

THE LARGEST POSSIBLE LINEAR SUBSPACES CONTAINED IN THE CLASSES OF DARBOUX-LIKE FUNCTIONS ON \mathbb{R}

by

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ABSTRACT

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Consider an arbitrary $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, where the family $\mathbb{R}^{\mathbb{R}}$ of all functions from \mathbb{R} to \mathbb{R} is considered as a linear space over \mathbb{R} . Does $\mathcal{F} \cup \{0\}$ contain a non-trivial lineal subspace? If so, how big the vector space can be? These questions are at the core of lineability theory. In particular, we say that a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is *lineable* (in $\mathbb{R}^{\mathbb{R}}$) provided there exists an infinite dimensional linear space contained in $\mathcal{F} \cup \{0\}$. There has been a lot of attention devoted to lineability problem of subsets of linear space of functions. For instance, the families of continuous nowhere differentiable functions and of differentiable nowhere monotone functions are lineable. It has also been known for a while that the class $D \subset \mathbb{R}^{\mathbb{R}}$ of Darboux functions (i.e., functions that satisfy the intermediate value property) is lineable. In fact, D is $2^{\mathfrak{c}}$ -lineable, that is, $D \cup \{0\}$ contains a subspace of dimension $2^{\mathfrak{c}}$, where $2^{\mathfrak{c}}$ is the cardinality of $\mathbb{R}^{\mathbb{R}}$. The goal of this work is to study the lineability of the subclasses of D that are in the algebra generated by D and seven of its subclasses (known as Darboux-like functions): extendable (Ext), almost continuous (AC), connectivity (Conn), peripherally continuous (PC), having perfect road (PR), having Cantor Intermediate Value Property (CIVP), and having Strong Cantor Intermediate Value Property (SCIVP).

This dissertation is arranged as follows. Chapter 1 focuses on presenting notations, definitions, and summary of all results contained in this work. In chapter 2, we give a general method to have \mathfrak{c} -lineable for all Darboux-like maps and even their restriction to Baire 2 class functions. In chapter 3, we will build some tools that allow us to show $2^{\mathfrak{c}}$ -lineability (i.e., maximal lineability) for all Darboux-like subclasses of $(PC \setminus D) \cup (AC \setminus Ext)$ in the algebra. In chapter 4, we are going to construct algebraically independent sets that shall be used to achieve the maximal lineability for all Darboux-like subclasses of $D \setminus Conn$. In the last part, in chapter 5, we will make some remarks on lineability and offer new possibilities for open problems.

To my parents, brother, and sisters

*To my wife **Jalala**, without her I would be completely lost.*

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Chapter 1

Introduction and preliminary results

1.1 Lineability: definition and background

1.1.1 Introduction

Over the last two decades, a lot of mathematicians have been interested in finding the largest possible vector spaces that are contained in various families of real functions, see e.g. survey [13], monograph [7], and the literature cited there. (More recent work in this direction include [6, 18, 33].) Specifically, given a (finite or infinite) cardinal number κ , a subset M of a vector space W is said to be κ -lineable (in W) provided there exists a linear space $V \subset M \cup \{0\}$ of dimension κ . This notion was first studied in 1966 by Vladimir Gurarii [43], even though he did not use the term lineability.

If a vector space W is also topological, then we say that $M \subset W$ is κ -spaceable there exists a linear space $V \subset M \cup \{0\}$ of dimension κ such that V closed in W . The problem of lineability and spaceability has a very long history and we will briefly mention to some of the old famous results for various classes of functions. One of these was due to Levine and Milman [50]. They showed the subset of $C[0, 1]$ of all continuous functions on $[0, 1]$ of bounded variation is not spaceable, where $C[0, 1]$ denotes to the Banach space of continuous functions $[0, 1] \rightarrow \mathbb{R}$ endowed with the supremum norm. Later on, Vladimir Gurarii [43] showed the set of real valued continuous nowhere differentiable functions on $[0, 1]$ is ω -lineable. Moreover, while it is clear that the set of everywhere differentiable functions on $[0, 1]$ is lineable, Gurarii showed that it is not ω -spaceable. In 2006, Juan B. Seoane in

his Ph.D. Dissertation [62], written under the supervision of R.M. Aron and Vladimir Gurariĭ showed that the set of differentiable functions on \mathbb{R} which are nowhere monotone is lineable in $C(\mathbb{R})$. Also, in the same work, he proved that the set of everywhere surjective on \mathbb{R} has the maximal lineability.

Our work here is a contribution to ongoing research concerning lineability problem. More specifically, we will examine maximal lineabilities of the classes of functions related to Darboux-like maps.

1.2 Notation

Our terminology is standard and follows [19]. In particular, the symbols \mathbb{N} , \mathbb{Q} , and \mathbb{R} stand for the sets of positive integers, rational numbers, and real numbers, respectively. A symbol \mathcal{B} denotes the standard countable basis $\{(p, q) : p < q \ \& \ p, q \in \mathbb{Q}\}$ of \mathbb{R} .

The cardinality of a set X is denoted by the symbol $|X|$. In particular, $|\mathbb{N}|$ is denoted by ω and $|\mathbb{R}|$ is denoted by \mathfrak{c} . For a cardinal number κ the symbol κ^+ stands for the cardinal successor of κ .

B_1 and B_2 stand for the families of Baire class 1 functions (i.e., pointwise limits of continuous functions) and Baire class 2 functions (i.e., pointwise limits of B_1 functions), respectively. We say that a set $B \subset \mathbb{R}$ is a Bernstein set if both B and $\mathbb{R} \setminus B$ intersect every nonempty perfect set. For a cardinal number κ , a set $A \subset \mathbb{R}$ is called κ -dense if $|A \cap I| \geq \kappa$ for every non-trivial interval I .

We consider only real-valued functions defined on subsets of \mathbb{R} . No distinction is made between a function and its graph. For a nonempty set X let \mathbb{R}^X be the class of all maps from X into the real line \mathbb{R} . We consider \mathbb{R}^X as a linear space over \mathbb{R} . For an $f \in \mathbb{R}^X$ its *support* is defined as

$$\text{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

Notice that we do not take the closure of the set above, even when X is a topological space. We write $f \upharpoonright A$ for the restriction of $f \in \mathbb{R}^X$ to the set $A \subset X$. The image of a set A under the function f is denoted by $f[A]$. For $B \subset \mathbb{R}$, χ_B denotes to the characteristic function of B . Also, for $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ we use the symbol $\neg\mathcal{F}$ to denote the complement of \mathcal{F} with respect to $\mathbb{R}^{\mathbb{R}}$, that is, $\neg\mathcal{F} := \mathbb{R}^{\mathbb{R}} \setminus \mathcal{F}$.

1.3 Definitions of Darboux like-functions

We begin this section by given a full definitions of eight classes of Darboux-like functions. For a metric space X , the main classes of Darboux-like functions from X to \mathbb{R} are defined as follows. (These notions will be used in this project mainly when X is an interval in \mathbb{R} .)

D_X of all *Darboux functions* $f \in \mathbb{R}^X$, that is, such that $f[C]$ is connected (i.e., an interval) for every connected $C \subset \mathbb{R}$. (Equivalently, $f \in D$ provided it has the intermediate value property). This class was first discovered by Jean Gaston Darboux (1842–1917) who, in his 1875 paper [35], showed that all derivatives, including those that are discontinuous, satisfy the intermediate value theorem.

PC_X of all *peripherally continuous functions* $f \in \mathbb{R}^X$, that is, such that for every $x \in X$, open interval $J \subset \mathbb{R}$ containing $f(x)$, and $\varepsilon > 0$, there exists an open neighborhood U of x of diameter $< \varepsilon$ such that $f[\text{bd}(U)] \subset J$, where $\text{bd}(U)$ is the boundary (or periphery) of U . Notice that for $X = \mathbb{R}$ this is equivalent to the statement that for every $x \in \mathbb{R}$ there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ with $\lim_{n \rightarrow \infty} f(s_n) = f(x) = \lim_{n \rightarrow \infty} f(t_n)$. This class was introduced in a 1907 paper [65] of John Wesley Young (1879–1932). The name comes from the papers [44, 45, 64]. Note that any function in $\mathbb{R}^{\mathbb{R}}$ with a graph dense in \mathbb{R}^2 is $PC_{\mathbb{R}}$.

PR_X of all functions $f \in \mathbb{R}^X$ *with perfect road*, that is, such that for every $x \in X$ there exists a perfect $P \subset X$ containing x such that x as a bilateral limit point of P (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) *when x is an interior point of X and that $f \upharpoonright P$ is continuous at x* . This class was introduced in a 1936 paper [52] of Isaie Maximoff, where he proved that $D \cap B_1 = PR \cap B_1$.

Conn_X of all *connectivity functions* $f \in \mathbb{R}^X$, that is, such that the graph of f restricted to any connected $C \subset X$ is a connected subset of $X \times \mathbb{R}$. Note that $f \in \text{Conn}_{\mathbb{R}}$ iff f is connected. This notion can be traced to a 1956 problem [55] stated by John Forbes Nash (1928–2015). We also refer to [45, 62].

AC_X of all *almost continuous functions* $f \in \mathbb{R}^X$ (in the sense of Stallings), that is, such that every open subset of $X \times \mathbb{R}$ containing the graph of f contains also the graph of a continuous function from X to \mathbb{R} . This class was first seriously studied in a 1959 paper [62] of John Robert Stallings (1935–2008); however, it appeared already in a 1957 paper [45] by Olan H. Hamilton (1899–1976). See also 1991 survey [56] by T. Natkaniec.

Ext_X of all *extendable functions* $f \in \mathbb{R}^X$, that is, such that there exists a connectivity function $g: X \times [0, 1] \rightarrow \mathbb{R}$ with $f(x) = g(x, 0)$ for all $x \in X$. The notion of extendable functions (without the name) first appeared in a 1959 paper [62] of J. Stallings, where he asks a question whether every connectivity function defined on $[0, 1]$ is extendable.

CIVP_X of all functions $f \in \mathbb{R}^X$ with *Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in X$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$. This class was first introduced in a 1982 paper [41] of Richard G. Gibson and Fred William Roush.

SCIVP_X of all functions $f \in \mathbb{R}^X$ with *Strong Cantor Intermediate Value Property*, that is, such that for all $p, q \in X$ with $p \neq q$ and $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$ and $f \upharpoonright P$ is continuous. This notion was introduced in a 1992 paper [59] of Harvey Rosen, R. Gibson, and F. Roush to help distinguish extendable and connectivity functions on \mathbb{R} .

We will drop the subscript X in this notation when $X = \mathbb{R}$ or X is clear from the context. Throughout this thesis, we will denote the collection of Darboux-like families by the symbol \mathbb{D} , that is, we put $\mathbb{D} := \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{SCIVP}, \text{CIVP}, \text{PR}, \text{PC}\}$. The diagram in Fig. 1.1 shows the relations between the classes in \mathbb{D} , when $X = \mathbb{R}$ or X is an interval. The arrows denote strict inclusions.

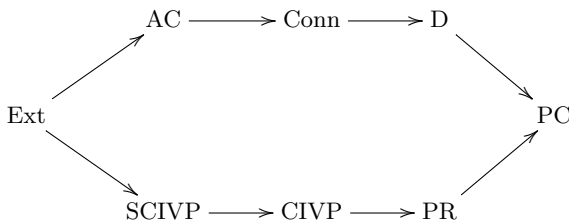


Figure 1.1: All inclusions, indicated by arrows, among the Darboux-like classes \mathbb{D} . The only inclusions among the intersections of these classes are those that follow trivially from this schema. (See [24, 40].)

The inclusions $\text{Conn} \subset \text{D} \subset \text{PC}$, $\text{PR} \subset \text{PC}$, and $\text{SCIVP} \subset \text{CIVP}$ are obvious from the previous definitions. On the other hand, the remaining inclusions are less obvious. Among them the inclusions $\text{Ext} \subset \text{AC} \subset \text{Conn}$ were proved by Stallings [62], while $\text{CIVP} \subset \text{PR}$ was stated without proof in [42] (although its proof can be found in [40, theorem 3.8]). The inclusion $\text{Ext} \subset \text{SCIVP}$ comes from [59].

The inclusions indicated in Fig. 1.1 are the only inclusions among these classes even when we add to the considerations the intersections of the classes from the top and bottom rows of Fig. 1.1. This is well described in the expository papers [20, 24, 40]. Specifically, $\text{AC} \setminus \text{CIVP} \neq \emptyset$ and $\text{CIVP} \setminus \text{AC} \neq \emptyset$ was shown in a 1982 paper [41]. The fact that $\text{Conn} \setminus \text{AC} \neq \emptyset$ is the trickiest to prove and is related to late 1960's papers: [58] of John Henderson Roberts, [34] of James L. Cornette, [48] of F. Burton Jones and Edward S. Thomas Jr., and [16] of J. Brown. The result $\text{D} \setminus \text{Conn} \neq \emptyset$ can be traced to 1965 paper [17] of Andrew M. Bruckner and Jack Gary Ceder (see also [16]), while examples for

$PC \setminus D \neq \emptyset$, $PR \setminus CIVP \neq \emptyset$, and $PC \setminus PR \neq \emptyset$ to a 2000 paper [24] of K. C. Ciesielski and Jan Jastrzębski.

The inclusions indicated in Fig. 1.1 suggest a natural split of \mathbb{D} into two subclasses: $\mathbb{U} := \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC}\}$ and $\mathbb{L} := \{\text{Ext}, \text{SCIVP}, \text{CIVP}, \text{PR}, \text{PC}\}$, each consisting of the families that are mutually comparable by inclusion. In particular, the algebra $\mathcal{A}(\mathbb{U})$ of subsets of PC generated by the classes in \mathbb{U} has 5 atoms:

$$\{\text{PC} \setminus \text{D}, \text{D} \setminus \text{Conn}, \text{Conn} \setminus \text{AC}, \text{AC} \setminus \text{Ext}, \text{Ext}\}.$$

Similarly, $\mathcal{A}(\mathbb{L})$ generated by the classes in \mathbb{L} has also 5 atoms:

$$\{\text{PC} \setminus \text{PR}, \text{PR} \setminus \text{CIVP}, \text{CIVP} \setminus \text{SCIVP}, \text{SCIVP} \setminus \text{Ext}, \text{Ext}\}.$$

This means that the algebra $\mathcal{A}(\mathbb{D})$ has theoretically 25 atoms, the intersections $L \cap U$, where $L \in \mathcal{A}(\mathbb{L})$ and $U \in \mathcal{A}(\mathbb{U})$. However, if $\text{Ext} \in \{U, L\}$, then $L \cap U = \emptyset$ unless $L = U = \text{Ext}$. Thus, $\mathcal{A}(\mathbb{D}) = \mathcal{A}(\mathbb{U} \cup \mathbb{L})$ has actually 17 atoms: Ext and the 16 atoms presented in Table 1.1.

\cap	$\text{PC} \setminus \text{PR}$	$\text{PR} \setminus \text{CIVP}$	$\text{CIVP} \setminus \text{SCIVP}$	$\text{SCIVP} \setminus \text{Ext}$
$\text{PC} \setminus \text{D}$	$\text{PC} \cap$ $\neg(\text{PR} \cup \text{D})$	$\text{PR} \cap$ $\neg(\text{CIVP} \cup \text{D})$	$\text{CIVP} \cap$ $\neg(\text{SCIVP} \cup \text{D})$	$\text{SCIVP} \setminus \text{D}$
$\text{D} \cap$ $\neg \text{Conn}$	$\text{D} \cap$ $\neg(\text{PR} \cup \text{Conn})$	$\text{D} \cap \text{PR} \cap$ $\neg(\text{CIVP} \cup \text{Conn})$	$\text{D} \cap \text{CIVP} \cap$ $\neg(\text{SCIVP} \cup \text{Conn})$	$\text{D} \cap \text{SCIVP} \cap$ $\neg \text{Conn}$
$\text{Conn} \cap$ $\neg \text{AC}$	$\text{Conn} \cap$ $\neg(\text{PR} \cup \text{AC})$	$\text{Conn} \cap \text{PR} \cap$ $\neg(\text{CIVP} \cup \text{AC})$	$\text{Conn} \cap \text{CIVP} \cap$ $\neg(\text{SCIVP} \cup \text{AC})$	$\text{Conn} \cap \text{SCIVP}$ $\cap \neg \text{AC}$
$\text{AC} \cap$ $\neg \text{Ext}$	$\text{AC} \setminus \text{PR}$	$\text{AC} \cap \text{PR} \cap$ $\neg \text{CIVP}$	$\text{AC} \cap \text{CIVP} \cap$ $\neg \text{SCIVP}$	$\text{AC} \cap \text{SCIVP} \cap$ $\neg \text{Ext}$

Table 1.1: All atoms of $\mathcal{A}(\mathbb{D})$ with exception of Ext .

In addition, we are going to consider various families of surjective functions. Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we say (see e.g. [22]) that:

- f is everywhere surjective, $f \in \text{ES}$, provided $f[G]$ is equal to \mathbb{R} for every non-empty open set $G \subset \mathbb{R}$;
- f is strongly everywhere surjective, $f \in \text{SES}$, provided $f^{-1}(y) \cap G$ has cardinality \mathfrak{c} for every

non-empty open set $G \subset \mathbb{R}$;

- f is perfectly everywhere surjective, $f \in \text{PES}$, provided $f[P]$ is equal to \mathbb{R} for every non-empty perfect set $P \subset \mathbb{R}$.
- f is Jones, $f \in \text{J}$, provided $f \cap K \neq \emptyset$ for every closed set $K \subset \mathbb{R}^2$ with uncountable projection on the x -axis.

The diagram in Fig 1.2 shows the relations between the above classes

$$\text{J} \longrightarrow \text{PES} \longrightarrow \text{SES} \longrightarrow \text{ES} \longrightarrow \text{D}$$

Figure 1.2: The arrows indicate strict inclusions.

We will end this section by a brief over review of what are previously known on lineability of Darboux like-maps and surjective functions.

The following results can be found, for instance, in [7].

Proposition 1.3.1. *The following classes are 2^c -lineable.*

1. All maps in Figures 1.1 and 1.2.
2. $\text{PES} \setminus \text{J}$, $\text{SES} \setminus \text{PES}$, $\text{D} \setminus \text{ES}$, $\text{AC} \setminus \text{Ext}$, and $\text{PC} \setminus \text{D}$.

1.4 Summary of our novel results

The following Table 1.2 summaries the results presented in this dissertation.

\cap	$\text{PC} \setminus \text{PR}$ 2^c	$\text{PR} \setminus \text{CIVP}$ 2^c	$\text{CIVP} \setminus \text{SCIVP}$ 2^c	$\text{SCIVP} \setminus \text{Ext}$ 2^c
$\text{PC} \setminus \text{D}$ 2^c , Thm 3.2.2	2^c Thm 3.2.2	2^c Thm 3.2.7	2^c Thm 3.2.12	2^c Thm 3.2.15
$\text{D} \setminus \text{Conn}$ 2^c , Thm 4.3.3	2^c Thm 4.3.3	2^c Thm 4.4.3	2^c Thm 4.4.5	2^c Thm 4.3.3
$\text{Conn} \setminus \text{AC}$ \mathfrak{c} , Cor 2.3.3	\mathfrak{c} Cor 2.3.3	\mathfrak{c} Cor 2.3.3	\mathfrak{c} Cor 2.3.3	\mathfrak{c} Cor 2.3.3
$\text{AC} \setminus \text{Ext}$ 2^c , Thm 3.2.5	2^c Thm 3.2.5	2^c Thm 3.2.8	2^c Thm 3.2.13	2^c Thm 3.2.17

Table 1.2: The values of lineabilities for all the classes in Table 1.1 and references to these results.

Chapter 2

\mathfrak{c} -lineability: A general method and its application to Darboux-like maps

2.1 Introduction

This chapter contains the materials that come from our paper [3]. We describe a simple class of \mathfrak{c} -dimensional linear subspaces of $\mathbb{R}^{\mathbb{R}}$ and show that many natural classes of real functions contain subspaces of this form, proving their \mathfrak{c} -lineability. The examples include all non-empty classes in the algebra of all Darboux-like functions, see Table 1.1, and of their restrictions to the Baire class 2 maps.

For a non-empty family $\mathcal{F} \subseteq \mathbb{R}^X$ of non-zero functions with pairwise disjoint supports consider the following vector subspace of \mathbb{R}^X :

$$\mathcal{L}_{\mathcal{F}} = \left\{ \sum_{f \in \mathcal{F}} s(f) \cdot f : s \in \mathbb{R}^{\mathcal{F}} \right\}.$$

The maps $\sum_{f \in \mathcal{F}} s(f) \cdot f$ are well defined, since the functions in \mathcal{F} have disjoint supports. The following proposition is obvious, unless $2^{|\mathcal{F}|} = \mathfrak{c}$.

Proposition 2.1.1. *If $\mathcal{F} \subseteq \mathbb{R}^X$ is an infinite family with pairwise disjoint supports, then $\mathcal{L}_{\mathcal{F}}$ has dimension $2^{|\mathcal{F}|}$.*

Proof. Let $\kappa = |\mathcal{F}|$ and B be a basis for $\mathcal{L}_{\mathcal{F}}$. If $2^\kappa > \mathfrak{c}$, then the conclusion is obvious, since otherwise $|B| < 2^\kappa$ and $2^\kappa = |\mathcal{L}_{\mathcal{F}}| = |\mathbb{R}| \cdot |B| = \mathfrak{c} \cdot |B| < 2^\kappa$, a contradiction. So, assume that $2^\kappa = \mathfrak{c}$. To finish the proof it is enough to find \mathfrak{c} -many linearly independent functions in $\mathcal{L}_{\mathcal{F}}$.

Let $\{f_k: k < \omega\}$ be a family of distinct non-zero functions in \mathcal{F} and \mathcal{A} be a family of \mathfrak{c} -many infinite pairwise almost disjoint subsets of ω . (See e.g. [51, proposition 5.26].) For every $A \in \mathcal{A}$ let $F_A = \sum_{k \in A} f_k \in \mathcal{L}_{\mathcal{F}}$ and notice that $\{F_A: A \in \mathcal{A}\} \subset \mathcal{L}_{\mathcal{F}}$ is linearly independent. To see this, choose $c_1, \dots, c_n \in \mathbb{R}$, distinct $A_1, \dots, A_n \in \mathcal{A}$, and notice that

$$c_1 F_{A_1} + c_2 F_{A_2} + \dots + c_n F_{A_n} \neq 0,$$

unless $c_1 = c_2 = \dots = c_n = 0$. Indeed, if $c_i \neq 0$, then there is a $k \in A_i \setminus \bigcup_{j \neq i} A_j$ and an $x \in \text{supp}(f_k)$ for which $(c_1 F_{A_1} + c_2 F_{A_2} + \dots + c_n F_{A_n})(x) = c_i f_k(x) \neq 0$. Thus, indeed the equation $2^\kappa = \mathfrak{c}$ implies that $\mathcal{L}_{\mathcal{F}}$ has dimension 2^κ . \square

2.2 \mathfrak{c} -lineability of the families of Baire 2 Darboux-like maps

Let $\mathcal{P}_X = \{\text{Ext}_X, \text{AC}_X, \text{Conn}_X, \text{D}_X, \text{PC}_X\}$ be the collection of the five main classes of Darboux-like functions. The topological nature of these classes and their format easily justifies the following properties.

Remark 2.2.1. Let $P \in \mathcal{P}$ and X and Y be metric spaces.

- (i) If $g \in P_Y$ and f is a homeomorphism from X to Y , then $g \circ f \in P_X$;
- (ii) If $g \in P_Y$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism, then $h \circ g \in P_Y$;
- (iii) If $g \in P_Y$, $P \neq \text{AC}$, and $B \subset Y$, then $g \upharpoonright B \in P_B$.

The format of the definition of the class AC does not allow to immediately conclude for it the above property (iii). In fact, in such generality the statement is false, see [56, example 2.1]. However, the following result for this family can be found in [56, corollary 2.2].

Proposition 2.2.2. *Let $f \in \mathbb{R}^{\mathbb{R}}$. Then*

$$f \in \text{AC} \text{ if, and only if, } f \upharpoonright [k, k+1] \in \text{AC}_{[k, k+1]} \text{ for every } k \in \mathbb{Z}.$$

The sufficiency condition in Proposition 2.2.2 will be also needed for the other three classes:

Lemma 2.2.3. *If $f \in \mathbb{R}^{\mathbb{R}}$, $P \in \{\text{AC}, \text{Conn}, \text{D}, \text{PC}\}$, and $f \upharpoonright [k, k+1] \in P_{[k, k+1]}$ for every $k \in \mathbb{Z}$, then $f \in P$.*

Proof. For $P = \text{AC}$ this is implied by Proposition 2.2.2.

For $P = \text{PC}$ the result is obvious from the definition of the class PC .

To see this for $P = \text{Conn}$ choose a connected $C \subset \mathbb{R}$, that is, an interval, and notice that the non-empty consecutive sets in the family $\{f \upharpoonright ([k, k+1] \cap C) : k \in \mathbb{Z}\}$ are connected (since $f \upharpoonright [k, k+1] \in \text{Conn}_{[k, k+1]}$ and $[k, k+1] \cap C$ is connected) and have non-empty intersections. So, by a well know result (see e.g. [53, theorem 23.3]) their union $f \upharpoonright C$ is connected, as needed, that is, indeed $f \in \text{Conn}$.

The argument for $P = \text{D}$ is similar, where for a connected $C \subset \mathbb{R}$ we just notice that $f[C]$, the union of the family $\{f[[k, k+1] \cap C] : k \in \mathbb{Z}\}$, is connected. \square

We know, see Figure 1.1, that the classes discussed above are related as follows

$$\text{Ext} \subsetneq \text{AC} \subsetneq \text{Conn} \subsetneq \text{D} \subsetneq \text{PC}. \quad (2.1)$$

It is also known that within the family B_1 all these classes coincide, see e.g. [24]. In contrast, within the family B_2 all inclusions presented in (2.1) remain strict. See e.g. [24]. This means that neither of the classes

$$B_2 \cap (\text{PC} \setminus \text{D}), B_2 \cap (\text{D} \setminus \text{Conn}), B_2 \cap (\text{Conn} \setminus \text{AC}), \text{ and } B_2 \cap (\text{AC} \setminus \text{Ext}), \quad (2.2)$$

is empty. The goal of this section is to show that each of these classes is \mathfrak{c} -lineable, which is the best result in this direction, since their cardinalities are bounded by $|B_2| = \mathfrak{c}$. To prove this lineability result, we recall the following fact. (Compare also citations in [24].)

Example 2.2.4. Each of the classes listed in (2.2) contains a Baire class 2 map $f: [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = f(1) = 0$.

Proof. For the class $B_2 \cap (\text{AC} \setminus \text{Ext})$ see [24, theorem 3.1].

For the class $B_2 \cap (\text{Conn} \setminus \text{AC})$ see [47].

For the class $B_2 \cap (\text{D} \setminus \text{Conn})$ see [15, example 2]. For the class $B_2 \cap (\text{PC} \setminus \text{D})$ take a function $F: [-1, 1] \rightarrow \mathbb{R}$ from [24, example 3.5] which is in $B_2 \setminus \text{D}$. It is PC , since it belongs to the class $\text{SCIVP} \subset \text{PC}$. Thus, if ℓ maps linearly $[0, 1]$ onto $[-1, 1]$, then, by Remark 2.2.1, the function f defined by $f(x) = F \circ \ell(x) - 1$ is as needed. \square

For a non-zero $f: [0, 1] \rightarrow \mathbb{R}$ with $f(0) = f(1) = 0$ let

$$\mathcal{F}_f := \{f_k : k \in \mathbb{Z}\},$$

where $f_0 \in \mathbb{R}^{\mathbb{R}}$ is an extension of f such that $f_0 \equiv 0$ on the complement of $[0, 1]$ and, generally, $f_k \in \mathbb{R}^{\mathbb{R}}$ is defined as $f_k(x) := f_0(x - k)$. Notice that \mathcal{F}_f is an infinite countable family of functions that have disjoint supports, since $\text{supp}(f_k) \subset (k, k + 1)$ for every $k \in \mathbb{Z}$. So, $\mathcal{L}_{\mathcal{F}_f}$ is well defined and has dimension $2^{|\mathcal{F}_f|} = \mathfrak{c}$.

Theorem 2.2.5. *Let $f: [0, 1] \rightarrow \mathbb{R}$ be such that $f(0) = f(1) = 0$. If $P, Q \in \mathcal{P}$ are such that $f \in P \setminus Q$, then $\mathcal{L}_{\mathcal{F}_f}$ justifies \mathfrak{c} -lineability of $P \setminus Q$.*

Proof. By Proposition 2.1.1, the space $\mathcal{L}_{\mathcal{F}_f}$ is well defined and has dimension \mathfrak{c} . Also, (2.1) and $P \setminus Q \neq \emptyset$ imply that $P \in \{\text{AC}, \text{Conn}, \text{D}, \text{PC}\}$ and $Q \in \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}\}$.

Next, take a non-zero $g \in \mathcal{L}_{\mathcal{F}_f}$. We need to show that $g \in P \setminus Q$. Indeed, $g = \sum_{k \in \mathbb{Z}} c_k f_k$ for some constants c_k not all zero. Moreover, $g \upharpoonright [k, k + 1] = c_k f \circ t_k$, where $t_k: [k, k + 1] \rightarrow [0, 1]$ is a translation given by $t_k(x) = x - k$. In particular, by Remark 2.2.1(i)&(ii), $g \upharpoonright [k, k + 1] \in P_{[k, k + 1]}$ for every $k \in \mathbb{Z}$. Thus, by Lemma 2.2.3, indeed $g \in P$.

Next, fix a $k \in \mathbb{Z}$ such that $c_k \neq 0$ and notice that, by Lemma 2.2.3 and the fact that $f_0 \upharpoonright [j, j + 1] \in Q$ for every non-zero $j \in \mathbb{Z}$, we have $f_0 \notin Q$. So, by Remark 2.2.1(i)&(ii), also $c_k f_k \notin Q$. In particular, the contrapositive version of Remark 2.2.1(iii) implies that $g \upharpoonright [k, k + 1] = c_k f_k \upharpoonright [k, k + 1] \notin Q_{[k, k + 1]}$. Thus, by Proposition 2.2.2 and contrapositive of Remark 2.2.1(iii), $g \notin Q$, finishing the proof. \square

Now, we are ready for the main result of this section.

Corollary 2.2.6. *Each of the classes listed in (2.2) is \mathfrak{c} -lineable.*

Proof. Choose $P, Q \in \mathcal{P}$ so that $B_2 \cap (P \setminus Q)$ is one of the classes listed in (2.2). By Example 2.2.4, there exists an $f: [0, 1] \rightarrow \mathbb{R}$ in $B_2 \cap (P \setminus Q)$ such that $f(0) = f(1) = 0$. By Theorem 2.2.5, the family $\mathcal{L}_{\mathcal{F}_f}$ justifies \mathfrak{c} -lineability of $P \setminus Q$. To finish the proof, it is enough to notice that if $f \in B_2$, then any map $g = \sum_{k \in \mathbb{Z}} c_k f_k$ from $\mathcal{L}_{\mathcal{F}_f}$ is also Baire class 2. \square

2.3 \mathfrak{c} -lineability of Darboux-like subclasses of $\text{Conn} \setminus \text{AC}$

We know, see Figure 1.1, that

$$\text{Ext} \subsetneq \text{SCIVP} \subsetneq \text{CIVP} \subsetneq \text{PR} \subsetneq \text{PC}. \quad (2.3)$$

Also, it is well known (see e.g. [24, theorem 1.2]) and easy to see that

$$\text{every Borel (so } B_2) \text{ map } f \in \mathbb{R}^{\mathbb{R}} \text{ is SCIVP}. \quad (2.4)$$

This, together with Example 2.2.4 and Corollary 2.2.6, implies immediately that

Corollary 2.3.1. *Each of the following classes (of column 4 of Table 1.1) is \mathfrak{c} -lineable:*

$$\text{SCIVP} \setminus \text{D}, \text{SCIVP} \cap \text{D} \setminus \text{Conn}, \text{SCIVP} \cap \text{Conn} \setminus \text{AC}, \text{SCIVP} \cap \text{AC} \setminus \text{Ext}. \quad (2.5)$$

The goal of this section is to prove \mathfrak{c} -lineability of the first three classes from the third row in Table 1.1, that is, the classes

$$\text{Conn} \setminus (\text{PR} \cup \text{AC}), \text{Conn} \cap \text{PR} \setminus (\text{CIVP} \cup \text{AC}), \text{Conn} \cap \text{CIVP} \setminus (\text{SCIVP} \cup \text{AC}), \quad (2.6)$$

for which nothing was known so far in this direction. Notice that each class in (2.6) is the intersection of the class $\text{Conn} \setminus \text{AC}$ with one of the following classes:

$$\text{PC} \setminus \text{PR}, \text{PR} \setminus \text{CIVP}, \text{and CIVP} \setminus \text{SCIVP}. \quad (2.7)$$

We will use the following variant of Theorem 2.2.5, with a very similar proof.

Theorem 2.3.2. *Let $f: [0, 1] \rightarrow \mathbb{R}$ be such that $f(0) = f(1) = 0$. If P and Q are among the classes in $\{\text{SCIVP}, \text{CIVP}, \text{PR}, \text{PC}\}$ and such that $f \in P \setminus Q$, then $\mathcal{L}_{\mathcal{F}_f}$ justifies \mathfrak{c} -lineability of $P \setminus Q$.*

Proof. First, notice that the analogues of Remark 2.2.1(i)–(iii) and Lemma 2.2.3 hold also for the classes PC, PR, CIVP, and SCIVP. By Proposition 2.1.1, the space $\mathcal{L}_{\mathcal{F}_f}$ is well defined and has dimension \mathfrak{c} . Also, (2.3) and $P \setminus Q \neq \emptyset$ imply that $P \in \{\text{CIVP}, \text{PR}, \text{PC}\}$ and $Q \in \{\text{SCIVP}, \text{CIVP}, \text{PR}\}$.

Take a non-zero $g \in \mathcal{L}_{\mathcal{F}_f}$, we need to show that $g \in P \setminus Q$. Indeed, $g = \sum_{k \in \mathbb{Z}} c_k f_k$ for some constants c_k not all zero. Moreover, $g \upharpoonright [k, k+1] = c_k f \circ t_k$, where $t_k: [k, k+1] \rightarrow [0, 1]$ is a translation given by $t_k(x) = x - k$. In particular, by above mentioned analog of Remark 2.2.1(i)&(ii),

$g \upharpoonright [k, k+1] \in P_{[k, k+1]}$ for every $k \in \mathbb{Z}$. Thus, by an analogue of Lemma 2.2.3, indeed $g \in P$. Also, if $k \in \mathbb{Z}$ is such that $c_k \neq 0$, then, by an analogue of Remark 2.2.1(ii), $g \upharpoonright [k, k+1] \notin Q_{[k, k+1]}$. So, by the analogues of Proposition 2.2.2 and of the contrapositive of Remark 2.2.1(iii), $g \notin Q$, finishing the proof. \square

Now, we are ready for the main result of this section.

Corollary 2.3.3. *Each of the classes listed in (2.6) is \mathfrak{c} -lineable.*

Proof. First notice that we have an analogue of Example 2.2.4 for the classes in (2.6):

- (a) each of the classes in (2.6) contains an $f: [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = f(1) = 0$.

To see this, first recall the results presented in the paper [28] that

- (b) each of the classes in (2.6) contains an $F: \mathbb{R} \rightarrow \mathbb{R}$ that belongs also to ES.

Indeed, the additivity coefficient A associated with each class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ (see e.g. [28, definition 1.1]) has the property that $A(\mathcal{F}) \geq 2$ if, and only if, $\mathcal{F} \neq \emptyset$ (see e.g. [28, proposition 1.2(i)]) while we have the following results:

$$[28, \text{theorem 6.3}] \quad A(\text{ES} \cap \text{Conn} \setminus (\text{PR} \cup \text{AC})) \geq \omega_1,$$

$$[28, \text{theorem 7.2}] \quad A(\text{ES} \cap \text{Conn} \cap \text{PR} \setminus (\text{CIVP} \cup \text{AC})) \geq \omega_1,$$

$$[28, \text{theorem 8.2}] \quad A(\text{ES} \cap \text{Conn} \cap \text{CIVP} \setminus (\text{SCIVP} \cup \text{AC})) \geq \omega_1.$$

This clearly implies (b). To see (a) let $F \in \text{ES}$ belong to one of the classes in (2.6), choose $a < b$ so that $F(a) = F(b) = 0$, and notice that $f := F \circ \ell_{[a, b]}$ is as needed for (a), where $\ell_{[a, b]}$ maps linearly $[0, 1]$ onto $[a, b]$.

To finish the proof, take an f as in (a) and notice that $\mathcal{L}_{\mathcal{F}_f}$ justifies \mathfrak{c} -lineability of an appropriate family from (2.6). Indeed, every non-zero map $g \in \mathcal{L}_{\mathcal{F}_f}$ is in $\text{Conn} \setminus \text{AC}$ by Theorem 2.2.5, while by Theorem 2.3.2 it belongs also to an appropriate class listed in (2.7). But this means that indeed g is in an appropriate class from (2.6). \square

We should also remark here, that technique used in the proof of Corollary 2.3.3 can be also used to prove that all other classes from Table 1.1 are \mathfrak{c} -lineable. All one needs for such proof is to notice that each class in the table contains a function as in (a) in the proof of Corollary 2.3.3. However, the \mathfrak{c} -lineability of other classes follows from the stronger lineability results as we will see in next two chapters. So, there is no reason for completing such argument here.

Chapter 3

On lineability of rows 1 and 4 in

Table 1.1

3.1 Introduction

This chapter is based on our paper [1] and its goal is to show that all Darboux-like subclasses of $(\text{PC} \setminus \text{D}) \cup (\text{AC} \setminus \text{Ext})$ in the algebra generated by \mathbb{D} are $2^{\mathfrak{c}}$ -lineable, that is, have maximal lineability. The following definition will constitute the main tool used in this chapter and next chapter.

Definition 3.1.1. For a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of functions with pairwise disjoint supports we define the *canonical linear space* $W_{\mathcal{F}}$ over \mathbb{R} , a subspace of $\mathbb{R}^{\mathbb{R}}$, as

$$W_{\mathcal{F}} := \bigcup_{n \in \mathbb{N}} \left\{ \sum_{i < n} a_i \varphi_i : a_i \in \mathbb{R} \ \& \ \varphi_i \in V_{\mathcal{F}} \text{ for every } i < n \right\},$$

where $V_{\mathcal{F}} = \left\{ \sum_{f \in \mathcal{F}} h(f) f : h \in \{0, 1\}^{\mathcal{F}} \right\}$. That is, $W_{\mathcal{F}}$ is spanned by $V_{\mathcal{F}}$.

For the remainder of this thesis, $W_{\mathcal{F}}$ refers to the canonical linear space over \mathbb{R} where \mathcal{F} is a family of functions from \mathbb{R} to \mathbb{R} that has pairwise disjoint supports.

Notice that each element in $V_{\mathcal{F}}$ is well defined, since maps in \mathcal{F} have pairwise disjoint supports. Also, if $|\mathcal{F}| = \mathfrak{c}$, then $|V_{\mathcal{F}}| = 2^{\mathfrak{c}}$. So, the following remark is obvious.

Remark 3.1.2. If $|\mathcal{F}| = \mathfrak{c}$, then $W_{\mathcal{F}}$ has dimension $2^{\mathfrak{c}}$.

For $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, notice that $W_{\mathcal{F}} \subset \mathcal{L}_{\mathcal{F}}$. In this chapter, we will repeatedly use the following simple fact that we will leave without a proof.

Remark 3.1.3. If $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is a family of functions with pairwise disjoint supports and $g \in W_{\mathcal{F}}$ is non-zero, then there is an $f \in \mathcal{F}$ and a non-zero $c \in \mathbb{R}$ such that $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$.

3.2 Results

3.2.1 Lineability of $\text{PC} \setminus (\text{D} \cup \text{PR})$

In this chapter, $\{B_{\xi}^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ is a fixed partition of \mathbb{R} into Bernstein sets. For a dense subset D of \mathbb{R} and $\xi < \mathfrak{c}$ define

$$(\alpha) \ \alpha_{\xi}^D := \sum_{d \in D} d \chi_{B_{\xi}^d}.$$

Clearly the supports of maps in the family $\mathcal{F}(\alpha^D) := \{\alpha_{\xi}^D : \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact that we will leave without a proof.

Fact 3.2.1. *If $D \subset \mathbb{R}$ is dense, $f \in \mathcal{F}(\alpha^D)$, $c \in \mathbb{R} \setminus \{0\}$, and $g \in \mathbb{R}^{\mathbb{R}}$ is such that $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$, then $g \upharpoonright P$ is dense in $P \times \mathbb{R}$ for every perfect $P \subset \mathbb{R}$. In particular, g has a dense graph and belongs to $\text{PC} \setminus \text{PR}$.*

Theorem 3.2.2. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{PC} \setminus (\text{D} \cup \text{PR})$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{PC} \setminus (\text{D} \cup \text{PR})$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\alpha^{\mathbb{Q}})$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3, there is an $f \in \mathcal{F}(\alpha^{\mathbb{Q}})$ and $c \in \mathbb{R} \setminus \{0\}$ with $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$. Thus, by Fact 3.2.1, g has a dense graph and belongs to $\text{PC} \setminus \text{PR}$.

Also, if $g = \sum_{i < \mathfrak{n}} a_i \varphi_i$, with $a_i \in \mathbb{R}$ and $\varphi_i \in V_{\mathcal{F}}$, then $g[\mathbb{R}]$ is contained in $a_1 \varphi_1[\mathbb{R}] + a_2 \varphi_2[\mathbb{R}] + \dots + a_n \varphi_n[\mathbb{R}] \subset a_1 \mathbb{Q} + a_2 \mathbb{Q} + \dots + a_n \mathbb{Q}$, a countable set. So $g[\mathbb{R}] \subsetneq \mathbb{R}$ which, together with the density of the graph of g , implies that $g \in \neg\text{D}$. Of course, by Remark 3.1.2, $W_{\mathcal{F}}$ has dimension $2^{\mathfrak{c}}$. □

3.2.2 Lineability of $\text{AC} \setminus \text{PR}$

A set $B \subset \mathbb{R}^2$ is a *blocking set* provided it is closed, meets the graph of every continuous function, and is disjoint with some (arbitrary) function $h \in \mathbb{R}^{\mathbb{R}}$. In this chapter, the family of all blocking sets will be denoted by \mathbb{B} . It is well known and easy to see that an $f \in \mathbb{R}^{\mathbb{R}}$ is in AC if, and only if, $f \cap K \neq \emptyset$ for every $K \in \mathbb{B}$. Recall that the x -axis projection of every blocking set contains a non-trivial interval, see e.g. [56]. (Compare also [28, lemma 5.1] and related history.)

As above, let $\{B_\xi^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ be a partition of \mathbb{R} into Bernstein sets.

Fact 3.2.3. *For any meager set $M \subset \mathbb{R}$ and $\xi < \mathfrak{c}$ there exists a map $\beta_\xi^M \in \mathbb{R}^{\mathbb{R}}$ with $\text{supp}(\beta_\xi^M) \subset \bigcup_{r \in \mathbb{R}} B_\xi^r \setminus M$ such that*

(β_1) $(\beta_\xi^M \upharpoonright (B_\xi^0 \setminus M)) \cap K \neq \emptyset$ for every $K \in \mathbb{B}$; and

(β_2) $\beta_\xi^M[\text{supp}(\beta_\xi^M) \cap P]$ is unbounded for every perfect P contained in $\mathbb{R} \setminus M$.

Proof. Take a function ϕ from an $E \subset B_\xi^0 \setminus M$ into \mathbb{R} such that $\phi \cap K \neq \emptyset$ for every $K \in \mathbb{B}$. Such a map can be constructed by an easy transfinite induction, see e.g. [56]. Let $\Phi_\xi \in \mathbb{R}^{\mathbb{R}}$ be an extension of ϕ such that $\Phi_\xi(x) = 0$ for every $x \in \mathbb{R} \setminus E$. Then $\beta_\xi^M := \Phi_\xi + \alpha_\xi^{\mathbb{Q}} \cdot \chi_{\mathbb{R} \setminus M}$, is as needed since Φ_ξ and $\alpha_\xi^{\mathbb{Q}}$ have disjoint supports, Φ_ξ ensures (β_1), and, by Fact 3.2.1, $\alpha_\xi^{\mathbb{Q}} \cdot \chi_{\mathbb{R} \setminus M}$ ensures (β_2). \square

Clearly the supports of maps in the family $\mathcal{F}(\beta^M) := \{\beta_\xi^M : \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact that we will leave without a proof.

Fact 3.2.4. *If $M \subset \mathbb{R}$ is meager, $f \in \mathcal{F}(\beta^M)$, $c \in \mathbb{R} \setminus \{0\}$, and $g \in \mathbb{R}^{\mathbb{R}}$ is such that $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$, then g has a dense graph, belongs to AC, and $g[P]$ is unbounded for every perfect P contained in $\mathbb{R} \setminus M$.*

Theorem 3.2.5. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{AC} \setminus \text{PR}$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{AC} \setminus \text{PR}$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\beta^{\emptyset})$ is as needed. This follows from Fact 3.2.4 and Remarks 3.1.2 and 3.1.3. Specifically, a non-zero $g \in W_{\mathcal{F}}$ is not in PR since, by Fact 3.2.4, $g[P]$ is unbounded for every perfect $P \subset \mathbb{R}$ and so, $g \upharpoonright P$ is discontinuous at every $x \in P$. \square

3.2.3 Lineability of $\text{PR} \setminus (\text{D} \cup \text{CIVP})$

For the rest of this chapter, $\{P^I \subset I : I \in \mathcal{B}\}$ is a family of pairwise disjoint nowhere dense perfect sets. For every $I \in \mathcal{B}$ let $\{P_\xi^I : \xi < \mathfrak{c}\}$ be an enumeration of some partition of P^I into perfect sets. For the use in the later part of this chapter it is convenient to put $\mathcal{P}^I := \{h_I[\{x\} \times 2^\omega] : x \in 2^\omega\}$, where h_I is a homeomorphism from $2^\omega \times 2^\omega$ onto P^I . Notice that the sets

$$M_\xi := \bigcup_{I \in \mathcal{B}} P_\xi^I,$$

are pairwise disjoint and that

$$M := \bigcup_{I \in \mathcal{B}} P^I = \bigcup_{\xi < \mathfrak{c}} M_\xi, \tag{3.1}$$

is meager. For every $\xi < \mathfrak{c}$ let

(γ) $\gamma_\xi := \sum_{I \in \mathcal{B}} \gamma_\xi^I$, where $\gamma_\xi^I: \mathbb{R} \rightarrow \mathbb{Q}$ has support contained in P_ξ^I and $(\gamma_\xi^I)^{-1}(q)$ contains non-empty perfect set for every $q \in \mathbb{Q}$.

Clearly the supports of maps in the family $\mathcal{F}(\gamma) := \{\gamma_\xi: \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact.

Fact 3.2.6. *If $f \in \mathcal{F}(\gamma)$ and $g \in \mathbb{R}^{\mathbb{R}}$ is such that $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$ for some $c \in \mathbb{R} \setminus \{0\}$, then g has a dense graph and belongs to PR.*

Proof. Clearly (γ) implies that $f \upharpoonright \text{supp}(f)$ is dense in \mathbb{R}^2 , so g has a dense graph. To see that $g \in \text{PR}$ choose an $x \in \mathbb{R}$ and a sequence $\langle q_n: n < \omega \rangle$ of non-zero rational numbers such that $c \cdot q_n \rightarrow_{n \rightarrow \infty} g(x)$.

Choose a sequence $\langle (a_n, b_n) \in \mathcal{B}: n < \omega \rangle$ such that $\lim_{n \rightarrow \infty} a_n = x$ and $a_0 < b_0 < a_2 < b_2 < \dots < x < \dots < a_3 < b_3 < a_1 < b_1$. By (γ), for every $n < \omega$ there exists a perfect set $P_n \subset (a_n, b_n)$ such that $f[P_{2n} \cup P_{2n+1}] = \{q_n\}$. Then $P := \{x\} \cup \bigcup_{n < \omega} P_n$ is a perfect set having x as a bilateral limit point and $g \upharpoonright P$ is continuous at x . \square

Theorem 3.2.7. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{PR} \setminus (\text{D} \cup \text{CIVP})$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{PR} \setminus (\text{D} \cup \text{CIVP})$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\gamma)$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3 and Fact 3.2.6, g has a dense graph and belongs to PR. Also, similarly as in the proof of Theorem 3.2.2 we see that $g[\mathbb{R}]$ is countable. This and the density of its graph imply that $g \in \neg \text{D}$.

Finally, to see that $g \in \neg \text{CIVP}$, using density of the graph of g , choose $p < q$ so that $g(p) < g(q)$. Since $g[\mathbb{R}]$ is countable, there is perfect $K \subset (g(p), g(q)) \setminus g[\mathbb{R}]$. Then there is no nonempty $P \subset (p, q)$ with $g[P] \subset K$, that is, indeed $g \in \neg \text{CIVP}$. \square

3.2.4 Lineability of $\text{AC} \cap \text{PR} \setminus \text{CIVP}$

Using the notation as formerly mentioned, for every $\xi < \mathfrak{c}$ define

$$(\delta) \quad \delta_\xi := \gamma_\xi + \beta_\xi^M.$$

Notice that the supports of γ_ξ and β_ξ^M are disjoint, the first contained in M , the second in $\mathbb{R} \setminus M$. It is also easy to see that the supports of maps in the family $\mathcal{F}(\delta) := \{\delta_\xi: \xi < \mathfrak{c}\}$ are pairwise disjoint.

Theorem 3.2.8. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{AC} \cap \text{PR} \setminus \text{CIVP}$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{AC} \cap \text{PR} \setminus \text{CIVP}$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\delta)$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3, there exist an $f = \delta_{\xi} \in \mathcal{F}(\delta)$ and a number $c \in \mathbb{R} \setminus \{0\}$ with $g \upharpoonright \text{supp}(\delta_{\xi}) = c \delta_{\xi} \upharpoonright \text{supp}(\delta_{\xi})$. Since $\text{supp}(\gamma_{\xi}) \subset \text{supp}(\delta_{\xi})$, this implies that $g \upharpoonright \text{supp}(\gamma_{\xi}) = c \delta_{\xi} \upharpoonright \text{supp}(\gamma_{\xi}) = c \gamma_{\xi} \upharpoonright \text{supp}(\gamma_{\xi})$ so by Fact 3.2.6, g has a dense graph and belongs to PR. Similarly $\text{supp}(\beta_{\xi}^M) \subset \text{supp}(\delta_{\xi})$, which implies that $g \upharpoonright \text{supp}(\beta_{\xi}^M) = c \delta_{\xi} \upharpoonright \text{supp}(\beta_{\xi}^M) = c \beta_{\xi}^M \upharpoonright \text{supp}(\beta_{\xi}^M)$ so, by Fact 3.2.4, $g \in \text{AC}$.

Finally, to see that $g \in \neg \text{CIVP}$, notice that $g[M] = \hat{g}[\mathbb{R}]$ for some $\hat{g} \in W_{\mathcal{F}(\gamma)}$ so, as in the proof of Theorem 3.2.7, $g[M] = \hat{g}[\mathbb{R}]$ is countable. Since g has a dense graph, we can choose $p < q$ so that $g(p) < g(q)$. Also, choose a nonempty bounded perfect $K \subset \mathbb{R} \setminus g[M]$. Take a nonempty perfect $P \subset (p, q)$. It is enough to prove that $g[P] \not\subset K$. So, by way of contradiction, assume that there is a perfect $P \subset \mathbb{R}$ with $g[P] \subset K$. Then, reducing P if necessary, we can assume that P is either contained in or disjoint with M .

But $P \subset \mathbb{R} \setminus M$ is impossible, since in such case Fact 3.2.3 implies that the set $g[P] \supset g[\text{supp}(\beta_{\xi}^M) \cap P] = c \beta_{\xi}^M[\text{supp}(\beta_{\xi}^M) \cap P]$ is unbounded, so it cannot be contained in bounded K .

Similarly, $P \subset M$ implies that $g[P] \subset g[M]$, which is disjoint with K , contradicting $g[P] \subset K$. Thus $g \in \neg \text{CIVP}$ as needed. \square

3.2.5 Lineability of $\text{CIVP} \setminus (\text{D} \cup \text{SCIVP})$

Here the families \mathcal{P}^I , used earlier to construct functions γ_{ξ} , will need to be chosen more carefully with the help of the following lemma.

Lemma 3.2.9. *For every $I \in \mathcal{B}$ there is a subfamily \mathcal{P}_0^I of \mathcal{P}^I with $|\mathcal{P}_0^I| = \mathfrak{c}$ such that if $\mathcal{P}_0 := \bigcup_{I \in \mathcal{B}} \mathcal{P}_0^I$, then for every perfect $P \subset \mathbb{R}$,*

- *if $|P \cap Q| \leq \omega$ for every $Q \in \mathcal{P}_0$, then $|P \setminus \bigcup \mathcal{P}_0| = \mathfrak{c}$.*

Proof. Let B be a Bernstein subset of 2^{ω} , that is, such that $B \cap Q \neq \emptyset \neq Q \setminus B$ for every perfect $Q \subset 2^{\omega}$. Clearly $|B| = \mathfrak{c}$. For every $I \in \mathcal{B}$ let h_I be a homeomorphism from $2^{\omega} \times 2^{\omega}$ onto P^I and let $\mathcal{P}_0^I := \{h_I[\{b\} \times 2^{\omega}] : b \in B\}$.

To see that this choice ensures \bullet , choose a perfect $P \subset \mathbb{R}$ so that $|P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$. We need to find a $b \in B$ and an $I \in \mathcal{B}$ so that $|P \cap h_I[\{b\} \times 2^{\omega}]| > \omega$.

Since $|P \setminus \bigcup_{I \in \mathcal{B}} P^I| \leq |P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$ there is an $I \in \mathcal{B}$ and a perfect $Q \subset P \cap P^I$. Let $\pi_1: 2^\omega \times 2^\omega \rightarrow 2^\omega$ be the projection onto the first coordinate. If the compact set $Q_0 := \pi_1[h_I^{-1}(Q)]$ is uncountable, then the set $Q_0 \setminus B$ has cardinality \mathfrak{c} and so has the set $h_I[\pi_1^{-1}(Q_0 \setminus B)] \subset P \setminus \bigcup \mathcal{P}_0$, contradicting our assumption that $|P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$. So, we can assume that Q_0 is countable. Then, for some $b \in Q_0$, the set $h_I[\pi_1^{-1}(b)] = h_I[\{b\} \times 2^\omega] \subset Q \subset P$ has cardinality \mathfrak{c} . By the assumption $|P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$ we must have $b \in B$, since otherwise $h_I[\pi_1^{-1}(b)] \subset P \setminus \bigcup \mathcal{P}_0$. So, $h_I[\pi_1^{-1}(b)] \subset P \cap h_I[\{b\} \times 2^\omega]$ is uncountable, as needed. \square

To ensure CIVP the range of our modified functions γ_ξ needs to intersect every perfect set while no generated function can be surjective. This will be achieved with the following lemma.

Lemma 3.2.10. *There exists a linear space $V \subset \mathbb{R}$ over \mathbb{Q} which intersects every non-empty perfect set $P \subset \mathbb{R}$ and such that*

- $a_1V + \dots + a_nV \neq \mathbb{R}$ for every $a_1, \dots, a_n \in \mathbb{R}$.

Proof. Let T be a transcendental basis that is also a Bernstein set¹ Choose countable infinite subset T_0 of T and let V be a vector space over \mathbb{Q} generated by $T \setminus T_0$. Notice that it is as needed.

To see •, let $F = \mathbb{Q}(T \setminus T_0)$ be a subfield of \mathbb{R} generated by $T \setminus T_0$. In particular T_0 is linearly independent over F , implying that the dimension of \mathbb{R} over F is infinite. Therefore, if $a_1, a_2, \dots, a_n \in \mathbb{R}$, then $a_1V + \dots + a_nV \subset a_1F + \dots + a_nF \subsetneq \mathbb{R}$. Clearly $V \supset T \setminus T_0$ intersects every non-empty perfect set $P \subset \mathbb{R}$. \square

Let \mathcal{P} be the family of all perfect subsets of \mathbb{R} and $\{P_\xi^{I,C} \subset P^I: \xi < \mathfrak{c} \ \& \ C \in \mathcal{P}\}$ be an enumeration of \mathcal{P}_0^I . For every $\xi < \mathfrak{c}$ let

$$(\kappa) \quad \kappa_\xi := \sum_{(I,C) \in \mathcal{B} \times \mathcal{P}} \kappa_\xi^{I,C}, \text{ where } \kappa_\xi^{I,C}: \mathbb{R} \rightarrow V \text{ has support contained in } P_\xi^{I,C}, \kappa_\xi^{I,C}[P_\xi^{I,C}] \subset C \cap V, \text{ and } \kappa_\xi^{I,C} \text{ is discontinuous on any perfect subset of } P_\xi^{I,C}.$$

Clearly the supports of the maps in the family $\mathcal{F}(\kappa) := \{\kappa_\xi: \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact.

Fact 3.2.11. *If $f \in \mathcal{F}(\kappa)$ and $g \in \mathbb{R}^\mathbb{R}$ is such that $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$ for some $c \in \mathbb{R} \setminus \{0\}$, then g has a dense graph and belongs to CIVP.*

Proof. This easily follows from our definition (κ) . \square

¹If $\{P_\xi: \xi < \mathfrak{c}\}$ is a list of all perfect subsets of \mathbb{R} and $t_\xi \in P_\xi \setminus \bar{\mathbb{Q}}(\{t_\zeta: \zeta < \xi\})$ for every $\xi < \mathfrak{c}$, then $T := \{t_\xi: \xi < \mathfrak{c}\}$ is as needed: it cannot contain any perfect $P \subset \mathbb{R}$, since then, for any $p \in P$, T would be disjoint from perfect $p + P$, see e.g. [21]

² $\kappa_\xi^{I,C} \upharpoonright P_\xi^{I,C}$ is just a Sierpiński-Zygmund function from $P_\xi^{I,C}$ into $C \cap V$, that is, a maps whose restriction to any set of cardinality \mathfrak{c} is discontinuous, see e.g. [32].

Theorem 3.2.12. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{CIVP} \setminus (\text{D} \cup \text{SCIVP})$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{CIVP} \setminus (\text{D} \cup \text{SCIVP})$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\kappa)$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3 and Fact 3.2.11, g has a dense graph and belongs to CIVP.

Also, if $g = \sum_{i < n} a_i \varphi_i$, with $a_i \in \mathbb{R}$ and $\varphi_i \in V_{\mathcal{F}}$, then $g[\mathbb{R}]$ is contained in $a_1 \varphi_1[\mathbb{R}] + a_2 \varphi_2[\mathbb{R}] + \dots + a_n \varphi_n[\mathbb{R}] \subset a_1 V + a_2 V + \dots + a_n V$ which, by Lemma 3.2.10, is strictly contained in \mathbb{R} . So $g[\mathbb{R}] \subsetneq \mathbb{R}$ which, together with the density of the graph of g , implies that $g \in \neg \text{D}$.

Finally, to see that $g \in \neg \text{SCIVP}$, using density of the graph of g , choose $p < q$ with $g(p) < g(q)$ and a nonempty perfect $K \subset (g(p), g(q)) \setminus \{0\}$. It is enough to show that for every perfect set $P \subset (p, q)$ with $g[P] \subset K$ the restriction $g \upharpoonright P$ is discontinuous. Indeed, $g[P] \subset K \not\ni 0$ implies that $P \subset \bigcup_{\xi < \mathfrak{c}} \text{supp}(\kappa_{\xi}) \subset \bigcup \mathcal{P}_0$. So, by Lemma 3.2.9, there is a $P_{\xi}^{I,C} \in \bigcup \mathcal{P}_0$ with $|P \cap P_{\xi}^{I,C}| > \omega$. In particular, there exists a perfect set $Q \subset P \cap P_{\xi}^{I,C}$. Notice that $g \upharpoonright \text{supp}(\kappa_{\xi}) = c \kappa_{\xi} \upharpoonright \text{supp}(\kappa_{\xi})$ and $c \neq 0$, since otherwise $g[Q] = \{0\} \notin K$. Since $\kappa_{\xi} \upharpoonright Q = \kappa_{\xi}^{I,C} \upharpoonright Q$ is discontinuous, as ensured in (κ) , $g \upharpoonright Q$ is discontinuous. \square

3.2.6 Lineability of $\text{AC} \cap \text{CIVP} \setminus \text{SCIVP}$

Using the notation as previously discussed, for every $\xi < \mathfrak{c}$ define

$$(\lambda) \quad \lambda_{\xi} := \kappa_{\xi} + \beta_{\xi}^M.$$

Notice that the supports of κ_{ξ} and β_{ξ}^M are disjoint, the first contained in M , the second in $\mathbb{R} \setminus M$. It is also easy to see that the supports of the maps in the family $\mathcal{F}(\lambda) := \{\lambda_{\xi} : \xi < \mathfrak{c}\}$ are pairwise disjoint.

Theorem 3.2.13. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{AC} \cap \text{CIVP} \setminus \text{SCIVP}$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{AC} \cap \text{CIVP} \setminus \text{SCIVP}$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\lambda)$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3, there is an $f = \lambda_{\xi} \in \mathcal{F}(\lambda)$ and $c \in \mathbb{R} \setminus \{0\}$ with $g \upharpoonright \text{supp}(\lambda_{\xi}) = c \lambda_{\xi} \upharpoonright \text{supp}(\lambda_{\xi})$. Since $\text{supp}(\kappa_{\xi}) \subset \text{supp}(\lambda_{\xi})$, this implies that $g \upharpoonright \text{supp}(\kappa_{\xi}) = c \lambda_{\xi} \upharpoonright \text{supp}(\kappa_{\xi}) = c \kappa_{\xi} \upharpoonright \text{supp}(\kappa_{\xi})$ so by Fact 3.2.11, g has a dense graph and belongs to CIVP. Similarly $\text{supp}(\beta_{\xi}^M) \subset \text{supp}(\lambda_{\xi})$, which implies that $g \upharpoonright \text{supp}(\beta_{\xi}^M) = c \lambda_{\xi} \upharpoonright \text{supp}(\beta_{\xi}^M) = c \beta_{\xi}^M \upharpoonright \text{supp}(\beta_{\xi}^M)$ so, by Fact 3.2.4, $g \in \text{AC}$.

Lastly, to see $g \in \neg\text{SCIVP}$, using density of the graph of g , choose $p < q$ with $g(p) < g(q)$ and a nonempty perfect $K \subset (g(p), g(q)) \setminus \{0\}$. It is enough to show that for every perfect set $P \subset (p, q)$ with $g[P] \subset K$ the restriction $g \upharpoonright P$ is discontinuous. As in the proof of Theorem 3.2.8, we can assume $P \subset M$. So, $P \subset \bigcup_{\xi < \mathfrak{c}} \text{supp}(\kappa_\xi) \subset \bigcup \mathcal{P}_0$ as $g[P] \subset K \not\equiv 0$. A similar argument as in Theorem 3.2.12 shows $g \upharpoonright P$ is discontinuous, as needed. \square

3.2.7 Lineability of $\text{SCIVP} \setminus \text{D}$

To ensure SCIVP the definition of $\kappa_\xi^{I,C}$ needs to be slightly changed, whereas no generated function can be surjective. For every $\xi < \mathfrak{c}$, let

$$(\mu) \quad \mu_\xi := \sum_{\langle I, C \rangle \in \mathcal{B} \times \mathcal{P}} \mu_\xi^{I,C}, \text{ where } \mu_\xi^{I,C}: \mathbb{R} \rightarrow C \cap V \text{ is defined as } \mu_\xi^{I,C} = a \chi_{P_\xi^{I,C}} \text{ for some } a \in C \cap V.$$

Notice that the support of $\mu_\xi^{I,C}$ is contained in $P_\xi^{I,C}$. So, the supports of maps in the family $\mathcal{F}(\mu) := \{\mu_\xi: \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact.

Fact 3.2.14. *If $f \in \mathcal{F}(\mu)$ and $g \in \mathbb{R}^\mathbb{R}$ is such that $g \upharpoonright \text{supp}(f) = c f \upharpoonright \text{supp}(f)$ for some $c \in \mathbb{R} \setminus \{0\}$, then g has a dense graph and belongs to SCIVP.*

Proof. It is straightforward from our definition (μ) . \square

Theorem 3.2.15. *There exists a family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{SCIVP} \setminus \text{D}$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{SCIVP} \setminus \text{D}$ is $2^\mathfrak{c}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\mu)$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3 and Fact 3.2.14, g has a dense graph and belongs to SCIVP. For $g \in \neg\text{D}$, the proof is an identical to that presented in Theorem 3.2.12. \square

3.2.8 Lineability of $\text{AC} \cap \text{SCIVP} \setminus \text{Ext}$

The hardest aspect of this argument will be ensuring that the functions in $W_{\mathcal{F}}$ are not extendable. For this, we recall the following useful result that was proved in [31].

Theorem 3.2.16. *If $f: \mathbb{R} \rightarrow \mathbb{R}$ is an extendable function with a dense graph, then for every $a, b \in \mathbb{R}$, $a < b$, and for each perfect set K between $f(a)$ and $f(b)$ there is a perfect set C between a and b such that $f[C] \subset K$ and the restriction $f \upharpoonright C$ is continuous strictly increasing.*

By using the notation as earlier stated, for every $\xi < \mathfrak{c}$

$$(\nu) \nu_\xi := \mu_\xi + \beta_\xi^M.$$

Notice that the supports of μ_ξ and β_ξ^M are disjoint, the first contained in M , the second in $\mathbb{R} \setminus M$. It is also easy to see that the supports of maps in the family $\mathcal{F}(\nu) := \{\nu_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint.

Theorem 3.2.17. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many functions with nonempty pairwise disjoint supports such that $g \in \text{AC} \cap \text{SCIVP} \setminus \text{Ext}$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\text{AC} \cap \text{SCIVP} \setminus \text{Ext}$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\nu)$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 3.1.3, there is an $f = \nu_\xi \in \mathcal{F}(\nu)$ and $c \in \mathbb{R} \setminus \{0\}$ with $g \upharpoonright \text{supp}(\nu_\xi) = c \nu_\xi \upharpoonright \text{supp}(\nu_\xi)$. Since $\text{supp}(\mu_\xi) \subset \text{supp}(\nu_\xi)$, this implies that $g \upharpoonright \text{supp}(\mu_\xi) = c \nu_\xi \upharpoonright \text{supp}(\mu_\xi) = c \mu_\xi \upharpoonright \text{supp}(\mu_\xi)$ so by Fact 3.2.14, g has a dense graph and belongs to SCIVP. Similarly $\text{supp}(\beta_\xi^M) \subset \text{supp}(\nu_\xi)$, which implies that $g \upharpoonright \text{supp}(\beta_\xi^M) = c \nu_\xi \upharpoonright \text{supp}(\beta_\xi^M) = c \beta_\xi^M \upharpoonright \text{supp}(\beta_\xi^M)$ so, by Fact 3.2.4, $g \in \text{AC}$.

Finally, to see $g \in \neg \text{Ext}$, using density of the graph of g , choose $p < q$ with $g(p) < g(q)$ and a nonempty perfect $K \subset (g(p), g(q)) \setminus \{0\}$. By Theorem 3.2.16, it is enough to show that for no perfect set $P \subset (p, q)$ with $g[P] \subset K$ the restriction $g \upharpoonright P$ is strictly increasing. As in the proof of Theorem 3.2.8, we can assume $P \subset M$. Since $g[P] \subset K \not\equiv 0$, which implies $P \subset \bigcup_{\xi < \mathfrak{c}} \text{supp}(\mu_\xi) \subset \bigcup \mathcal{P}_0$. So, by Lemma 3.2.9, there is a $P_\xi^{I,C} \in \bigcup \mathcal{P}_0$ with $|P \cap P_\xi^{I,C}| > \omega$. Notice that $P \cap P_\xi^{I,C} \subset P_\xi^{I,C} \subset \text{supp}(\mu_\xi) \subset \text{supp}(\nu_\xi)$. So, $\mu_\xi \upharpoonright P$ is not strictly increasing and the same is true for $\nu_\xi \upharpoonright P$ and $g \upharpoonright P$. Thus, $g \in \neg \text{Ext}$, as needed. \square

Chapter 4

On lineability of row 2 in Table 1.1

4.1 Introduction

The content of this chapter comes from the three papers [2, 4, 5]. Our aim is to show 2^c -lineability of all non-empty classes of functions in the algebra $\mathcal{A}(\mathbb{D})$ of Darboux-like maps that are contained in the family $\mathbb{D} \setminus \text{Conn}$. More precisely, the four classes in $\mathcal{A}(\mathbb{D})$ we are interested in can be written as

$$\begin{aligned} \mathbb{D} \cap \neg \text{Conn} \cap \neg \text{PR} & \quad \mathbb{D} \cap \text{SCIVP} \cap \neg \text{Conn} \\ \mathbb{D} \cap \text{PR} \cap \neg \text{Conn} \cap \neg \text{CIVP} & \quad \mathbb{D} \cap \text{CIVP} \cap \neg \text{Conn} \cap \neg \text{SCIVP}. \end{aligned} \tag{4.1}$$

The arguments for these classes stand apart from the proofs of 2^c -lineability of other classes in the algebra $\mathcal{A}(\mathbb{D})$: the proofs we present in this chapter are considerably more delicate and heavily rely on the existence of algebraically independent subsets of \mathbb{R} having different structures. The presented results generalize recently published proof of 2^c -lineability of the class $\mathbb{D} \setminus \text{Conn}$, see [21, theorem 2.1].

For $G \subset \mathbb{R}$ and a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ we define

$$\text{supp}(\mathcal{F}) := \bigcup_{f \in \mathcal{F}} \text{supp}(f) \quad \text{and} \quad \mathcal{F} \upharpoonright G := \{f \cdot \chi_G : f \in \mathcal{F}\}.$$

For an $A \subset \mathbb{R}$ the symbol $\mathbb{Q}(A)$ denotes the subfield of \mathbb{R} generated by A , that is, $\mathbb{Q}(A)$ is the intersection of all subfields of \mathbb{R} that contain A . By $\bar{\mathbb{Q}}(A)$ we denote the algebraic closure of $\mathbb{Q}(A)$ in \mathbb{R} , that is, $\bar{\mathbb{Q}}(A)$ is the set of $x \in \mathbb{R}$ that are algebraic over $\mathbb{Q}(A)$. We say that an $S \subset \mathbb{R}$ is *algebraically independent* when it is algebraically independent over \mathbb{Q} ; it is a *transcendental basis* provided it is a maximal algebraically independent subset of \mathbb{R} . For every algebraically independent

set $S \subset \mathbb{R}$ there exists a transcendental basis T with $S \subset T$, see e.g. [46]. If T is a transcendental basis, then every $x \in \mathbb{R}$ is algebraic over $\mathbb{Q}(T)$, that is, $\bar{\mathbb{Q}}(T) = \mathbb{R}$. For an $A \subset \mathbb{R}$, a *transcendence degree* of \mathbb{R} over $\bar{\mathbb{Q}}(A)$ is the cardinality of any transcendental basis of \mathbb{R} over $\bar{\mathbb{Q}}(A)$. Notice that if $S \subset \mathbb{R}$ is algebraically independent such that $\bar{\mathbb{Q}}(S) = \bar{\mathbb{Q}}(A)$ and $T \supset S$ is a transcendental basis, then the transcendence degree of \mathbb{R} over $\bar{\mathbb{Q}}(A)$ equals the cardinality of the set $T \setminus S$.

In what follows we will repeatedly use the following simple fact.

Remark 4.1.1. Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be a family of functions having pairwise disjoint supports. If $g \in W_{\mathcal{F}}$, then there is a finite set $A_g \subset \mathbb{R}$ such that

- for every $f \in \mathcal{F}$ there exists an $a_f \in \mathbb{Q}(A_g)$ so that $g = a_f \cdot f$ on $\text{supp}(f)$.

Proof. Let $g = \sum_{i < n} a_i \varphi_i \in W_{\mathcal{F}}$ with $\varphi_i = \sum_{f \in \mathcal{F}} h^i(f) f \in V(\mathcal{F})$ for every $i < n$. Then, $A_g := \{a_i : i < n\}$ is as needed. Indeed, for every $f \in \mathcal{F}$ and $x \in \text{supp}(f)$ we have

$$g(x) = \sum_{i < n} a_i \varphi_i(x) = \sum_{i < n} a_i h^i(f) f(x) = a_f f(x),$$

where $a_f := \sum_{i < n} a_i h^i(f) \in \mathbb{Q}(A_g)$. □

All non-zero functions in $\mathbb{R}^{\mathbb{R}}$ we will consider in this chapter will have dense graphs. In particular, the following simple remark will be useful for us.

Remark 4.1.2. If $f \in \mathbb{R}^{\mathbb{R}}$ has a dense graph in \mathbb{R}^2 , then $f \in \mathbb{D}$ if, and only if, $f \in \text{ES}$.

4.2 Families \mathcal{H} with $W_{\mathcal{H}}$

4.2.1 Families \mathcal{H} for which $W_{\mathcal{H}}$ ensures lineability of classes in \mathbb{D}

Proposition 4.2.1. Let $\mathcal{F}, \mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$ be families of functions with pairwise disjoint supports such that all maps in \mathcal{F} have graphs dense in \mathbb{R}^2 . Assume that $B \subset \mathbb{R}$ is such that $\text{supp}(\mathcal{F}) \subset B$ and that $\mathcal{F} \upharpoonright B = \mathcal{H} \upharpoonright B$.

- (i) The graph of every non-zero $g \in W_{\mathcal{H}}$ is dense in \mathbb{R}^2 ;
- (ii) If $\mathcal{F} \subset \text{CIVP}$, then $W_{\mathcal{H}} \subset \text{CIVP}$;
- (iii) If $\mathcal{F} \subset \text{SCIVP}$, then $W_{\mathcal{H}} \subset \text{SCIVP}$;
- (iv) If $\mathcal{F} \subset \mathbb{D}$ and $B \setminus \text{supp}(\mathcal{F})$ is dense in \mathbb{R} ; then $W_{\mathcal{H}} \subset \text{ES} \cup \{0\}$;

(v) If $\mathcal{F} \subset \text{PES}$ and $B \setminus \text{supp}(\mathcal{F})$ intersects every perfect subset of \mathbb{R} , then $W_{\mathcal{H}} \subset \text{PES} \cup \{0\}$.

Proof. Choose a non-zero $g \in W_{\mathcal{H}}$. By Remark 4.1.1, there exists an $h \in \mathcal{H}$ and non-zero $c \in \mathbb{R}$ such that $g = c \cdot h$ on $\text{supp}(h)$. By our assumption, there exists an $f \in \mathcal{F}$ such that $f \upharpoonright B = h \upharpoonright B$. In particular, $g = c \cdot f$ on $\text{supp}(f)$.

To see (i) notice that $f \upharpoonright \text{supp}(f)$, as well as its multiplication by c , has graph dense in \mathbb{R}^2 .

To see (ii) notice that the assumption that $f \in \text{CIVP}$ implies that $c \cdot f \in \text{CIVP}$ and in the definition of the class CIVP we can restrict our attention to perfect sets $K \subset \mathbb{R} \setminus \{0\}$ in which case the condition $c \cdot f[P] \subset K$ is achieved only for $P \subset \text{supp}(f)$, so that $g[P] = c \cdot f[P] \subset K$, as needed. The argument for (iii) is essentially the same as for (ii).

To see (iv) first notice that, by Remark 4.1.2, $\mathcal{F} \subset \text{ES}$. In particular, for every $r \in \mathbb{R} \setminus \{0\}$ we have $g^{-1}(r) \supset (c \cdot f)^{-1}(r)$ and this last set is dense, since $c \cdot f \in \text{ES}$. Finally, $g^{-1}(0)$ is dense, since it contains $B \setminus \text{supp}(\mathcal{F})$. The argument for (v) is identical the same as for (iv). \square

Notice, in particular, that all non-zero functions in the considered spaces $W_{\mathcal{F}}$ (which will have dense graphs) will be in ES.

4.2.2 Families \mathcal{H} for which $W_{\mathcal{H}}$ ensures lineability of $\neg \text{Conn}$

Let $\text{id}_{\star} \in \mathbb{R}^{\mathbb{R}}$ be defined as $\text{id}_{\star}(x) = x$ for $x \neq 0$ and $\text{id}_{\star}(0) = 1$. Notice that this ensures that the function $1/\text{id}_{\star}$ is well defined at all points, including $x = 0$. The following lemma is considered one of the essential tools to prove the main results in the this chapter.

Lemma 4.2.2. *Let $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$ be a family of functions with pairwise disjoint supports and graphs dense in \mathbb{R}^2 . If $(g/\text{id}_{\star})[\mathbb{R}] \neq \mathbb{R}$ for every $g \in W_{\mathcal{H}}$, then $W_{\mathcal{H}} \subset \neg \text{Conn} \cup \{0\}$.*

Proof. Let $g \in W_{\mathcal{H}} \setminus \{0\}$. Then, by Proposition 4.2.1(i), g has a dense graph. To see that $g \in \neg \text{Conn}$ choose an $a \in \mathbb{R} \setminus (g/\text{id}_{\star})[\mathbb{R}]$. This means that $(g/\text{id}_{\star})(x) \neq a$ for every $x \in \mathbb{R}$. In particular, $g(x) \neq a \text{id}_{\star}(x) = ax$ for every $x \neq 0$. Since g has a dense graph, there exist $q > p > 0$ such that $g(p) > ap$ and $g(q) < aq$. But this implies that the three-segment set $(\{p\} \times (-\infty, ap]) \cup \{(x, ax) : x \in [p, q]\} \cup (\{q\} \times [aq, \infty))$ separates the graph of g . So, indeed $g \in \neg \text{Conn}$. \square

In Lemma 4.2.2 we assume that no function in $(1/\text{id}_{\star}) \cdot W_{\mathcal{H}}$ is surjective. But how to ensure this together with the needed property that $W_{\mathcal{H}} \subset \text{ES} \cup \{0\}$? To see this first notice that $(1/\text{id}_{\star}) \cdot W_{\mathcal{H}} = W_{\mathcal{G}}$ where $\mathcal{G} = (1/\text{id}_{\star}) \cdot \mathcal{H}$. To ensure that no function in $W_{\mathcal{G}}$ is surjective for such \mathcal{G} , we will use the following simple lemma.

Lemma 4.2.3. *Let $S_0 \subset \mathbb{R}$ be such that \mathbb{R} has infinite transcendence degree over $\bar{\mathbb{Q}}(S_0)$. If $\mathcal{G} \subset \mathbb{Q}(S_0)^{\mathbb{R}}$ is a family of functions with pairwise disjoint supports, then no function in $W_{\mathcal{G}}$ is surjective.*

Proof. Let S be a transcendental base of $\bar{\mathbb{Q}}(S_0)$ (over \mathbb{Q}) and T be a transcendental base with $S \subset T$. Fix a $g \in W_{\mathcal{G}}$ and choose a finite $S_g \subset T$ such that $A_g \subset \mathbb{Q}(S_g)$, where A_g as in the Remark 4.1.1. Then, the range of g is contained in $\bar{\mathbb{Q}}(S_g \cup S)$, while $\bar{\mathbb{Q}}(S_g \cup S) \neq \bar{\mathbb{Q}}(T) = \mathbb{R}$ by our assumption that \mathbb{R} has infinite transcendence degree over $\bar{\mathbb{Q}}(S_0) = \bar{\mathbb{Q}}(S)$. So, indeed g is not surjective. \square

4.2.3 Families \mathcal{H} with $W_{\mathcal{H}}$ ensuring lineability of $\text{ES} \cap \neg \text{Conn}$

$2^{\mathfrak{c}}$ -lineability of ES is established via well known and perhaps the easiest construction presented in this thesis. Specifically, for a family $\Delta := \{D_{\xi}^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ of pairwise disjoint dense subsets of \mathbb{R} define the functions

$$(\varphi) \ \varphi_{\xi} := \sum_{r \in \mathbb{R}} r \cdot \chi_{D_{\xi}^r}$$

and let $\mathcal{F}(\varphi) := \{\varphi_{\xi} : \xi < \mathfrak{c}\}$. Clearly functions in $\mathcal{F}(\varphi)$ are ES and have pairwise disjoint supports. So, $W_{\mathcal{F}(\varphi)}$ is well defined. To ensure that $W_{\mathcal{F}(\varphi)}$ is contained in $\neg \text{Conn} \cup \{0\}$ we will consider the sets D_{ξ}^r of the form $\text{id}_{\star}(r) \cdot S_{\xi}^r$ for some dense sets $S_{\xi}^r \subset \mathbb{R}$.

Proposition 4.2.4. *Let $\mathcal{S} = \{S_{\xi}^r \subset \mathbb{R} : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ be a family of dense sets and put $S := \bigcup \mathcal{S}$. For $r \in \mathbb{R}$ and $\xi < \mathfrak{c}$ let $D_{\xi}^r := \text{id}_{\star}(r) \cdot S_{\xi}^r$. Then every function φ_{ξ} is ES and $\varphi_{\xi}/\text{id}_{\star}$ has range contained in $\mathbb{Q}(S)$.*

In particular, if \mathbb{R} has infinite transcendence degree over $\bar{\mathbb{Q}}(S)$, the sets in

$$\Delta := \{\text{id}_{\star}(r) \cdot S_{\xi}^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}, \tag{4.2}$$

are pairwise disjoint, and $\mathcal{F}(\varphi) := \{\varphi_{\xi} : \xi < \mathfrak{c}\}$, then the $W_{\mathcal{F}(\varphi)}$ is well defined and $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$ -lineability of $\text{ES} \setminus \text{Conn}$.

Proof. Clearly $\varphi_{\xi} \in \text{ES}$, since each set D_{ξ}^r is dense. To see $(\varphi_{\xi}/\text{id}_{\star})[\mathbb{R}] \subset \mathbb{Q}(S)$ notice that

$$\begin{aligned} (\varphi_{\xi}/\text{id}_{\star})[\mathbb{R}] &= \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \frac{r \cdot \chi_{D_{\xi}^r}}{\text{id}_{\star}}[\mathbb{R}] = \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \left\{ \frac{r}{\text{id}_{\star}(x)} : x \in D_{\xi}^r \right\} \\ &= \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \left\{ \frac{r}{\text{id}_{\star}(\text{id}_{\star}(r)x)} : x \in S_{\xi}^r \right\} = \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \left\{ \frac{1}{x} : x \in S_{\xi}^r \right\}. \end{aligned}$$

So, the first part is proved. To see the second part, notice that the family $W_{\mathcal{F}(\varphi)}$ is well defined and that, by Remark 3.1.2, it has dimension $2^{\mathfrak{c}}$. The fact that $W_{\mathcal{F}(\varphi)} \subset \text{ES} \cup \{0\}$ is justified by Proposition 4.2.1(iv), used with $B = \mathbb{R}$ and $\mathcal{H} = \mathcal{F}(\varphi)$, while $W_{\mathcal{F}(\varphi)} \subset \neg\text{Conn} \cup \{0\}$ from Lemmas 4.2.3 and 4.2.2. \square

To successfully use Proposition 4.2.4 to show $2^{\mathfrak{c}}$ -lineability of $\text{ES} \setminus \text{Conn}$ we still need to find the families \mathcal{S} satisfying its assumptions. This is a relatively easy task, if we ignore the requirement that the sets in the family Δ from (4.2) need to be pairwise disjoint. In fact, such a family with considerably stronger properties (including \mathfrak{c} -density of each set S_ξ^r) is constructed in Lemma 4.4.1. The refining of such family to one ensuring also pairwise disjointness of the sets in Δ can then be found using the following result.

Lemma 4.2.5. *Let \mathcal{S} be a family of pairwise disjoint sets such that $\bigcup \mathcal{S}$ is algebraically independent and for every $S \in \mathcal{S}$ let $r_S \in \mathbb{R} \setminus \{0\}$. Then for every $S \in \mathcal{S}$ there exists a set $N_S \subset \bigcup \mathcal{S}$ of cardinality less than \mathfrak{c} such that the sets in $\Delta := \{r_S \cdot (S \setminus N_S) : S \in \mathcal{S}\}$ are pairwise disjoint. Moreover, if $r_S \in \mathbb{Q}$, then $N_S = \emptyset$.*

Proof. The proof of this lemma is implicitly included in the proof of [21, theorem 2.1]. (See (a) and (b) in that proof.) Specifically, if $T = \{t_\xi : \xi < \mathfrak{c}\}$ is a transcendental basis extending $\bigcup \mathcal{S}$ and $\eta_S < \mathfrak{c}$ is the smallest such that $r_S \in \bar{\mathbb{Q}}(\{t_\xi : \xi < \eta_S\})$, then the sets $N_S := \bar{\mathbb{Q}}(\{t_\xi : \xi < \eta_S\}) \cap \bigcup \mathcal{S}$ are as needed. For more details see [21]. \square

4.3 $2^{\mathfrak{c}}$ -lineability of $\text{ES} \cap \neg\text{Conn} \cap \text{SCIVP}$ and $\text{ES} \cap \neg\text{Conn} \cap \neg\text{PR}$

We start with examining the two classes from the top row of (4.1).

4.3.1 The class $\text{ES} \cap \neg\text{Conn} \cap \neg\text{PR}$

The $2^{\mathfrak{c}}$ -lineability of this class is an easy corollary of the main result from [21].

Theorem 4.3.1. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of \mathfrak{c} -many maps with pairwise disjoint supports such that $W_{\mathcal{F}} \subset (\text{PES} \cap \neg\text{Conn}) \cup \{0\}$. In particular, $\text{ES} \cap \neg\text{Conn} \cap \neg\text{PR}$ is $2^{\mathfrak{c}}$ -lineable.*

Proof. The existence of an \mathcal{F} satisfying the first part of the theorem was proved in [21, theorem 2.1]. (See also Subsection 4.3.3.) To finish the proof, it is enough to show that $\text{PES} \subset \neg\text{PR}$. Indeed, if $f \in \text{PES}$, then $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}$. On the other hand, for any perfect $P_0 \subset \mathbb{R}$ the continuity of $f \upharpoonright P_0$ at any $x \in P_0$ implies that $f[P]$ is bounded for some perfect $P \subset P_0$. So, $f \notin \text{PR}$. \square

4.3.2 The class $\text{ES} \cap \neg \text{Conn} \cap \text{SCIVP}$

In a 2021 paper [5] we proved that $\text{ES} \cap \text{SCIVP} \setminus \text{Conn}$ is \mathfrak{c}^+ -lineable assuming that \mathfrak{c} is regular¹. The following theorem improves this result in two ways. It removes the assumption that \mathfrak{c} is regular and replaces the cardinal \mathfrak{c}^+ with the best possible value of $2^\mathfrak{c}$. In the proof of theorem we will use the following well known fact.

Lemma 4.3.2. *There exists a family $\mathcal{M}_0 = \{P^I \subset I : I \in \mathcal{B}\}$ of pairwise disjoint perfect sets such that $\bigcup \mathcal{M}_0$ is algebraically independent and the transcendence degree of \mathbb{R} over $\bar{\mathbb{Q}}(\bigcup \mathcal{M}_0)$ is \mathfrak{c} .*

Proof. Let $K \subset \mathbb{R}$ be an algebraically independent compact perfect set, see [63] or [54]. Choose a family $\{K^I \subset K : I \in \mathcal{B}\}$ of pairwise disjoint perfect sets such that the set $K \setminus \bigcup_{I \in \mathcal{B}} K^I$ has cardinality \mathfrak{c} . For every $I \in \mathcal{B}$ choose non-zero $p_I, q_I \in \mathbb{Q}$ such that the set $P^I := p_I + q_I K^I$ is contained in I . Then $\mathcal{M}_0 := \{P^I \subset I : I \in \mathcal{B}\}$ is as needed. \square

Next, let $\mathcal{M}_0 = \{P^I \subset I : I \in \mathcal{B}\}$ be as above. For every $I \in \mathcal{B}$ let $\{P_\xi^{I,r} \subset P^I : \xi < \mathfrak{c} \text{ \& } r \in \mathbb{R}\}$ be a family of pairwise disjoint perfect sets and put $\mathcal{S} := \bigcup_{I \in \mathcal{B}} \{P_\xi^{I,r} \subset P^I : \xi < \mathfrak{c} \text{ \& } r \in \mathbb{R}\}$. Applying Lemma 4.2.5 to \mathcal{S} and numbers $r_S = \text{id}_\star(r)$ for $S = P_\xi^{I,r}$ we can reduce each $P_\xi^{I,r}$, if necessary, to a perfect set such that the sets in the family $\{\text{id}_\star(r) \cdot P_\xi^{I,r} : r \in \mathbb{R} \text{ \& } \xi < \mathfrak{c} \text{ \& } I \in \mathcal{B}\}$ are pairwise disjoint. For every $\xi < \mathfrak{c}$ and $r \in \mathbb{R}$ put $S_\xi^r := \bigcup_{I \in \mathcal{B}} P_\xi^{I,r}$ and $D_\xi^r := \text{id}_\star(r) \cdot S_\xi^r$.

Let $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$, where the functions φ_ξ are defined as in (φ) , that is, $\varphi_\xi := \sum_{r \in \mathbb{R}} r \cdot \chi_{D_\xi^r}$ for the sets D_ξ^r as above.

Theorem 4.3.3. *If $\mathcal{F}(\varphi)$ is defined as above, then $W_{\mathcal{F}(\varphi)}$ justifies $2^\mathfrak{c}$ -lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{SCIVP}$.*

Proof. $W_{\mathcal{F}(\varphi)}$ has dimension $2^\mathfrak{c}$ by Remark 3.1.2. Also, each $\varphi_\xi \in \mathcal{F}(\varphi)$ is in SCIVP. Indeed, every set S_ξ^r contains a perfect subset of every $I \in \mathcal{B}$ (namely $P_\xi^{I,r}$), so the same is true for each set $D_\xi^r = \text{id}_\star(r) \cdot S_\xi^r$. In particular, for every $I \in \mathcal{B}$ and perfect $K \subset \mathbb{R}$ there exists a perfect $P \subset I$ such that $\varphi_\xi \upharpoonright P$ is constant (so continuous) with value $r \in K$, proving that indeed $\varphi_\xi \in \text{SCIVP}$. Therefore, $\mathcal{F}(\varphi) \subset \text{SCIVP}$, so by Proposition 4.2.1(iii) used with $B := \mathbb{R}$ and $\mathcal{H} = \mathcal{F}(\varphi)$, $W_{\mathcal{F}(\varphi)}$ justifies $2^\mathfrak{c}$ -lineability of SCIVP.

Finally, notice that $\mathcal{S} = \{S_\xi^r \subset \mathbb{R} : r \in \mathbb{R} \text{ \& } \xi < \mathfrak{c}\}$ and $\Delta := \{\text{id}_\star(r) \cdot S_\xi^r : r \in \mathbb{R} \text{ \& } \xi < \mathfrak{c}\}$ satisfy the assumptions of Proposition 4.2.4. Hence $W_{\mathcal{F}(\varphi)}$ justifies $2^\mathfrak{c}$ -lineability of $\text{ES} \setminus \text{Conn}$. \square

¹Recall that \mathfrak{c} is regular when the union of less than \mathfrak{c} -many sets, each of which has cardinality less than \mathfrak{c} , has cardinality less than \mathfrak{c} .

The consistency of the $2^{\mathfrak{c}}$ -lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{SCIVP}$ can be also deduced from the theorem proved in [5] that this class is \mathfrak{c}^+ -lineable if \mathfrak{c} is a regular cardinal. Thus, Theorem 4.3.3 can be viewed as a generalization of the result from [5].

4.3.3 Proof of $2^{\mathfrak{c}}$ -lineability of $\text{PES} \cap \neg \text{Conn}$

It is worth to notice that the developed machinery, which we used to prove the previous theorem, gives also a direct proof of [21, Theorem 2.1].

To see this, let T be an algebraically independent set intersecting every perfect set (i.e., T is a Bernstein set). Let $\{S_\xi^r \subset T: \xi < \mathfrak{c} \ \& \ r \in \mathbb{R}\}$ be a family of pairwise disjoint Bernstein sets. Applying Lemma 4.2.5 we can ensure that the sets in the family $\Delta := \{\text{id}_*(r) \cdot S_\xi^r: r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ are pairwise disjoint. Let $\mathcal{F}(\varphi) := \{\varphi_\xi: \xi < \mathfrak{c}\}$, where functions φ_ξ are defined as in (φ) , that is, $\varphi_\xi := \sum_{r \in \mathbb{R}} r \cdot \chi_{D_\xi^r}$ for the sets D_ξ^r as above. Then the family $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$ -lineability of $\text{PES} \cap \neg \text{Conn}$.

Indeed the argument used in the proof of Theorem 4.3.3 immediately implies that $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$ -lineability of $\neg \text{Conn}$. Also, the fact that all sets S_ξ^r and $D_\xi^r = \text{id}_*(r) \cdot S_\xi^r$ are Bernstein, ensures that $\mathcal{F}(\varphi) \subset \text{PES}$. So, by Proposition 4.2.1(v) used with $B = \mathbb{R}$ and $\mathcal{H} = \mathcal{F}$, $W_{\mathcal{F}(\varphi)} \subset \text{PES} \cup \{0\}$, as needed.

4.4 Lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{PR} \cap \neg \text{CIVP}$ and $\text{ES} \cap \neg \text{Conn} \cap \text{CIVP} \cap \neg \text{SCIVP}$

The following lemma will be used to establish $2^{\mathfrak{c}}$ -lineability of both of these classes.

Lemma 4.4.1. *Let $\mathcal{M}_0 = \{P^I \subset I: I \in \mathcal{B}\}$ be a family of pairwise disjoint perfect sets as in Lemma 4.3.2, that is, such that $\bigcup \mathcal{M}_0$ is algebraically independent and the transcendence degree of \mathbb{R} over $\bar{\mathbb{Q}}(\bigcup \mathcal{M}_0)$ is \mathfrak{c} . Then there exist a family $\mathcal{S}_0 = \{S^r \subset \mathbb{R} \setminus \bigcup \mathcal{M}_0: r \in \mathbb{R}\}$ of pairwise disjoint \mathfrak{c} -dense sets and a set $Z \subset \mathbb{R} \setminus \mathbb{Q}(\bigcup \mathcal{S}_0)$ such that*

- (i) $\bigcup(\mathcal{M}_0 \cup \mathcal{S}_0)$ is algebraically independent;
- (ii) $\bigcup\{\text{id}_*(r) \cdot S^r: r \in \mathbb{R}\}$ contains no perfect set;
- (iii) Z intersects every perfect set and $Z \cup \bigcup \mathcal{S}_0$ is algebraically independent.

Proof. Let $\langle\langle r_\xi, J_\xi \rangle\rangle: \xi < \mathfrak{c}$ be an enumeration of $\mathbb{R} \times \mathcal{B}$ with \mathfrak{c} -many repetitions and $\{P_\xi: \xi < \mathfrak{c}\}$ be an

enumeration of all perfect subsets of \mathbb{R} . By induction on $\xi < \mathfrak{c}$ choose a sequence $\langle \langle x_\xi, y_\xi, z_\xi \rangle : \xi < \mathfrak{c} \rangle$ so that

$$(A_\xi) \quad x_\xi \in J_\xi \setminus \bar{\mathbb{Q}}(\bigcup \mathcal{M}_0 \cup \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta / \text{id}_*(r_\zeta) : \zeta < \xi\} \cup \{z_\zeta : \zeta < \xi\});$$

$$(C_\xi) \quad y_\xi \in P_\xi \setminus \{\text{id}_*(r_\zeta) \cdot x_\zeta : \zeta \leq \xi\};$$

$$(Z_\xi) \quad z_\xi \in P_\xi \setminus \bar{\mathbb{Q}}(\{x_\zeta : \zeta \leq \xi\} \cup \{z_\zeta : \zeta < \xi\}).$$

Such x_ξ can be chosen, as otherwise the transcendence degree of \mathbb{R} over $\bar{\mathbb{Q}}(\bigcup \mathcal{M}_0)$ would be less than \mathfrak{c} .

For each $r \in \mathbb{R}$ define $S^r := \{x_\xi : r_\xi = r\}$ and let $Z = \{z_\xi : \xi < \mathfrak{c}\}$. We claim that these definitions ensure that $\mathcal{S}_0 = \{S^r : r \in \mathbb{R}\}$ and Z are as needed.

Indeed, clearly the sets in \mathcal{S}_0 are pairwise disjoint and \mathfrak{c} -dense, since the sequence $\langle x_\xi : \xi < \mathfrak{c} \rangle$ is one-to-one and each pair $\langle r, J \rangle \in \mathbb{R} \times \mathcal{B}$ appears in the sequence $\langle \langle r_\xi, J_\xi \rangle : \xi < \mathfrak{c} \rangle$ \mathfrak{c} -many times. Also $\bigcup \mathcal{S}_0 = \{x_\xi : \xi < \mathfrak{c}\}$ is contained in $\mathbb{R} \setminus \bar{\mathbb{Q}}(\bigcup \mathcal{M}_0)$. So, (A_ξ) ensures (i).

To show (ii), for a perfect set $P \subset \mathbb{R}$ choose a $\xi < \mathfrak{c}$ such that $P_\xi = P$ and notice that $y_\xi \in P_\xi = P$ does not belong to $\bigcup \{\text{id}_*(r) \cdot S^r : r \in \mathbb{R}\} = \{\text{id}_*(r_\zeta) \cdot x_\zeta : \zeta < \mathfrak{c}\}$: $y_\xi \notin \{\text{id}_*(r_\zeta) \cdot x_\zeta : \zeta \leq \xi\}$ is ensured by (C_ξ) , while $y_\xi \notin \{\text{id}_*(r_\zeta) \cdot x_\zeta : \xi < \zeta\}$ by the conditions (A_ζ) with $\zeta > \xi$.

Finally, the first part of (iii)—the fact that Z intersects every perfect set—is ensured by (Z_ξ) , while its second part—an algebraic independence of the set $Z \cup \bigcup \mathcal{S}_0 = \{z_\xi : \xi < \mathfrak{c}\} \cup \{x_\xi : \xi < \mathfrak{c}\}$ —by the choice as in (A_ξ) and (Z_ξ) . \square

In what follows $\mathcal{M}_0 = \{P^I \subset I : I \in \mathcal{B}\}$, $\mathcal{S}_0 = \{S^r \subset \mathbb{R} \setminus \bigcup \mathcal{M}_0 : r \in \mathbb{R}\}$, and Z are always as in Lemma 4.4.1. Removing from each set S^r one number, if necessary, we can assume that

$$(A) \quad \text{the transcendence degree of } \mathbb{R} \text{ over both } \bar{\mathbb{Q}}(\bigcup(\mathcal{M}_0 \cup \mathcal{S}_0)) \text{ and } \bar{\mathbb{Q}}(Z \cup \bigcup \mathcal{S}_0) \text{ is } \mathfrak{c}.$$

Notice that

$$(B) \quad M := \bigcup \mathcal{M}_0 \text{ is a meager } F_\sigma\text{-set and that } \bigcup \mathcal{S}_0 \text{ is contained in } M^c := \mathbb{R} \setminus M.$$

For every $r \in \mathbb{R}$ let $\{S_\xi^r \subset S^r : \xi < \mathfrak{c}\}$ be a family of pairwise disjoint \mathfrak{c} -dense sets. Since $\bigcup(\mathcal{M}_0 \cup \mathcal{S}_0)$ is algebraically independent, we can apply Lemma 4.2.5 to the family $\mathcal{S} := \{S_\xi^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \cup \{M\}$ and numbers $r_S = \text{id}_*(r)$ for $S = S_\xi^r$ and $r_M = 1$ to slightly reduce sets S_ξ^r , if necessary, to ensure that the sets in the family $\Delta := \{\text{id}_*(r) \cdot S_\xi^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \cup \{M\}$ are pairwise disjoint. In summary,

$$(C) \quad \text{the sets in } \{S_\xi^r \subset \bigcup \mathcal{S}_0 : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \text{ are } \mathfrak{c}\text{-dense, pairwise disjoint, and the sets in } \Delta := \{\text{id}_*(r) \cdot S_\xi^r : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \cup \{M\} \text{ are pairwise disjoint.}$$

(D) let $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$, where functions $\varphi_\xi := \sum_{r \in \mathbb{R}} r \cdot \chi_{D_\xi^r}$ are as in (φ) with $D_\xi^r := \text{id}_*(r) \cdot S_\xi^r$ for the sets S_ξ^r as in (C).

The following fact will be used in the proofs of our two remaining theorems.

Proposition 4.4.2. *If $\mathcal{F}(\varphi)$ is as in (D) and functions in $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$ having pairwise disjoint support are such that $\mathcal{H} \upharpoonright M^c = \mathcal{F}(\varphi) \upharpoonright M^c$, then $W_{\mathcal{H}} \subset \text{ES} \cup \{0\}$.*

Proof. Our definition of $\mathcal{F}(\varphi)$ ensures that $\mathcal{F}(\varphi) \subset \text{D}$. Also, $\text{supp}(\mathcal{F}(\varphi)) \subset M^c$ and $M^c \setminus \text{supp}(\mathcal{F}(\varphi))$ is dense in \mathbb{R} , since it contains dense sets S_ξ^0 . So, Proposition 4.2.1(iv) with $B = M^c$ implies that $W_{\mathcal{H}} \subset \text{ES} \cup \{0\}$. \square

4.4.1 $2^{\mathfrak{c}}$ -lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{PR} \cap \neg \text{CIVP}$

For every $I \in \mathcal{B}$ choose a family $\{P_\xi^{I,q} \subset P^I : \xi < \mathfrak{c} \ \& \ q \in \mathbb{Q}\}$ of pairwise disjoint perfect sets and for every $\xi < \mathfrak{c}$ define

$$\gamma_\xi := \sum_{(I,q) \in \mathcal{B} \times \mathbb{Q}} q \chi_{P_\xi^{I,q}} \quad \text{and} \quad h_\xi = \varphi_\xi + \gamma_\xi, \quad (4.3)$$

where maps φ_ξ are from the family $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$ from Proposition 4.4.2. Clearly the supports of maps in the family $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint as functions in $\mathcal{F}(\varphi)$ have disjoint supports contained in M^c , while the maps in $\{\gamma_\xi : \xi < \mathfrak{c}\}$ have disjoint supports contained in M .

Theorem 4.4.3. *If $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ for the functions h_ξ from (4.3), then $W_{\mathcal{H}}$ justifies $2^{\mathfrak{c}}$ -lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{PR} \cap \neg \text{CIVP}$.*

Proof. $W_{\mathcal{H}}$ has dimension $2^{\mathfrak{c}}$ by Remark 3.1.2. The inclusion $W_{\mathcal{H}} \subset \text{ES} \cup \{0\}$ is ensured by Proposition 4.4.2, since $\mathcal{H} \upharpoonright M^c = \mathcal{F}(\varphi) \upharpoonright M^c$.

To see that $W_{\mathcal{H}} \subset \neg \text{Conn} \cup \{0\}$ notice that for every $\xi < \mathfrak{c}$

$$(\gamma_\xi / \text{id}_*)[\mathbb{R}] \subset \left\{ \frac{q}{r} : q \in \mathbb{Q} \ \& \ r \in \bigcup \mathcal{M}_0 \right\} \subset \mathbb{Q}(\bigcup \mathcal{M}_0),$$

and, by (D) and Proposition 4.2.4, $(\varphi_\xi / \text{id}_*)[\mathbb{R}] \subset \mathbb{Q}(\bigcup \mathcal{S}_0)$. Therefore, we have $(h_\xi / \text{id}_*)[\mathbb{R}] \subset (\varphi_\xi / \text{id}_*)[\mathbb{R}] \cup (\gamma_\xi / \text{id}_*)[\mathbb{R}] \subset \mathbb{Q}(\bigcup (\mathcal{M}_0 \cup \mathcal{S}_0))$. Thus, by (A) and Lemmas 4.2.3 and 4.2.2, indeed $W_{\mathcal{H}} \subset \neg \text{Conn} \cup \{0\}$.

Next notice that every non-zero $g \in W_{\mathcal{H}}$ is in PR. This argument is a variation of the Fact 3.2.6. Indeed, by Remark 4.1.1, there exists a $\xi < \mathfrak{c}$ and non-zero $c \in \mathbb{R}$ such that $g = c \cdot h_\xi$ on $\text{supp}(h_\xi)$.

In particular, for every $x \in \mathbb{R}$, choose a sequence $\langle q_n \in \mathbb{Q} : n \in \mathbb{N} \rangle$ converging to $g(x)/c$ and disjoint intervals $I_n = (a_n, b_n) \in \mathcal{B}$ so that $a_{2n} \nearrow_n x$ and $a_{2n+1} \searrow_n x$. Then $P := \{x\} \cup \bigcup_{n \in \mathbb{N}} P_\xi^{I_n, q_n}$ is perfect having x as a bilateral limit point and $g \upharpoonright P$ is continuous at x , since

$$\lim_{p \rightarrow x, p \in P} g(p) = \lim_{p \rightarrow x, p \in P} c \cdot h_\xi(p) = \lim_{n \rightarrow \infty} c \cdot q_n = g(x).$$

So, indeed $g \in \text{PR}$.

To finish the proof, it is enough to show that $W_{\mathcal{H}} \subset \neg \text{CIVP} \cup \{0\}$. So, choose a non-zero $g \in W_{\mathcal{H}}$. By Remark 4.1.1, there exists a finite set $A_g \subset \mathbb{R}$ such that

$$g \subset \bigcup_{a \in \mathbb{Q}(A_g)} \bigcup_{\xi < c} a \cdot h_\xi. \quad (4.4)$$

Choose a perfect $K \subset \mathbb{R} \setminus \mathbb{Q}(A_g)$. Since the graph of g is dense so it is enough to show that $g[P] \not\subset K$ for every perfect $P \subset \mathbb{R}$. By way of contradiction, assume that $g[P] \subset K$ for some perfect $P \subset \mathbb{R}$. Since M is Borel, we can assume that either $P \subset M$ or $P \subset M^c$.

However, if $P \subset M$, then $h_\xi[P] = \gamma_\xi[P] \subset \mathbb{Q}$. So, by (4.4) and the fact that our definition ensures $\gamma_\xi[\mathbb{R}] \subset \mathbb{Q}$, we have $g[P] \subset \bigcup_{a \in \mathbb{Q}(A)} \bigcup_{\xi < c} a \cdot \gamma_\xi[P] \subset \mathbb{Q}(A_g) \subset K^c$, contradicting our assumption that $g[P] \subset K$.

This would mean that $P \subset M^c$. But this is also impossible, since the fact that $\bigcup \{\text{id}_*(r) \cdot S^r : r \in \mathbb{R}\}$ contains no perfect set (ensured by part (ii) of Lemma 4.4.1) implies that there exists an $x \in P \setminus \text{supp}(\mathcal{H})$ so that $0 = g(x) \in g[P]$, while $0 \notin K$, a contradiction. \square

4.4.2 $2^\mathfrak{c}$ -lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{CIVP} \cap \neg \text{SCIVP}$

Although there is a considerable similarity of the argument in this case to the previous one, we need to choose the family $\{P_\xi^{I, q} \subset P^I : \xi < \mathfrak{c} \ \& \ q \in \mathbb{Q}\}$ with considerable more care. For this we will use the following lemma which is a slight modification of Lemma 3.2.9. Let \mathcal{P} be the family of all non-empty perfect subsets of \mathbb{R} .

Lemma 4.4.4. *For every $P^I \in \mathcal{P}$ there is a family \mathcal{P}^I of continuum many pairwise disjoint perfect subsets of P^I such that if $P \in \mathcal{P}$ is contained in $\bigcup \mathcal{P}^I$, then there is a $\hat{P} \in \mathcal{P}^I$ such that $P \cap \hat{P}$ is uncountable.*

Proof. Choosing a subset, if necessary, we can assume that P^I is homeomorphic to 2^ω . Let B be a Bernstein subset of 2^ω and $h_I : 2^\omega \times 2^\omega \rightarrow P^I$ be an embedding. Then the family $\mathcal{P}^I :=$

$\{h_I[\{b\} \times 2^\omega] : b \in B\}$ is as needed. Indeed, assume that $P \subset \bigcup \mathcal{P}^I$ is perfect and let $Q_0 := \{x \in 2^\omega : h_I[\{x\} \times 2^\omega] \cap P \neq \emptyset\}$, that is, Q_0 is the projection of $h_I^{-1}(P \cap P^I)$ onto the first coordinate. The compact set Q_0 must be countable, since otherwise $Q_0 \setminus B \neq \emptyset$, that is, $P \not\subset \bigcup \mathcal{P}^I$, a contradiction. Therefore there is a $b \in B$ so that the intersection of P and $\hat{P} = h_I[\{b\} \times 2^\omega] \in \mathcal{P}^I$ is uncountable. \square

For every $I \in \mathcal{B}$ let \mathcal{P}^I be as in Lemma 4.4.4 and let $\{P_\xi^{I,K} \subset P^I : \xi < \mathfrak{c} \ \& \ K \in \mathcal{P}\}$ be its enumeration. Let Z be as in (A). Recall that Z intersects every $P \in \mathcal{P}$. For every $\xi < \mathfrak{c}$ let

$$(\kappa) \ \kappa_\xi := \sum_{(I,K) \in \mathcal{B} \times \mathcal{P}} \kappa_\xi^{I,K}, \text{ where } \kappa_\xi^{I,K} : \mathbb{R} \rightarrow Z \text{ has support contained in } P_\xi^{I,K}, \kappa_\xi^{I,K}[P_\xi^{I,K}] \subset K \cap Z, \text{ and } \kappa_\xi^{I,K} \text{ is discontinuous on any perfect } Q \subset P_\xi^{I,K}.$$

Let

$$h_\xi = \varphi_\xi + \kappa_\xi, \tag{4.5}$$

where maps φ_ξ are from the family $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$ from Proposition 4.4.2. Clearly the supports of maps in the family $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint as functions in $\mathcal{F}(\varphi)$ have disjoint supports contained in M^c , while the maps in $\{\kappa_\xi : \xi < \mathfrak{c}\}$ have disjoint supports contained in M .

Theorem 4.4.5. *If $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ for the functions h_ξ from (4.5), then $W_{\mathcal{H}}$ justifies $2^{\mathfrak{c}}$ -lineability of $\text{ES} \cap \neg \text{Conn} \cap \text{CIVP} \cap \neg \text{SCIVP}$.*

Proof. Similarly as in the proof of Theorem 4.4.3 the space $W_{\mathcal{H}}$ has dimension $2^{\mathfrak{c}}$ by Remark 3.1.2 and $W_{\mathcal{H}} \subset \text{ES} \cup \{0\}$ is ensured by Proposition 4.4.2, since we have $\mathcal{H} \upharpoonright M^c = \mathcal{F}(\varphi) \upharpoonright M^c$.

To see that $W_{\mathcal{H}} \subset \neg \text{Conn} \cup \{0\}$ notice that for every $\xi < \mathfrak{c}$,

$$(\kappa_\xi / \text{id}_\star)[\mathbb{R}] \subset \left\{ \frac{z}{r} : z \in Z \ \& \ r \in \bigcup \mathcal{M}_0 \right\} \subset \mathbb{Q}(Z \cup \bigcup \mathcal{S}_0),$$

and, by (D) and Proposition 4.2.4, we have $(\varphi_\xi / \text{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(\bigcup \mathcal{S}_0)$. Therefore, $(h_\xi / \text{id}_\star)[\mathbb{R}] \subset (\varphi_\xi / \text{id}_\star)[\mathbb{R}] \cup (\kappa_\xi / \text{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(Z \cup \bigcup \mathcal{S}_0)$. Thus, by (A) and Lemmas 4.2.3 and 4.2.2, indeed $W_{\mathcal{H}} \subset \neg \text{Conn} \cup \{0\}$.

Next notice that every h_ξ is in CIVP. Indeed, for every $I \in \mathcal{B}$ and $K \in \mathcal{P}$ the perfect set $P_\xi^{I,K}$ is contained in I and $h_\xi[P_\xi^{I,K}] = \kappa_\xi[P_\xi^{I,K}] \subset K$. Thus, $\mathcal{H} \subset \text{CIVP}$ so, by Proposition 4.2.1(ii) used with $B = M$ and $\mathcal{F} = \{\kappa_\xi : \xi < \mathfrak{c}\}$, $W_{\mathcal{H}} \subset \text{CIVP}$.

To finish the proof, it is enough to show that $W_{\mathcal{H}} \subset \neg \text{SCIVP} \cup \{0\}$. So, choose a non-zero $g \in W_{\mathcal{H}}$ and a perfect $K \subset \mathbb{R} \setminus \mathbb{Q}$. Since the graph of g is dense it is enough to show that for every

${}^2 \kappa_\xi^{I,K} \upharpoonright P_\xi^{I,K}$ can be chosen as a Sierpiński-Zygmund function from $P_\xi^{I,K}$ into $K \cap Z$.

perfect $P \subset \mathbb{R}$, if $g[P] \subset K$, then $g \upharpoonright P$ is discontinuous. By way of contradiction, assume that there exists a perfect $P \subset \mathbb{R}$ such that $g[P] \subset K$ and $g \upharpoonright P$ is continuous. Since M is Borel, we can assume that either $P \subset M$ or $P \subset M^c$.

But $P \subset M^c$ is impossible, since this and the fact that $\bigcup\{\text{id}_*(r) \cdot S^r : r \in \mathbb{R}\}$ contains no perfect set would imply that there exists an $x \in P \setminus \text{supp}(\mathcal{H})$ with $0 = g(x) \in g[P]$, while $0 \notin K$, a contradiction.

However, the inclusion $P \subset M$ together with $g[P] \subset K \subset \mathbb{R} \setminus \{0\}$ imply that $P \subset M \cap \text{supp}(\{\kappa_\xi : \xi < \mathfrak{c}\}) \subset \bigcup_{I \in \mathcal{B}} P^I$. So, choosing perfect subset of P , if necessary, we can assume that $P \subset P^I$ for some $I \in \mathcal{B}$. But this means that $P \subset \bigcup \mathcal{P}^I$. Since \mathcal{P}^I is as in Lemma 4.4.4, there exists a $P_\xi^{I, K'} \in \mathcal{P}^I$ such that $P \cap P_\xi^{I, K'}$ is uncountable. In particular, there exists a perfect set $Q \subset P \cap P_\xi^{I, K'}$. Then, by Remark 4.1.1, there exists an $a \in \mathbb{R}$ such that $g \upharpoonright Q = a h_\xi \upharpoonright Q = a \kappa_\xi \upharpoonright Q$. But this is impossible, since $a = 0$ implies $g[Q] = \{0\} \subset K^c$, while $a \neq 0$ implies that $g \upharpoonright Q = a \kappa_\xi \upharpoonright Q$ is discontinuous by the choice of functions κ_ξ , contradicting our assumption that $g \upharpoonright P$ is continuous. \square

Chapter 5

Open problems and our related results

5.1 Open problems

As we have previously seen, all the non-empty classes \mathcal{G} in $\mathcal{A}(\mathbb{D})$ disjoint with $(\text{Conn} \setminus \text{AC})$ are 2^c -lineable. On the other hand, for the non-empty classes $\mathcal{G} \in \mathcal{A}(\mathbb{D})$ with $\mathcal{G} \subset \text{Conn} \setminus \text{AC}$ it is only known that \mathcal{G} is \mathfrak{c} -lineable while potentially each of these classes can be 2^c -lineable. This naturally leads to the following problem.

Problem 5.1.1. Is the class $\text{Conn} \setminus \text{AC}$ 2^c -lineable? Is the same true for all non-empty classes $\mathcal{G} \in \mathcal{A}(\mathbb{D})$ contained in $\text{Conn} \setminus \text{AC}$? What about \mathfrak{c}^+ -lineability of these classes?

The problem above is not the end of the story: one may go further and ask for stronger results. More precisely, let

- $\mathcal{F} \in \{\text{D} \setminus \text{ES}, \text{ES} \setminus \text{SES}, \text{SES} \setminus \text{PES}, \text{PES} \setminus \text{J}, \text{J}\},$
- $\mathcal{G} \in \{\text{PC} \setminus \text{D}, \text{D} \setminus \text{Conn}, \text{Conn} \setminus \text{AC}, \text{AC} \setminus \text{Ext}\},$
- $\mathcal{H} \in \{\text{PC} \setminus \text{PR}, \text{PR} \setminus \text{CIVP}, \text{CIVP} \setminus \text{SCIVP}, \text{SCIVP} \setminus \text{Ext}\}$

Problem 5.1.2. Are the non-empty classes of the form $\mathcal{F} \cap \mathcal{H}$, $\mathcal{G} \cap \mathcal{H}$, and $\mathcal{F} \cap \mathcal{G} \cap \mathcal{H}$ 2^c -lineable? What about their \mathfrak{c}^+ -lineability?

Of course, an examination of the examples provided in Chapters 2-4 shows that several of these classes are indeed 2^c -lineable. However, the full the general study indicated in Problem 5.1.2 was

never undertaken and indicates possible further extension of presented work.

5.2 Our related results not included in this work

We have also some works concerning the intersections of nonempty classes in $\mathcal{A}(\mathbb{D})$ with the class of Sierpiński-Zygmund functions. Recall that for $X \subset \mathbb{R}$, a map $f: X \rightarrow \mathbb{R}$ is a Sierpiński-Zygmund function (or just SZ-function) provided $f \upharpoonright S$ is discontinuous for any $S \subset X$ of cardinality \mathfrak{c} . The first example of an SZ-function was constructed by Sierpiński and Zygmund in their 1923 paper [61]. Compare also the recent survey [32] on the SZ-maps. The functions in SZ are as far from being continuous as possible: by a 1922 theorem of Blumberg [14] for every $f: \mathbb{R} \rightarrow \mathbb{R}$ there exists a (countable) dense subset D of \mathbb{R} with $f \upharpoonright D$ being continuous. Thus, one might expect that no SZ-map can be continuous in a generalized sense, e.g., to be Darboux. Surprisingly, it has been proved, see [9], that this last statement is independent of the usual axioms ZFC of set theory.

Recall that $\text{cov}(\mathcal{M}) = \mathfrak{c}$ is the statement, consistent with ZFC, that \mathbb{R} is not a union of less than \mathfrak{c} -many meager sets, where \mathcal{M} denotes the ideal of all meager subsets of \mathbb{R} .

Now, we record some of our major results about the lineability of the intersection of Darboux-like and SZ functions.

Theorem 5.2.1. *Assume that \mathfrak{c} is a regular cardinal and $\text{cov}(\mathcal{M}) = \mathfrak{c}$. Then the family $\text{SZ} \cap \text{ES} \setminus \text{Conn}$ is \mathfrak{c}^+ -lineable and so also $\text{SZ} \cap \text{D} \setminus \text{Conn}$.*

Notice that, as mentioned above, it is consistent with ZFC that $\text{SZ} \cap \text{ES} \subset \text{SZ} \cap \text{D} = \emptyset$, in which case $\text{SZ} \cap \text{ES} \setminus \text{Conn}$ cannot be even 1-lineable. Therefore, some extra set theoretical assumption is necessary in Theorem 5.2.1.

In addition, we proved the following strong result.

Theorem 5.2.2. *Assume that \mathfrak{c} is a regular cardinal and $\text{cov}(\mathcal{M}) = \mathfrak{c}$. Then the family $\text{SZ} \cap \text{CIVP} \cap \text{ES} \setminus \text{Conn}$ is \mathfrak{c}^+ -lineable and so also $\text{SZ} \cap \text{CIVP} \cap \text{D} \setminus \text{Conn}$.*

Bibliography

- [1] Gbrel M. Albkwe and K.C. Ciesielski, *Maximal lineability of several subclasses of Darboux-like maps on \mathbb{R}* , Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (2022), no. 1, 24 (13 pages), DOI 10.1007/s13398-021-01169-2.
- [2] ———, *Maximal lineability of the classes of functions in the algebra Darboux-like maps that are Darboux but not connectivity*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (2022), no. 2, 83 (13 pages), DOI 10.1007/s13398-022-01226-4.
- [3] Gbrel M. Albkwe and Krzysztof Chris Ciesielski, *c-lineability: A general method and its application to Darboux-like maps*, J. Math. Anal. Appl. **505** (2022), no. 2, Paper No. 125611, DOI 10.1016/j.jmaa.2021.125611.
- [4] Gbrel M. Albkwe, K. C. Ciesielski, and Jerzy Wojciechowski, *Lineability of the functions that are Sierpiński–Zygmund, Darboux, but not connectivity*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), no. 3, 145 (10 pages), DOI 10.1007/s13398-020-00881-9.
- [5] Gbrel Albkwe, K.C. Ciesielski, and J. Wojciechowski, *c^+ -lineability of the class of Darboux maps with the strong Cantor intermediate value property which are not connectivity*, Topology Proc. **60** (2022), 127 (10 pages).
- [6] G. Araújo, L. Bernal-González, G. A. Muñoz-Fernández, J. A. Prado-Bassas, and J. B. Seoane-Sepúlveda, *Lineability in sequence and function spaces*, Studia Math. **237** (2017), no. 2, 119–136, DOI 10.4064/sm8358-10-2016.
- [7] R.M. Aron, L. Bernal González, D.M. Pellegrino, and J.B. Seoane Sepúlveda, *Lineability: the search for linearity in mathematics*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016.
- [8] R.M. Aron, V.I. Gurariy, and J.B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on \mathbb{R}* , Proc. Amer. Math. Soc. **133** (2005), no. 3, 795–803, DOI 10.1090/S0002-9939-04-07533-1.
- [9] Marek Balcerzak, Krzysztof Ciesielski, and Tomasz Natkaniec, *Sierpiński-Zygmund functions that are Darboux, almost continuous, or have a perfect road*, Arch. Math. Logic **37** (1997), no. 1, 29–35, DOI 10.1007/s001530050080.
- [10] Artur Bartoszewicz and Szymon Głąb, *Additivity and lineability in vector spaces*, Linear Algebra Appl. **439** (2013), no. 7, 2123–2130, DOI 10.1016/j.laa.2013.06.007. MR3090459
- [11] L. Bernal-González, *Algebraic genericity of strict-order integrability*, Studia Mathematica, 199, 279-293. (2010).
- [12] L. Bernal-González and Manuel Ordóñez Cabrera, *Lineability criteria, with applications*, J. Funct. Anal. **266** (2014), no. 6, 3997–4025, DOI 10.1016/j.jfa.2013.11.014.
- [13] L. Bernal-González, D. Pellegrino, and J.B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 1, 71–130.

- [14] H. Blumberg, *New properties of all real functions*, Trans. Amer. Math. Soc. **24** (1922), no. 2, 113–128, DOI 10.2307/1989037.
- [15] Jack B. Brown, *Nowhere dense Darboux graphs*, Colloq. Math. **20** (1969), 243–247, DOI 10.4064/cm-20-2-243-247. MR255740
- [16] ———, *Connectivity, semi-continuity, and the Darboux property*, Duke Math. J. **36** (1969), 559–562.
- [17] A.M. Bruckner and J.G. Ceder, *Darboux continuity*, Jber. Deutsch. Math.-Verein. **67** (1964/1965), no. Abt. 1, 93–117.
- [18] M. C. Calderón-Moreno, P. J. Gerlach-Mena, and J. A. Prado-Bassas, *Lineability and modes of convergence*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), no. 1, Paper No. 18, 12, DOI 10.1007/s13398-019-00743-z.
- [19] K. Ciesielski, *Set theory for the working mathematician*, London Mathematical Society Student Texts, vol. 39, Cambridge University Press, Cambridge, 1997.
- [20] K.C. Ciesielski, *Set-theoretic real analysis*, J. Appl. Anal. **3** (1997), no. 2, 143–190, DOI 10.1515/JAA.1997.143.
- [21] ———, *Maximal lineability of the class of Darboux not connectivity maps on \mathbb{R}* , Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. accepted, available at <https://math.wvu.edu/~kciesiel/publications.html>.
- [22] K. C. Ciesielski, J. L. Gámez-Merino, L. Mazza, and J. B. Seoane-Sepúlveda, *Cardinal coefficients related to surjectivity, Darboux, and Sierpiński-Zygmund maps*, Proc. Amer. Math. Soc. **145** (2017), no. 3, 1041–1052, DOI 10.1090/proc/13294.
- [23] Krzysztof Chris Ciesielski, José L. Gámez-Merino, Daniel Pellegrino, and Juan B. Seoane-Sepúlveda, *Lineability, spaceability, and additivity cardinals for Darboux-like functions*, Linear Algebra Appl. **440** (2014), 307–317, DOI 10.1016/j.laa.2013.10.033.
- [24] K.C. Ciesielski and J. Jastrzębski, *Darboux-like functions within the classes of Baire one, Baire two, and additive functions*, Topology Appl. **103** (2000), no. 2, 203–219, DOI 10.1016/S0166-8641(98)00169-2.
- [25] K.C. Ciesielski and A. Maliszewski, *Cardinal invariants concerning bounded families of extendable and almost continuous functions*, Proceedings of the American Mathematical Society **126** (1998), no. 2, 471–479.
- [26] K.C. Ciesielski and A.W. Miller, *Cardinal concerning functions whose sum is almost continuous*, Real Anal. Exchange **20** (1994/95), no. 2, 657–672.
- [27] K.C. Ciesielski and T. Natkaniec, *Algebraic properties of the class of Sierpiński-Zygmund functions*, Topology Appl. **79** (1997), no. 1, 75–99.
- [28] K. C. Ciesielski, T. Natkaniec, D. L. Rodríguez-Vidanes, and J. B. Seoane-Sepúlveda, *Additivity coefficients for all classes in the algebra of Darboux-like maps on \mathbb{R}* , Results Math. **76** (2021), no. 1, Paper No. 7, 38, DOI 10.1007/s00025-020-01287-0.
- [29] Krzysztof Chris Ciesielski and Cheng-Han Pan, *Examples of Sierpiński-Zygmund maps in the class of Darboux-like functions*, Banach J. Math. Anal. **14** (2020), no. 2, 433–449, DOI 10.1007/s43037-019-00001-9.
- [30] K.C. Ciesielski and I. Reclaw, *Cardinal invariants concerning extendable and peripherally continuous functions*, Real Anal. Exchange **21** (1995/96), no. 2, 459–472.
- [31] Krzysztof Ciesielski and Andrzej Roslanowski, *Two examples concerning almost continuous functions*, Topology Appl. **103** (2000), no. 2, 187–202, DOI 10.1016/S0166-8641(98)00168-0.

- [32] K. C. Ciesielski and J. B. Seoane-Sepúlveda, *A century of Sierpiński–Zygmund functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 4, 3863–3901, DOI 10.1007/s13398-019-00726-0.
- [33] J. Alberto Conejero, Mar Fenoy, Marina Murillo-Arcila, and Juan B. Seoane-Sepúlveda, *Lineability within probability theory settings*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 3, 673–684, DOI 10.1007/s13398-016-0318-y.
- [34] J.L. Cornette, *Connectivity functions and images on Peano continua*, Fund. Math. **58** (1966), 183–192, DOI 10.4064/fm-58-2-183-192.
- [35] G. Darboux, *Mémoire sur les fonctions discontinues*, Ann. Sci. École Norm. Sup. (2) **4** (1875), 57–112 (French).
- [36] H. Fast, *Une remarque sur la propriété de Weierstrass*, Colloq. Math. **7** (1959), 75–77, DOI 10.4064/cm-7-1-75-77 (French).
- [37] José L. Gámez-Merino, Gustavo A. Muñoz-Fernández, Víctor M. Sánchez, and Juan B. Seoane-Sepúlveda, *Sierpiński–Zygmund functions and other problems on lineability*, Proc. Amer. Math. Soc. **138** (2010), no. 11, 3863–3876, DOI 10.1090/S0002-9939-2010-10420-3. MR2679609
- [38] J.L. Gámez-Merino, G.A. Muñoz-Fernández, and J.B. Seoane-Sepúlveda, *Lineability and additivity in $\mathbb{R}^{\mathbb{R}}$* , J. Math. Anal. Appl. **369** (2010), no. 1, 265–272.
- [39] José L. Gámez-Merino, *Large algebraic structures inside the set of surjective functions*, Bull. Belg. Math. Soc. Simon Stevin **18** (2011), no. 2, 297–300.
- [40] R.G. Gibson and T. Natkaniec, *Darboux like functions*, Real Anal. Exchange **22** (1996/97), no. 2, 492–533.
- [41] R.G. Gibson and F. Roush, *The Cantor intermediate value property*, Proceedings of the 1982 Topology Conference (Annapolis, Md., 1982), 1982, pp. 55–62.
- [42] ———, *Connectivity functions with a perfect road*, Real Anal. Exchange **11** (1985/86), no. 1, 260–264. The ninth summer real analysis symposium (Louisville, Ky., 1985).
- [43] V.I. Gurariĭ, *Subspaces and bases in spaces of continuous functions*, Dokl. Akad. Nauk SSSR **167** (1966), 971–973 (Russian).
- [44] M.R. Hagan, *Equivalence of connectivity maps and peripherally continuous transformations*, Proc. Amer. Math. Soc. **17** (1966), 175–177.
- [45] O.H. Hamilton, *Fixed points for certain noncontinuous transformations*, Proc. Amer. Math. Soc. **8** (1957), 750–756.
- [46] Thomas W. Hungerford, *Algebra*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1974.
- [47] J. Jastrzębski, *An answer to a question of R. G. Gibson and F. Roush*, Real Anal. Exchange **15** (1989/90), no. 1, 340–341.
- [48] F.B. Jones and E.S. Thomas Jr., *Connected G_δ graphs*, Duke Math. J. **33** (1966), 341–345.
- [49] K.R. Kellum, *Sums and limits of almost continuous functions*, Colloq. Math. **31** (1974), 125–128, DOI 10.4064/cm-31-1-125-128.
- [50] B Levine and D Milman, *On linear sets in space C consisting of functions of bounded variation*, Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.](4) **16** (1940), 102–105.
- [51] Azriel Levy, *Basic set theory*, Vol. 13, Courier Corporation, 2002.

- [52] I. Maximoff, *Sur les fonctions ayant la propriété de Darboux*, Prace Mat. Fiz. **43** (1936), 241–265.
- [53] J.R. Munkres, *Topology*, Prentice-Hall, 2nd ed., 2000.
- [54] J. Mycielski, *Independent sets in topological algebras*, Fund. Math. **55** (1964), 139–147, DOI 10.4064/fm-55-2-139-147.
- [55] J. Nash, *Generalized Brouwer theorem*, Bull. Amer. Math. Soc. **62** (1956), no. 1, 76.
- [56] Tomasz Natkaniec, *Almost continuity*, Real Analysis Exchange **17** (1991), no. 2, 462–520.
- [57] ———, *New cardinal invariants in real analysis*, Bulletin of the Polish Academy of Sciences-Mathematics **44** (1996), no. 2, 251–256.
- [58] J.H. Roberts, *Zero-dimensional sets blocking connectivity functions*, Fund. Math. **57** (1965), 173–179.
- [59] H. Rosen, R.G. Gibson, and F. Roush, *Extendable functions and almost continuous functions with a perfect road*, Real Anal. Exchange **17** (1991/92), no. 1, 248–257.
- [60] J.B. Seoane-Sepúlveda, *Chaos and lineability of pathological phenomena in analysis*, ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)—Kent State University.
- [61] W. Sierpiński and A. Zygmund, *Sur une fonction qui est discontinue sur tout ensemble de puissance du continu*, Fund. Math. **4** (1923), 316–318.
- [62] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.
- [63] J. v. Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann. **99** (1928), no. 1, 134–141, DOI 10.1007/BF01459089 (German).
- [64] G.T. Whyburn, *Connectivity of peripherally continuous functions*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 1040–1041.
- [65] J. Young, *A theorem in the theory of functions of a real variable*, Rend. Circ. Mat. Palermo **24** (1907), 187–192.