Chapter I: The Real Numbers

Don't play poker at someone's house unless you know the rules of the house. And don't play mathematics unless you know the rules of the subject – the axioms.

Accordingly, it is only fair that we set down the rules we will play by: the axioms for the real numbers. This is simple enough to do. However, some basic consequences of the axioms should also be presented so that you know how some "rules" you have been taught, which are not axioms, follow from the axioms. For example, "a minus times a minus is a plus", "zero times any number is zero","$0 < 1$", "$\frac{x}{y} + \frac{z}{w} = \frac{xw+yz}{yw}$" are not "rules" and formulas to be committed to memory for future use; they all follow from the axioms. Once you understand *how* they follow from the axioms, you will understand them better; put another way, the axioms will focus your understanding.

We can not verify *all* the familiar consequences of the axioms. We verify some of the more prominent consequences of the axioms; we hope that what we do, and what you are asked to do in exercises, is enough to make you feel that all the arithmetic you use could, indeed, be verified from the axioms.

All this having been said, I have to admit to being slightly disingenuous. I am referring to certain facts about the natural numbers $1, 2, 3, ...$ stated in section 4 (1.18). These facts are not consequences of the axioms for the real numbers listed in section 1; instead, they come from a way the natural numbers can be constructed. We do not do the construction, but the facts are easy for you to accept based on your past experience with the natural numbers. In effect, we will accept certain facts about the natural numbers as though they are axioms, but we postpone mentioning them until section 4.

1. The Axioms

We denote the set of real numbers by R^1.

We state the axioms for the real numbers. You are familiar with most of the axioms as the "rules of arithmetic"; the exception may be the Completeness Axiom. The Completeness Axiom is necessary since the rational numbers satisfy all the other axioms – thus, without the Completeness Axiom, we are not guaranteed that $\sqrt{2}$, π, etc. are real numbers (we prove that positive numbers have square roots in section 5).

(A) Addition Axioms. There is a function, $+$, defined on the Cartesian product $\mathsf{R}^1 \times \mathsf{R}^1$ satisfying A1 - A5 below (we write $a + b$ to stand for $+(a, b)$):

A1: For any $a, b \in \mathsf{R}^1$, $a + b \in \mathsf{R}^1$. (Closure)

A2: For any $a, b \in \mathsf{R}^1$, $a + b = b + a$. (Commutativity)

A3: For any $a, b, c \in \mathsf{R}^1$, $(a + b) + c = a + (b + c)$. (Associativity)

A4: There is a real number, denoted by 0, such that $a+0 = a$ for all $a \in \mathsf{R}^1$. (Identity)

A5: For each $a \in \mathsf{R}^1$, there is a real number, denoted by $-a$, such that $a + (-a) = 0$. (Inverse)

(M) Multiplication Axioms. There is a function, $\cdot$, defined on the Cartesian product $\mathsf{R}^1 \times \mathsf{R}^1$ satisfying M1 - M5 below (we write $a \cdot b$ to stand for $\cdot(a, b)$):

M1: For any $a, b \in \mathsf{R}^1$, $a \cdot b \in \mathsf{R}^1$. (Closure)

M2: For any $a, b \in \mathsf{R}^1$, $a \cdot b = b \cdot a$. (Commutativity)

M3: For any $a, b, c \in \mathsf{R}^1$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (Associativity)

M4: There is a real number, denoted by 1, such that $1 \neq 0$ and $a \cdot 1 = a$ for all $a \in \mathsf{R}^1$. (Identity)

M5: For each $a \in \mathsf{R}^1$ such that $a \neq 0$, there is a real number, denoted by a^{-1} or by $\frac{1}{a}$, such that $a \cdot a^{-1} = 1$, equivalently, $a \cdot \frac{1}{a} = 1$. (Inverse)

(D) Distributive Axiom. For all $a, b, c \in \mathsf{R}^1$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

(O) Order Axioms. There is a relation, $<$, on R^1 satisfying O1 - O4 below ($x < y$ is read x is less than y):

O1: For any $x, y \in \mathsf{R}^1$, one and only one of the following holds: $x < y$, $y < x$, or $x = y$. (Trichotomy)

O2: If $x, y, z \in \mathsf{R}^1$ and if $x < y$ and $y < z$, then $x < z$. (Transitivity)

O3: If $x, y, z \in \mathsf{R}^1$ and if $x < y$, then $x + z < y + z$.

O4: If $x, y, z \in \mathsf{R}^1$ and if $x < y$ and $0 < z$, then $x \cdot z < y \cdot z$.

We write $x \leq y$ to mean $x < y$ or $x = y$. We sometimes write $y > x$ or $y \geq x$ to mean $x < y$ or $x \leq y$, respectively. A number x is said to be *positive* if $x > 0$, *negative* if $x < 0$ and *nonnegative* if $x \geq 0$.

(C) Completeness Axiom.[1] If A is a nonempty subset of R^1 such that A has an upper bound u (i.e., $a \leq u$ for all $a \in A$), then A has a least upper bound ℓ (i.e., ℓ is an upper bound for A and $\ell \leq u$ for all upper bounds u for A). (See Remarks about the Completeness Axiom below.)

We summarize the axioms: Axioms (A), (M) and (D) say that R^1 is a field; axioms (A), (M), (D) and (O) say that R^1 is an ordered field; axioms (A), (M), (D), (O) and (C) say that R^1 is a complete ordered field. It can be shown that there is a complete ordered field and that there is only one complete ordered field (up to isomorphism). Thus, the reals are the unique complete ordered field.

Remarks about the Completeness Axiom. We make three clarifying observations about the Completeness Axiom.

First, the word *has* in the axiom is not intended to be possessive: the open interval $(0, 1)$ has 1 as an upper bound, but $1 \notin (0, 1)$.

Second, a nonempty set A that has an upper bound has *only one* least upper bound: If ℓ_1 and ℓ_2 were two different least upper bounds of A, then $\ell_1 < \ell_2$ (since ℓ_2 is an upper bound) and $\ell_2 < \ell_1$, in contradiction to O1.

Third, the requirement in the axiom that A be nonempty is necessary. This is because *every* real number is an upper bound of the empty set $\emptyset$, which we see as follows: If a real number x is not an upper bound of $\emptyset$, then $a \not\leq x$ for some $a \in \emptyset$, which is impossible since there is no point a in $\emptyset$.

[1] The Completeness Axiom is often called the Least Upper Bound Axiom. We will say more about this in section 8.

2. Some Notation, Intervals

Eventually, we adopt all the notation used in arithmetic and algebra. For now, we minimize our notation to the notation in section 1 together with the following convenient extensions of that notation:

- $\frac{a}{c}$ stands for $a \cdot \frac{1}{c}$ (assuming $c \neq 0$ – recall M5); in particular, $\frac{a+b}{c}$ stands for $(a+b) \cdot \frac{1}{c}$ and $\frac{a \cdot b}{c}$ stands for $(a \cdot b) \cdot \frac{1}{c}$.
- $a^n = \underbrace{a \cdot a \cdot \ \cdots \ \cdot a}_{n \text{ terms}}$ for $a \in \mathsf{R}^1$ and $n = 1, 2, \ldots$. (We discuss $a^{\frac{1}{n}}$ in section 5, where we show $a^{\frac{1}{n}}$ is a number for all $a \geq 0$ and $n = 1, 2, \ldots$.)
- We frequently juxtapose order relations: for example, $a < b \leq c$ means $a < b$ and $b \leq c$.

We use the usual notation for intervals. We divide intervals into three kinds:

1. Open intervals: (a, b), (a, ∞), $(-\infty, a)$ and $(-\infty, \infty) = \mathsf{R}^1$, where

$$(a, b) = \{x \in \mathsf{R}^1 : a < x < b\}, \quad a < b;$$
$$(a, \infty) = \{x \in \mathsf{R}^1 : x > a\};$$
$$(-\infty, a) = \{x \in \mathsf{R}^1 : x < a\}.$$

2. Closed intervals: $[a, b]$, $[a, \infty)$ and $(-\infty, a]$, where

$$[a, b] = \{x \in \mathsf{R}^1 : a \leq x \leq b\}, \quad a \leq b;$$
$$[a, \infty) = \{x \in \mathsf{R}^1 : x \geq a\};$$
$$(-\infty, a] = \{x \in \mathsf{R}^1 : x \leq a\}.$$

3. Half-open (or half-closed) bounded intervals: $[a, b)$ and $(a, b]$, where

$$[a, b) = \{x \in \mathsf{R}^1 : a \leq x < b\}, \quad a < b;$$
$$(a, b] = \{x \in \mathsf{R}^1 : a < x \leq b\}, \quad a < b.$$

The notation for an open interval and an ordered pair is the same; nevertheless, the context will prevent confusion. In the notation for intervals, we used ∞ and $-\infty$ only as abstract symbols; in particular, the symbols ∞ and $-\infty$ never denote real numbers.

3. Algebra and Arithmetic

After stating the axioms in section 1, we remarked that the axioms say that the reals are the *unique* complete ordered field. Thus, if your previous experience with real numbers leads you to believe that real numbers satisfy the axioms, then you should believe that the axioms yield all the "facts" you have used about real numbers all your life (except for what we have said about the natural numbers in the introduction to the chapter). We show how some of these facts are consequences of the axioms; many other facts are left as exercises for you to do.

Our first theorem verifies cancellation for addition.

Theorem 1.1: Let $x, y, z \in \mathsf{R}^1$. If $x + y = x + z$, then $y = z$.

Proof: Since $x + y = x + z$ and $-x$ is a real number (by A5), clearly $(-x) + (x + y) = (-x) + (x + z)$. Hence, by A3,

$$((-x) + x) + y = ((-x) + x) + z).$$

Thus, by A2, $(x + (-x)) + y = (x + (-x)) + z$. Hence, by A5, $0 + y = 0 + z$. Hence, by A2, $y + 0 = z + 0$. Therefore, by A4, $y = z$. ¥

Theorem 1.2: If $x, y \in \mathsf{R}^1$ and $x + y = 0$, then $y = -x$. In other words, the additive inverse $-a$ of a in A5 is unique.

Proof: Since $x + y = 0$ by assumption and $x + (-x) = 0$ by A5, we have that

$$x + y = x + (-x).$$

Therefore, by Theorem 1.1, $y = -x$. ¥

The following corollary shows that the familiar adage "a minus times a minus is a plus" is true:

Corollary 1.3: For any $x \in \mathsf{R}^1$, $-(-x) = x$.

Proof: By A5, $x + (-x) = 0$. Hence, by A2, $(-x) + x = 0$. Therefore, by Theorem 1.2, $x = -(-x)$. ¥

Theorem 1.4: For any $x \in \mathsf{R}^1$, $x \cdot 0 = 0$.

Proof: We have

$$x + (x \cdot 0) \overset{\text{M4}}{=} (x \cdot 1) + (x \cdot 0) \overset{\text{D}}{=} x \cdot (1 + 0) \overset{\text{A4}}{=} x \cdot 1 \overset{\text{M4}}{=} x \overset{\text{A4}}{=} x + 0.$$

Therefore, by Theorem 1.1, $x \cdot 0 = 0$. ¥

Theorem 1.5: For any $x \in \mathsf{R}^1$, $-x = (-1) \cdot x$.

Proof: First, note that

$$x + (-1) \cdot x \overset{\text{M4}}{=} (x \cdot 1) + ((-1) \cdot x) \overset{\text{M2}}{=} (x \cdot 1) + (x \cdot (-1))$$
$$\overset{\text{D}}{=} x \cdot (1 + (-1)) \overset{\text{A5}}{=} x \cdot 0 \overset{1.4}{=} 0.$$

Therefore, by Theorem 1.2, $(-1) \cdot x = -x$. ¥

Before we give our next theorem, we comment about the statement and the proof of the theorem.

The conclusion of our next theorem is a compound statement, where the two parts are connected with the word *or*. When two statements are connected by *or*, we include the possibility that both statements may be true. This is not always the case in common usage: "At 7:00 P.M., I will be in New Orleans or I will be in Boston" obviously excludes both statements from being true. We use "either ... or" when we mean one or the other but not both. If P and Q are statements, then our meaning for the statement "P or Q" is called the *inclusive* *disjunction of* P *and* Q; the statement "either P or Q" is called the *exclusive disjunction of* P *and* Q.

The proof of our next theorem illustrates a logical principle: To prove that the disjunction of two statements P and Q is true, it is sufficient to assume one of the statements is false and prove the other statement is true.

As you know from experience, the following theorem is useful in finding solutions to equations and inequalities.

Theorem 1.6: If $x, y \in \mathsf{R}^1$ and $x \cdot y = 0$, then $x = 0$ or $y = 0$.

Proof: Assume that $x \neq 0$. Then $\frac{1}{x}$ is a real number by M5. Thus, since $x \cdot y = 0$ by assumption and since $\frac{1}{x} \cdot 0 = 0$ by Theorem 1.4, we have that

$$\tfrac{1}{x} \cdot (x \cdot y) = 0.$$

Hence, by M3, $(\frac{1}{x} \cdot x) \cdot y = 0$. Thus, by M2, $(x \cdot \frac{1}{x}) \cdot y = 0$. Therefore, by M5, $1 \cdot y = 0$. Hence, by M2, $y \cdot 1 = 0$. Therefore, by M4, $y = 0$. ¥

The following theorem verifies cancellation for multiplication.

Theorem 1.7: If $x, y \in \mathsf{R}^1$ and $y \neq 0$, then $y \cdot \frac{x}{y} = x$.

Proof: Recall from the notation in section 2 that $\frac{x}{y} = x \cdot \frac{1}{y}$. Therefore,

$$y \cdot \tfrac{x}{y} = y \cdot (x \cdot \tfrac{1}{y}) \overset{\text{M2}}{=} y \cdot (\tfrac{1}{y} \cdot x) \overset{\text{M3}}{=} (y \cdot \tfrac{1}{y}) \cdot x \overset{\text{M5}}{=} 1 \cdot x$$
$$\overset{\text{M2}}{=} x \cdot 1 \overset{\text{M4}}{=} x. \text{ ¥}$$

We now come to the familiar formula for adding fractions.

Theorem 1.8: If $x, y, z, w \in \mathsf{R}^1$ such that $y \neq 0$ and $w \neq 0$, then

$$\tfrac{x}{y} + \tfrac{z}{w} = \tfrac{x \cdot w + y \cdot z}{y \cdot w}.$$

Proof: Recall from section 2 that the right-hand side of the equation is shorthand for $(x \cdot w + y \cdot z) \cdot \frac{1}{y \cdot w}$. Thus, we must first know that $\frac{1}{y \cdot w}$ is a real number: Since $y \neq 0$ and $w \neq 0$, $y \cdot w \neq 0$ by Theorem 1.6; therefore, $\frac{1}{y \cdot w}$ is a real number by M5.

Now,

$$(y \cdot w) \cdot (\tfrac{x}{y} + \tfrac{z}{w}) \overset{\text{D}}{=} [(y \cdot w) \cdot \tfrac{x}{y}] + [(y \cdot w) \cdot \tfrac{z}{w}]$$
$$\overset{\text{M2}}{=} [(w \cdot y) \cdot \tfrac{x}{y}] + [(y \cdot w) \cdot \tfrac{z}{w}] \overset{\text{M3}}{=} [w \cdot (y \cdot \tfrac{x}{y})] + [y \cdot (w \cdot \tfrac{z}{w})]$$
$$\overset{1.7}{=} w \cdot x + y \cdot z \overset{\text{M2}}{=} x \cdot w + y \cdot z.$$

Hence,

$$(*)\ \tfrac{1}{y \cdot w} \cdot [(y \cdot w) \cdot (\tfrac{x}{y} + \tfrac{z}{w})] = \tfrac{1}{y \cdot w} \cdot [x \cdot w + y \cdot z]$$

Our theorem follows from (*) since the left - hand side of (*) is

$$\tfrac{1}{y\cdot w}\cdot[(y\cdot w)\cdot(\tfrac{x}{y}+\tfrac{z}{w})]\overset{\text{M3}}{=}[\tfrac{1}{y\cdot w}\cdot(y\cdot w)]\cdot(\tfrac{x}{y}+\tfrac{z}{w})]$$

$$\overset{\text{M2}}{=}[(y\cdot w)\cdot\tfrac{1}{y\cdot w}]\cdot(\tfrac{x}{y}+\tfrac{z}{w})]\overset{\text{M5}}{=}1\cdot(\tfrac{x}{y}+\tfrac{z}{w})\overset{\text{M2}}{=}(\tfrac{x}{y}+\tfrac{z}{w})\cdot 1$$

$$\overset{\text{M4}}{=}\tfrac{x}{y}+\tfrac{z}{w}$$

and the right - hand side of (*) is

$$\tfrac{1}{y\cdot w}\cdot[x\cdot w+y\cdot z]\overset{\text{M2}}{=}[x\cdot w+y\cdot z]\cdot\tfrac{1}{y\cdot w}=\tfrac{x\cdot w+y\cdot z}{y\cdot w}.$$ ¥

Theorem 1.9: $0<1$.

Proof: By M4, $0\neq 1$. Hence, by O1, either $1<0$ or $0<1$ (not both). Assume by way of contradiction that $1<0$. Then, by 03,

$$1+(-1)<0+(-1).$$

Thus, since $1+(-1)=0$ by A5 and since $0+(-1)\overset{\text{A2}}{=}(-1)+0\overset{\text{A4}}{=}-1$, we have that $0<-1$. Hence, by O4, $0\cdot(-1)<(-1)\cdot(-1)$. Therefore, by M2, $(-1)\cdot 0<(-1)\cdot(-1)$. Hence, by Theorem 1.4, $0<(-1)\cdot(-1)$. Thus, since $(-1)\cdot(-1)=-(-1)$ by Theorem 1.5, $0<-(-1)$. Hence, by Corollary 1.3, $0<1$. Therefore, we have a contradiction to our assumption that $1<0$ (since O1 says $1<0$ and $0<1$ can not *both* occur). ¥

Corollary 1.10: For any $x\in\mathsf{R}^1$, $x<x+1$.

Proof: By Theorem 1.9, $0<1$. Hence, by O3, $0+x<1+x$. Thus, by A2, $x+0<x+1$. Therefore, since $x+0=x$ by A4, $x<x+1$. ¥

Exercise 1.11: For any $x\in\mathsf{R}^1$, $x+(-1)<x$.

Exercise 1.12: Let $x,y,z\in\mathsf{R}^1$ such that $x\neq 0$. If $x\cdot y=x\cdot z$, then $y=z$.

Exercise 1.13: If $x\in\mathsf{R}^1$ such that $x\neq 0$, then $(x^{-1})^{-1}=x$.

Exercise 1.14: If $x>0$, then $-x<0$ and $\frac{1}{x}>0$.

Exercise 1.15: If $x,y,z,w\in\mathsf{R}^1$ such that $y\neq 0$ and $w\neq 0$, then

$$\tfrac{x}{y}\cdot\tfrac{z}{w}=\tfrac{x\cdot z}{y\cdot w}.$$

Exercise 1.16: If $x<y$ and $z<0$, then $x\cdot z>y\cdot z$.

Exercise 1.17: For any $x\in\mathsf{R}^1$ such that $x\neq 0$, $x\cdot x>0$.

4. The Natural Numbers

As mentioned in section 1, the reals are the unique complete ordered field. The existence of a complete ordered field is proved by constructing one. The process of constructing a complete ordered field often begins with constructing what will become the natural numbers. These are the numbers you have always seen denoted by $1, 2, 3, \ldots$. We are not going to construct the natural numbers; we merely assume the natural numbers are the numbers $1, 2, 3, \ldots$ and denote the set of all natural numbers by N.

To attempt a little rigor, note that 1 is a real number by M4; then, by A1, $1 + 1$ is a real number which we denote by 2; then, by A1, $(1 + 1) + 1$ is a real number which we denote by 3; then, by A1, $((1 + 1) + 1) + 1$ is a real number which we denote by 4; and so on. However, what do we mean by "and so on"? This is troublesome since we are indicating an infinite process that is, heretofore, not well defined. How are we assured we have defined a set of objects? We return to this later (after the proof of Theorem 1.20).

Even though we will not construct the natural numbers, we need to assume facts about the natural numbers that are by-products of the construction. The facts are easy to accept since they seem obvious from our experience with the natural numbers. However, we emphsize that the facts are not merely intuitive – they come from the construction of the natural numbers and they hold for the "numbers" that are (properly) designated as the natural numbers in any construction of a complete ordered field.

1.18 Facts Assumed about N:

- $1 \in \mathsf{N}$ and $1 \leq n$ for all $n \in \mathsf{N}$.
- If $n, m \in \mathsf{N}$, then $n + m \in \mathsf{N}$ and $n \cdot m \in \mathsf{N}$. (Closure)
- If $n \in \mathsf{N}$, then $n > 0$ and $n + (-1) < n$.
- If $n \in \mathsf{N}$ and $n > 1$, then $n + (-1) \in \mathsf{N}$.
- **Well Ordering Principle:** Every nonempty subset S of N has a least member ℓ (i.e., there exists $\ell \in S$ such that $\ell \leq s$ for all $s \in S$).

We illustrate the usefulness of the Well Ordering Principle by proving the following seemingly obvious result (think about how you might prove the result without using the Well Ordering Principle):

Theorem 1.19: There is no natural number between 0 and 1.

Proof: Let $S = \{x \in \mathsf{N} : 0 < x < 1\}$, and assume by way of contradiction that $S \neq \emptyset$. Then, by the Well Ordering Principle, there is a least member ℓ of S. Since $\ell \in S$, $0 < \ell < 1$. Hence, by O4,

$$0 \cdot \ell < \ell \cdot \ell < 1 \cdot \ell;$$

furthermore, $0 \cdot \ell = 0$ (by M2 and Theorem 1.4), and $1 \cdot \ell = \ell$ (by M2 and M4). Thus, $0 < \ell \cdot \ell < \ell$. Combining this with the fact that $\ell < 1$, we have

$$(*)\ 0 < \ell \cdot \ell < \ell < 1.$$

Since $\ell \in S$, $\ell \in \mathsf{N}$. Hence, by our assumption that N is closed under multiplication (1.18), $\ell \cdot \ell \in \mathsf{N}$. Thus, since $0 < \ell \cdot \ell < 1$ (by (*)), $\ell \cdot \ell \in S$. Therefore, since $\ell \cdot \ell < \ell$ (by (*)), ℓ is not the least member of S. This is a contradiction to our choice of ℓ. ¥

We now prove an important consequence of the Well Ordering Principle.

Theorem 1.20 (Induction Principle): For each $n \in \mathsf{N}$, let P_n be a statement. If P_1 is true and if P_{n+1} is true whenever P_n is true, then P_n is true for all $n \in \mathsf{N}$.

Proof: Let $S = \{n \in \mathsf{N} : P_n \text{ is false}\}$, and assume by way of contradiction that $S \neq \emptyset$. Then, by the Well Ordering Principle, there is a least member ℓ of S. Since P_1 is true, $1 \notin S$; thus, $\ell \neq 1$. Also, since $\ell > 0$ (1.18), we see from Theorem 1.19 that $\ell \not< 1$. Hence, by O1, $\ell > 1$. Thus, $\ell + (-1) \in \mathsf{N}$ (1.18); also, $\ell + (-1) < \ell$ (1.18). Thus, since ℓ is the least member of S, $P_{\ell+(-1)}$ is true. Hence, by assumption in our theorem, $P_{[\ell+(-1)]+1}$ is true (note: $P_{[\ell+(-1)]+1}$ is indeed one of the statements in our theorem, for since $\ell + (-1) \in \mathsf{N}$ and since $1 \in \mathsf{N}$ (1.18), we know that $[\ell + (-1)] + 1 \in \mathsf{N}$ (1.18)). This says that P_ℓ is true since

$$[\ell + (-1)] + 1 \overset{\text{A3}}{=} \ell + [(-1) + 1] \overset{\text{A2}}{=} \ell + [1 + (-1)] \overset{\text{A5}}{=} \ell + 0 \overset{\text{A4}}{=} \ell.$$

Therefore, having proved that P_ℓ is true, we have that $\ell \notin S$. This establishes a contradiction. ¥

Recall our attempt at rigor in the second paragraph of the section. You can now answer the questions we asked there: I invite you to use the Induction Principle to define the set of natural numbers in the manner indicated.

We prove one more theorem about the natural numbers. First, we motivate the importance of the theorem by showing that it is needed in elementary situations.

We recall the proof that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0 as presented in most calculus books (the proof that follows is taken verbatim from Edwards and Penney, *Calculus*, Prentice Hall, fifth edition, p. 627):

"Suppose that we want to establish rigorously the intuitively evident fact that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ coverges to zero,

$$\lim_{n\to\infty} \tfrac{1}{n} = 0.$$

Because $L = 0$ here, we only need to convince ourselves that to each positive number ϵ there corresponds an integer N such that

$$\left|\tfrac{1}{n}\right| = \tfrac{1}{n} < \epsilon \quad \text{if } n \geq N.$$

But evidently it suffices to choose any fixed integer $N > \frac{1}{\epsilon}$. Then $n \geq N$ implies immediately that

$$\tfrac{1}{n} \leq \tfrac{1}{N} < \epsilon,$$

as desired (Fig. 11.2.3)."

The proof is "correct" but incomplete. In fact, the essential point is missed: How do you know there is an integer N such that $N > \frac{1}{\epsilon}$? We correct the deficiency with our next theorem, which gives an important property of the natural numbers. First, we prove a lemma; the lemma is obvious based on our experience with the natural numbers.

Lemma 1.21: N has no upper bound.

Proof: Suppose by way of contradiction that N has an upper bound. Then, since $\mathsf{N} \neq \emptyset$ (because $1 \in \mathsf{N}$ (1.18)), the Completeness Axiom says that N has a least upper bound ℓ. By Exercise 1.11, $\ell + (-1) < \ell$. Hence, $\ell + (-1)$ can not be an upper bound for N (since ℓ is the *least* upper bound for N). Thus, there exists $k \in \mathsf{N}$ such that $k \not\leq \ell + (-1)$; hence, by O1, $\ell + (-1) < k$. Hence, $\ell < k + (-(-1))$ (use O3, and A3 - A5). Thus, since $-(-1) = 1$ by Corollary 1.3, $\ell < k + 1$. Hence, by O1,

$$\ell \not\geq k + 1;$$

furthermore, $k + 1 \in \mathsf{N}$ (by 1.18 since $k \in \mathsf{N}$ and $1 \in \mathsf{N}$). Therefore, ℓ is not an upper bound of N. This is a contradition (since ℓ *is* an upper bound of N). ¥

Theorem 1.22 (Archimedean Property): If $x, y \in \mathsf{R}^1$ and $x > 0$, then there exists $n \in \mathsf{N}$ such that $y < n \cdot x$.

Proof: Since $x \in \mathsf{R}^1$ and $x \neq 0$ (by O1), $\frac{1}{x} \in \mathsf{R}^1$ (by M5). Thus, $y \cdot \frac{1}{x} \in \mathsf{R}^1$ (by M1). Hence, by Lemma 1.21, there exists $n \in \mathsf{N}$ such that $y \cdot \frac{1}{x} < n$ (we are also using O1 here). Thus, since $x > 0$, we have by O4 that

$$(y \cdot \tfrac{1}{x}) \cdot x < n \cdot x.$$

Therefore, since

$$(y \cdot \tfrac{1}{x}) \cdot x \overset{\text{M3}}{=} y \cdot (\tfrac{1}{x} \cdot x) \overset{\text{M2}}{=} y \cdot (x \cdot \tfrac{1}{x}) \overset{\text{M5}}{=} y \cdot 1 \overset{\text{M4}}{=} y,$$

we have that $y < n \cdot x$. ¥

Exercise 1.23: In line with the discussion preceding Lemma 1.21, prove that for any given $\epsilon > 0$, there exists $N \in \mathsf{N}$ such that $N > \frac{1}{\epsilon}$.

Let $\epsilon > 0$, and let $n_i \in \mathsf{N}$ for each $i \in \mathsf{N}$ such that $n_i < n_{i+1}$ for each i. Prove that there exists N such that $\frac{1}{n_i} < \epsilon$ for all $i \geq N$.

Exercise 1.24: 1 is the least upper bound of the open interval $(0, 1)$.

5. Proof That Nonnegative Numbers Have nth Roots

We presented the axioms for the real numbers in section 1 and made certain assumptions, which we take as axioms, about the natural numbers in section 4 (1.18). We claimed at the beginning of the chapter that all properties of real numbers that you are familiar with can be proved on the basis of the axioms in section 1 and the facts assumed about N in section 4 (1.18). We supported the claim in section 3. We support the claim further here by proving an important theorem that everyone believes, but, perhaps, only from experience.

Let $a \geq 0$ and let n be a natural number. We are all familiar with the symbols $\sqrt[n]{a}$ and $a^{\frac{1}{n}}$ as representing the n^{th} root of a. From past experience, we are use to thinking of $\sqrt[n]{a}$ as a number. We show that $\sqrt[n]{a}$ is a number and that this result follows from the axioms we have given. The proof of the theorem is very long – about six pages. The reason for the length of the proof is not the difficulty of the proof, but rather the fact that we carry out the details of many computations in the proof. Our purpose for including detailed computations is to give you further assurance that manipulations with arithmetic and inequalities that you already know can be verified rigorously. The main parts of the proof are broken down into steps that will help you understand how the proof is organized.

For a real number $a \geq 0$ and a natural number n, we define an n^{th} *root of* a to be a nonnegative real number b such that $b^n = a$. We will see that there is only one n^{th} root of each number $a \geq 0$; thus, we can say *the* n^{th} root of a rather than *an* n^{th} root of a. We denote the n^{th} of a by $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$. (A further discussion of roots of numbers is in section 4 of Chapter VIII, where we include odd roots of negative numbers.)

The proof of our theorem is based on the Completeness Axiom. At the beginning of section 1 we said that without the Completeness Axiom we are not guaranteed that $\sqrt{2}$ is a real number. The proof of our theorem shows that *with* the Completeness Axiom we can prove that $\sqrt{2}$ *is* a real number.

Theorem 1.25: For any real number $a \geq 0$, there is a unique real number $b \geq 0$ such that $b^n = a$. In other words, every nonnegative real number has a unique (nonnegative) n^{th} root.

Proof: We prove the theorem only for square roots. The proof for n^{th} roots uses similar ideas (with some computations aided by the Binomial Theorem (Theorem 21.41), whose proof you can read now). Another proof for n^{th} roots is in section 4 of Chapter VIII, but that proof depends on previous material.

We first dispense with the case when $a = 0$: By Theorem 1.4, 0 is a square root of 0 and, by Exercise 1.17 and O1, 0 is the only square root of 0.

Now, fix $a > 0$. The proof centers around considering the following set:

$$S = \{x \geq 0 : x^2 \leq a\}.$$

We show that S has a least upper bound b; then we show that b is the square root of a.

Step 1: Proof that S has a least upper bound b

First, note that $S \neq \emptyset$ since $0 \in S$ by Theorem 1.4.

Next, we show that $a + 1$ is an upper bound for S. Assume by way of contradiction that there is an $x \in S$ such that $x \not\leq a + 1$. Then, by O1, $a + 1 < x$. Hence,

$$0 < a \overset{1.10}{<} a + 1 < x.$$

Thus, $a+1>0$ and $x>0$ (by O2); therefore, since $a+1<x$, we see that

$$(a+1)^2 \overset{\text{O4}}{<} x\cdot(a+1) \overset{\text{M2}}{=} (a+1)\cdot x \overset{\text{O4}}{<} x^2;$$

also, using the Distributive Axiom (D) and various axioms in (A) and (M), we see that

$$(a+1)^2 = (a^2+1)+(a+a).$$

Hence, we have

(1) $(a^2+1)+(a+a)<x^2$.

We obtain a contradiction to our assumption that $x\in S$ by proving

(2) $a<x^2$.

Proof of (2)*:* Since $a^2>0$ by Exercise 1.17, $a^2+1>0$ by Corollary 1.10 and O2. Hence,

$$(a^2+1)+(a+a) \overset{\text{O3}}{>} 0+(a+a) \overset{\text{A2}}{=} (a+a)+0 \overset{\text{A4}}{=} (a+a);$$

furthermore, since $a>0$,

$$a+a \overset{\text{O3}}{>} 0+a \overset{\text{A2}}{=} a+0 \overset{\text{A4}}{=} a.$$

Combining the last two inequalities using O2, we obtain that

$$a<(a^2+1)+(a+a).$$

Therefore, by (1) and O2, $a<x^2$. This proves (2).

By (2) and O1, $x^2 \not\leq a$. Hence, $x\notin S$. This is a contradiction. Therefore, $a+1$ is an upper bound for S.

We have proved that $S\neq\emptyset$ and that S has an upper bound. Therefore, by the Completeness Axiom, S has a least upper bound b. This completes Step 1.

We note the following fact for use several times: Since $n>0$ for all $n\in\mathsf{N}$ (1.18), we have from Exercise 1.14 that

(3) $\frac{1}{n}>0$ for all $n\in\mathsf{N}$.

Step 2: Proof that $b>0$

We prove that $b>0$ by finding a positive number in S that is smaller than b.

Since $a>0$, Theorem 1.22 says there exists $n\in\mathsf{N}$ such that $1<n\cdot a$. Thus, by (3) and O4, $1\cdot\frac{1}{n}<(n\cdot a)\cdot\frac{1}{n}$. Hence, $\frac{1}{n}<a$ (use M2 - M5). Thus, by (3) and O4, we have that

(4) $\left(\frac{1}{n}\right)^2<a\cdot\frac{1}{n}$.

We prove that the right - hand side of (4) is $\leq a$ as follows: Since $1\leq n$ (1.18) and $a>0$, we have by O4 that $1\cdot a\leq n\cdot a$. Hence, $a\leq a\cdot n$ (by M2 and M4). Thus, since $\frac{1}{n}>0$ (by (3)),

$$a \cdot \frac{1}{n} \overset{\text{O4}}{\leq} (a \cdot n) \cdot \frac{1}{n} \overset{\text{M3}}{=} a \cdot (n \cdot \frac{1}{n}) \overset{\text{M5}}{=} a \cdot 1 \overset{\text{M4}}{=} a,$$

which proves that $a \cdot (\frac{1}{n}) \leq a$.

Now, having proved that $a \cdot (\frac{1}{n}) \leq a$, we have by (4) and O2 that

$$\left(\frac{1}{n}\right)^2 < a.$$

Therefore, since $\frac{1}{n} > 0$ (by (3)) we have proved that $\frac{1}{n} \in S$. Thus, since b is an upper bound for S, $b \geq \frac{1}{n}$. Therefore, since $\frac{1}{n} > 0$, $b > 0$ (by O2). This completes Step 2.

We show that $b^2 = a$ by showing that $b^2 \not> a$ and that $b^2 \not< a$ (and then applying O1).

Step 3: Proof that $b^2 \not> a$

Since $b > 0$ by Step 2, $b + b \overset{\text{O3}}{>} 0 + b \overset{\text{A2}}{=} b + 0 \overset{\text{A4}}{=} b > 0$; therefore, by O2, we have that

(5) $b + b > 0$.

Now, suppose by way of contradiction that $b^2 > a$. Note from (5) and O1 that $b + b \neq 0$ and, hence, that $\frac{1}{b+b}$ is a number by M5. We prove that

(6) $\frac{b^2 + (-a)}{b+b} > 0$.

Proof of (6)*:* By our assumption that $b^2 > a$ (by assumption), we see that

$$b^2 + (-a) \overset{\text{O3}}{>} a + (-a) \overset{\text{A5}}{=} 0;$$

thus, since $\frac{1}{b+b} > 0$ (by (5) and Exercise 1.14), we have by O4 that

$$[b^2 + (-a)] \cdot \frac{1}{b+b} > 0 \cdot \frac{1}{b+b}.$$

Therefore, since $0 \cdot \frac{1}{b+b} = 0$ (by M2 and Theorem 1.4), we have proved (6).

By (6) and Theorem 1.22, there exists $k \in \mathsf{N}$ such that

$$1 < k \cdot \frac{b^2 + (-a)}{b+b}.$$

Thus, since $\frac{1}{k} > 0$ (by (3)), we see from O4 (using M2 - M5) that

(7) $\frac{1}{k} < \frac{b^2 + (-a)}{b+b}$.

We show that

(8) $b + (-\frac{1}{k}) > 0$.

Proof of (8)*:* Since $a > 0$,

$$a + b^2 \overset{\text{O3}}{>} 0 + b^2 \overset{\text{A2}}{=} b^2 + 0 \overset{\text{A4}}{=} b^2;$$

hence, by O3 and A2 - A5, $b^2 > b^2 + (-a)$. Thus, since $\frac{1}{b+b} > 0$ (by (5) and Exercise 1.14), we have by O4 that

(8i) $\frac{b^2+(-a)}{b+b} < \frac{b^2}{b+b}$;

Since $b > 0$ (by Step 2), $b + b \overset{\text{O3}}{>} 0 + b \overset{\text{A2}}{=} b + 0 \overset{\text{A4}}{=} b$; thus, again since $b > 0$, $(b + b) \cdot b \overset{\text{O4}}{>} b^2$. Therefore, since $\frac{1}{b+b} > 0$ (by (5) and Exercise 1.14), we have

$$[(b + b) \cdot b] \cdot \tfrac{1}{b+b} \overset{\text{O4}}{>} b^2 \cdot \tfrac{1}{b+b} = \tfrac{b^2}{b+b};$$

Thus, since $[(b + b) \cdot b] \cdot \frac{1}{b+b} = b$ (by M2 - M5),

$$b > \tfrac{b^2}{b+b}.$$

Hence, by (8i) and O2, $\frac{b^2+(-a)}{b+b} < b$. Thus, by (7) and O2, $\frac{1}{k} < b$. Hence,

$$\tfrac{1}{k} + (-\tfrac{1}{k}) \overset{\text{O3}}{<} b + (-\tfrac{1}{k});$$

Therefore, by A5, $0 < b + (-\frac{1}{k})$. This proves (8).

Next, we show that

(9) $\big(b + (-\frac{1}{k})\big)^2 > a$.

Proof of (9)*:* Since $b + b > 0$ (by (5)), we see from (7) and O4 that

$$\tfrac{1}{k} \cdot (b + b) < \tfrac{b^2+(-a)}{b+b} \cdot (b + b);$$

furthermore, using M2 - M5, we see that

$$\tfrac{b^2+(-a)}{b+b} \cdot (b + b) = b^2 + (-a).$$

Hence,

$$\tfrac{1}{k} \cdot (b + b) < b^2 + (-a).$$

Thus, since $\frac{1}{k} \cdot (b + b) \overset{\text{M2}}{=} (b + b) \cdot \frac{1}{k} = \frac{b+b}{k}$, we have

$$\tfrac{b+b}{k} + [a + (-\tfrac{b+b}{k})] \overset{\text{O3}}{<} [b^2 + (-a)] + [a + (-\tfrac{b+b}{k})].$$

Hence, using A2 - A5, we see that

(9i) $a < b^2 + (-\frac{b+b}{k})$.

Since $\frac{1}{k} > 0$ (by (3)), $\frac{1}{k} \neq 0$ (by O1); hence, $\big(\frac{1}{k}\big)^2 > 0$ (by Exercise 1.17). Thus,

$$\big(\tfrac{1}{k}\big)^2 + [b^2 + (-\tfrac{b+b}{k})] \overset{\text{O3}}{>} 0 + [b^2 + (-\tfrac{b+b}{k})];$$

hence, by A2 and A4,

$$\left(\tfrac{1}{k}\right)^2 + [b^2 + (-\tfrac{b+b}{k})] > b^2 + (-\tfrac{b+b}{k});$$

thus, by (9i) and O2, we have that

(9ii) $\left(\frac{1}{k}\right)^2 + [b^2 + (-\frac{b+b}{k})] > a.$

We show that the left-hand side of (9ii) is $\left(b + (-\frac{1}{k})\right)^2$, which completes the proof of (9). The computations that follow are tedious, but are done so that you can see that what we prove depends only on the axioms (and some previous theorems). We note that when we change from $\frac{x}{y}$ to $x \cdot \frac{1}{y}$ or vice versa, this change is justified by the notation in section 2; in particular, the change does not use an axiom or a theorem. Now,

$$\begin{aligned}
&\left(b + (-\tfrac{1}{k})\right)^2 \overset{\text{D, M2}}{=} \left[b^2 + (b \cdot (-\tfrac{1}{k}))\right] + \left[(b \cdot (-\tfrac{1}{k})) + (-\tfrac{1}{k})^2\right] \\
&\overset{1.5}{=} \left[b^2 + \{b \cdot ((-1) \cdot \tfrac{1}{k})\}\right] + \left[\{b \cdot ((-1) \cdot \tfrac{1}{k})\} + ((-1) \cdot \tfrac{1}{k})^2\right] \\
&\overset{\text{A3, M2, M3}}{=} b^2 + \left[\left(((-1) \cdot \tfrac{b}{k}) + ((-1) \cdot \tfrac{b}{k})\right) + \left(\{(-1) \cdot (-1)\} \cdot \{\tfrac{1}{k}\}^2\right)\right] \\
&\overset{1.5, 1.3}{=} b^2 + \left[\left(((-1) \cdot \tfrac{b}{k}) + ((-1) \cdot \tfrac{b}{k})\right) + \left(1 \cdot \{\tfrac{1}{k}\}^2\right)\right] \\
&\overset{\text{D, M2}}{=} b^2 + \left[\{(-1) \cdot (\tfrac{b}{k} + \tfrac{b}{k})\} + (\{\tfrac{1}{k}\}^2 \cdot 1)\right] \\
&\overset{1.8, \text{M4}}{=} b^2 + \left[\{(-1) \cdot \tfrac{(k \cdot b) + (k \cdot b)}{k^2}\} + \{\tfrac{1}{k}\}^2\right] \\
&\overset{\text{D}, 1.5}{=} b^2 + [(-\tfrac{k \cdot (b+b)}{k^2}) + \left(\tfrac{1}{k}\right)^2] \overset{1.15}{=} b^2 + [-(\tfrac{k}{k} \cdot \tfrac{b+b}{k}) + \left(\tfrac{1}{k}\right)^2] \\
&\overset{\text{M5, M4}}{=} b^2 + [-\tfrac{b+b}{k} + \left(\tfrac{1}{k}\right)^2] \overset{\text{A2, A3}}{=} \left(\tfrac{1}{k}\right)^2 + [b^2 + (-\tfrac{b+b}{k})].
\end{aligned}$$

Therefore, as remaked after (9ii), we have proved (9).

We now show that

(10) $b + (-\frac{1}{k})$ is an upper bound for S.

Proof of (10)*:* Let $y \in \mathsf{R}^1$ such that $y > b + (-\frac{1}{k})$. We show that $y \notin S$ (which proves (10) contrapositively by O1).

Since $y > b + (-\frac{1}{k})$ and $b + (-\frac{1}{k}) > 0$ (by (8)), we see from O2 that $y > 0$. Thus, since $y > b + (-\frac{1}{k})$, we have that

$$y^2 \overset{04}{>} (b + (-\tfrac{1}{k})) \cdot y \overset{\text{M2}}{=} y \cdot (b + (-\tfrac{1}{k}));$$

also, since $b + (-\frac{1}{k}) > 0$ (by (8)) and $y > b + (-\frac{1}{k})$, we have by O4 that

$$y \cdot (b + (-\tfrac{1}{k})) > \left((b + (-\tfrac{1}{k})\right)^2.$$

Therefore, applying O2 to the two inequalities above, we have

$$y^2 > \big((b + (-\tfrac{1}{k})\big)^2.$$

Hence, by (9) and O2, $y^2 > a$. Thus, $y^2 \not\leq a$ (by O1). Hence, $y \notin S$. Therefore, by the comment at the beginning of the proof of (10), we have proved (10).

Finally, we complete the proof for Step 3. Specifically, we obtain a contradiction to b being the *least* upper bound for S by showing that $b + (-\frac{1}{k}) < b$ and applying (10).

Since $\frac{1}{k} > 0$ by (3), $-\frac{1}{k} < 0$ by Exercise 1.14. Hence, by O3, $(-\frac{1}{k}) + b < 0 + b$; thus, by A2 and A4, $b + (-\frac{1}{k}) < b$. Hence, by O1, $b \not\leq b + (-\frac{1}{k})$. Therefore, by (10), b is not the *least* upper bound for S. This is a contradiction. The contradiction is the result of our supposition near the beginning of the proof of Step 3 that $b^2 > a$. Therefore, $b^2 \not> a$. This completes Step 3.

Step 4: Proof that $b^2 \not< a$

We omit references to the use of axioms in section 1, some theorems and exercises in section 3, and the assumptions in 1.18. In other words, we use the familiar "rules" of arithmetic without reference; we invite the reader to fill in the details (which are similar to the details in Step 3).

Suppose by way of contradiction that $b^2 < a$. Then, since $\frac{a-b^2}{(b+b)+1} > 0$, we see from Theorem 1.22 that there exists $m \in \mathsf{N}$ such that

$$1 < m \cdot \tfrac{a-b^2}{(b+b)+1}.$$

Hence, it follows that

(11) $b^2 + \frac{(b+b)+1}{m} < a$.

Now, note that

$$(b + \tfrac{1}{m})^2 = b^2 + \tfrac{b+b}{m} + (\tfrac{1}{m})^2 \leq b^2 + \tfrac{b+b}{m} + \tfrac{1}{m} = b^2 + \tfrac{(b+b)+1}{m}.$$

Hence, by (11),

$$(b + \tfrac{1}{m})^2 < a;$$

furthermore, $b + \frac{1}{m} > 0$ (since b is an upper bound of S and $0 \in S$). Therefore, $b + \frac{1}{m} \in S$. However, since $b < b + \frac{1}{m}$, this contradicts that b is an upper bound for S. The contradiction comes from our assumption that $b^2 < a$. Therefore, $b^2 \not< a$. This completes Step 4.

Step 5: Completing the proof

We know from Steps 3 and 4 that $b^2 \not> a$, and that $b^2 \not< a$. Therefore, by O1, $b^2 = a$. By Step 2, $b > 0$. Therefore, it only remains to prove the uniqueness part of our theorem, that is, that b is the only nonnegative number such that $b^2 = a$. As in the proof of Step 4, we omit references to the use of the axioms in section 1, etc.

Assume that $c \geq 0$ and that $c^2 = a$. We show that $b = c$. Since $b^2 = c^2$, $b^2 + (-[c^2]) = 0$. Thus, since $b^2 + (-[c^2]) = (b + c)(b + [-c])$, we have that

$$(b+c)(b+[-c])=0.$$

Thus, since $b+c \neq 0$ (recall from Step 2 that $b>0$), we have by Theorem 1.6 that $b+[-c]=0$. Therefore, $b=c$. ¥

6. The Betweenness Property for the Rational Numbers

The *rational numbers* are the numbers that can be written in the form $\frac{m}{n}$ or $-\frac{m}{n}$, where $m,n \in \mathsf{N}$, together with the number 0. We denote the set of all rational numbers by Q.

We prove a fundamental result about the rational numbers: There is a rational number between any two (different) real numbers.

Theorem 1.26: If $a,b \in \mathsf{R}^1$ such that $a<b$, then there is a rational number r such that $a<r<b$.

Proof: As we did in the latter part of the proof of Theorem 1.25, we omit references to the use of axioms in section 1, some theorems and exercises in section 3, and assumptions in 1.18 (except when we use the Well Ordering Principle).

The theorem is obvious when $a<0<b$ (take $r=0$). Thus, there are only two cases to consider: $a \geq 0$ and $b \leq 0$.

Assume first that $a \geq 0$. Since $a<b$, $b+(-a)>0$. Hence,by Theorem 1.22, there exists $n \in \mathsf{N}$ such that $1<n \cdot (b+(-a))$. (To envision the proof that follows, rewrite the inequality as $\frac{1}{n}<b+(-a)$.)

Let

$$S=\{k \in \mathsf{N} : a<\tfrac{k}{n}\}.$$

By Lemma 1.21, $n \cdot a<k$ for some $k \in \mathsf{N}$; thus, $S \neq \emptyset$. Hence, by the Well Ordering Principle for N (1.18), S has a least member ℓ. Since $\ell, n \in \mathsf{N}$, $\frac{\ell}{n} \in \mathsf{Q}$.

Since $\ell \in S$, $a<\frac{\ell}{n}$. We complete the proof for the case when $a \geq 0$ by proving that

$$(*) \ \tfrac{\ell}{n}<b.$$

Proof of (*)*:* We first prove that

(1) $\frac{\ell+(-1)}{n} \leq a$.

Proof of (1)*:* Suppose by way of contradiction that $\frac{\ell+(-1)}{n}>a$. Then, since $n>0$, $\ell+(-1)>n \cdot a$. Thus, since $n \cdot a \geq 0$ (here is where we use that $a \geq 0$), $\ell+(-1)>0$. Hence, $\ell>1$. Thus, since $\ell \in \mathsf{N}$, $\ell+(-1) \in \mathsf{N}$. Therefore, since $\frac{\ell+(-1)}{n}>a$, we have that $\ell+(-1) \in S$; however, since $\ell+(-1)<\ell$ (by Exercise 1.11), this contradicts the fact that ℓ is the least member of S. Therefore, we have proved (1).

Now, to prove (*), note from (1) that $\frac{\ell}{n}+(-\frac{1}{n}) \leq a$. Hence, $\frac{\ell}{n} \leq a+\frac{1}{n}$. Also, $\frac{1}{n}<b+(-a)$ by our choice of n. Therefore,

$$\frac{\ell}{n} \leq a + \frac{1}{n} < a + (b + (-a)) = b.$$

This proves (*) and, therefore, completes the proof of our theorem for the case when $a \geq 0$.

Finally, consider the case when $b \leq 0$. Then $0 \leq -b < -a$; hence, having already proved the theorem for this type of situation, there exists $q \in \mathsf{Q}$ such that $-b < q < -a$. Therefore, $a < -q < b$ and $-q \in \mathsf{Q}$. ¥

Exercise 1.27: If $a, b \in \mathsf{R}^1$ such that $a < b$, then there are infinitely many rational numbers in the open interval (a, b).

7. Absolute value

For any number a, the *absolute value of* a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & , \text{if } a \geq 0 \\ -a & , \text{if } a < 0. \end{cases}$$

Intuitively, $|a|$ is how far a is from the origin 0 (without regard to whether a is positive or negative). Thus, $|a + (-b)|$ can be thought of as the distance between a and b. Therefore, we call $|a + (-b)|$ the *distance between* a *and* b (or the *distance from* a *to* b).

We note four basic properties of absolute value (the reader should supply proofs that the properties hold). The properties can be used to show that the function that assigns to an ordered pair (a, b) of real numbers the number $|a + (-b)|$ satisfies the general definition of a distance function (see Exercise 1.30).

1. $|a| \geq 0$ for all $a \in \mathsf{R}^1$.
2. $|a| = 0$ if and only if $a = 0$.
3. $|a \cdot b| = |a| \cdot |b|$ for all $a, b \in \mathsf{R}^1$.
4. $|a + b| \leq |a| + |b|$ for all $a, b \in \mathsf{R}^1$ (Triangle Inequality).

Exercise 1.28: If $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.

Exercise 1.29: $\big||a| - |b|\big| \leq |a - b|$ for all $a, b \in \mathsf{R}^1$.

Exercise 1.30: Let X be a set. A *distance function* (or *metric*) *for* X is a real-valued function d defined on the Cartesian product $X \times X$ that satisfies the following four conditions:

(1) $d(x, y) \geq 0$ for all $x, y \in X$;
(2) $d(x, y) = 0$ if and only if $x = y$;
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (Symmetry);
(4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (Triangle Inequality).

Define $d(a, b) = |a + (-b)|$ for all $a, b \in \mathsf{R}^1$. Prove that d is a distance function (in the general sense just defined).

8. Concluding Comments

The reader should now be convinced that familiar manipulations and properties of the real numbers can be verified on the basis of the axioms for the real numbers and various theorems in previous sections. Thus, we will no longer refer to many of the axioms and theorems when we use them. There are some exceptions: We will refer to the Completeness Axiom, the Well Ordering Principle, the Induction Principle and the Archimedean Principle when we use them.

From now on we use all the usual notation associated with arithmetic. For example, we write $a - b$ to mean $a + (-b)$ and ab to mean $a \cdot b$. In an expression involving combinations of addition and multiplication, we assume that the juxtaposed multiplications are carried out first; for example, $ab + c$ means $(ab) + c$.

We use the usual symbols for specific numbers. In particular, we assume the natural numbers are the numbers $1, 2, 3, ...$ with their familiar properties.

We assume standard arithmetic – the multiplication tables, long division, etc. – without proof.

To summarize, having taken a bath, we have now drained away all the dirty water!

We conclude by stating a dual to the Completeness Axiom.

The Completeness Axiom stated in section 1 is frequently called the Least Upper Bound Axiom. We will often have occasion to use an equivalent formulation of the axiom called the Greatest Lower Bound Axiom, which we state after giving relevant terminology.

Let $A \subset \mathsf{R}^1$. A *lower bound for* A is a number x such that $x \leq a$ for all $a \in A$. A *greatest lower bound for* A is a lower bound g for A such that $g \geq x$ for all lower bounds x for A.

Greatest Lower Bound Axiom. If A is a nonempty subset of R^1 such that A has an lower bound, then A has a greatest lower bound.

It is easy to prove the Greatest Lower Bound Axiom is equivalent to the Completeness Axiom in section 1. Thus, from the point of view of our development, the Greatest Lower Bound Axiom is actually a theorem.

The analogues of remarks we made about the Completeness Axiom at the end of section 1 apply to the Greatest Lower Bound Axiom. In particular, a nonempty set with a lower bound has *only one* greatest lower bound.

Notation:

- lub, **sup:** We denote the least upper bound of a set A by *lub* A or by $\sup A$ (sup stands for supremum).
- glb, **inf:** We denote the greatest lower bound of a set A by *glb* A or by $\inf A$ (inf stands for infimum).
- **max**, **min:** We sometimes use $\max A$ and $\min A$ to stand for $\sup A$ and *glb* A, respectively. (We usually use max and min only when we know the set A is finite.)

Chapter II: The Notion of Arbitrary Closeness

In the first two sections we study the notion of arbitrary closeness. We then show that the notion leads in a natural intuitive way to the idea of continuous functions.

1. Introduction to Arbitrary Closeness

We define what it means for a real number to be arbitrarily close to a nonempty set of real numbers. Then we present some examples and two basic theorems.

If p is a real number and A is a nonempty set of real numbers, then the *infimum distance of p to A*, denoted by $dist(p, A)$, is defined by

$$dist(p, A) = glb\,\{|p - a| : a \in A\}.$$

Definition. Let $p \in \mathsf{R}^1$ and let $A \subset \mathsf{R}^1$ such that $A \neq \emptyset$. We say that p *is arbitrarily close to A*, denoted by writing $p \sim A$, provided that $A \neq \emptyset$ and $dist(p, A) = 0$.

We write $p \not\sim A$ to mean that the number p is not arbitrarily close to the set A.

Example 2.1: $0 \sim (0, 1)$, $p \sim (0, 1)$ if $p \in (0, 1)$, and $2 \not\sim (0, 1)$.

Example 2.2: Every real number is arbitrarily close to the set Q of rational numbers.

Theorem 2.3: Let $p \in \mathsf{R}^1$ and let $A \subset \mathsf{R}^1$ such that $A \neq \emptyset$. Then $p \sim A$ if and only if for each open interval I such that $p \in I$, $I \cap A \neq \emptyset$.

Proof: Assume that there is an open interval $I = (a, b)$ such that $p \in I$ and $I \cap A = \emptyset$. Then

$$A \subset (-\infty, a] \cup [b, \infty);$$

thus, since $a < p < b$,

$$dist(p, A) \geq \min\{p - a, b - p\} > 0.$$

Therefore, $p \not\sim A$.

Conversely, assume that $p \not\sim A$. Then $dist(p, A) > 0$. Let J be the open interval given by

$$J = (p - dist(p, A), p + dist(p, A)).$$

Then $p \in J$ since $dist(p, A) > 0$. In addition, $J \cap A = \emptyset$ since if $x \in J$, then $|p - x| < dist(p, A)$ and, therefore, $x \notin A$. ¥

Lemma 2.4: Let $p \in \mathsf{R}^1$ and let (a, b) be an open interval such that $p \in (a, b)$. Then there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset (a, b)$.

Proof: Let

$$\epsilon = \min\{p-a, b-p\}.$$

Since $p \in (a,b)$, $\epsilon > 0$. Since $\epsilon \le p-a$, $a \le p-\epsilon$ and, since $\epsilon \le b-p$, $p+\epsilon \le b$; therefore, $(p-\epsilon, p+\epsilon) \subset (a,b)$. ¥

Theorem 2.5: Let $p \in \mathsf{R}^1$ and let $A \subset \mathsf{R}^1$ such that $A \ne \emptyset$. Then $p \sim A$ if and only if for each $\epsilon > 0$, $(p-\epsilon, p+\epsilon) \cap A \ne \emptyset$.

Proof: It follows easily from Lemma 2.4 that the condition involving ϵ here is equivalent to the condition involving I in Theorem 2.3. Therefore, Theorem 2.5 follows from (and is actually a reformulation of) Theorem 2.3. ¥

2. The Set of Points Arbitrarily Close to a Set

If A is a nonempty set of real numbers, we let $A^\sim$ denote the set of all real numbers that are arbitrarily close to A; in other words,

$$A^\sim = \{x \in \mathsf{R}^1 : x \sim A\}.$$

It is convenient to extend the notation to the empty set $\emptyset$ by making the intuitively reasonable assumption that no real number is arbitrarily close to the empty set; in symbols, $\emptyset^\sim = \emptyset$.

Exercise 2.6: For any open interval (a,b), what is $(a,b)^\sim$? (As for all exercises, prove that your answer is correct).

Theorem 2.7: For any $A \subset \mathsf{R}^1$, $A \subset A^\sim$.

Proof: Since $\emptyset^\sim = \emptyset$ (by definition), the theorem is true when $A = \emptyset$. So, assume that $A \ne \emptyset$. Let $p \in A$. If I is an open interval such that $p \in I$, then $p \in I \cap A$ and, hence, $I \cap A \ne \emptyset$. Therefore, by Theorem 2.3, $p \in A^\sim$. ¥

Example 2.8: If A is a finite subset of R^1, then $A^\sim = A$.

Example 2.9: $\mathsf{N}^\sim = \mathsf{N}$; if $A = \{\frac{1}{n} : n \in \mathsf{N}\}$, then $A^\sim = A \cup \{0\}$.

Exercise 2.10: If $A \subset B$, then $A^\sim \subset B^\sim$.

Theorem 2.11: For any $A, B \subset \mathsf{R}^1$, $(A \cup B)^\sim = A^\sim \cup B^\sim$.

Proof: Since $A \subset A \cup B$, $A^\sim \subset (A \cup B)^\sim$ by Exercise 2.10; similarly, $B^\sim \subset (A \cup B)^\sim$. Therefore, $A^\sim \cup B^\sim \subset (A \cup B)^\sim$.

We prove the reverse containment, namely, $(A \cup B)^\sim \subset A^\sim \cup B^\sim$.

Assume first that $A = \emptyset$. Then $A \cup B = B$ and, hence,

$$(A \cup B)^\sim = B^\sim = \emptyset^\sim \cup B^\sim = A^\sim \cup B^\sim.$$

Similarly, if $B = \emptyset$, then $(A \cup B)^\sim = A^\sim \cup B^\sim$. This proves that if $A = \emptyset$ or $B = \emptyset$, then $(A \cup B)^\sim = A^\sim \cup B^\sim$.

So, we assume from now on that $A \ne \emptyset$ and $B \ne \emptyset$. We prove that $(A \cup B)^\sim \subset A^\sim \cup B^\sim$ with a contrapositive argument (a direct argument can not be done with the present methods: see Exercise 2.12).

Assume that $p \in \mathsf{R}^1$ such that $p \notin A^\sim \cup B^\sim$. Then, by Theorem 2.5, there exist $\epsilon_1, \epsilon_2 > 0$ such that