important limits are considered when p is *not* a point of X. For example, the derivative of a function f at a point p is $\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$; the expression $\frac{f(p+h)-f(p)}{h}$ defines a function of h for which 0 is not in its domain.

We also note that the requirement that p be a limit point of X is important in the definition. For if p is not a limit point of X, then any number whatsoever is a limit of f as x approaches p, even when $p \in X$; this is seen by taking $\delta = dist(p, X - \{p\})$ (try this for any function $f : \mathsf{N} \to \mathsf{R}^1$ and any choice of L). What we are suggesting here is that the requirement that p be a limit point of X makes the limit unique (if the limit exists); we now prove that this is the case.

Theorem 3.1: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If $\lim_{x\to p} f(x) = L_1$ and $\lim_{x\to p} f(x) = L_2$, then $L_1 = L_2$.

Proof: Suppose by way of contradiction that $L_1 \neq L_2$. Let $\epsilon = \frac{|L_1 - L_2|}{2}$, and note that $\epsilon > 0$. Since $\lim_{x\to p} f(x) = L_i$ for each $i = 1$ and 2, there exist $\delta_1, \delta_2 > 0$ satisfying the following for each $i = 1$ and 2:

(*) For all $x \in X - \{p\}$ such that $|x - p| < \delta_i$, $|f(x) - L_i| < \epsilon$.

Now, let $\delta = \min\{\delta_1, \delta_2\}$, and note that $\delta > 0$. Thus, since $p \sim X - \{p\}$, we have by Theorem 2.5 that

$$(p - \delta, p + \delta) \cap (X - \{p\}) \neq \emptyset.$$

Hence, there is a point $x_0 \in (p - \delta, p + \delta) \cap (X - \{p\})$. Thus, since $\delta \leq \delta_i$ for each $i = 1$ and 2, we see from (*) that

$$|f(x_0) - L_i| < \epsilon \text{ for each } i.$$

Therefore,

$$|L_1 - L_2| = |L_1 - f(x_0) + f(x_0) - L_2| \leq |L_1 - f(x_0)| + |f(x_0) - L_2| < 2\epsilon.$$

Thus, since $\epsilon = \frac{|L_1 - L_2|}{2}$, $|L_1 - L_2| < |L_1 - L_2|$; however, this is impossible. ¥

It is convenient to have the following general agreement: *When we consider an algebraic expression as being a function, we assume, often without saying so, that the domain of the function is the largest set of real numbers for which the expression makes sense (unless we say otherwise).*

In the example below, we illustrate the thought process for computing limits of specific functions. The thought process is important even though we establish general theorems for evaluating limits in the next chapter.

Example 3.2: $\lim_{x \to 7} \frac{1}{x-4} = \frac{1}{3}$. To prove this, let $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $x \in \mathsf{R}^1 - \{4\}$ (which is the understood domain of the function $f(x) = \frac{1}{x-4}$),

$$(*) \ \left|\frac{1}{x-4} - \frac{1}{3}\right| < \epsilon \text{ when } x \neq 7 \text{ and } |x - 7| < \delta.$$

We start our search for δ by writing $\left|\frac{1}{x-4} - \frac{1}{3}\right|$ in a way that tells us how its value depends on $|x - 7|$:

$$(1) \ \left|\frac{1}{x-4} - \frac{1}{3}\right| = \left|\frac{3-(x-4)}{3(x-4)}\right| = \left|\frac{-x+7}{3(x-4)}\right| = \frac{|x-7|}{3|x-4|}.$$

Next, we make an initial restriction on δ so that we can bound the size of the last expression in (1) when $|x - 7| < \delta$. This means we want δ small enough so that if $|x - 7| < \delta$, then x is bounded away from 4. This happens for any fixed $\delta < 3$. So, we assume temporarily that $\delta \leq 1$ and, of course, that $\delta > 0$. (We will see when we make our final choice for δ why we do not simply take $\delta = 1$ here).

Now, we examine what our assumption $|x - 7| < \delta \leq 1$ says about the size of $\left|\frac{1}{x-4} - \frac{1}{3}\right|$. Since $|x - 7| < \delta \leq 1$, we see that $x > 6$ and, thus, $2 < |x - 4|$. Hence,

$$\frac{1}{|x-4|} < \frac{1}{2}.$$

Thus, $\frac{|x-7|}{3|x-4|} < \frac{|x-7|}{6}$. Therefore, by (1), we have that

(2) $\left|\frac{1}{x-4} - \frac{1}{3}\right| < \frac{|x-7|}{6}$ when $0 < \delta \leq 1$.

We now make our final choice for δ and verify that our choice works. Note that $\frac{|x-7|}{6} < \epsilon$ if $|x-7| < 6\epsilon$, and let

$$\delta = \min\{1, 6\epsilon\}.$$

Then, for all $x \in \mathsf{R}^1 - \{4\}$ such that $x \neq 7$ and $|x-7| < \delta$, we have

$$\left|\frac{1}{x-4} - \frac{1}{3}\right| \overset{(2)}{<} \frac{|x-7|}{6} < \frac{6\epsilon}{6} = \epsilon.$$

This proves (*).

Exercise 3.3: Prove that $\lim_{x \to 1} \frac{x}{x-3} = \frac{-1}{2}$.

Exercise 3.4: Prove that $\lim_{x \to 2} 4x + 5 = 13$.

Exercise 3.5: Prove that $\lim_{x \to 4} \frac{x-4}{x^2-2x-8} = \frac{1}{6}$.

Exercise 3.6: Prove that $\lim_{x \to p} |x| = |p|$.

Exercise 3.7: Prove that $\lim_{x \to p} \sqrt{x} = \sqrt{p}$ for all $p \geq 0$. (Note: $f(x) = \sqrt{x}$ is a function on $[0, \infty)$ by Theorem 1.25.)

Exercise 3.8: Prove that $\lim_{x \to -3} \frac{|x+3|}{x+3}$ does not exist.

Exercise 3.9: Assume that $\lim_{x \to p} f(x) = \sqrt{82} - 9$, where p is a limit point of the domain X of f. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in X - \{p\}$ such that $|x - p| < \delta$. If $p \in X$, must $f(p) > 0$?

Exercise 3.10: Give an example of functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ such that $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0} g(x) = 0$, and $\lim_{x \to 0} \frac{f(x)}{g(x)} = 23$.

2. Limits in Terms of Arbitrary Closeness

We reformulate the definition of limit entirely in terms of the notion arbitrary closeness. We use the reformulation in the next section.

Theorem 3.11: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. Then $\lim_{x \to p} f(x) = L$ if and only if whenever $A \subset X$ such that $p \sim A - \{p\}$, then $L \sim f(A - \{p\})$.

Proof: Assume that $\lim_{x \to p} f(x) = L$. Let $A \subset X$ such that $p \sim A - \{p\}$. We show that $L \sim f(A - \{p\})$ by using Theorem 2.5. Let $\epsilon > 0$. Then, since $\lim_{x \to p} f(x) = L$, there exists $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$, we have

$$|f(x) - L| < \epsilon.$$

Since $p \sim A - \{p\}$, there is a point $x_0 \in (p - \delta, p + \delta) \cap (A - \{p\})$ by Theorem 2.5. Hence, $|x_0 - p| < \delta$ and $x_0 \in X - \{p\}$. Thus, $|f(x_0) - L| < \epsilon$; also, since $x_0 \in A - \{p\}$, $f(x_0) \in f(A - \{p\})$. Hence,

$$f(x_0) \in (L - \epsilon, L + \epsilon) \cap f(A - \{p\}).$$

We have shown that for any $\epsilon > 0$, $(L - \epsilon, L + \epsilon) \cap f(A - \{p\}) \neq \emptyset$. Therefore, by Theorem 2.5, $L \sim f(A - \{p\})$.

Conversely, assume that $\lim_{x \to p} f(x) \neq L$. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there is a point $x_\delta \in X - \{p\}$ such that $|x_\delta - p| < \delta$ and $|f(x_\delta) - L| \geq \epsilon$. In other words, the following set is nonempty for each $\delta > 0$:

$$A_\delta = \{x \in X - \{p\} : |x - p| < \delta \text{ and } |f(x) - L| \geq \epsilon\}.$$

Now, let $A = \cup_{\delta > 0} A_\delta$. Since $A_\delta \neq \emptyset$ for each $\delta > 0$, we see that

$$(p - \delta, p + \delta) \cap A \neq \emptyset \text{ for each } \delta > 0.$$

Hence, by Theorem 2.5, $p \sim A$. Thus, since $p \notin A$, we have that

(1) $p \sim A - \{p\}$.

Since $|f(x) - L| \geq \epsilon$ for all $x \in A$,

$$(L - \epsilon, L + \epsilon) \cap f(A) = \emptyset,$$

which gives $(L - \epsilon, L + \epsilon) \cap f(A - \{p\}) = \emptyset$. Thus, by Theorem 2.5, we have that

(2) $L \not\sim f(A - \{p\})$.

Finally, we see from (1) and (2) that the condition in the second part of our theorem is false for the set A we have defined. ¥

3. The Limit Characterization of Continuity

We show that our definition of continuity in the preceding chapter is equivalent to the definition of continuity as presented in calculus. In other words, the standard definition of continuity (in terms of limits) is, for us, a theorem. The reason for this seemingly strange development is discussed in section 4.

Theorem 3.12: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$ such that p is a limit point of X. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Proof: Assume that f is continuous at p. Then, for any $A \subset X$ such that $p \sim A - \{p\}$, we see from our definition of continuity that $f(p) \sim f(A - \{p\})$. Therefore, by Theorem 3.11, $\lim_{x \to p} f(x) = f(p)$.

Conversely, assume that f is not continuous at p. Then, by our definition of continuity, there exists $A \subset X$ such that $p \sim A$ but $f(p) \not\sim f(A)$.

Since $f(p) \not\sim f(A)$, $f(p) \notin f(A)$ (by Theorem 2.7); hence, $p \notin A$, which shows that $A = A - \{p\}$. Thus, since $p \sim A$ and $f(p) \not\sim f(A)$, we have that

$p \sim A-\{p\}$ and $f(p) \not\sim f(A-\{p\})$. Therefore, $\lim_{x\to p} f(x) \neq f(p)$ by Theorem 3.11. ¥

The theorem we just proved characterizes continuity only at limit points of X. The following corollary completes the characterization.

Corollary 3.13: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$. Then f is continuous at p if and only if p is an isolated point of X or $\lim_{x\to p} f(x) = f(p)$ when p is a limit point of X.

Proof: Assume that f is continuous at p and that p is not an isolated point of X. Then p is a limit point of X and, hence, $\lim_{x\to p} f(x) = f(p)$ by Theorem 3.12. This proves that continuity at p implies the second conditions in the corollary.

Conversely, if p is an isolated point of X, then f is continuous at p by Theorem 2.35. If p is a limit point of X and if $\lim_{x\to p} f(x) = f(p)$, then f is continuous at p by Theorem 3.12. ¥

Exercise 3.14: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in X$. Then f is continuous at p if and only if for any open interval I such that $f(p) \in I$, there is an open interval J such that $p \in J$ and $f(J) \subset I$.

4. Limits in Terms of Continuity

In all calculus books, limits are defined before continuity and continuity is then defined in terms of limits. In our presentation, we have reversed the order for introducing these ideas. The reason we have done this is our realization that in trying to understand limits, you are really trying to understand continuity; the theorem below explains this. It is my opinion that continuity is simpler and easier to understand than limits. Thus, why *not* introduce continuity first and use it as a vehicle for building up intuition for the more subtle idea of limits.

In general terms, the following theorem says that $\lim_{x\to p} f(x)$ exists if and only if the function f can be defined or redefined at p so that the resulting function is continuous at p.

Theorem 3.15: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. Then $\lim_{x\to p} f(x) = L$ if and only if the function $g : X \cup \{p\} \to \mathsf{R}^1$ given by

$$g(x) = \begin{cases} f(x) & , \text{if } x \in X \\ L & , \text{if } x = p \end{cases}$$

is continuous at p.

Proof: Note that $g(x) = f(x)$ for all $x \in X - \{p\}$. Thus, we see easily from the definition of limit (section 1) that $\lim_{x\to p} f(x) = L$ if and only if $\lim_{x\to p} g(x) = L$. Thus, since $L = g(p)$, $\lim_{x\to p} f(x) = L$ if and only if $\lim_{x\to p} g(x) = g(p)$. Therefore, by Theorem 3.12, $\lim_{x\to p} f(x) = L$ if and only if g is continuous at p. ¥

5. One - sided Limits

A point in the real line can be "approached" from the left and from the right. This simple observation leads us to a way to break limits down into two cases – limits from the left and limits from the right. Considering the two cases separately is sometimes helpful in computing limits or in showing limits do not exist. This is especially true when a function is defined by a formula that changes at a point (the change can happen explicitly or implicitly – compare Exercises 3.17 and 3.18). We prove a theorem that can be applied in such situations.

We note the definition of the restriction of a function. Let X and Y be sets, and let $f : X \to Y$ be a function. For any set $X' \subset X$, the *restriction of* f *to* X', denoted by $f|X'$, is the function from X' to Y defined in the following simple way:

$$(f|X')(x') = f(x'), \text{ all } x' \in X.$$

We define one - sided limits:

Definition. Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a limit point of $X \cap (-\infty, p]$. We say *L is the limit of f as x approaches p from the left*, or the *left - hand limit of f as x approaches p*, written $\lim_{x \to p^-} f(x) = L$, provided that

$$\lim_{x \to p}(f|X \cap (-\infty, p])(x) = L.$$

Similarly, assuming that p is a limit point of $X \cap (p, \infty]$, we say *L is the limit of f as x approaches p from the right*, or the *right - hand limit of f as x approaches p*, written $\lim_{x \to p^+} f(x) = L$, provided that

$$\lim_{x \to p}(f|X \cap [p, \infty))(x) = L.$$

The following terminology is descriptive and will help make statements succinct: Let $X \subset \mathsf{R}^1$ and let $p \in \mathsf{R}^1$; we call p a *two - sided limit point of X* provided that p is a limit point of $X \cap (-\infty, p]$ and p is a limit point of $X \cap [p, \infty)$.

Theorem 3.16: Let $X \subset \mathsf{R}^1$, let $f : X \to \mathsf{R}^1$ be a function, and let $p \in \mathsf{R}^1$ such that p is a two - sided limit point of X. Then $\lim_{x \to p} f(x) = L$ if and only if

$$\lim_{x \to p^-} f(x) = L = \lim_{x \to p^+} f(x).$$

Proof: Assume that $\lim_{x \to p} f(x) = L$. Let $\epsilon > 0$. Then, by the definition of limit, there exists $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$,

$$|f(x) - L| < \epsilon.$$

Therefore, it is clear that $|f(x) - L| < \epsilon$ for all $x \in X \cap (-\infty, p)$, as well as for all $x \in X \cap (p, \infty)$, such that $|x - p| < \delta$. This proves that

$$\lim_{x \to p^-} f(x) = L = \lim_{x \to p^+} f(x).$$

Conversely, assume that $\lim_{x\to p^-} f(x) = L = \lim_{x\to p^+} f(x)$. Let $\epsilon > 0$. Since $\lim_{x\to p^-} f(x) = L$ and since (by definition)

$$\lim_{x\to p^-} f(x) = \lim_{x\to p}(f|X \cap (-\infty, p])(x),$$

there exists $\delta_1 > 0$ such that for all $x \in X \cap (-\infty, p)$ such that $|x - p| < \delta_1$,

$$|(f|X \cap [p, \infty])(x) - L| < \epsilon;$$

similarly, since $\lim_{x\to p^+} f(x) = L$, there exists $\delta_2 > 0$ such that for all $x \in X \cap (p, \infty)$ such that $|x - p| < \delta_2$,

$$|(f|X \cap [p, \infty])(x) - L| < \epsilon.$$

Therefore, letting $\delta = \min\{\delta_1, \delta_2\}$, we see that for all $x \in X - \{p\}$ such that $|x - p| < \delta$,

$$|f(x) - L| < \epsilon.$$

This proves that $\lim_{x\to p} f(x) = L$ (note: for us to conclude that $\lim_{x\to p} f(x) = L$, the definition of limit in section 1 requires us to know that p is a limit point of X; this follows from Exercise 2.30 since p is a limit point of $X \cap (-\infty, p]$). ¥

We conclude with comments about limits and one-sided limits. When we defined $\lim_{x\to p} f(x)$ in section 1, we did not make the common assumption that the point p lies in an open interval contained in the domain of f. Thus, for example, we can properly write $\lim_{x\to p} \sqrt{x}$ even when $p = 0$, whereas common practice forces authors to write $\lim_{x\to 0^+} \sqrt{x}$. In general, when the domain of f is an interval, we write $\lim_{x\to p} f(x)$ whether p is an end point of the interval or not, whereas other authors *are forced* to make the distinction. In this situation, we consider the distinction between limits and one-sided limits a distraction – a nuisance – rather than substantive. On the other hand, there are situations in which it is important to consider one-sided limits. By defining limits as we did, all our general theorems about limits in the next chapter *automatically* hold for their one-sided analogues.

Exercise 3.17: Find $\lim_{x\to 3} f(x)$ (if the limit exists) when

$$f(x) = \begin{cases} x + 1 & \text{, if } x \leq 3 \\ -4x + 16 & \text{, if } x > 3. \end{cases}$$

Exercise 3.18: Find $\lim_{x\to 4} \frac{|x-4|}{x-4}$ (if the limit exists).

Exercise 3.19: Find $\lim_{x\to 0} \frac{x^2}{|x|}$ (if the limit exists).

Exercise 3.20: Find $\lim_{x\to 1} \frac{x-1}{|x^2+x-2|}$ (if the limit exists).

Chapter IV: Limit Theorems

We prove theorems about limits of sums, differences, products and quotients of functions whose limits separately exist. We obtain general results about continuity as corollaries; as consequences, we show that all polynomials are continuous and that all rational functions are continuous (on their domains). We then prove theorems about limits of compositions of functions, including the Substitution Theorem. Next, we prove the simple but useful Squeeze Theorem. Finally, we briefly discuss limits of sequences.

All our theorems concerning limits hold for one-sided limits (see the comments at the end of the last section of Chapter III). We keep this in mind rather than stating the one-sided versions of the theorems.

1. Limits for Sums and Differences

We prove theorems about limits and continuity of sums and differences of two functions. We then extend the sum theorems to finitely many functions.

Definition. Let $X \subset \mathsf{R}^1$, and let $f, g : X \to \mathsf{R}^1$ be functions. The *sum of f and g* is the function $f + g : X \to \mathsf{R}^1$ defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in X.$$

Similarly, the *difference of f and g* is the function $f - g : X \to \mathsf{R}^1$ defined by

$$(f-g)(x) = f(x) - g(x) \quad \text{for all } x \in X.$$

We first prove that the limit of the sum of two functions whose limits separately exist is the sum of the limits of the two functions. Note that this shows, in particular, that the limit of the sum exists (provided that the separate limits exist).

Theorem 4.1: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x\to p} f(x) = L \quad \text{and} \quad \lim_{x\to p} g(x) = M,$$

then $\lim_{x\to p}(f+g)(x) = L + M$.

Proof: Let $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$, $|(f+g)(x) - (L+M)| < \epsilon$.

The clue to how to find δ comes from rewriting $|(f+g)(x) - (L+M)|$ so that expressions related to different assumptions in the theorem are grouped together:

$$\begin{aligned} |(f+g)(x) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M|. \end{aligned}$$

Thus, we want to find a $\delta > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$ for all $x \in X - \{p\}$ such that $|x - p| < \delta$. It is fairly easy to find such a δ; we now prove the theorem using what we have just observed as a guide (a cheat sheet!).

Since $\lim_{x\to p} f(x) = L$, there is a $\delta_1 > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta_1$,

$$|f(x) - L| < \tfrac{\epsilon}{2}.$$

Since $\lim_{x \to p} g(x) = M$, there is a $\delta_2 > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta_2$,

$$|g(x) - M| < \tfrac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and for all $x \in X - \{p\}$ such that $|x - p| < \delta$,

$$|(f + g)(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \text{ ¥}$$

Our next theorem is the analogue of Theorem 4.1 for the difference of two functions.

Theorem 4.2: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x \to p} f(x) = L \quad \text{and} \quad \lim_{x \to p} g(x) = M,$$

then $\lim_{x \to p}(f - g)(x) = L - M$.

Exercise 4.3: Prove Theorem 4.2.

Corollary 4.4: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions, and let $p \in X$. If f and g are continuous at p, then $f + g$ and $f - g$ are continuous at p.

Proof: The corollary follows immediately from Theorem 4.1 and Theorem 4.2 using Corollary 3.13. ¥

We extend Theorem 4.1 to the sum of finitely many functions. The *sum of finitely many functions* is defined inductively: Having already defined the sum of two functions, assume inductively that we have defined the sum of n functions (with the same domain) for some natural number $n \geq 2$; then, for any $n + 1$ functions with the same domain, define $f_1 + \cdots + f_n + f_{n+1}$ to be the function $(f_1 + \cdots + f_n) + f_{n+1}$ (see Theorem 1.20).

Theorem 4.5: Let $X \subset \mathsf{R}^1$, let $f_i : X \to \mathsf{R}^1$ be a function for each $i = 1, 2, ..., n$, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x \to p} f_i(x) = L_i \text{ for each } i = 1, 2, ..., n,$$

then $\lim_{x \to p}(f_1 + f_2 + \cdots + f_n)(x) = L_1 + L_2 + \cdots + L_n$.

Proof: We prove the theorem by induction on the number n of functions. The Induction Principle is Theorem 1.20.

The theorem is obviously true when $n = 1$.

Assume inductively that for some natural number k, the theorem is true for *any* k functions.

Let $f_1, f_2, ..., f_{k+1}$ be any $k + 1$ functions satisfying the assumptions in the theorem; that is, for each $i = 1, 2, ..., k + 1$, f_i is a function from X to R^1 such that $\lim_{x \to p} f_i(x) = L_i$. Then, by our inductive assumption,

$$\lim_{x\to p}(f_1 + f_2 + \cdots + f_k)(x) = L_1 + L_2 + \cdots + L_k.$$

Thus, since $\lim_{x\to p} f_{k+1}(x) = L_{k+1}$, Theorem 4.1 gives us that

$$\lim_{x\to p}((f_1 + f_2 + \cdots + f_k) + f_{k+1})(x) = (L_1 + L_2 + \cdots + L_k) + L_{k+1}.$$

Therefore, by our definition of finite sums of functions,

$$\lim_{x\to p}(f_1 + f_2 + \cdots + f_k + f_{k+1})(x) = L_1 + L_2 + \cdots + L_k + L_{k+1}.$$

This proves the theorem is true for $k+1$ functions under the assumption that it is true for k functions.

The theorem now follows from the Induction Principle. ¥

Corollary 4.6: Let $X \subset \mathsf{R}^1$, and let $p \in X$. If each of finitely many functions is continuous at p, then the sum function is continuous at p.

Proof: Apply Theorem 4.5 and Corollary 3.13. ¥

Exercise 4.7: Give an example of two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p}(f+g)(x)$ exists but $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ do not exist.

Exercise 4.8: Are there two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p}(f+g)(x)$ and $\lim_{x\to p} f(x)$ both exist but $\lim_{x\to p} g(x)$ does not exist?

2. Limits for Products

We prove theorems about limits and continuity of products of finitely many functions.

Definition. Let $X \subset \mathsf{R}^1$, and let $f, g : X \to \mathsf{R}^1$ be functions. The *product of* f *and* g is the function $f \cdot g : X \to \mathsf{R}^1$ defined by

$$(f \cdot g)(x) = f(x)g(x) \quad \text{for all } x \in X.$$

We first prove that the limit of the product of two functions whose limits separately exist is the product of the limits of the two functions.

Theorem 4.9: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x\to p} f(x) = L \quad \text{and} \quad \lim_{x\to p} g(x) = M,$$

then $\lim_{x\to p}(f \cdot g)(x) = LM$.

Proof: Let $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$, $|(f \cdot g)(x) - LM| < \epsilon$.

As in the proof of Theorem 4.1, the clue for finding δ comes from rewriting $|(f \cdot g)(x) - LM|$ so that expressions related to different assumptions in the theorem are grouped together. To group the proper expressions, we employ the trick of subtracting and adding an expression, namely, $Lg(x)$:

$$\begin{aligned}|(f \cdot g)(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &\leq |g(x)(f(x) - L)| + |L(g(x) - M)| \\ &= |g(x)|\,|f(x) - L| + |L|\,|g(x) - M|.\end{aligned}$$

Thus, we want to find a $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$,

$$(*)\ |g(x)|\,|f(x) - L| < \tfrac{\epsilon}{2} \quad \text{and} \quad (**)\ |L|\,|g(x) - M| < \tfrac{\epsilon}{2}.$$

We show how to find such a δ as follows.

We first bound $|g(x)|$: Since $\lim_{x \to p} g(x) = M$, there is a $\delta_1 > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta_1$, $|g(x) - M| < 1$; hence, by Exercise 1.29, $\big||g(x)| - |M|\big| < 1$. Thus, we have that

(1) $|g(x)| < 1 + |M|$ for all $x \in X - \{p\}$ such that $|x - p| < \delta_1$.

Next, with (1) and (*) in mind, we note that since $\lim_{x \to p} f(x) = L$, there is a $\delta_2 > 0$ such that

(2) $|f(x) - L| < \frac{\epsilon}{2(1+|M|)}$ for all $x \in X - \{p\}$ such that $|x - p| < \delta_2$.

Then we see from (1) and (2) that $\min\{\delta_1, \delta_2\}$ is a δ that makes (*) hold for all $x \in X - \{p\}$ such that $|x - p| < \delta$.

Next, we find a $\delta_3 > 0$ that makes (**) hold for all $x \in X - \{p\}$ such that $|x - p| < \delta_3$. Our immediate inclination is to use that $\lim_{x \to p} g(x) = M$ to choose $\delta_3 > 0$ such that $|g(x) - M| < \frac{\epsilon}{2|L|}$ for the relevant points x, hence (**) holds. However, this obviously does not work when $L = 0$; nevertheless, if $L = 0$, then any $\delta_3 > 0$ makes (**) hold for the relevant points x. Thus, we can take two cases in defining δ_3 – the case when $L \neq 0$ and the case when $L = 0$ – or we can use the trick of considering the positive number $\frac{\epsilon}{2(1+|L|)}$. We choose the latter: Since $\lim_{x \to p} g(x) = M$, there is a $\delta_3 > 0$ such that

(3) $|g(x) - M| < \frac{\epsilon}{2(1+|L|)}$ for all $x \in X - \{p\}$ such that $|x - p| < \delta_3$.

Finally, let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\delta > 0$ and for all $x \in X - \{p\}$ such that $|x - p| < \delta$, we see using (1), (2) and (3) that

$$\begin{aligned}|(f \cdot g)(x) - LM| &\leq |g(x)|\,|f(x) - L| + |L|\,|g(x) - M| \\ &< (1 + |M|)\tfrac{\epsilon}{2(1+|M|)} + |L|\,\tfrac{\epsilon}{2(1+|L|)} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$ ¥

Corollary 4.10: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions, and let $p \in X$. If f and g are continuous at p, then $f \cdot g$ is continuous at p.

Proof: Simply apply Theorem 4.9 and Corollary 3.13. ¥

We extend Theorem 4.9 to the product of finitely many functions. The *product of finitely many functions* is defined inductively in the same way that we defined the sum of finitely many functions in the preceding section.

Theorem 4.11: Let $X \subset \mathsf{R}^1$, let $f_i : X \to \mathsf{R}^1$ be a function for each $i = 1, 2, ..., n$, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x\to p} f_i(x) = L_i \text{ for each } i = 1, 2, ..., n\,,$$

then $\lim_{x\to p}(f_1 \cdot f_2 \cdot \cdots \cdot f_n)(x) = L_1 L_2 \cdots L_n$.

Exercise 4.12: Prove Theorem 4.11.

Corollary 4.13: Let $X \subset \mathsf{R}^1$, and let $p \in X$. If each of finitely many functions is continuous at p, then the product function is continuous at p.

Proof: Apply Theorem 4.11 and Corollary 3.13. ¥

Exercise 4.14: Give an example of two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p}(f \cdot g)(x)$ exists but $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ do not exist.

Exercise 4.15: Are there two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$ such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p}(f \cdot g)(x)$ and $\lim_{x\to p} f(x)$ both exist but $\lim_{x\to p} g(x)$ does not exist?

3. Continuity of Polynomials

We are now in a position to easily prove the important fact that all polynomials are continuous.

Definition. A *polynomial* is a function f that can be written in the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n, \text{ all } x \in \mathsf{R}^1,$$

where $c_0, c_1, ..., c_n$ are constants.

The constants $c_0, c_1, ..., c_n$ are called the *coefficients of the polynomial* f; c_i is called the i^{th} *coefficient of* f. If $c_n \neq 0$, we say that f is a *polynomial of degree* n.

Note that we say f is a polynomial if it *can be written* in the form indicated. Thus, for example, the function f defined by $f(x) = 3(x-4)(x^6 + 5x^2)^3$ is a polynomial.

We use the following functions in the proof that polynomials are continuous: A *constant function* is a function all of whose values are the same (i.e., a polynomial of degree 0); the *identity function* is the function f given by $f(x) = x$ for all $x \in \mathsf{R}^1$.

Theorem 4.16: All polynomials are continuous on R^1.

Proof: Any constant function and the identity function are continuous, as we showed in Example 2.23. Thus, for any fixed real number c and for any fixed natural number k, the function $f(x) = cx^k$ (all $x \in \mathsf{R}^1$) is continuous by Corollary 4.13. Our theorem now follows from Corollary 4.6. ¥

Theorems really make life easy: Can you imagine proving with epsilons and deltas, without theorems about limits, that the function f given by $f(x) = 6x^{89} + \frac{168}{31}x^{25} - \sqrt{17}x^{13} + 49$ is continuous?

Exercise 4.17: At which real numbers p is the function f given by $f(x) = \frac{8x^3-64}{2(x-2)}$ continuous?

Exercise 4.18: Is the function f given by $f(x) = \frac{x^2-x}{x}$ a polynomial?

4. Limits for Quotients

We prove theorems about limits and continuity of quotients of two functions.

Definition. Let $X \subset \mathsf{R}^1$, and let $f, g : X \to \mathsf{R}^1$ be functions such that $g(x) \neq 0$ for any $x \in X$. The *quotient of* f *and* g is the function $\frac{f}{g} : X \to \mathsf{R}^1$ defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in X.$$

We prove that the limit of the quotient of two functions whose limits separately exist is the quotient of the limits of the two functions provided, of course, that the limit of the function in the denominator is not zero. When the limit of the denominator *is* zero, the limit of the quotient may or may not exist: $\lim_{x\to 0} \frac{1}{x}$ does not exist and $\lim_{x\to 0} \frac{x}{x} = 1$.

We prove a lemma about reciprocals; then our theorem about limits of quotients follows easily using the theorem about limits of products (Theorem 4.9).

Lemma 4.19: Let $X \subset \mathsf{R}^1$, let $g : X \to \mathsf{R}^1$ be a function such that $g(x) \neq 0$ for any $x \in X$, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x\to p} g(x) = M \neq 0,$$

then $\lim_{x\to p} \frac{1}{g}(x) = \frac{1}{M}$.

Proof: Let $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$, $\left|\frac{1}{g}(x) - \frac{1}{M}\right| < \epsilon$.

As we did in proofs of previous theorems of this type, let us first examine what is involved in finding δ. We rewrite $\left|\frac{1}{g}(x) - \frac{1}{M}\right|$ so that the expression that we know can be made small, namely $|g(x) - M|$, is by itself (and hope that we can take care of the rest):

$$\left|\frac{1}{g}(x) - \frac{1}{M}\right| = \left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M-g(x)}{Mg(x)}\right| = \frac{1}{|M|}\frac{1}{|g(x)|}\,|g(x) - M|.$$

Hence, we want to find a $\delta > 0$ such that for all $x \in X - \{p\}$ such that $|x - p| < \delta$,

$$\frac{1}{|M|}\frac{1}{|g(x)|}\,|g(x) - M| < \epsilon.$$

We now proceed with the proof, using what we have written as a guide.

Since $\lim_{x\to p} g(x) = M$, we see easily using Exercise 1.29 that

$$\lim_{x\to p} |g(x)| = |M|.$$

Thus, since $M \neq 0$, there is a $\delta_1 > 0$ such that

(1) $|g(x)| > \frac{|M|}{2}$ for all $x \in X - \{p\}$ such that $|x - p| < \delta_1$.

Since $M \neq 0$, $\frac{M^2\epsilon}{2} > 0$ (use of the quantity $\frac{M^2\epsilon}{2}$ comes from (1) and the observations we referred to as a guide). Thus, since $\lim_{x\to p} g(x) = M$, there is a $\delta_2 > 0$ such that

(2) $|g(x) - M| < \frac{M^2\epsilon}{2}$ for all $x \in X - \{p\}$ such that $|x - p| < \delta_2$.

Now, let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and for all $x \in X - \{p\}$ such that $|x - p| < \delta$, we see from (1) and (2) that

$$\left|\frac{1}{g}(x) - \frac{1}{M}\right| = \frac{1}{|M|}\frac{1}{|g(x)|}|g(x) - M| < \frac{1}{|M|}\frac{2}{|M|}\frac{M^2\epsilon}{2} = \epsilon. \text{ ¥}$$

Theorem 4.20: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions such that $g(x) \neq 0$ for any $x \in X$, and let $p \in \mathsf{R}^1$ such that p is a limit point of X. If

$$\lim_{x\to p} f(x) = L \text{ and } \lim_{x\to p} g(x) = M \neq 0,$$

then $\lim_{x\to p} \frac{f}{g}(x) = \frac{L}{M}$.

Proof: Observe that $\frac{f}{g} = f \cdot \frac{1}{g}$; then use Lemma 4.19 to apply Theorem 4.9 to the product $f \cdot \frac{1}{g}$. ¥

Corollary 4.21: Let $X \subset \mathsf{R}^1$, let $f, g : X \to \mathsf{R}^1$ be functions such that $g(x) \neq 0$ for any $x \in X$, and let $p \in X$. If f and g are continuous at p, then $\frac{f}{g}$ is continuous at p.

Proof: Use Theorem 4.20 and Corollary 3.13. ¥

Exercise 4.22: Give an example of two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$, $g(x) \neq 0$ for all $x \in \mathsf{R}^1$, such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p} \frac{f}{g}(x)$ exists but $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ do not exist.

Exercise 4.23: Are there two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$, $g(x) \neq 0$ for all $x \in \mathsf{R}^1$, such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p} \frac{f}{g}(x)$ and $\lim_{x\to p} f(x)$ both exist but $\lim_{x\to p} g(x)$ does not exist?

Exercise 4.24: Are there two functions $f, g : \mathsf{R}^1 \to \mathsf{R}^1$, $g(x) \neq 0$ for all $x \in \mathsf{R}^1$, such that for some point $p \in \mathsf{R}^1$, $\lim_{x\to p} \frac{f}{g}(x)$ and $\lim_{x\to p} g(x)$ both exist but $\lim_{x\to p} f(x)$ does not exist?

5. Continuity of Rational Functions

Definition. A *rational function* is a function that can be written as a quotient of two polynomials.

The following theorem is trivial to prove in view of what we have already done.

Theorem 4.25: Every rational function is continuous on its domain.

Proof: By Theorem 4.16, polynomials are continuous on R^1. Therefore, our theorem follows from Corollary 4.21. ¥

Exercise 4.26: Is the function f given by $f(x) = \frac{1}{(\frac{1}{x})}$ (all $x \neq 0$) a rational function?

6. Compositions of Functions and Limits

We prove a theorem about the continuity of compositions of functions and a generalization concerning limits of compositions.

Definition. Let X, Y, and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions. The *composition f followed by g* is the function from X to Z denoted by $g \circ f$ and defined by letting

$$(g \circ f)(x) = g(f(x)), \text{ all } x \in X.$$

We often use the phrase *the composition of f and g* when the context makes it clear (or unimportant) which function is first.[3]

Perhaps you have never drawn the graph of a composition of two specific functions. If not, try the following exercise:

Exercise 4.27: Let $f, g : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x + \frac{1}{2} & , \text{if } 0 \leq x \leq \frac{1}{2} \\ -2x + 2 & , \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}, \quad g(x) = \begin{cases} -x + \frac{1}{2} & , \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & , \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Draw the graphs of $f \circ f$, $g \circ f$ and $f \circ g$.

Our first theorem concerns the continuity of the composition of two functions. The theorem is simple to prove using only the definition of continuity (above Example 2.23).

Theorem 4.28: Let $X, Y, Z \subset \mathsf{R}^1$, and let $f : X \to Y$ and $g : Y \to Z$ be functions. If f is continuous at p and g is continuous at $f(p)$, then $g \circ f$ is continuous at p.

Proof: Let $A \subset X$ such that $p \sim A$. Then, by the definition of continuity, $f(p) \sim f(A)$. Thus, since g is continuous at $f(p)$, $g(f(p)) \sim g(f(A))$. Hence, we have proved that for any $A \subset X$ such that $p \sim A$,

$$(g \circ f)(p) \sim (g \circ f)(A).$$

Therefore, $g \circ f$ is continuous at p. ¥

Our next theorem is called the Substitution Theorem because it says that under certain conditions, $\lim_{x \to p}(g \circ f)(x)$ can be found by substituting $\lim_{x \to p} f(x)$

[3] In the definition of composition, the order of the functions is important: $\mathsf{f} \circ \mathsf{g}$ is not defined on all of Y when $\mathsf{g}(\mathsf{Y}) \not\subset \mathsf{X}$; furthermore, even if $\mathsf{X} = \mathsf{Y} = \mathsf{Z}$, $\mathsf{g} \circ \mathsf{f}$ is almost always different from $\mathsf{f} \circ \mathsf{g}$.