5. Intermediate Value Property for Derivatives

When we sketched graphs of specific functions, we determined the sign of a derivative or a second derivative on an interval (complementary to the critical points) using the following procedure: We checked the sign at one point in the interval and then appealed to the Intermediate Value Theorem (Theorem 5.2) to conclude that the sign was the same throughout the interval. This works fine as long as the derivatives are continuous. Can a derivative fail to be continuous? If so, is there a systematic way to check signs for such a derivative in order to apply various tests easily? (I am referring to the tests in Theorem 10.19 and Corollary 10.31.)

The answer to the first question is yes, a derivative can fail to be continuous. The answer to the second question is that the answer to the first question is irrelevant: We can determine the sign of a derivative on an interval the way as we always did – by checking the sign at only one point of the interval – whether the derivative is continuous or not! In other words, derivatives do not change sign on an interval on which they are defined without having value zero at some point of the interval.

We give an example that verifies our answer to the first question, and we give a theorem that explains our answer to the second question.

Example 10.48: We give an example of a differentiable function $f : \mathsf{R}^1 \to \mathsf{R}^1$ such that its derivative is not continuous. Define f by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0. \end{cases}$$

Using various results in Chapter VII and Theorem 8.20, we see that f is differentiable at every point $x \neq 0$ and that

$$f'(x) = x^2[\cos(\tfrac{1}{x})](\tfrac{-1}{x^2}) + 2x\sin(\tfrac{1}{x}) = 2x\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x}), \ \ x \neq 0.$$

furthermore, we see that f is differentiable at $x = 0$ as follows: For $x \neq 0$,

$$0 \leq \left|\tfrac{f(x)-f(0)}{x-0}\right| = \left|x\sin(\tfrac{1}{x})\right| \leq |x|;$$

thus, since $\lim_{x\to 0}|x| = 0$, the Squeeze Theorem (Theorem 4.34) applies to give us that

$$\lim_{x\to 0}\left|\tfrac{f(x)-f(0)}{x-0}\right| = 0.$$

This proves that $f'(0) = 0$ (recall Exercise 6.10).

Finally, we show that f' is not continuous at 0 by showing that $\lim_{x\to 0} f'(x)$ does not exist. Recall that

$$f'(x) = x^2[\cos(\tfrac{1}{x})](\tfrac{-1}{x^2}) + 2x\sin(\tfrac{1}{x}) = 2x\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x}), \ \ x \neq 0.$$

Note that

$$0 \le \left|2x\sin(\tfrac{1}{x})\right| \le |2x|, \quad x \ne 0;$$

thus, since $\lim_{x\to 0}|2x| = 0$, we have by the Squeeze Theorem (Theorem 4.34) that

$$\lim_{x\to 0} 2x\sin(\tfrac{1}{x}) = 0.$$

Hence, if $\lim_{x\to 0} f'(x)$ existed, then we would have

$$\lim_{x\to 0}\cos(\tfrac{1}{x}) \overset{4.2}{=} \lim_{x\to 0} 2x\sin(\tfrac{1}{x}) - \lim_{x\to 0} f'(x),$$

which is impossible (since, as is clear, $\lim_{x\to 0}\cos(\frac{1}{x})$ does not exist).

Next, we show why, even though derivatives may not be continuous, we can determine the sign of a derivative on an interval complementary to the critical points by checking the sign at only one point of the interval. The reason is simple enough – derivatives, continuous or not, satisfy the conclusion to the Intermediate Value Theorem (Theorem 5.2). We prove this in Theorem 10.50. First, we introduce relevant terminology and discuss the notion we define. (The terminology carries the name of the French mathematician G. Darboux (1842-1917) who proved the theorem we will prove.)

Definition: Let I be an interval, and let $f : I \to \mathsf{R}^1$ be a function. We say that f is a *Darboux function* provided that for any two points $p, q \in I$ and any point y between $f(p)$ and $f(q)$, there is a point x between p and q such that $f(x) = y$ (i.e., for any subinterval J of I, $f(J)$ is an interval).

There are fairly simple functions that are Darboux but not continuous: For example, let

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{, if } x \ne 0 \\ 0 & \text{, if } x = 0. \end{cases}$$

Actually, the derivative f' of the function in Example 10.48 is another example of a discontinuous Darboux function. This fact about the function in Example 10.48 illustrates the content of the theorem we will prove: Any derivative on an interval is a Darboux function.

We use the following lemma in the proof of our theorem.

Lemma 10.49: Let f be a continuous function on an interval I, and let C denote the set of all slopes of chords joining any two points on the graph of f; that is,

$$C = \{\tfrac{f(s)-f(r)}{s-r} : s, r \in I \text{ and } s \ne r\}.$$

Then C is an interval.

Proof: Fix $p \in C$, say

$$p = \tfrac{f(a)-f(b)}{a-b}, \quad a < b.$$

We show that there is an interval in C joining p to any other point of C. To this end, let $z \in C$, say

$$z = \tfrac{f(u)-f(v)}{u-v}, \quad u < v.$$

Note that since $a-b<0$ and $u-v<0$, $[1-t](a-b)+t(u-v) \neq 0$ for all $t \in [0,1]$; in anticipation of what comes next, we write this as follows:

$$[1-t]a + tu - [1-t]b - tv \neq 0 \quad \text{for all } t \in [0,1].$$

Hence, the following formula defines a function $\sigma : [0,1] \to C$ such that $\sigma(0) = p$ and $\sigma(1) = z$ as follows:

$$\sigma(t) = \tfrac{f([1-t]a+tu)-f([1-t]b+tv)}{[1-t]a+tu-[1-t]b-tv}, \quad \text{all } t \in [0,1].$$

By the continuity of f and by various theorems about continuity in Chapter IV (notably, 4.4, 4.21 and 4.28), we see that σ is continuous. Thus, by the Intermediate Value Theorem (Theorem 5.2), $\sigma([0,1])$ is an interval. Therefore, since $\sigma(0) = p$ and $\sigma(1) = z$, we have proved that p and any other point z of C lie in an interval in C. It now follows easily that C is an interval. ¥

Theorem 10.50: If f is a differentiable function on an interval I, then f' is a Darboux function.

Proof: Let D be the set of all values of the first derivative of f on I,

$$D = \{f'(x) : x \in I\}.$$

We prove that D is an interval, which is simply another way of stating the theorem we are proving.

Let C be as in Lemma 10.49. Since f is continuous by Theorem 6.14, C is an interval by Lemma 10.49. Let E denote the set of end points of C (E may be empty).

The Mean Value Theorem (Theorem 10.2) says that $C \subset D$. The definition of the derivative says that every value of the first derivative of f is a limit of slopes of chords; hence, $D \subset C \cup E$ (since $C \cup E = C^{\sim}$, where $C^{\sim}$ is the set of all points arbitrarily close to C, as defined in sections 1 and 2 of Chapter II).

We have proved that

$$C \text{ is an interval and } C \subset D \subset C \cup E.$$

Therefore, it follows at once that D is an interval. ¥

In Exercise 10.16 you were asked if a certain function with a simple discontinuity was a derivative of some function. You probably worked the exercise in a fairly computational way. Theorem 10.50 yields the solution to Exercise 10.16 immediately and furnishes a completely different perspective on the exercise. We briefly discuss the situation in general.

Let f be a function defined on an open interval I. Then f is said to have a *simple discontinuity at a point* $p \in I$, sometimes called a *discontinuity of the first kind*, provided that f is not continuous at p and $\lim_{x\to p^-} f(x)$ and $\lim_{x\to p^+} f(x)$ exist. The function f is said to have a *discontinuity of the second kind at* p provided that f is not continuous at p and f does not have a simple discontinuity at p.

There are exactly two ways a function can have a simple discontinuity at p: Either $\lim_{x\to p^-} f(x) \neq \lim_{x\to p^+} f(x)$ or $\lim_{x\to p^-} f(x) = \lim_{x\to p^+} f(x) \neq f(p)$.

Corollary 10.51: If f is a differentiable function on an open interval I, then f' has no simple discontinuities.

Proof: Left as the first exercise below. ¥

Exercise 10.52: Prove Corollary 10.51. In fact, prove that the corollary extends to all Darboux functions; that is, any discontinuity of a Darboux function on an open interval is a discontinuity of the second kind.

Exercise 10.53: A differentiable function on R^1 can have derivative equal to zero at a point and yet not have a local extremum at the point (e.g., $f(x) = x^3$). Similarly, a differentiable function on R^1 can have a positive derivative at a point without being strictly increasing in any neighborhood of the point (compare with Theorem 10.17): Modify the function in Example 10.48 to give an example.

(*Hint:* Think geometrically: modify the graph of the function in Example 10.48.)

Exercise 10.54: Define $f : [\frac{9}{10}, \frac{21}{10}] \to \mathsf{R}^1$ by $f(x) = x^4 - 6x^3 + 12x^2$. Find

$$D = \{f'(x) : x \in [\tfrac{9}{10}, \tfrac{21}{10}]\}, \;\; C = \{\tfrac{f(s)-f(r)}{s-r} : s, r \in [\tfrac{9}{10}, \tfrac{21}{10}] \text{ and } s \neq r\};$$

D and C are the sets in the proof of Theorem 10.50.

Exercise 10.55: Let $f : \mathsf{R}^1 \to \mathsf{R}^1$ be a polynomial of odd degree. Theorem 10.50 implies that the set D of all values of the first derivative of f is an interval. What types of intervals can D be? What types of intervals can the set C in Lemma 10.49 be?

Exercise 10.56: Repeat Exercise 10.55 for the case when f is a polynomial of even degree.

Exercise 10.57: Prove that at most one of the functions f and g below can be a derivative of a function:

$$f(x) = \begin{cases} \sin(\dfrac{1}{x}) & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases} \qquad g(x) = \begin{cases} \sin(\dfrac{1}{x}) & \text{, if } x \neq 0 \\ 1 & \text{, if } x = 0. \end{cases}$$

Chapter XI: Area

The chapter is a bridge between previous chapters and the topic of subsequent chapters (the integral). We simply present an informal, nonrigorous discussion of an aspect of area for the purpose of motivating the integral. Our discussion connects derivatives with area!

Consider the continuous function f whose graph we have drawn in Figure 1 below. We want to find the area between the graph of f and the interval $[a, b]$ on x - axis.

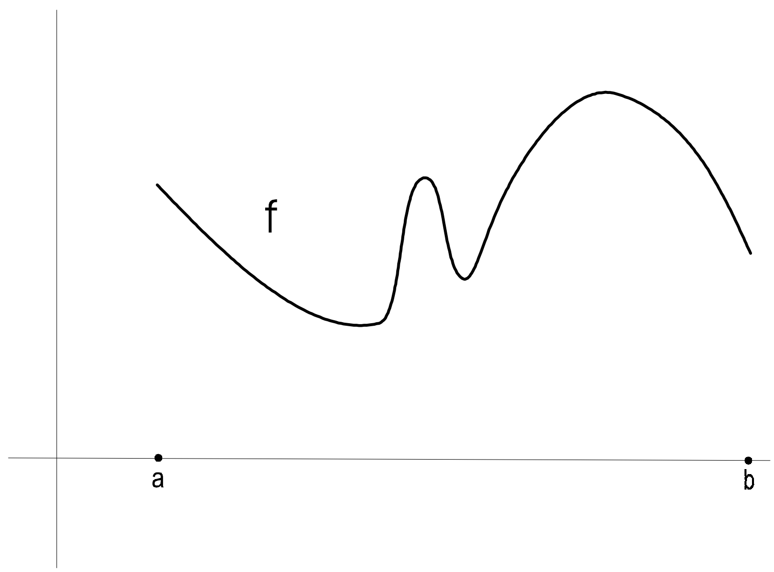

Figure 1

There is an obvious question here: What do we mean by area (referring to the area between the graph of f and the interval $[a, b]$)? We will answer the question in a precise way in Chapter XIV. Here we answer the question somewhat intuitively, and then we describe how to compute the area.

We start by dividing the interval $[a, b]$ into n intervals whose end points are

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b.$$

We think of each of the intervals $[x_{i-1}, x_i]$ as being small, and we consider the rectangles R_i of height $f(x_i)$ and width $x_i - x_{i-1}$, as in Figure 2 (we use $f(x_i)$ as a matter of convenience; we could use $f(t_i)$ for any $t_i \in [x_{i-1}, x_i]$).

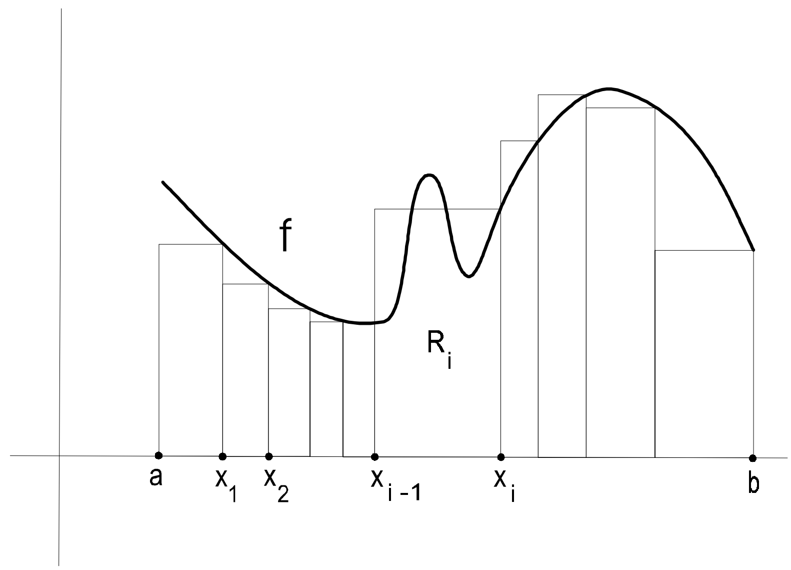

Figure 2

We know from elementary geometry that the area of each rectangle R_i is $f(x_i)(x_i - x_{i-1})$. Thus, the sum $S = \Sigma_{i=1}^{n} f(x_i)(x_i - x_{i-1})$ represents the area of the region covered by all the rectangles. Observe that if x_{i-1} and x_i are very close to one another for each i, then the sum S is very close to what we would call the area between the graph of f and the interval $[a, b]$. Consider dividing the interval $[a, b]$ into more and more subintervals in such a way that the end points x_{i-1} and x_i of the intervals get closer and closer together: If we can compute the "limit" of the sums S associated with the subdivisions, then we will have computed what we would call the area between the graph of f and the interval $[a, b]$.[6]

Now, having indicated what we mean by the area between the graph of f and the interval $[a, b]$, we give a procedure for computing the area. The method is so ingenious that it stands as a monument to human thought.

We make use of the *area function* $A : [a, b] \to \mathsf{R}^1$, defined as follows: For each $x \in [a, b]$, $A(x)$ is the area between the graph of $f|[a, x]$ and the interval $[a, x]$. (We will see in section 2 of Chapter XIV that $A(x)$ is the integral of f over the interval $[a, x]$.)

If we knew a formula for A, computing the area between the graph of f and the interval $[a, b]$ would be easy – we would simply plug b into the formula. Thus, we want to find a formula for A, or at least enough information about A to find $A(b)$.

[6] Note that the "limit" mentioned here is not a limit as we defined the term in Chapter III since each sum S depends on many points $\mathsf{x_i}$. In other words, S is not a function of a single real variable. We have used the term "limit" in an intuitive way – to conjure up a picture in the reader's mind. We give a rigorous definition in section 2 of Chapter XIV.

We "show" that the area function A is differentiable by "computing" its derivative (the quotes mean we show and compute as best as we can without a mathematically precise definition of area). Then we discover what the derivative of A has to do with finding the area we want.

Fix $x \in [a, b]$. In order to find

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h},$$

it is clear that we must write the numerator with a factor of h.

We first examine the numerator $A(x + h) - A(x)$ for some given $h > 0$; we assume h to be near enough to 0 so that $x + h < b$ (if $x = b$, we only consider the case when $h < 0$, which we will consider later for any x).

We see from Figure 3 that $A(x+h) - A(x)$ is the area between the graph of $f|[x, x + h]$ and the interval $[x, x + h]$.

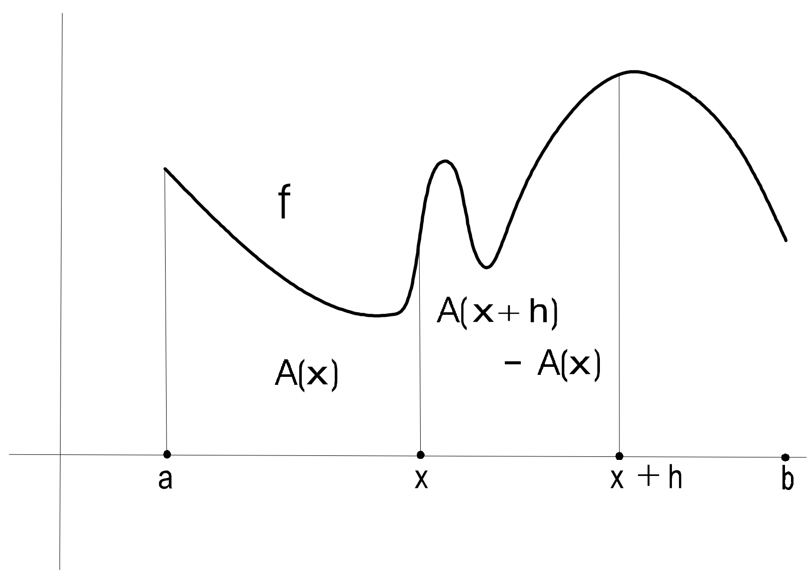

Figure 3

The continuous function f has a maximum value M and a minimum value m on $[x, x+h]$ (by Theorem 5.13). Consider the function $\varphi : [m, M] \to \mathsf{R}^1$ that assigns to a point $t \in [m, M]$ the area of the rectangle $[x, x + h] \times [0, t]$ (see Figure 4); since the height of the rectangle is t and its width is h,

$$\varphi(t) = th \text{ for each } t \in [m, M].$$

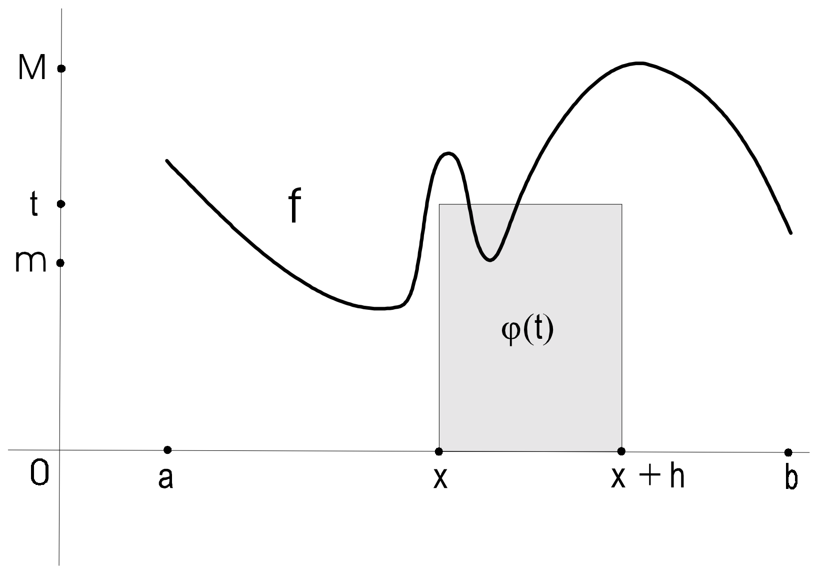

Figure 4

We know that the function φ is continuous (see Example 2.23); furthermore, since $A(x+h) - A(x)$ is the area between the graph of $f|[x, x+h]$ and the interval $[x, x+h]$, we know that

$$\varphi(m) \leq A(x+h) - A(x) \leq \varphi(M).$$

Hence, there is a point $t_h \in [m, M]$ such that $\varphi(t_h) = A(x+h) - A(x)$; in other words,

$$t_h h = A(x+h) - A(x).$$

Now, note that f is continuous on $[x, x+h]$ (by Exercise 5.3); thus, since $t_h \in [m, M]$ and since m and M are values of f on $[x, x+h]$, there is a point $x_h \in [x, x+h]$ such that $f(x_h) = t_h$ (by Theorem 5.2). Therefore, by the previous displayed item, we have

$$(*)\ f(x_h)h = A(x+h) - A(x).$$

The equality in (*) also holds when $h < 0$ (and near enough to 0 so that $x + h > a$): For then the area between the graph of $f|[x+h, x]$ and the interval $[x+h, x]$ is $A(x) - A(x+h)$, and the rectangle $[x+h, x] \times [0, t]$ has width $-h$ for any $t \in [m, M]$; hence, by the analogue of the argument above (in this case, $\varphi(t) = t(-h)$), there is a point $t_h \in [m, M]$ such that

$$t_h(-h) = A(x) - A(x+h),$$

and there is a point $x_h \in [x+h, h]$ such that $f(x_h) = t_h$, thus

$$f(x_h)(-h) = A(x) - A(x+h),$$

which is the same as (*).

We are ready to compute the derivative of A at x: Using that (*) holds whether h is positive or negative, we have that

$$\frac{A(x+h)-A(x)}{h} = \frac{f(x_h)h}{h} = f(x_h), \text{ where } x_h \text{ lies between } x \text{ and } x+h.$$

Hence,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = \lim_{h\to 0} f(x_h);$$

furthermore, since $\lim_{h\to 0} x_h = x$ by the Squeeze Theorem (Theorem 4.34) and since f is continuous at x, we see that $\lim_{h\to 0} f(x_h) = f(x)$ (by Theorem 4.29 by considering the function $h \mapsto x_h$). Therefore,

$$A'(x) = f(x).$$

So, the derivative of the area function is f; but what does that have to do with computing the area between the graph of f and the interval $[a,b]$? Think about it before reading further. Here is a hint: The area we want to compute is $A(b)$, and $A(b) = A(b) - A(a)$.

We show the way to compute $A(b)$. The method is theoretical, but after we discuss the method we will illustrate that it works quite well in practice.

Let g be *any* function whose derivative on $[a,b]$ is f. Then, since $g' = A'$, g and A differ by a constant (by Theorem 10.8), say $A - g = C$. Thus,

$$A(b) - A(a) = \Big(g(b) + C\Big) - \Big(g(a) + C\Big) = g(b) - g(a).$$

Therefore, since $A(b) = A(b) - A(a)$, we can now conclude the following:

(#) *To find the area between the graph of f and the interval $[a,b]$, we need only find a function g whose derivative on $[a,b]$ is f; then the area between the graph of f and $[a,b]$ is $g(b) - g(a)$.*

We give two examples to illustrate how easy it is to apply the procedure we have found.

Example 11.1: We find the area between the graph of $f(x) = x^2$ and the interval $[1,3]$. The function $g(x) = \frac{x^3}{3}$ has derivative f (by Lemma 7.11); therefore, by (#), the area between the graph of f and the interval $[1,3]$ is

$$g(3) - g(1) = 9 - \tfrac{1}{3} = \tfrac{26}{3}.$$

Example 11.2: We find the area between the graph of $f(x) = x^{\frac{2}{5}} + 3x^3$ and the interval $[1,3]$. The function $g(x) = \frac{5}{7}x^{\frac{7}{5}} + \frac{3}{4}x^4$ has derivative f (by Theorem 7.1 and Theorem 8.16); hence, by (#), the area between the graph of f and the interval $[1,3]$ is

$$g(3) - g(1) = \frac{5}{7}3^{\frac{7}{5}} + \frac{243}{4} - \frac{41}{28}.$$

How do we know that the procedure in (#) really does give the area? The most reasonable way to check this is to see if the procedure gives various areas that are known from geometry. We offer the following exercise as a start:

Exercise 11.3: Show that the procedure in (#) gives the formulas from geometry for the areas of rectangles, triangles and circles.

(*Hint:* In the case of a circle of radius r about the origin, consider the function $g(x) = \frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}(\frac{x}{2})$.)

When more complicated figures (than those in Exercise 11.3) whose areas are known from geometry are analyzed using the procedure in (#), the answer is always the same: Applying (#) results in arriving at the known areas. In the end, therefore, we will be jusified in *defining* area in terms of the integral and using the procedure in (#) to find the area – see section 2 of Chapter XIV.

We conclude with a few exercises.

Exercise 11.4: Find the area between the graph of $f(x) = \sin(x)$ and the interval $[0, \pi]$.

Exercise 11.5: Find the area between the graph of $f(x) = \frac{1}{\sqrt{1-x^2}}$ and the interval $[0, \frac{1}{2}]$.

Exercise 11.6: Find formulas for the area functions for Examples 11.1 and 11.2.

Exercise 11.7: Using the intuitive observation that the area of two nonoverlapping regions is the sum of the areas of the two regions, find the area above the interval $[0, 1]$ between the graphs of the two functions $f_1(x) = x^4$ and $f_2(x) = x^5$.

Chapter XII: The Integral

In the first part of preceding chapter, we intuitively discussed a way of defining area in order to provide a tangible picture to keep in mind when studying the integral. In this chapter, we begin a rigorous treatment of the integral. This is the first of four chapters concerned directly with the theory of the integral. (There are many types of integrals; we will only study one type – the Riemann integral – which we simply refer to as the integral.)

After presenting preliminary notions and results, we define the integral in section 3. In section 4, we prove an existence theorem that gives a necessary and sufficient condition for a function to be integrable (Theorem 12.15); we also prove a theorem that provides a way (albeit limited) to evaluate the integral (Theorem 12.17). In section 5, we use the existence theorem in section 4 to prove that all continuous functions are integrable.

1. Partitions

In this section (and the next) we present a rigorous and systematic treatment of some of the ideas that we introduced informally in the preceding chapter. Thus, we consider the preceding chapter as motivation for what follows.

Definition. A *partition of* $[a,b]$ when $a<b$ is a finite subset P of $[a,b]$ that can be indexed so that $P=\{x_0,x_1,...,x_n\}$, where

$$x_0=a<x_1<x_2<\cdots<x_n=b, \text{ some } n\geq 1.$$

It is also to be understood that the interval $[a,a]$ has a (unique) partition, namely, $\{a\}$.

For example, $\{0,1\}$ and $\{0,\frac{1}{3},\frac{1}{2},1\}$ are partitions of $[0,1]$. Obviously, every interval $[a,b]$ has a partition.

Whenever P is a partition and we write $P=\{x_0,x_1,...,x_n\}$, we assume (without explicitly saying so) that the points x_i satisfy the condition in the definition above. We prove all results that involve partitions, directly or indirectly (as in the case of integrals), assuming that $a<b$. It will be evident that the results hold when $a=b$.

Definition. Let P_1 and P_2 be partitions of $[a,b]$. We say that P_2 *is a refinement of* P_1, written $P_2\preceq P_1$, provided that $P_2\supset P_1$.

We can think of a refinement of a partition P as being obtained from P by adding points to P (although, of course, a partition is a refinement of itself). Obviously, every partition of $[a,b]$ is a refinement of $\{a,b\}$.

Exercise 12.1: Give an example of two partitions of $[a,b]$ such that neither one is a refinement of the other.

A relation $\ll$ between elements of a set S is a *partial order on* S provided that the relation is reflexive ($s\ll s$ for all $s\in S$), antisymmetric (if $s_1\ll s_2$ and $s_2\ll s_1$, then $s_1=s_2$), and transitive (if $s_1\ll s_2$ and $s_2\ll s_3$, then $s_1\ll s_3$).

For example, $\leq$ is a partial order on R^1 by axioms O1 and O2 in section 1 of Chapter I.

Note the following simple fact:

Exercise 12.2: The relation $\preceq$ of refinement on the collection $\mathcal{P}$ of all partitions of a given interval $[a, b]$ is a partial order.

Definition. Let P_1 and P_2 be refinements of $[a, b]$. A *common refinement of* P_1 *and* P_2 is a partition P of $[a, b]$ such that $P \preceq P_1$ and $P \preceq P_2$.

Exercise 12.3: For any two partitions P_1 and P_2 of $[a, b]$, there is a smallest common refinement of P_1 and P_2; that is, there is a common refinement, P, of P_1 and P_2 such that every common refinement of P_1 and P_2 contains P.

2. Upper and Lower Sums

We continue with our presentation of the background necessary for defining the integral and understanding the definition.

We adopt the following notation: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function, and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. For each $i = 1, 2, ..., n$,

$$\Delta x_i = x_i - x_{i-1}, \quad M_i(f) = lub\, f([x_{i-1}, x_i]), \quad m_i(f) = glb\, f([x_{i-1}, x_i]).$$

Definition. Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function, and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$.

- The *upper sum of* f *with respect to* P, denoted by $U_P(f)$, is defined by

$$U_P(f) = \Sigma_{i=1}^n M_i(f)\Delta x_i.$$

- The *lower sum of* f *with respect to* P, denoted by $L_P(f)$, is defined by

$$L_P(f) = \Sigma_{i=1}^n m_i(f)\Delta x_i.$$

Exercise 12.4: Define $f : [-4, 4] \to \mathsf{R}^1$ by $f(x) = x^3 - 12x$. Evaluate $U_P(f)$ and $L_P(f)$ for the partition $P = \{-4, 1, 4\}$.

Exercise 12.5: Define $f : [0, 4] \to \mathsf{R}^1$ by $f(x) = x^3 - 9x^2 + 26x - 24$. Evaluate $U_P(f)$ and $L_P(f)$ for the partition $P = \{0, 1, 3, 4\}$.

Lemma 12.6: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function. For any partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$, $L_P(f) \leq U_P(f)$.

Proof: For each i, $m_i(f) \leq M_i(f)$ and $\Delta x_i > 0$, hence $m_i(f)\Delta x_i \leq M_i(f)\Delta x_i$. Therefore, the lemma follows immediately by summing over i. ¥

Lemma 12.7: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function. Let P be a partition of $[a, b]$, and let q be a point of $[a, b]$ such that $q \notin P$. Let $Q = P \cup \{q\}$ (considered as a partition of $[a, b]$). Then

$$U_Q(f) \leq U_P(f) \quad \text{and} \quad L_Q(f) \geq L_P(f).$$

Proof: Assume that $P = \{x_0, x_1, ..., x_n\}$. Let k be such that $x_k < q < x_{k+1}$. Then, letting

$$\alpha = \Big(lub\, f([x_k, q])\Big)(q - x_k) + \Big(lub\, f([q, x_{k+1}])\Big)(x_{k+1} - q),$$

we have that $U_Q(f) = \Sigma_{i \neq k+1} M_i(f)\Delta x_i + \alpha$. Also, since $lub(A) \leq lub(B)$ when $A \subset B$,

$$\begin{aligned}\alpha &= \Big(lub\, f([x_k, q])\Big)(q - x_k) + \Big(lub\, f([q, x_{k+1}])\Big)(x_{k+1} - q) \\ &\leq \Big(lub\, f([x_k, x_{k+1}])\Big)(q - x_k) + \Big(lub\, f([x_k, x_{k+1}])\Big)(x_{k+1} - q) \\ &= \Big(lub\, f([x_k, x_{k+1}])\Big)(x_{k+1} - x_k).\end{aligned}$$

Therefore,

$$U_Q(f) = \Sigma_{i \neq k+1} M_i(f)\Delta x_i + \alpha \leq \Sigma_{i=1}^{n} M_i(f)\Delta x_i = U_P(f).$$

Similarly, $L_Q(f) \geq L_P(f)$. ¥

Lemma 12.8: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function, and let P_1 and P_2 be partitions of $[a, b]$ such that $P_2 \preceq P_1$. Then

$$U_{P_2}(f) \leq U_{P_1}(f) \quad \text{and} \quad L_{P_2}(f) \geq L_{P_1}(f).$$

Proof: Let $y_1, y_2, ..., y_m$ be the points in $P_2 - P_1$ (we assume that $P_1 \neq P_2$ since, otherwise, the lemma is obvious). We successively define partitions Q_j, $j = 1, ..., m$, of $[a, b]$ as follows:

$$Q_1 = P_1, \;\; Q_2 = Q_1 \cup \{y_1\}, \;\; Q_3 = Q_2 \cup \{y_2\}, \; ... \;, \;\; Q_m = P_2.$$

Since Q_{j+1} has exactly one more point than Q_j for each j, each successive inequality below follows at once from Lemma 12.7:

$$U_{P_2}(f) = U_{Q_m}(f) \leq U_{Q_{m-1}}(f) \leq \cdots \leq U_{Q_2}(f) \leq U_{Q_1}(f) = U_{P_1}(f)$$

and

$$L_{P_2}(f) = L_{Q_m}(f) \geq L_{Q_{m-1}}(f) \geq \cdots \geq L_{Q_2}(f) \geq L_{Q_1}(f) = L_{P_1}(f). \text{ ¥}$$

Lemma 12.9: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function, and let P_1 and P_2 be partitions of $[a, b]$. Then

$$L_{P_1}(f) \leq U_{P_2}(f).$$

Proof: Let P be a common refinement of P_1 and P_2 (see Exercise 12.3). Then

$$L_{P_1}(f) \overset{12.8}{\leq} L_P(f) \overset{12.6}{\leq} U_P(f) \overset{12.8}{\leq} U_{P_2}(f). \text{ ¥}$$

The numbers $lub_{P\in\mathcal{P}}L_P(f)$ and $glb_{P\in\mathcal{P}}U_P(f)$ in the next lemma are the basis for our definition of the integral in the next section.

Lemma 12.10: Let $f:[a,b]\to \mathsf{R}^1$ be a bounded function, and let $\mathcal{P}$ denote the collection of all partitions of $[a,b]$. Then $lub_{P\in\mathcal{P}}L_P(f)$ and $glb_{P\in\mathcal{P}}U_P(f)$ exist and

$$lub_{P\in\mathcal{P}}L_P(f)\le glb_{P\in\mathcal{P}}U_P(f).$$

Proof: There is a partition P_1 of $[a,b]$. By Lemma 12.9, $L_{P_1}(f)$ is a lower bound for the set of all upper sums of f with respect to all partitions of $[a,b]$. Hence, by the Greatest Lower Bound Axiom (section 8 of Chapter I), $glb_{P\in\mathcal{P}}U_P(f)$ exists, and

$$(*)\ L_{P_1}(f)\le glb_{P\in\mathcal{P}}U_P(f).$$

Note that we have proved (*) for any partition P_1 of $[a,b]$. Hence, $glb_{P\in\mathcal{P}}U_P(f)$ is an upper bound for the set of all lower sums of f with respect to all partitions of $[a,b]$. Therefore, by the Least Upper Bound Axiom (Completeness Axiom), $lub_{P\in\mathcal{P}}L_P(f)$ exists, and it is clear that

$$lub_{P\in\mathcal{P}}L_P(f)\le glb_{P\in\mathcal{P}}U_P(f).\ ¥$$

Except for very simple functions, it is difficult to directly compute the numbers $lub_{P\in\mathcal{P}}L_P(f)$ and $glb_{P\in\mathcal{P}}U_P(f)$ in Lemma 12.10. For example, the reader might try to compute the numbers in Lemma 12.10 for the case when f is the function on $[0,1]$ defined by $f(x)=x$. In fact, computing the numbers $lub_{P\in\mathcal{P}}L_P(f)$ and $glb_{P\in\mathcal{P}}U_P(f)$ is actually evaluating integrals or showing integrals do not exist, as we will see from the definition of the integral (in the next section). Nevertheless, we can at this time compute the numbers in Lemma 12.10 for a few functions. We illustrate how to do this in the two examples below. In the first example, $lub_{P\in\mathcal{P}}L_P(f)=glb_{P\in\mathcal{P}}U_P(f)$; in the second example, $lub_{P\in\mathcal{P}}L_P(f)\neq glb_{P\in\mathcal{P}}U_P(f)$.

Example 12.11: Define $f:[0,2]\to\mathsf{R}^1$ by

$$f(x)=\begin{cases}1 & \text{, if } x\neq 1\\ 2 & \text{, if } x=1.\end{cases}$$

Let $\mathcal{P}$ denote the collection of all partitions of $[0,2]$. We show that

$$lub_{P\in\mathcal{P}}L_P(f)=glb_{P\in\mathcal{P}}U_P(f)=2.$$

Let $P=\{x_0,x_1,...,x_n\}$ be a partition of $[0,2]$. Note that each of the intervals $[x_{i-1},x_i]$ contains a point different from 1; hence, $m_i(f)=1$ for each i. Thus,

$$L_P(f)=\Sigma_{i=1}^n\Delta x_i=x_n-x_0=2-0=2.$$

Therefore, $lub_{P\in\mathcal{P}}L_P(f)=2$.

We now show that $glb_{P\in\mathcal{P}}U_P(f)=2$. Let $\epsilon>0$ such that $\epsilon<1$. Consider the following very simple partition Q of $[0,2]$:

$$Q = \{0, 1-\epsilon, 1+\epsilon, 2\}.$$

We compute $U_Q(f)$:

$$U_Q(f) = 1([1-\epsilon]-0) + 2([1+\epsilon]-[1-\epsilon]) + 1(2-[1+\epsilon]) = 2+2\epsilon.$$

Thus, since ϵ can be as close to zero as we like, we have proved that

$$glb_{P\in\mathcal{P}} U_P(f) \leq 2.$$

Also, having proved above that $lub_{P\in\mathcal{P}} L_P(f) = 2$, we know from Lemma 12.10 that $2 \leq glb_{P\in\mathcal{P}} U_P(f)$. Therefore,

$$glb_{P\in\mathcal{P}} U_P(f) = 2 = lub_{P\in\mathcal{P}} L_P(f).$$

Example 12.12: Define $f : [0,1] \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} 0 & \text{, if } x \text{ is rational} \\ 1 & \text{, if } x \text{ is irrational.} \end{cases}$$

Let $\mathcal{P}$ denote the collection of all partitions of $[0,1]$. We show that

$$lub_{P\in\mathcal{P}} L_P(f) = 0 \text{ and } glb_{P\in\mathcal{P}} U_P(f) = 1.$$

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0,1]$. By Theorem 1.26 (and its analogue for irrational numbers), there is a rational number and an irrational number in each of the intervals $[x_{i-1}, x_i]$. Hence,

$$L_P(f) = \Sigma_{i=1}^n (0)\Delta x_i = 0$$

and

$$\begin{aligned} U_P(f) &= \Sigma_{i=1}^n (1)\Delta x_i = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= x_n - x_0 = 1 - 0 = 1. \end{aligned}$$

Therefore, $lub_{P\in\mathcal{P}} L_P(f) = 0$ and $glb_{P\in\mathcal{P}} U_P(f) = 1$.

The cancellation that gave $\Sigma_{i=1}^n \Delta x_i = x_n - x_0$ in Example 12.12 is trivial but has far - reaching generalizations in multi - dimensional calculus (for example, in the proof of Green's Theorem).

Exercise 12.13: Let f be a constant function on an interval $[a,b]$, say $f(x) = c$ for all $x \in [a,b]$. Compute $lub_{P\in\mathcal{P}} L_P(f)$ and $glb_{P\in\mathcal{P}} U_P(f)$.

Exercise 12.14: Define $f : [0,2] \to \mathsf{R}^1$ by

$$f(x) = \begin{cases} 1 & \text{, if } 0 \leq x < 1 \\ 3 & \text{, if } 1 \leq x \leq 2. \end{cases}$$

Compute $lub_{P\in\mathcal{P}} L_P(f)$ and $glb_{P\in\mathcal{P}} U_P(f)$.

3. Definition of the Integral

We are ready to define the integral.

Definition. Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function, and let $\mathcal{P}$ denote the collection of all partitions of $[a, b]$. Recall that we showed in Lemma 12.10 that the numbers $glb_{P\in\mathcal{P}}U_P(f)$ and $lub_{P\in\mathcal{P}}L_P(f)$ exist.

- The *upper integral of f over* $[a, b]$ is $glb_{P\in\mathcal{P}}U_P(f)$, which we denote from now on by $\overline{\int}_a^b f$.
- The *lower integral of f over* $[a, b]$ is $lub_{P\in\mathcal{P}}L_P(f)$, which we denote from now on by $\underline{\int}_a^b f$.
- We say that f *is integrable over* $[a, b]$ provided that $\overline{\int}_a^b f = \underline{\int}_a^b f$, in which case we call the common value $\overline{\int}_a^b f = \underline{\int}_a^b f$ *the integral of f over* $[a, b]$ (or *the integral of f from a to b*). We denote the integral of f over $[a, b]$ by $\int_a^b f$ or by $\int_a^b f(x)dx$. The notation $\int_a^b f(x)dx$ is read *integral of f over $[a, b]$ with respect to the variable x.*[7]

In the expressions $\overline{\int}_a^b f$, $\underline{\int}_a^b f$ and $\int_a^b f$, the numbers a and b are referred to as *the limits of integration* (a being *the lower limit of integration* and b being *the upper limit of integration*) The function f is called the *integrand*.

From what we showed in Example 12.11, we can now say that the function f in the example is integrable and $\int_0^2 f = 2$. On the other hand, from what we showed in Example 12.12, the function f in Example 12.12 is not integrable.

We prove results about integrals over $[a, b]$ as though $a < b$ without saying so. The reader can easily check that the results are true when $a = b$ ($\int_a^a f = 0$ since $\{a\}$ is the only partition of the interval $[a, a]$).

4. Two Theorems about Integrability

We prove two theorems about integrability and show how the theorems can be applied.

Our first theorem is useful for proving that a function is integrable; we illustrate this for a specific function after we prove the theorem. We use the theorem in the next section to prove that all continuous functions are integrable, and we use the theorem in many other places as well.

[7] Regarding the notation $\int_a^b f(x)dx$, the symbol dx has absolutely no mathematical content other than to indicate the variable with respect to which the integration is being performed. Thus, the symbol dx can be used to clarify situations when the expression being integrated contains two or more letters as symbols; for example, simply writing $\int_a^b t^2x^3$ puts in doubt whether we are integrating with respect to t or with respect to x, whereas writing $\int_a^b t^2x^3dx$ and $\int_a^b t^2x^3dt$ makes it clear what the variable of integration is in each case.

Theorem 12.15: Let $f : [a,b] \to \mathsf{R}^1$ be a bounded function. Then f is integrable over $[a,b]$ if and only if for each $\epsilon > 0$, there is a partition P of $[a,b]$ such that

$$U_P(f) - L_P(f) < \epsilon.$$

Proof: Assume that f is integrable over $[a,b]$. Let $\epsilon > 0$. Since

$$\int_a^b f = \overline{\int}_a^b f = glb_{P \in \mathcal{P}} U_P(f) \quad \text{and} \quad \int_a^b f = \underline{\int}_a^b f = lub_{P \in \mathcal{P}} L_P(f),$$

there are a partitions P_1 and P_2 of $[a,b]$ such that

(1) $U_{P_1}(f) < \int_a^b f + \frac{\epsilon}{2}$ and $L_{P_2}(f) > \int_a^b f - \frac{\epsilon}{2}$.

Let P be a common refinement of P_1 and P_2 (see Exercise 12.3). Then, by Lemma 12.6 and Lemma 12.8 , we have

(2) $L_{P_2}(f) \leq L_P(f) \leq U_P(f) \leq U_{P_1}(f)$.

Now,

$$U_P(f) - L_P(f) \overset{(2)}{\leq} U_{P_1}(f) - L_{P_2}(f) \overset{(1)}{<} \int_a^b f + \frac{\epsilon}{2} - (\int_a^b f - \frac{\epsilon}{2}) = \epsilon.$$

This proves that P is as required in the theorem.

Conversely, assume that for each $\epsilon > 0$, there is a partition P_ϵ of $[a,b]$ such that

$$U_{P_\epsilon}(f) - L_{P_\epsilon}(f) < \epsilon.$$

Then, since $\overline{\int}_a^b f = glb_{P \in \mathcal{P}} U_P(f)$ and $\underline{\int}_a^b f = lub_{P \in \mathcal{P}} L_P(f)$,

$$0 \overset{12.10}{\leq} \overline{\int}_a^b f - \underline{\int}_a^b f \leq U_{P_\epsilon}(f) - L_{P_\epsilon}(f) < \epsilon \text{ for all } \epsilon > 0.$$

Hence, $\overline{\int}_a^b f - \underline{\int}_a^b f = 0$ (it follows from the axioms in section 1 of Chapter I that if $0 \leq x < \epsilon$ for all $\epsilon > 0$, then $x = 0$). Therefore, $\overline{\int}_a^b f = \underline{\int}_a^b f$, which proves that f is integrable. ¥

Lest it escape us without notice, we point out that Theorem 12.15 says that we need only find *one* appropriate partition for each $\epsilon > 0$ in order to show a function is integrable. This feature of Theorem 12.15 makes it significantly easier to show a function is integrable than it would be to show the function is integrable using the definition of integrability directly. We illustrate this with the following example:

Example 12.16: Define $f : [0,2] \to \mathsf{R}^1$ by $f(x) = x^2$. We show that f is integrable over $[0,2]$ by applying Theorem 12.15.

Let $\epsilon > 0$. Let n be a natural number such that $\frac{4}{n} < \epsilon$ (the number n exists by the Archimedean Property (Theorem 1.22)). Let P be the partition of $[0,2]$ given by

$$P = \{x_0 = 0, x_1 = \frac{1}{n}, ..., x_i = \frac{i}{n}, ..., x_{2n} = 2\}.$$

Note that f is strictly increasing (by Theorem 10.17 since $f'(x) = 2x > 0$ for all $x \in [0, 2]$). Hence,

$$M_i(f) = x_i^2, \;\; m_i(f) = x_{i-1}^2, \;\text{ each } i = 1, 2, ..., 2n.$$

Thus, since $\Delta x_i = \frac{1}{n}$ for each i,

$$\begin{aligned} U_P(f) - L_P(f) &= \Sigma_{i=1}^{2n} x_i^2 \tfrac{1}{n} - \Sigma_{i=1}^{2n} x_{i-1}^2 \tfrac{1}{n} = \tfrac{1}{n}(\Sigma_{i=1}^{2n} x_i^2 - \Sigma_{i=1}^{2n} x_{i-1}^2) \\ &= \tfrac{1}{n}(x_{2n}^2 - x_0^2) = \tfrac{1}{n}(4 - 0) < \epsilon. \end{aligned}$$

Therefore, by Theorem 12.15, f is integrable over $[0, 2]$.

Note that we did not evaluate the integral in Example 12.16 – Theorem 12.15 is not set up to evaluate integrals. Our next theorem gives a condition that can be used to evaluate integrals (in practice, however, the theorem has very limited use for this purpose). After we prove the theorem, we apply the theorem to evaluate the integral in the example above.

We note that the limits in the following theorem are limits of sequences, which we discussed in section 8 of Chapter IV.

Theorem 12.17: Let $f : [a, b] \to \mathsf{R}^1$ be a bounded function. Assume that $P_1, P_2, ..., P_n, ...$ are partitions of $[a, b]$ such that

$$\lim_{n\to\infty} U_{P_n}(f) = \lim_{n\to\infty} L_{P_n}(f) = c.$$

Then f is integrable over $[a, b]$ and $\int_a^b f = c$.

Proof: By definition, $\underline{\int}_a^b f = lub_{P\in\mathcal{P}} L_P(f)$ and $\overline{\int}_a^b f = glb_{P\in\mathcal{P}} U_P(f)$; hence,

$$L_{P_n}(f) \le \underline{\int}_a^b f \overset{12.10}{\le} \overline{\int}_a^b f \le U_{P_n}(f), \;\text{ all } n = 1, 2, ...$$

Thus, by the Squeeze Theorem (Theorem 4.34), which holds for sequences by Theorem 4.38, we have that

$$\underline{\int}_a^b f = c \;\text{ and }\; \overline{\int}_a^b f = c.$$

Therefore, f is integrable and $\int_a^b f = c$. ¥

Example 12.18: We use Theorem 12.17 to evaluate the integral of the function in Example 12.16; we show that $\int_0^2 x^2 = \frac{8}{3}$.

We use following formula; the formula can be verified by induction (we leave the verification for the reader in Exercise 12.19):

$$(*)\ \Sigma_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \text{ for each } n = 1, 2, ...$$

For each $n = 1, 2, ...$, let P_n be the partition of $[0, 2]$ given by

$$P_n = \{x_0 = 0, x_1 = \tfrac{1}{n}, ..., x_i = \tfrac{i}{n}, ..., x_{2n} = 2\}.$$

Then, since $M_i(f) = x_i^2$ and $m_i(f) = x_{i-1}^2$ for each i (as in Example 12.16),

$$U_{P_n}(f) = \Sigma_{i=1}^{2n} x_i^2 \tfrac{1}{n} \quad \text{and} \quad L_{P_n}(f) = \Sigma_{i=1}^{2n} x_{i-1}^2 \tfrac{1}{n} \quad \text{for each } n.$$

Hence, for each n,

$$\begin{aligned} U_{P_n}(f) &= \tfrac{1}{n}\Sigma_{i=1}^{2n}(\tfrac{i}{n})^2 = \tfrac{1}{n^3}\Sigma_{i=1}^{2n} i^2 \overset{(*)}{=} \tfrac{1}{n^3}\tfrac{2n(2n+1)(4n+1)}{6} \\ &= \tfrac{(2n+1)(4n+1)}{3n^2} = \tfrac{8}{3} + \tfrac{2}{n} + \tfrac{1}{3n^2} \end{aligned}$$

and

$$\begin{aligned} L_{P_n}(f) &= \Sigma_{i=1}^{2n}(\tfrac{i-1}{n})^2 \tfrac{1}{n} = \tfrac{1}{n^3}\Sigma_{i=1}^{2n}(i-1)^2 = \tfrac{1}{n^3}\Sigma_{i=1}^{2n-1} i^2 \\ &\overset{(*)}{=} \tfrac{1}{n^3}\tfrac{(2n-1)(2n)(4n-1)}{6} = \tfrac{(2n-1)(4n-1)}{3n^2} = \tfrac{8}{3} - \tfrac{2}{n} + \tfrac{1}{3n^2}\,. \end{aligned}$$

Thus, $\lim_{n\to\infty} U_{P_n}(f) = \frac{8}{3}$ and $\lim_{n\to\infty} L_{P_n}(f) = \frac{8}{3}$. Therefore, by Theorem 12.17, $\int_0^2 x^2 = \frac{8}{3}$.

Exercise 12.19: Verify that $\Sigma_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for each $n = 1, 2, ...$ by using induction (Theorem 1.20). (We used the formula in Example 12.18.)

In Examples 12.16 and 12.18, we used partitions that divide the interval of integration into intervals of equal length. These types of partitions are useful because we can factor Δx_i out of summations when computing upper and lower sums. We call a partition of an interval $[a, b]$ that divides $[a, b]$ into intervals of equal length Δx_i a *regular partition*.

Exercise 12.20: Evaluate $\int_a^b x$ for any $a \le b$.
(*Hint:* First prove that $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$ for each $n = 1, 2, \ldots$.)

Exercise 12.21: Determine if f is integrable, where $f : [0, 1] \to \mathsf{R}^1$ is defined as follows (Q denotes the set of all rational numbers; for integers m and n, $\frac{m}{n}$ *in lowest terms* means m and n have no common divisor other than ± 1):

$$f(x) = \begin{cases} 0 & \text{, if } x \text{ is irrational} \\ 1 & \text{, if } x = 0 \\ \frac{1}{n} & \text{, if } x \in \mathsf{Q} - \{0\} \text{ and } x = \frac{m}{n} \text{ in lowest terms.} \end{cases}$$

Exercise 12.22: Assume that $f(x) \le g(x) \le h(x)$ for all $x \in [a, b]$ and that f and h are integrable over $[a, b]$. If $\int_a^b f = \int_a^b h$, then g is integrable and $\int_a^b g$ is equal to $\int_a^b f = \int_a^b h$.

Exercise 12.23: If f is increasing on $[a, b]$ or decreasing on $[a, b]$, then f is integrable over $[a, b]$.

Exercise 12.24: In connection with Exercise 12.23, is every one-to-one bounded function on an interval $[a, b]$ integrable over $[a, b]$?

Exercise 12.25: If $f : [a,b] \to \mathsf{R}^1$ is a nonnegative function that is integrable over $[a,b]$, then $\int_a^b f \geq 0$.

Exercise 12.26: Let $f : [a,b] \to \mathsf{R}^1$ be a nonnegative function that is integrable over $[a,b]$. Then $\int_a^b f = 0$ if and only if $glbf(I) = 0$ for each open interval I in $[a,b]$.

Exercise 12.27: Let $f : [a,b] \to \mathsf{R}^1$ be a function that is integrable over $[a,b]$, and let $g : [a,b] \to \mathsf{R}^1$ be a function that agrees with f except at finitely many points. Is g integrable over $[a,b]$?

5. Continuous Functions Are Integrable

We prove that any continuous function defined on a closed and bounded interval is integrable. This is an existence theorem – it does not show how to evaluate the integral. We will be able to evaluate integrals of many simple continuous functions using the Fundamental Theorem of Calculus, which we prove in Chapter XIV. However, evaluating integrals of most continuous functions is difficult, usually impossible; ad hoc methods can sometimes be employed, but most often one has to settle for approximate evaluations by numerical methods.

The following notion is of general importance and is the key idea that we use to prove our theorem:

Definition: Let $X \subset \mathsf{R}^1$, and let $f : X \to \mathsf{R}^1$ be a function. We say that f is *uniformly continuous on* X provided that for any $\epsilon > 0$, there is a $\delta > 0$ such that if $x_1, x_2 \in X$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.

Exercise 12.28: Let $X \subset \mathsf{R}^1$. If $f : X \to \mathsf{R}^1$ is uniformly continuous, then f is continuous.

Exercise 12.29: The converse of the result in Exercise 12.28 is false: The function $f : \mathsf{R}^1 \to \mathsf{R}^1$ given by $f(x) = x^2$ is continuous but not uniformly continuous.

Exercise 12.30: Any linear function f (i.e., $f(x) = mx + b$) is uniformly continuous on R^1. More generally, if f is differentiable on an interval I and the derivative f' is bounded on I, then f is uniformly continuous on I.

The following theorem is not concerned with integrals, but it is the basis of our proof that continuous functions are integrable. The theorem is so important in all of mathematics that even though it plays the role of a lemma here, we can not bring ourselves to call the theorem a lemma. The theorem shows that the converse of the result in Exercise 12.28 is true when X is a closed and bounded interval.

Theorem 12.31: If $f : [a,b] \to \mathsf{R}^1$ is continuous, then f is uniformly continuous.

Proof: Suppose by way of contradiction that f is not uniformly continuous. Then, for some $\epsilon > 0$, there are points $x_n, y_n \in [a,b]$ for each $n \in \mathsf{N}$ such that