(1) $|x_n - y_n| < \frac{1}{n}$ for each $n \in \mathsf{N}$

and

(2) $|f(x_n) - f(y_n)| \geq \epsilon$ for each $n \in \mathsf{N}$.

Let

$$X = \{x_n : n \in \mathsf{N}\}.$$

We prove that X is an infinite set. Suppose that the set X is finite. Then there is a point $q \in X$ such that $q = x_n$ for infinitely many n. Hence, we can assume that

(3) $q = x_{n_i}$, where $n_i < n_{i+1}$ for each $i \in \mathsf{N}$.

Then, by (3) and (1), we have

(4) $|q - y_{n_i}| < \frac{1}{n_i}$ for each $i \in \mathsf{N}$

and, by (3) and (2), we have

(5) $|f(q) - f(y_{n_i})| \geq \epsilon$ for each $i \in \mathsf{N}$.

Let

$$Y = \{y_{n_i} : i \in \mathsf{N}\}.$$

Recall from (3) that $n_i < n_{i+1}$ for each $i \in \mathsf{N}$; hence, by (4) and the second part of Exercise 1.23, the sequence $\{y_{n_i}\}_{i=1}^{\infty}$ converges to q. Thus, $q \sim Y$ (definition in section 1 of Chapter II). However, by (5), $f(q) \not\sim f(Y)$. Hence, f is not continuous at q by the definition of continuity (section 3 of Chapter II). This contradicts the assumption in our theorem. Therefore, we have proved that X is an infinite set.

Now, since X is a bounded infinite set, X has a limit point p in R^1 (by Exercise 5.16); furthermore, since $X \subset [a,b]$ and $p \sim X$, it follows easily that $p \in [a,b]$ (if $p \notin [a,b]$, then $p \not\sim [a,b]$ by Theorem 2.5; hence, $p \not\sim X$ by Exercise 2.10, a contradiction).

Since $p \in [a,b]$, f is continuous at p. Hence, by Theorem 3.12, there is a $\delta > 0$ such that

(6) $|f(x) - f(p)| < \frac{\epsilon}{2}$ whenever $x \in [a,b]$ and $|x - p| < \delta$.

Since p is a limit point of X, we have by Exercise 2.33 that

$$|x_n - p| < \tfrac{\delta}{2} \text{ for infinitely many } n.$$

Hence, by the Archimedean Property (Theorem 1.22), there is a natural number k such that $\frac{1}{k} < \frac{\delta}{2}$ and $|x_k - p| < \frac{\delta}{2}$. Thus,

$$|y_k - p| \leq |y_k - x_k| + |x_k - p| \overset{(1)}{<} \tfrac{1}{k} + \tfrac{\delta}{2} < \delta.$$

Since $|x_k - p| < \frac{\delta}{2} < \delta$ and $|y_k - p| < \delta$,

$$|f(x_k) - f(y_k)| \leq |f(x_k) - f(p)| + |f(p) - f(y_k)| \overset{(6)}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This contradicts (2). ¥

We note the following terminology, which calls attention to an important idea in connection with integrals.

Definition: The *norm of a partition* $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$, which we denote by $\|P\|$, is defined by

$$\|P\| = \max\{\Delta x_i : i = 1, 2, ..., n\}.$$

Exercise 12.32: For any $\eta > 0$, there is a partition P of $[a, b]$ such that $\|P\| < \eta$.

We now prove our theorem.

Theorem 12.33: If $f : [a, b] \to \mathsf{R}^1$ is a continuous function, then f is integrable over $[a, b]$.

Proof: We will apply Theorem 12.15. Let $\epsilon > 0$. By Theorem 12.31, f is uniformly continuous. Hence, there is a $\delta > 0$ such that

(1) $|f(y) - f(z)| < \frac{\epsilon}{b-a}$ whenever $y, z \in [a, b]$ and $|y - z| < \delta$.

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$ such that $\|P\| < \delta$ (P exists by Exercise 12.32). For each $i = 1, 2, ..., n$, $f|[x_{i-1}, x_i]$ is continuous (by Exercise 5.3); hence, by the Maximum - Minimum Theorem (Theorem 5.13), $f|[x_{i-1}, x_i]$ attains its maximum value at a point $y_i \in [x_{i-1}, x_i]$ and its minimum value at a point $z_i \in [x_{i-1}, x_i]$; in other words, we have that

(2) $M_i(f) = f(y_i)$ and $m_i(f) = f(z_i)$ for each $i = 1, 2, ..., n$.

Now, we show that P satisfies the condtion in Theorem 12.15:

$$U_P(f) - L_P(f) = \Sigma_{i=1}^n M_i(f)\Delta x_i - \Sigma_{i=1}^n m_i(f)\Delta x_i$$

$$= \Sigma_{i=1}^n [M_i(f) - m_i(f)]\Delta x_i \overset{(2)}{=} \Sigma_{i=1}^n [f(y_i) - f(z_i)]\Delta x_i$$

$$\overset{(1)}{<} \Sigma_{i=1}^n \frac{\epsilon}{b-a}\Delta x_i = \frac{\epsilon}{b-a}\Sigma_{i=1}^n \Delta x_i = \frac{\epsilon}{b-a}\Sigma_{i=1}^n (x_i - x_{i-1})$$

$$= \frac{\epsilon}{b-a}(x_n - x_0) = \frac{\epsilon}{b-a}(b - a) = \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, we have by Theorem 12.15 that f is integrable over $[a, b]$. ¥

Exercise 12.34: If $f : [a, b] \to \mathsf{R}^1$ is a bounded function that is continuous at all but finitely many points, then f is integrable over $[a, b]$.

Chapter XIII: The Algebra of the Integral

We show that sums, differences, products, quotients (with a condition), and absolute values of integrable functions are integrable. Finally, we examine integrals over subintervals (which we discuss again in the first section of Chapter XVI).

We remark that most results in this chapter follow immediately from a characterization of integrability in Chapter XV. In addition, the best general theorem about quotients follows easily from the characterization in Chapter XV (we were unable to find a proof using the methods we use in the present chapter; compare Theorem 13.36 with the result in Exercise 15.34). Nevertheless, this chapter is important for two reasons: First, it is always a good idea to understand why theorems are true from the most basic point of view; second, several results we prove here are necessary for proving the Fundamental Theorem of Calculus, which we want to prove as soon as possible, and some of those results are not consequences of the characterization in Chapter XV (e.g., the inequality in Theorem 13.17 and Theorem 13.40).

1. Integrability of Sums

We prove that the sum of two integrable functions is integrable and that the integral of the sum is the sum of the integrals (Theorem 13.3).

Lemma 13.1: Let f and g be bounded functions defined on a nonempty set X. Then

(1) $lub_{x\in X}(f(x)+g(x)) \leq lub_{x\in X} f(x) + lub_{x\in X} g(x)$

and

(2) $glb_{x\in X}(f(x)+g(x)) \geq glb_{x\in X} f(x) + glb_{x\in X} g(x)$.

Proof: For any $y \in X$,

$$f(y)+g(y) \leq lub_{x\in X} f(x) + lub_{x\in X} g(x);$$

hence, $lub_{x\in X} f(x) + lub_{x\in X} g(x)$ is an upper bound for $\{f(x)+g(x) : x \in X\}$. Therefore, by the Completeness Axiom, $lub_{x\in X}(f(x)+g(x))$ exists and, clearly,

$$lub_{x\in X}(f(x)+g(x)) \leq lub_{x\in X} f(x) + lub_{x\in X} g(x),$$

which proves (1). The proof of (2) is similar. ¥

The inequalities in Lemma 13.1 are in general strict, as can be seen, for example, by taking $f(x) = x$ and $g(x) = -\frac{1}{2}x + \frac{1}{2}$ for all $x \in [0,1]$.

Lemma 13.2: Let $f, g : [a,b] \to \mathsf{R}^1$ be bounded functions, and let P be a partition of $[a,b]$. Then

(1) $U_P(f+g) \leq U_P(f) + U_P(g)$

and

(2) $L_P(f+g) \geq L_P(f) + L_P(g)$.

Proof: Assume that $P = \{x_0, x_1, ..., x_n\}$. By Lemma 13.1,

(*) $M_i(f+g) \le M_i(f) + M_i(g)$ and $m_i(f+g) \ge m_i(f) + m_i(g)$, all i.

Therefore,

$$\begin{aligned} U_P(f+g) &= \Sigma_{i=1}^n M_i(f+g)\Delta x_i \overset{(*)}{\le} \Sigma_{i=1}^n [M_i(f) + M_i(g)]\Delta x_i \\ &= \Sigma_{i=1}^n M_i(f)\Delta x_i + \Sigma_{i=1}^n M_i(g)\Delta x_i = U_P(f) + U_P(g) \end{aligned}$$

and, similarly, $L_P(f+g) \ge L_P(f) + L_P(g)$. ¥

We now prove our theorem.

Theorem 13.3: If f and g are integrable over $[a,b]$, then $f+g$ is integrable over $[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof: Let $\epsilon > 0$. Then, by the definition of the integrals of f and g (section 3 of Chapter XII), there are partitions P_1 and P_2 of $[a,b]$ such that

$$U_{P_1}(f) < \int_a^b f + \tfrac{\epsilon}{2} \quad \text{and} \quad U_{P_2}(g) < \int_a^b g + \tfrac{\epsilon}{2}.$$

Let P be a common refinement of P_1 and P_2 (see Exercise 12.3). Then, by Lemma 12.8, $U_P(f) \le U_{P_1}(f)$ and $U_P(g) \le U_{P_2}(g)$. Hence,

$$U_P(f) < \int_a^b f + \tfrac{\epsilon}{2} \quad \text{and} \quad U_P(g) < \int_a^b g + \tfrac{\epsilon}{2}.$$

Thus, since $\overline{\int}_a^b (f+g) \le U_P(f+g) \overset{13.2}{\le} U_P(f) + U_P(g)$, we have proved that

(1) $\overline{\int}_a^b (f+g) < \left(\int_a^b f + \int_a^b g\right) + \epsilon.$

Similarly, there are partitions Q_1 and Q_2 of $[a,b]$ such that

$$L_{Q_1}(f) > \int_a^b f - \tfrac{\epsilon}{2} \quad \text{and} \quad L_{Q_2}(g) > \int_a^b g - \tfrac{\epsilon}{2},$$

and, for a common refinement, Q, of Q_1 and Q_2, Lemma 12.8 shows that $L_Q(f) \ge L_{Q_1}(f)$ and $L_Q(g) \ge L_{Q_2}(g)$; hence,

$$L_Q(f) > \int_a^b f - \tfrac{\epsilon}{2} \quad \text{and} \quad L_Q(g) > \int_a^b g - \tfrac{\epsilon}{2}.$$

Thus, since $\underline{\int}_a^b (f+g) \ge L_Q(f+g) \overset{13.2}{\ge} L_Q(f) + L_Q(g)$, we have proved that

(2) $\underline{\int}_a^b (f+g) > \left(\int_a^b f + \int_a^b g\right) - \epsilon.$

We now have that

$$\left(\int_a^b f + \int_a^b g\right) - \epsilon \overset{(2)}{<} \underline{\int}_a^b (f+g) \overset{12.10}{\le} \overline{\int}_a^b (f+g) \overset{(1)}{<} \left(\int_a^b f + \int_a^b g\right) + \epsilon.$$

Therefore, since ϵ was an arbitrary positive number, we see that

$$\underline{\int}_a^b (f+g) = \overline{\int}_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Thus, $f+g$ is integrable (by the first equality) and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. ¥

Corollary 13.4: If each of finitely many functions $f_1, f_2, ..., f_n$ is integrable over $[a,b]$, then their sum is integrable over $[a,b]$ and

$$\int_a^b (f_1 + f_2 + ... + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

Proof: Left as an exercise. ¥

Exercise 13.5: Prove Corollary 13.4.

In analogy with Theorem 13.3, the difference of two integrable functions is integrable and the integral of the difference is the difference of the integrals. We prove this result in the next section (Corollary 13.12).

Exercise 13.6: Define $f : [0,2] \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} 2 & \text{, if } 0 \le x < 1 \\ 5 & \text{, if } x = 1 \\ 4 & \text{, if } 1 < x \le 2. \end{cases}$$

Using Example 12.11 and Exercise 12.14, evaluate $\int_0^2 f$.

2. Integrability of Scalar Products

A *scalar product of a function* $f : X \to \mathsf{R}^1$ is the function $\lambda f : X \to \mathsf{R}^1$ obtained by multiplying each value of f by a fixed real number λ; that is,

$$(\lambda f)(x) = \lambda f(x) \text{ for all } x \in X.$$

The term *scalar product* is from vector spaces, where it refers to the product of a vector by a field element. The terminology is, therefore, appropriate here since the set of all real-valued functions defined on a nonempty set X forms a vector space under pointwise addition of functions (the vectors) and scalar product as defined above.

We prove that a scalar product λf of an integrable function f on $[a,b]$ is integrable and that the expected formula holds:

$$\int_a^b \lambda f = \lambda \int_a^b f.$$

Combining this result with the result about sums in the preceding section (Theorem 13.3), we have that the set V of all integrable functions defined on $[a,b]$ is a vector space and that $\int_a^b$ is a linear transformation from V to the vector space R^1; in other words,

$$\int_a^b (\lambda_1 f + \lambda_2 g) = \lambda_1 \int_a^b f + \lambda_2 \int_a^b g, \text{ all } f, g \in V \text{ and } \lambda_1, \lambda_2 \in \mathsf{R}^1.$$

In connection with $\int_a^b$ being a linear transformation, note that Exercise 12.26 characterizes all the nonnegative integrable functions in the null space of $\int_a^b$.

We remark that our theorem about the integrability of scalar products is a special case of the theorem about products that we will prove in section 4.

For any subset X of R^1 and any real number λ, we let λX denote the set defined by

$$\lambda X = \{\lambda x : x \in X\}.$$

Lemma 13.7: Let X be a nonempty bounded subset of R^1.

(1) If $\lambda \geq 0$, then $lub\lambda X = \lambda lubX$ and $glb\lambda X = \lambda glbX$.

(2) If $\lambda < 0$, then $lub\lambda X = \lambda glbX$ and $glb\lambda X = \lambda lubX$.

Proof: We prove part (1). Since (1) is trivial when $\lambda = 0$, we assume that $\lambda > 0$.

Since $x \leq lubX$ for all $x \in X$ and since $\lambda > 0$, it is clear that $\lambda x \leq \lambda lubX$ for all $x \in X$. Hence, $\lambda lubX$ is an upper bound for λX; thus, by the Completeness Axiom, $lub\lambda X$ exists and, obviously,

(a) $lub\lambda X \leq \lambda lubX$.

Since $\lambda x \leq lub\lambda X$ for all $x \in X$ and since $\lambda > 0$, we have that $x \leq \frac{1}{\lambda} lub\lambda X$ for all $x \in X$. Hence, $lubX \leq \frac{1}{\lambda} lub\lambda X$; thus, since $\lambda > 0$, we have

(b) $\lambda lubX \leq lub\lambda X$.

By (a) and (b), $lub\lambda X = \lambda lubX$. Similarly, by replacing lub with glb and reversing inequalities in the argument above, we obtain that $glb\lambda X = \lambda glbX$. This proves part (1).

We leave the proof of part (2) as an exercise. ¥

Exercise 13.8: Prove part (2) of Lemma 13.7.

Lemma 13.9: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function, and let P be a partition of $[a, b]$.

(1) For any $\lambda \geq 0$, $U_P(\lambda f) = \lambda U_P(f)$ and $L_P(\lambda f) = \lambda L_P(f)$.

(2) For any $\lambda < 0$, $U_P(\lambda f) = \lambda L_P(f)$ and $L_P(\lambda f) = \lambda U_P(f)$.

Proof: Assume that $P = \{x_0, x_1, ..., x_n\}$.

We prove part (1). Let $\lambda \geq 0$. Then, by the first part of Lemma 13.7,

$$(*)\ M_i(\lambda f) = \lambda M_i(f) \quad \text{and} \quad m_i(\lambda f) = \lambda m_i(f), \ \text{ all } i.$$

Therefore,

$$U_P(\lambda f) = \Sigma_{i=1}^n M_i(\lambda f)\Delta x_i \overset{(*)}{=} \Sigma_{i=1}^n \lambda M_i(f)\Delta x_i = \lambda U_P(f)$$

and

$$L_P(\lambda f) = \Sigma_{i=1}^n m_i(\lambda f)\Delta x_i \overset{(*)}{=} \Sigma_{i=1}^n \lambda m_i(f)\Delta x_i = \lambda L_P(f).$$

This proves part (1).

We leave the proof of part (2) as an exercise. ¥

Exercise 13.10: Prove part (2) of Lemma 13.9.

Theorem 13.11: Let $\lambda \in \mathsf{R}^1$. If f is integrable over $[a,b]$, then λf is integrable over $[a,b]$ and

$$\int_a^b \lambda f = \lambda \int_a^b f.$$

Proof: Let $\mathcal{P}$ denote the collection of all partitions of $[a,b]$. By the definitions of the upper and lower integrals (section 3 of Chapter XII),

$$(1)\ \overline{\int}_a^b \lambda f = glb_{P\in\mathcal{P}} U_P(\lambda f),\ \ \lambda \overline{\int}_a^b f = \lambda glb_{P\in\mathcal{P}} U_P(f)$$

and

$$(2)\ \underline{\int}_a^b \lambda f = lub_{P\in\mathcal{P}} L_P(\lambda f),\ \ \lambda \underline{\int}_a^b f = \lambda lub_{P\in\mathcal{P}} L_P(f)$$

We use the first part of Lemma 13.9 and then the first part of Lemma 13.7 in each of the following:

$$(3)\ glb_{P\in\mathcal{P}} U_P(\lambda f) = glb_{P\in\mathcal{P}} \lambda U_P(f) = \lambda glb_{P\in\mathcal{P}} U_P(f),\ \text{ if } \lambda \geq 0;$$

$$(4)\ lub_{P\in\mathcal{P}} L_P(\lambda f) = lub_{P\in\mathcal{P}} \lambda L_P(f) = \lambda lub_{P\in\mathcal{P}} L_P(f),\ \text{ if } \lambda \geq 0.$$

We use the second part of Lemma 13.9 and then the second part of Lemma 13.7 in each of the following:

$$(5)\ glb_{P\in\mathcal{P}} U_P(\lambda f) = glb_{P\in\mathcal{P}} \lambda L_P(f) = \lambda lub_{P\in\mathcal{P}} L_P(f),\ \text{ if } \lambda < 0;$$

$$(6)\ lub_{P\in\mathcal{P}} L_P(\lambda f) = lub_{P\in\mathcal{P}} \lambda U_P(f) = \lambda glb_{P\in\mathcal{P}} U_P(f),\ \text{ if } \lambda < 0.$$

Now, assume that $\lambda \geq 0$. Then, using that f is integrable over $[a,b]$ for the last equalities below,

$$\overline{\int}_a^b \lambda f \overset{(1)}{=} glb_{P\in\mathcal{P}} U_P(\lambda f) \overset{(3)}{=} \lambda glb_{P\in\mathcal{P}} U_P(f) \overset{(1)}{=} \lambda \overline{\int}_a^b f = \lambda \int_a^b f$$

and

$$\underline{\int}_a^b \lambda f \overset{(2)}{=} lub_{P\in\mathcal{P}} L_P(\lambda f) \overset{(4)}{=} \lambda lub_{P\in\mathcal{P}} L_P(f) \overset{(2)}{=} \lambda \underline{\int}_a^b f = \lambda \int_a^b f;$$

therefore, $\overline{\int}_a^b \lambda f = \lambda \int_a^b f = \underline{\int}_a^b \lambda f$, which proves the lemma when $\lambda \geq 0$.

Finally, assume that $\lambda < 0$. Then, using that f is integrable over $[a,b]$ for the last equalities below,

$$\overline{\int}_a^b \lambda f \overset{(1)}{=} glb_{P\in\mathcal{P}} U_P(\lambda f) \overset{(5)}{=} \lambda lub_{P\in\mathcal{P}} L_P(f) \overset{(2)}{=} \lambda \underline{\int}_a^b f = \lambda \int_a^b f$$

and

$$\underline{\int}_a^b \lambda f \overset{(2)}{=} lub_{P\in\mathcal{P}} L_P(\lambda f) \overset{(6)}{=} \lambda glb_{P\in\mathcal{P}} U_P(f) \overset{(1)}{=} \lambda \overline{\int}_a^b f = \lambda \int_a^b f;$$

therefore, $\overline{\int}_a^b \lambda f = \lambda \int_a^b f = \underline{\int}_a^b \lambda f$, which proves the lemma when $\lambda < 0$. ¥

We can now easily obtain the theorem for the difference of two integrable functions that is analogous to the theorem for the sum of two integrable functions in the preceding section (Theorem 13.3):

Corollary 13.12: If f and g are integrable over $[a, b]$, then $f - g$ is integrable over $[a, b]$ and

$$\int_a^b (f - g) = \int_a^b f - \int_a^b g.$$

Proof: By Theorem 13.11, $-g$ is integrable over $[a, b]$ and $\int_a^b -g = -\int_a^b g$. Therefore, since $f - g = f + (-g)$, we have by Theorem 13.3 that $f - g$ is integrable over $[a, b]$ and

$$\int_a^b (f - g) = \int_a^b (f + (-g)) \overset{13.3}{=} \int_a^b f + \int_a^b -g \overset{13.11}{=} \int_a^b f - \int_a^b g. \text{ ¥}$$

Exercise 13.13: Using Exercise 12.13, Example 12.18 and Exercise 12.20, evaluate $\int_0^2 (5x^2 - 3x + 4)$.

3. Integrability of Absolute Values

The *absolute value of a function* $f : X \to \mathsf{R}^1$ is the function $|f| : X \to \mathsf{R}^1$ defined by

$$|f|(x) = |f(x)| \text{ for all } x \in X.$$

We prove that the absolute value $|f|$ of an integrable function f on $[a, b]$ is integrable and that

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

The inequality is what we would expect in view of the definition of upper and lower sums and the Triangle Inequality for absolute values (above Exercise 1.28); however, our proof of the inequality is along different lines.

Lemma 13.14: Assume that f and g are integrable over $[a, b]$ and that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f \leq \int_a^b g.$$

Proof: Let $h = g - f$. Then, by Corollary 13.12, h is integrable and

$$\int_a^b h = \int_a^b g - \int_a^b f.$$

Also, since $h(x) \geq 0$ for all $x \in [a, b]$, $\int_a^b h \geq 0$ (by Exercise 12.25). Therefore,

$$\int_a^b g - \int_a^b f \geq 0,$$

which proves the lemma. ¥

Exercise 13.15: If f is integrable over $[a, b]$ and $\alpha \leq f(x) \leq \beta$ for all $x \in [a, b]$, then

$$\alpha(b - a) \leq \int_a^b f \leq \beta(b - a).$$

Lemma 13.16: Let X be a nonempty set, and let $f : X \to \mathsf{R}^1$ be a bounded function. Then

$$lub_{x\in X}\,|f(x)| - glb_{x\in X}\,|f(x)| \le lub_{x\in X} f(x) - glb_{x\in X} f(x).$$

Proof: Let $p, q \in X$. Then

$$\begin{aligned}|f(p)| - |f(q)| &\overset{1.29}{\le} |f(p) - f(q)| = \max\{f(p), f(q)\} - \min\{f(p), f(q)\} \\ &\le lub_{x\in X} f(x) - glb_{x\in X} f(x),\end{aligned}$$

which says

$$|f(p)| \le lub_{x\in X} f(x) - glb_{x\in X} f(x) + |f(q)|.$$

Since this inequality holds for all points $p \in X$, we have that

$$lub_{x\in X}\,|f(x)| \le lub_{x\in X} f(x) - glb_{x\in X} f(x) + |f(q)|.$$

Hence,

$$lub_{x\in X}\,|f(x)| - lub_{x\in X} f(x) + glb_{x\in X} f(x) \le |f(q)|.$$

Since this inequality holds for all points $q \in X$, we now have that

$$lub_{x\in X}\,|f(x)| - lub_{x\in X} f(x) + glb_{x\in X} f(x) \le glb_{x\in X}\,|f(x)|,$$

which proves the lemma. ¥

We now prove our theorem.

Theorem 13.17: If f is integrable over $[a, b]$, then $|f|$ is integrable over $[a, b]$ and

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Proof: Let $\epsilon > 0$. Since f is integrable over $[a, b]$, we know from Theorem 12.15 that there is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that

$$U_P(f) - L_P(f) < \epsilon.$$

Hence, using Lemma 13.16 on each of the intervals $[x_{i-1}, x_i]$ for the inequality is the third step below, we have

$$\begin{aligned}U_P(|f|) - L_P(|f|) &= \Sigma_{i=1}^n [M_i(|f|) - m_i(|f|)]\Delta x_i \\ &\le \Sigma_{i=1}^n [M_i(f) - m_i(f)]\Delta x_i = U_P(f) - L_P(f) < \epsilon.\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, we have by Theorem 12.15 that $|f|$ is integrable over $[a, b]$.

Finally, we prove the inequality in the theorem. Having just proved that $|f|$ is integrable over $[a, b]$, we know from Theorem 13.11 that $-|f|$ is integrable over $[a, b]$. Therefore,

$$-\int_a^b |f| \overset{13.11}{=} \int_a^b -|f| \overset{13.14}{\leq} \int_a^b f \overset{13.14}{\leq} \int_a^b |f|,$$

which shows that $\left|\int_a^b f\right| \leq \int_a^b |f|$. ¥

Exercise 13.18: If $|f|$ is integrable over $[a,b]$, then is f integrable over $[a,b]$?

Exercise 13.19: If f and g are integrable over $[a,b]$, then the maximum function $f \bigvee g$ and the minimum function $f \bigwedge g$ are integrable over $[a,b]$. (We defined $f \bigvee g$ and $f \bigwedge g$ in Exercise 4.33.)

(*Hint:* The same as the hint for Exercise 4.33.)

Exercise 13.20: We defined the notion of a distance function for a set in Exercise 1.30. Let $\mathcal{I}([a,b])$ denote the set of all functions that are integrable over $[a,b]$. For any $f, g \in \mathcal{I}([a,b])$, let

$$d(f,g) = \int_a^b |f - g|.$$

Determine whether d is a metric for $\mathcal{I}([a,b])$.

4. Integrability of Products

We prove that the product of two integrable functions is integrable. It is not possible to give a formula for the integral of the product; in particular, the integral of the product of two integrable functions is not necessarily the product of the integrals (e.g., $\int_0^2 x^2 \neq (\int_0^2 x)(\int_0^2 x)$ by Example 12.18 and Exercise 12.20).

Our theorems about products (along with other results) can be used to show that all polynomials are integrable over any closed and bounded interval $[a,b]$, $a < b$ (Exercise 13.28).

We first prove that the product of two integrable functions is integrable for the case when the functions are nonnegative; we then apply Exercise 13.19 to obtain the general result (Theorem 13.26).

Note the following exercise for use in Lemma 13.22.

Exercise 13.21: Using axioms in section 1 of Chapter I, prove that if $0 \leq a \leq b$ and $0 \leq c \leq d$, then $ac \leq bd$.

Lemma 13.22: Let f and g be bounded nonnegative functions defined on a nonempty set X. Then

(1) $[glb_{x\in X} f(x)][glb_{x\in X} g(x)] \leq glb_{x\in X}(f \cdot g)(x)$

and

(2) $lub_{x\in X}(f \cdot g)(x) \leq [lub_{x\in X} f(x)][lub_{x\in X} g(x)]$.

Proof: By Exercise 13.21, we have

$$[glb_{x\in X} f(x)][glb_{x\in X} g(x)] \leq f(y)g(y), \quad \text{all } y \in X.$$

Hence, $[glb_{x\in X} f(x)][glb_{x\in X} g(x)]$ is a lower bound for $\{(f \cdot g)(x) : x \in X\}$. Therefore, $glb_{x\in X}(f \cdot g)(x)$ exists by the Greatest Lower Bound Axiom (section 8 of Chapter I) and

$$[glb_{x\in X}f(x)][glb_{x\in X}g(x)] \leq glb_{x\in X}(f\cdot g)(x).$$

This proves (1). The proof of (2) is similar. ¥

Lemma 13.23: Let $f, g : [a,b] \to \mathsf{R}^1$ be bounded nonnegative functions, and let s be an upper bound for both $f([a,b])$ and $g([a,b])$. If P is a partition of $[a,b]$, then

$$U_P(f\cdot g) - L_P(f\cdot g) \leq s[U_P(f) - L_P(f)] + s[U_P(g) - L_P(g)].$$

Proof: Let $P = \{x_0, x_1, ..., x_n\}$. We see from Lemma 13.22 that

$$(*)\ m_i(f)m_i(g) \leq m_i(f\cdot g) \leq M_i(f\cdot g) \leq M_i(f)M_i(g), \text{ all } i.$$

Hence,

$$\begin{aligned} U_P(f\cdot g) - L_P(f\cdot g) &= \Sigma_{i=1}^n [M_i(f\cdot g) - m_i(f\cdot g)]\Delta x_i \\ &\overset{(*)}{\leq} \Sigma_{i=1}^n [M_i(f)M_i(g) - m_i(f)m_i(g)]\Delta x_i \\ &= \Sigma_{i=1}^n [M_i(f)M_i(g) - m_i(f)M_i(g) + m_i(f)M_i(g) - m_i(f)m_i(g)]\Delta x_i \\ &= \Sigma_{i=1}^n M_i(g)[M_i(f) - m_i(f)]\Delta x_i + \Sigma_{i=1}^n m_i(f)[M_i(g) - m_i(g)]\Delta x_i \\ &\leq s\Sigma_{i=1}^n [M_i(f) - m_i(f)]\Delta x_i + s\Sigma_{i=1}^n [M_i(g) - m_i(g)]\Delta x_i \\ &= s[U_P(f) - L_P(f)] + s[U_P(g) - L_P(g)]. \end{aligned}$$ ¥

Our next lemma is the product theorem for nonnegative functions.

Lemma 13.24: If f and g are nonnegative functions that are integrable over $[a,b]$, then $f\cdot g$ is integrable over $[a,b]$.

Proof: Let $\epsilon > 0$ (we will use Theorem 12.15). Since f and g are integrable over $[a,b]$, f and g are bounded; thus, we have an upper bound $s > 0$ for both $f([a,b])$ and $g([a,b])$. Now, by Theorem 12.15, there are partitions P_1 and P_2 of $[a,b]$ such that

(1) $U_{P_1}(f) - L_{P_1}(f) < \frac{\epsilon}{2s}$ and $U_{P_2}(g) - L_{P_2}(g) < \frac{\epsilon}{2s}$.

Let P be a common refinement of P_1 and P_2 (see Exercise 12.3). Then, by Lemma 12.6 and Lemma 12.8, we have

(2) $L_{P_1}(f) \leq L_P(f) \leq U_P(f) \leq U_{P_1}(f)$

and

(3) $L_{P_2}(g) \leq L_P(g) \leq U_P(g) \leq U_{P_2}(g)$.

Now,

$$U_P(f) - L_P(f) \overset{(2)}{\leq} U_{P_1}(f) - L_{P_1}(f) \overset{(1)}{<} \frac{\epsilon}{2s}$$

and, similarly,

$$U_P(g) - L_P(g) \overset{(3)}{\leq} U_{P_2}(g) - L_{P_2}(g) \overset{(1)}{<} \tfrac{\epsilon}{2s}.$$

Hence, by Lemma 13.23 (and since $s > 0$),

$$U_P(f \cdot g) - L_P(f \cdot g) < s\tfrac{\epsilon}{2s} + s\tfrac{\epsilon}{2s} = \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, we have by Theorem 12.15 that $f \cdot g$ is integrable over $[a, b]$. ¥

Exercise 13.25: If f and g are integrable over $[a, b]$ and neither f nor g changes sign on $[a, b]$, then $f \cdot g$ is integrable over $[a, b]$.

We are ready to prove the general theorem about products.

Theorem 13.26: If f and g are integrable over $[a, b]$, then $f \cdot g$ is integrable over $[a, b]$.

Proof: Let $\overline{0}$ denote the zero function on $[a, b]$ (i.e., $\overline{0}(x) = 0$ for all $x \in [a, b]$). Consider the maximum and minimum functions of f and $\overline{0}$ and of g and $\overline{0}$ (as defined in Exercise 4.33): $f \bigvee \overline{0}$, $f \bigwedge \overline{0}$, $g \bigvee \overline{0}$, and $g \bigwedge \overline{0}$. By taking the four cases involving the possible signs of $f(x)$ and $g(x)$ for a fixed (but arbitrary) point x, we see that

(1) $f \cdot g = (f \bigvee \overline{0}) \cdot (g \bigvee \overline{0}) + (f \bigvee \overline{0}) \cdot (g \bigwedge \overline{0})$

$$+ (f \bigwedge \overline{0}) \cdot (g \bigvee \overline{0}) + (f \bigwedge \overline{0}) \cdot (g \bigwedge \overline{0}).$$

Each of the functions $f \bigvee \overline{0}$, $f \bigwedge \overline{0}$, $g \bigvee \overline{0}$, and $g \bigwedge \overline{0}$ is integrable over $[a, b]$ by Exercise 13.19; furthermore, none of these functions changes sign on $[a, b]$. Hence, by Exercise 13.25, each of the four product functions on the right-hand side of (1) is integrable on $[a, b]$. Therefore, by (1) and Corollary 13.4, $f \cdot g$ is integrable on $[a, b]$. ¥

Corollary 13.27: If each of finitely many functions is integrable over $[a, b]$, then their product is integrable over $[a, b]$.

Proof: The corollary follows from Theorem 13.26 by a straightforward induction (Theorem 1.20). ¥

Exercise 13.28: Use results in this chapter to prove that polynomials are integrable over any closed and bounded interval. (Note: The result also follows from Theorem 12.33 since polynomials are continuous by Theorem 4.16.)

Exercise 13.29: If the product of two bounded functions is integrable over $[a, b]$, then must each of the functions be integrable over $[a, b]$? In other words, is the converse of Theorem 13.26 true?

Exercise 13.30: True or false: If f and g are integrable over $[a, b]$, then

$$\left|\int_a^b f \cdot g\right| \leq \left|\int_a^b f\right| \left|\int_a^b g\right|.$$

5. Integrability of Quotients

Let f and g be integrable functions defined on an interval $[a, b]$ such that g is never zero. The quotient $\frac{f}{g}$ is not necessarily integrable since $\frac{f}{g}$ may not be bounded. However, $\frac{f}{g}$ is integrable when $\frac{f}{g}$ is bounded. We do not know how to prove this theorem without using a characterization theorem in Chapter XV. At this time, we prove that $\frac{f}{g}$ is integrable when g is bounded away from zero (Theorem 13.36). You will be asked to prove the general theorem later (in Exercise 15.34).

If X is a nonempty set and $g : X \to \mathsf{R}^1$ is a function, then we say that g is *bounded away from zero* provided that there is an $\alpha > 0$ such that $|g(x)| > \alpha$ for all $x \in X$.

We first prove the theorem about integrability of quotients for the case of reciprocals (Theorem 13.35); then the theorem about quotients follows easily using our previous theorem about the integrability of products (Theorem 13.26). This pattern is analogous to what we did to obtain theorems about limits of quotients and derivatives of quotients in previous chapters.

We prove three lemmas. We use the first two lemmas to prove the third lemma, which we use to obtain the result about reciprocals.

Lemma 13.31: Let g be bounded function defined on a nonempty set X such that $glb_{x \in X} g(x) > 0$. Then

(1) $glb_{x \in X} \frac{1}{g(x)} = \frac{1}{lub_{x \in X} g(x)}$

and

(2) $lub_{x \in X} \frac{1}{g(x)} = \frac{1}{glb_{x \in X} g(x)}$.

Proof: Since $0 < g(y) \leq lub_{x \in X} g(x)$ for all $y \in X$, we have

$$\frac{1}{lub_{x \in X} g(x)} \leq \frac{1}{g(y)} \text{ for all } y \in X.$$

Hence, $\frac{1}{lub_{x \in X} g(x)}$ is a lower bound for $\{\frac{1}{g(x)} : x \in X\}$. Therefore, $glb_{x \in X} \frac{1}{g(x)}$ exists by the Greatest Lower Bound Axiom (section 8 of Chapter I) and

$$\frac{1}{lub_{x \in X} g(x)} \leq glb_{x \in X} \frac{1}{g(x)}.$$

Therefore, $\frac{1}{lub_{x \in X} g(x)} = glb_{x \in X} \frac{1}{g(x)}$ since if $\frac{1}{lub_{x \in X} g(x)} < glb_{x \in X} \frac{1}{g(x)}$, then there exists $z \in X$ such that $\frac{1}{g(z)} < glb_{x \in X} \frac{1}{g(x)}$, a contradiction. This proves (1). The proof of (2) is similar (and is left as an exercise). ¥

Exercise 13.32: Prove part (2) of Lemma 13.31.

Lemma 13.33: Let g be bounded function defined on a nonempty set X such that $lub_{x \in X} g(x) < 0$. Then

(1) $glb_{x \in X} \frac{1}{g(x)} = \frac{1}{lub_{x \in X} g(x)}$

and

(2) $lub_{x \in X} \frac{1}{g(x)} = \frac{1}{glb_{x \in X} g(x)}$.

Proof: Let $h = -g$. Then, since $glb_{x\in X} h(x) \overset{13.7}{=} -lub_{x\in X} g(x) > 0$, we can apply 13.31 to h to obtain that

(a) $glb_{x\in X} \frac{1}{h(x)} = \frac{1}{lub_{x\in X} h(x)}$

and

(b) $lub_{x\in X} \frac{1}{h(x)} = \frac{1}{glb_{x\in X} h(x)}$.

Therefore,

$$glb_{x\in X} \frac{1}{g(x)} \overset{13.7}{=} -lub \frac{1}{h(x)} \overset{(b)}{=} \frac{-1}{glb_{x\in X} h(x)} \overset{13.7}{=} \frac{1}{lub_{x\in X} g(x)}$$

and

$$lub_{x\in X} \frac{1}{g(x)} \overset{13.7}{=} -glb_{x\in X} \frac{1}{h(x)} \overset{(a)}{=} \frac{-1}{lub_{x\in X} h(x)} \overset{13.7}{=} \frac{1}{glb_{x\in X} g(x)}. \text{ ¥}$$

Lemma 13.34: Let g be bounded function defined on a nonempty set X such that g is bounded away from zero, say $|g(x)| > \alpha$ for all $x \in X$. Let

$$M = lub_{x\in X} \frac{1}{g(x)}, \quad m = glb_{x\in X} \frac{1}{g(x)}$$

Then $M - m < \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{\alpha^2}$.

Proof: Let

$$X^+ = \{x \in X : g(x) > 0\}, \quad X^- = \{x \in X : g(x) < 0\}.$$

We prove the lemma by considering three cases.

Case 1*:* $X = X^+$. Then

$$\begin{aligned} M - m &\overset{13.31}{=} \frac{1}{glb_{x\in X} g(x)} - \frac{1}{lub_{x\in X} g(x)} \\ &= \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{[glb_{x\in X} g(x)][lub_{x\in X} g(x)]} < \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{\alpha^2}. \end{aligned}$$

Case 2*:* $X = X^-$. Then

$$\begin{aligned} M - m &\overset{13.33}{=} \frac{1}{glb_{x\in X} g(x)} - \frac{1}{lub_{x\in X} g(x)} \\ &= \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{[glb_{x\in X} g(x)][lub_{x\in X} g(x)]} < \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{\alpha^2}. \end{aligned}$$

Case 3*:* $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. Then we can let

$$\gamma^+ = glb\, g(X^+), \quad \gamma^- = lub\, g(X^-).$$

We see that

$$\begin{aligned} M - m &\overset{13.31}{=} \frac{1}{\gamma^+} - m \overset{13.33}{=} \frac{1}{\gamma^+} - \frac{1}{\gamma^-} = \frac{\gamma^- - \gamma^+}{\gamma^+\gamma^-} = \frac{\gamma^+ - \gamma^-}{|\gamma^+\gamma^-|} \\ &\leq \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{|\gamma^+\gamma^-|} < \frac{lub_{x\in X} g(x) - glb_{x\in X} g(x)}{\alpha^2}. \text{ ¥} \end{aligned}$$

Theorem 13.35: If g is integrable over $[a, b]$ and g is bounded away from zero, then $\frac{1}{g}$ is integrable over $[a, b]$.

Proof: Since g is bounded away from zero, there exists $\alpha > 0$ such that $|g(x)| > \alpha$ for all $x \in [a,b]$.

Let $\epsilon > 0$. Since g is integrable over $[a,b]$, we have by Theorem 12.15 that there is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that

(1) $U_P(g) - L_P(g) < \alpha^2 \epsilon$.

Now,

$$\begin{aligned} U_P(\tfrac{1}{g}) - L_P(\tfrac{1}{g}) &= \Sigma_{i=1}^n [M_i(\tfrac{1}{g}) - m_i(\tfrac{1}{g})]\Delta x_i \\ &\overset{13.34}{<} \Sigma_{i=1}^n \tfrac{M_i(g) - m_i(g)}{\alpha^2}\Delta x_i = \tfrac{1}{\alpha^2}\Sigma_{i=1}^n [M_i(g) - m_i(g)]\Delta x_i \\ &= \tfrac{1}{\alpha^2}[U_P(g) - L_P(g)] \overset{(1)}{<} \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, we have by Theorem 12.15 that $\frac{1}{g}$ is integrable over $[a,b]$. ¥

We are ready to prove our main result.

Theorem 13.36: If f and g are are integrable over $[a,b]$ and g is bounded away from zero, then $\frac{f}{g}$ is integrable over $[a,b]$.

Proof: By Theorem 13.35, $\frac{1}{g}$ is integrable over $[a,b]$. Therefore, since $\frac{f}{g} = f \cdot \frac{1}{g}$, $\frac{f}{g}$ is integrable over $[a,b]$ by Theorem 13.26. ¥

Exercise 13.37: Rational functions are integrable over any closed and bounded interval contained in their domain. Prove this without using Theorem 12.33.

Exercise 13.38: Give an example of integrable functions $f, g : [0,1] \to \mathsf{R}^1$ such that $\frac{f}{g}$ is integrable over $[0,1]$ but g is not bounded away from zero.

6. Integrability Over Subintervals

We prove that if f is integrable over $[a,b]$ and if $[c,d]$ is a subinterval of $[a,b]$, then the restriction of f to $[c,d]$ is integrable over $[c,d]$.[8] As a consequence, we obtain the following sum formula (where c is a point of $[a,b]$):

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 13.39: If f is integrable over $[a,b]$ and $[c,d]$ is a subinterval of $[a,b]$, then the restricted function $f|[c,d]$ is integrable over $[c,d]$.

Proof: Let $\epsilon > 0$. Since f is integrable over $[a,b]$, Theorem 12.15 gives us a partition P of $[a,b]$ such that

(1) $U_P(f) - L_P(f) < \epsilon$.

Let $Q = P \cup \{c,d\}$, a partition of $[a,b]$. Then, since Q is a refinement of P,

[8] To avoid cumbersome notation, we write $\int_c^d f$ instead of $\int_c^d f|[c,d]$ for the integral of f over $[c.d]$.

$$L_P(f) \overset{12.8}{\leq} L_Q(f) \overset{12.6}{\leq} U_Q(f) \overset{12.8}{\leq} U_P(f).$$

Hence, by (1), we have that

(2) $U_Q(f) - L_Q(f) < \epsilon$.

Let $R = Q \cap [c,d]$. Since $c, d \in Q$ and Q is a partition of $[a,b]$, R is a partition of $[c,d]$; furthermore, each term in the sum for $U_R(f|[c,d]) - L_R(f|[c,d])$ is a term in the sum for $U_Q(f) - L_Q(f)$, and all terms in the sums are positive (since each term is of the form $[M_i(f) - m_i(f)]\Delta x_i$). Thus,

$$U_R(f|[c,d]) - L_R(f|[c,d]) \leq U_Q(f) - L_Q(f).$$

Hence, by (2),

$$U_R(f|[c,d]) - L_R(f|[c,d]) < \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, we have by Theorem 12.15 that $f|[c,d]$ is integrable over $[c,d]$. ¥

Theorem 13.40: If f is integrable over $[a,b]$ and c is any point of $[a,b]$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: Let $\epsilon > 0$. By Theorem 13.39, $\int_a^c f$ and $\int_c^b f$ exist. Thus, since the integral of a function is, by definition, the common value of the upper and the lower integrals of the function, there are partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

(1) $U_{P_1}(f|[a,c]) < \int_a^c f + \frac{\epsilon}{2}, \ \ L_{P_1}(f|[a,c]) > \int_a^c f - \frac{\epsilon}{2},$

$U_{P_2}(f|[c,b]) < \int_c^b f + \frac{\epsilon}{2}, \ \ L_{P_2}(f|[c,b]) > \int_c^b f - \frac{\epsilon}{2}.$

Let $P = P_1 \cup P_2$, a partition of $[a,b]$. Then

$$\left(\int_a^c f - \tfrac{\epsilon}{2}\right) + \left(\int_c^b f - \tfrac{\epsilon}{2}\right) \overset{(1)}{<} L_{P_1}(f|[a,c]) + L_{P_2}(f|[c,b]) = L_P(f) \leq \int_a^b f$$

and

$$\int_a^b f \leq U_P(f) = U_{P_1}(f|[a,c]) + U_{P_2}(f|[c,b]) \overset{(1)}{<} \left(\int_a^c f + \tfrac{\epsilon}{2}\right) + \left(\int_c^b f + \tfrac{\epsilon}{2}\right).$$

The first and last parts of the expressions give us that

$$\int_a^c f + \int_c^b f - \epsilon < \int_a^b f < \int_a^c f + \int_c^b f + \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, we have that

$$\int_a^b f = \int_a^c f + \int_c^b f. \text{ ¥}$$

A useful result in the reverse direction to Theorem 13.39 is in Exercise 13.43.

Exercise 13.41: Evaluate $\int_{-2}^{2} f$ when

$$f(x) = \begin{cases} x & \text{, if } -2 \leq x \leq 0 \\ x^2 & \text{, if } 0 \leq x \leq 2. \end{cases}$$

Be sure to explain why f is integrable over $[-2, 2]$.

Exercise 13.42: Evaluate $\int_{-2}^{2} f$ when $f(x) = |x + 1|$. Be sure to explain why f is integrable over $[-2, 2]$.

Exercise 13.43: If f is integrable over $[a, c]$ and f is integrable over $[c, b]$, then f is integrable over $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Exercise 13.44: Evaluate $\int_1^3 f$ when

$$f(x) = \begin{cases} x & \text{, if } 1 \leq x \leq 2 \\ x + 1 & \text{, if } 2 < x \leq 3. \end{cases}$$

Be sure to explain why f is integrable over $[1, 3]$.

Chapter XIV: The Fundamental Theorem of Calculus

We prove the Fundamental Theorem of Calculus. This beautiful theorem shows a surprising connection between derivatives and integrals – in short, the theorem unifies the subject of calculus. Thus, it is only appropriate that the theorem stand alone, in a chapter all by itself. Nevertheless, we include an application that illustrates a geometric aspect of the theorem; namely (in section 2), we discuss the use of the theorem in connection with computing area, which puts our informal discussion in Chapter XI on a firm (rigorous) foundation.

1. The Fundamental Theorem

We prove the Fundamental Theorem of Calculus after we prove the following technical lemma.

Lemma 14.1: Assume that f is integrable over $[a, b]$, where $a < b$. Let $p \in [a, b]$, and let $h \neq 0$ be a real number such that $p + h \in [a, b]$. For each $x \in [a, b]$, let $F(x) = \int_a^x f$ (which exists by Theorem 13.39).

(1) If $h > 0$, then $\left|\frac{F(p+h)-F(p)}{h} - f(p)\right| \leq \frac{1}{h}\int_p^{p+h} |f - f(p)|$.

(2) If $h < 0$, then $\left|\frac{F(p+h)-F(p)}{h} - f(p)\right| \leq \frac{-1}{h}\int_{p+h}^{p} |f - f(p)|$.

Proof: To prove (1), assume that $h > 0$. Then, since $a \leq p < p + h$,

$$\int_a^{p+h} f \stackrel{13.40}{=} \int_a^p f + \int_p^{p+h} f.$$

Thus,

$$\frac{F(p+h)-F(p)}{h} = \frac{1}{h}\left(\int_a^{p+h} f - \int_a^p f\right) = \frac{1}{h}\int_p^{p+h} f;$$

also, since $\int_p^{p+h} f(p) = hf(p)$ (by Exercise 12.13),

$$f(p) = \frac{1}{h}\int_p^{p+h} f(p).$$

Hence,

$$\left|\frac{F(p+h)-F(p)}{h} - f(p)\right| = \left|\frac{1}{h}\int_p^{p+h} f - \frac{1}{h}\int_p^{p+h} f(p)\right|$$

$$\stackrel{13.12}{=} \left|\frac{1}{h}\int_p^{p+h}(f - f(p))\right| = \frac{1}{h}\left|\int_p^{p+h}(f - f(p))\right| \stackrel{13.17}{\leq} \frac{1}{h}\int_p^{p+h} |f - f(p)|.$$

This proves (1).

To prove (2), assume that $h < 0$. Then, since $a \leq p + h < p$,

$$\int_a^p f \stackrel{13.40}{=} \int_a^{p+h} f + \int_{p+h}^p f.$$

Thus,

$$\tfrac{F(p+h)-F(p)}{h} = \tfrac{1}{h}\left(\int_a^{p+h} f - \int_a^p f\right) = \tfrac{-1}{h}\int_{p+h}^p f;$$

also, since $\int_{p+h}^p f(p) = -hf(p)$ (by Exercise 12.13),

$$f(p) = \tfrac{-1}{h}\int_{p+h}^p f(p).$$

Hence (note for equality in third row below that $\frac{-1}{h} > 0$),

$$\begin{aligned} \left|\tfrac{F(p+h)-F(p)}{h} - f(p)\right| &= \left|\tfrac{-1}{h}\int_{p+h}^p f + \tfrac{1}{h}\int_{p+h}^p f(p)\right| \\ &= \left|\tfrac{-1}{h}\left(\int_{p+h}^p f - \int_{p+h}^p f(p)\right)\right| \overset{13.12}{=} \left|\tfrac{-1}{h}\int_{p+h}^p (f - f(p))\right| \\ &= \tfrac{-1}{h}\left|\int_{p+h}^p (f - f(p))\right| \overset{13.17}{\leq} \tfrac{-1}{h}\int_{p+h}^p |f - f(p)|. \end{aligned}$$

This proves (2). ¥

Theorem 14.2 (The Fundamental Theorem of Calculus): Assume that $a < b$ and that $f : [a,b] \to \mathsf{R}^1$ is a continuous function.

(1) The function F given by $F(x) = \int_a^x f$ for each $x \in [a,b]$ is differentiable on $[a,b]$ and $F' = f$.

(2) If g is any differentiable function on $[a,b]$ such that $g' = f$, then

$$\int_a^b f = g(b) - g(a).$$

Proof: To prove part (1), first note that the function F in (1) is, indeed, defined for each $x \in [a,b]$ by Theorem 12.33 (since $f|[a,x]$ is continuous by Exercise 5.3).

Now, fix a point $p \in [a,b]$. We want to show that $F'(p) = f(p)$. We show this using the definition of the derivative (in section 1 of Chapter VI),

$$F'(p) = \lim_{h\to 0} \tfrac{F(p+h)-F(p)}{h} \quad \text{(if the limit exists).}$$

Specifically (recalling the definition of limit in section 1 of Chapter III), we show that for any $\epsilon > 0$, there is a $\delta > 0$ such that

(*) $\left|\frac{F(p+h)-F(p)}{h} - f(p)\right| < \epsilon$ when $h \neq 0$, $p+h \in [a,b]$, and $|h| < \delta$.

Proof of (*): Let $\epsilon > 0$. Then, since f is continuous at p, Theorem 3.12 gives us a $\delta > 0$ such that

(i) $|f(x) - f(p)| < \frac{\epsilon}{2}$ for all $x \in [a,b]$ such that $|x - p| < \delta$.

We prove that this choice of δ satisfies (*).

Fix $h \neq 0$ such that $p + h \in [a,b]$ and $|h| < \delta$.

Assume first that $h > 0$. Then, by (i), $|f(x) - f(p)| < \frac{\epsilon}{2}$ for all $x \in [p, p+h]$; hence,

$$\int_p^{p+h} |f - f(p)| \overset{13.14}{\leq} \int_p^{p+h} \tfrac{\epsilon}{2} \overset{12.13}{=} h\tfrac{\epsilon}{2}.$$

Thus,

$$\left|\tfrac{F(p+h)-F(p)}{h} - f(p)\right| \overset{14.1}{\leq} \tfrac{1}{h}\int_p^{p+h} |f - f(p)| \leq \tfrac{1}{h}(h\tfrac{\epsilon}{2}) = \tfrac{\epsilon}{2} < \epsilon.$$

This proves (*) when $h > 0$.

Assume next that $h < 0$ (the proof is the same as when $h > 0$, as we will see). Then, by (i), $|f(x) - f(p)| < \frac{\epsilon}{2}$ for all $x \in [p+h, p]$; hence,

$$\int_{p+h}^{p} |f - f(p)| \overset{13.14}{\leq} \int_{p+h}^{p} \tfrac{\epsilon}{2} \overset{12.13}{=} -h\tfrac{\epsilon}{2}.$$

Thus,

$$\left|\tfrac{F(p+h)-F(p)}{h} - f(p)\right| \overset{14.1\text{ (part (2))}}{\leq} \tfrac{-1}{h}\int_{p+h}^{p} |f - f(p)| \leq \tfrac{-1}{h}(-h\tfrac{\epsilon}{2}) = \tfrac{\epsilon}{2} < \epsilon.$$

This proves (*) when $h < 0$.

Therefore, we have proved that for any $\epsilon > 0$, there is a $\delta > 0$ such that (*) holds.

Hence, $F'(p) = f(p)$. This completes the proof of part (1) of our theorem.

To prove part (2), let g be any differentiable function on $[a, b]$ such that $g' = f$. Then, by part (1) of our theorem, $g' = F'$. Hence, by Theorem 10.8, g and F differ by a constant, say

$$F(x) - g(x) = C \text{ for all } x \in [a, b].$$

Let's evaluate C: Since $F(a) = g(a) + C$ and $F(a) = \int_a^a f = 0$ (by the definition of F), we have that $C = -g(a)$. Therefore,

$$\int_a^b f = F(b) = g(b) + C = g(b) - g(a).$$

This proves part (2) of our theorem. ¥

It is easy to apply the Fundamental Theorem of Calculus to evaluate integrals of many continuous functions whose integrals we could not have evaluated up until now. For example,

$$\int_1^3 x^4 = \tfrac{3^5}{5} - \tfrac{1^5}{5} = \tfrac{242}{5}, \quad \int_1^4 \sqrt{x} = \tfrac{2}{3}4^{\frac{3}{2}} - \tfrac{2}{3}1^{\frac{3}{2}} = \tfrac{14}{3},$$
$$\int_0^{\frac{\pi}{3}} \sin(x) = -\cos(\tfrac{\pi}{3}) + \cos(0) = \tfrac{1}{2},$$

and so on. However, there are numerous continuous functions whose integrals we still can not evaluate: For example, $\int_1^2 \frac{x^2+1}{\sqrt{x^5+3x+2}}$ or even one as simple as $\int_1^2 \frac{1}{x}$. We will never be able to evaluate the first integral; however, we will see in

Chapter XVI that the second integral is ln(2) and, thus, that logarithm tables can be used to approximate the value of the second integral.

Exercise 14.3: The assumption that f is continuous in part (1) of the Fundamental Theorem of Calculus is necessary: Give an example to show that part (1) of the theorem would be false if we had only assumed that f is integrable over $[a, b]$.

(Note: Part (2) of the Fundamental Theorem of Calculus generalizes to functions f that are only assumed to be integrable over $[a, b]$. We will not prove this.)

Exercise 14.4: Let $f : [0, 6] \to \mathsf{R}^1$ be the continuous function whose graph is drawn in Figure 14.4 below. Let F be the function in part (1) of Theorem 14.2, $F(x) = \int_0^x f$ for all $x \in [0, 6]$. At which points (on the x-axis) does F have local or global extrema? What type of extremum occurs at each such point? Sketch a rough graph of F (showing where F has inflection points).

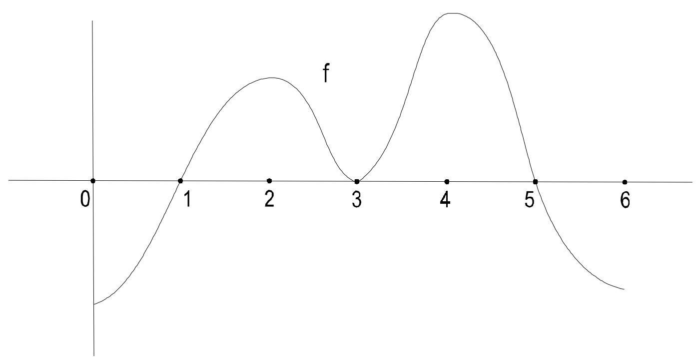

Figure 14.4

Exercise 14.5: If f is a continuous on $[a, b]$ and $a < b$, then there is a point $p \in (a, b)$ such that

$$f(p) = \frac{\int_a^b f}{b-a}.$$

This result is called the Mean Value Theorem for Integrals, and $f(p) = \frac{\int_a^b f}{b-a}$ is often referred to as the *average value of* f *over* $[a, b]$. (The next three exercises are follow ups to this one.)

Exercise 14.6: Find the average value of $f(x) = \sin(2x)$ over $[0, \frac{\pi}{2}]$. (Average value is as defined in Exercise 14.5.)

The mean daily temperature in Morgantown t months after May 1 ($t \leq 6$) is given by the formula $f(t) = 63 + 30\sin(\frac{\pi t}{12})$. Determine the average value of the temperature between June 1 and September 1.

Exercise 14.7: Give an example to show that the result in Exercise 14.5 would fail if we had only assumed that f is integrable on $[a, b]$.

Exercise 14.8: Assume that f is continuous on $[a, b]$ and that $a < b$. For any $x \in [a, b]$ with $x > a$, the average value of f over $[a, x]$ is $\frac{\int_a^x f}{x-a}$ (Exercise 14.5); on the other hand, the average of the values $f(a)$ and $f(x)$ is $\frac{f(a)+f(x)}{2}$.

Determine all the continuous functions f on $[a, b]$ such that for all $x \in [a, b]$ with $x > a$, the average value of f over $[a, x]$ is the average of the values $f(a)$ and $f(x)$.

2. Area Again

Let f be a continuous nonnegative function on an interval $[a, b]$. In Chapter XI, we intuitively discussed the idea of the area between the graph of f and the interval $[a, b]$, and we indicated how to compute the area. It is almost evident that the Fundamental Theorem of Calculus gives us a rigorous definition for the area function A that we used in Chapter XI and is the theorem behind the procedure we arrived at for computing area in Chapter XI. It is only *almost evident* because our approach to area in Chapter XI was slightly different than our approach to the integral in Chapter XII. We show in this section that the two approaches are actually equivalent.

We temporarily disregard the approach to area in Chapter XI. In its place, we define the area between the graph of any integrable function f and the interval $[a, b]$ on which f is defined in terms of the integral. This general definition does not require f to be continuous or to be nonnegative (as was required in Chapter XI).

Definition: Let f be an integrable function on the interval $[a, b]$. We define the *area between the graph of* f *and the interval* $[a, b]$ to be $\int_a^b |f|$. (Recall that $|f|$ is integrable over $[a, b]$ by Theorem 13.17.)

In the definition we assume f is integrable, not just that $|f|$ is integrable even though the area is the integral of $|f|$. By doing so, Theorem 13.3 assures us that the *existence* of area is invariant under vertical translation; this is obviously a property that any notion called area should have. In fact, this property would fail if we had only assumed in the definition that $|f|$ is integrable: For example, if f is defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{, if } x \text{ is rational} \\ -1 & \text{, if } x \text{ is irrational} \end{cases}$$

then $\int_0^1 |f| = 1$ but $\int_0^1 |f + 1|$ does not exist (just like in Example 12.12).

Next, we bring the definition of area above in sync with the approach to area in Chapter XI. We first provide terminology for the types of sums we used in Chapter XI.

Definition: Let $f : [a, b] \to \mathsf{R}^1$ be a function, and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. A *Riemann sum for* f *with respect to* P is a sum of the form $\Sigma_{i=1}^n f(t_i)\Delta x_i$ for any choice of points $t_i \in [x_{i-1}, x_i]$ for each i. We denote any such Riemann sum by $R_P(f)$ (without reference to the points t_i).

We now define the notion of limit for Riemann sums. The definition gives rigorous meaning to the intuitive idea for limits of sums that we worked with in Chapter XI (see the footnote on the second page of Chapter XI). Recall that the norm, $\|P\|$, of a partition P is defined above Exercise 12.32.

Definition: Let $f : [a, b] \to \mathsf{R}^1$ be a function. We say that L *is the limit of the Riemann sums for f as the norms of the partitions of $[a, b]$ go to* 0, written

$$\lim_{\|P\|\to 0} R_P(f) = L,$$

provided that for each $\epsilon > 0$, there is a $\delta > 0$ such that if $P = \{x_0, x_1, ..., x_n\}$ is any partition of $[a, b]$ and $\|P\| < \delta$, then $|R_P(f) - L| < \epsilon$, meaning that

$$|\Sigma_{i=1}^n f(t_i)\Delta x_i - L| < \epsilon \text{ for all choices of points } t_i \in [x_{i-1}, x_i].$$

Finally, the following theorem will show that our approach to area in Chapter XI is the same as area defined in terms of the integral at the beginning of this section (see comments following the proof):

Theorem 14.9: If f is a continuous function on $[a, b]$, then

$$\int_a^b f = \lim_{\|P\|\to 0} R_P(f).$$

Proof: Let $\epsilon > 0$. Then, since f is uniformly continuous (by Theorem 12.31), there is a $\delta > 0$ such that

$$|f(y) - f(z)| < \tfrac{\epsilon}{b-a} \text{ whenever } y, z \in [a, b] \text{ and } |y - z| < \delta.$$

Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[a, b]$ such that $\|P\| < \delta$. Then, since δ satisfies the condition for δ in the proof of Theorem 12.33, the calculations in the proof of Theorem 12.33 show that

$$(*)\ U_P(f) - L_P(f) < \epsilon.$$

Now, choose any points $t_i \in [x_{i-1}, x_i]$ for each i. Since $m_i(f) \leq f(t_i) \leq M_i(f)$ for each i, it is clear from the definitions of $L_P(f)$ and $U_P(f)$ (section 2 of Chapter XII) that

$$L_P(f) \leq \Sigma_{i=1}^n f(t_i)\Delta x_i \leq U_P(f);$$

also, by the definition of the integral (section 3 of Chapter XII),

$$L_P(f) \leq \int_a^b f \leq U_P(f).$$

Thus, by (*),

$$\left|\Sigma_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f\right| < \epsilon.$$

Therefore, since the points t_i are any points in the intervals $[x_{i-1}, x_i]$,

$$\left|R_P(f) - \int_a^b f\right| < \epsilon. \text{ ¥}$$

As in Chapter XI, let f be a continuous nonnegative function on an interval $[a, b]$. By Theorem 14.9, we can now conclude that the area function A in Chapter XI is the function in part (1) of the Fundamental Theorem of Calculus; that is,

$$A(x) = \int_a^x f \text{ for each } x \in [a, b].$$

We also see that the procedure for computing area in Chapter XI, which is summarized in (#) above Example 11.1, is justified by part (2) of the Fundamental Theorem of Calculus.

Exercise 14.10: Let $f(x) = x^2 + x - 2$. Find the area between the graph of f and the interval $[-2, 3]$.

Exercise 14.11: Let $f(x) = \frac{1}{96+x^3}$. Find $c > 0$ such that the area between the graph of f and the interval $[c, 3c]$ is largest.

Exercise 14.12: Using only Theorem 14.9 and the Mean Value Theorem (Theorem 10.2), give a short, elegant proof of part (2) of the Fundamental Theorem of Calculus.

Exercise 14.13: If $f : [a, b] \to \mathsf{R}^1$ is a function such that $\lim_{\|P\|\to 0} R_P(f)$ exists, then f is bounded on $[a, b]$.

Exercise 14.14: If $f : [a, b] \to \mathsf{R}^1$ is a function such that $\lim_{\|P\|\to 0} R_P(f)$ exists, then f is integrable over $[a, b]$ and

$$\int_a^b f = \lim_{\|P\|\to 0} R_P(f).$$

(*Hint:* Let $L = \lim_{\|P\|\to 0} R_P(f)$. Let $\epsilon > 0$. Give reasons for each of the following statements: There is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that $L - \frac{\epsilon}{2} < \Sigma_{i=1}^{n} f(t_i)\Delta x_i < L + \frac{\epsilon}{2}$ for all $t_i \in [x_{i-1}, x_i]$. For each i, there exist $p_i, q_i \in [x_{i-1}, x_i]$ such that $M_i(f) - \frac{\epsilon}{2(b-a)} < f(p_i)$ and $f(q_i) < m_i(f) + \frac{\epsilon}{2(b-a)}$ (note that $M_i(f)$ and $m_i(f)$ exist by Exercise 14.13). Then $U_P(f) - \frac{\epsilon}{2} = \Sigma_{i=1}^{n}[M_i(f) - \frac{\epsilon}{2(b-a)}]\Delta x_i < L + \frac{\epsilon}{2}$, hence $U_P(f) < L + \epsilon$; similarly, $L_P(f) > L - \epsilon$. Thus, $\overline{\int_a^b} f < L + \epsilon$ and $L - \epsilon < \underline{\int_a^b} f$. The result now follows.)

The converse of the first part of Exercise 14.14 is true: If f is integrable over $[a, b]$, then $\lim_{\|P\|\to 0} R_P(f)$ exists. Thus, a function f is integrable over $[a, b]$ if and only if $\lim_{\|P\|\to 0} R_P(f)$ exists, in which case $\int_a^b f = \lim_{\|P\|\to 0} R_P(f)$. This equivalence justifies the somewhat common practice of *defining* the integral in terms of Riemann sums.

Riemann sums are useful for envisioning how to set up an integral to solve a mathematical or physical problem. One illustration of this is in Chapter XI – it was only natural to use Riemann sums to arrive at the notion of area. We give another illustration in the following exercise:

Exercise 14.15: Let f be a continuous nonnegative function on $[a, b]$. Using Riemann sums, find a reasonable formula for the volume of the solid obtained by revolving the graph of f about the x-axis.

Indicate that your formula is reasonable by showing that it gives the known value ($\frac{4}{3}\pi r^3$) for the volume of the sphere of radius r centered at the origin in 3-space.