Consequences of the Covering Property Axiom CPA

Based on *Covering Property Axiom CPA*, by K. Ciesielski and J. Pawlikowski, to appear in **Cambridge Tracts in Mathematics**, Cambridge Univ. Press.

Under CPA we have $\mathfrak{c} = \omega_2$; 2^{ω_1} can be arbitrarily large.

Real Functions

(F1) There exists a family $\mathcal{G}$ of uniformly continuous functions from $\mathbb{R}$ to $[0,1]$ such that $|\mathcal{G}| = \omega_1$ and for every $S \in [\mathbb{R}]^{\mathfrak{c}}$ there exists a $g \in \mathcal{G}$ with $g[S] = [0,1]$.

(F2) There exists a family $\mathcal{F}$ of less than continuum many $\mathcal{C}^1$ functions from $\mathbb{R}$ to $\mathbb{R}$ such that the plain $\mathbb{R}^2$ is covered by functions from $\mathcal{F}$ and their inverses (i.e., each $f \in \mathcal{F}$ is a used as a function on a horizontal axis and on a vertical axis.)

(F3) For every Borel function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a family $\mathcal{F}$ of less than continuum many "$\mathcal{C}^1$" functions (i.e., differentiable functions with continuous derivatives, where derivative can be infinite) whose graphs cover the graph of f.

(F4) For an arbitrary function h from a subset S of a Polish space X *onto* a Polish space Y there exists a uniformly continuous function f from a subset of X into Y such that $|f \cap h| = \mathfrak{c}$. In particular,

- there is no Darboux Sierpiński-Zygmund function $f\colon \mathbb{R} \to \mathbb{R}$, that is, for every Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ there is a subset Y of $\mathbb{R}$ of cardinality $\mathfrak{c}$ such that $f \restriction Y$ is continuous;
- for any function h from a subset S of $\mathbb{R}$ onto a perfect subset of $\mathbb{R}$ there exists a function $f \in$ "$\mathcal{C}^\infty_{\text{perf}}$" such that $|f \cap h| = \mathfrak{c}$ and f can be extended to a function $\bar{f} \in$ "$\mathcal{C}^1(\mathbb{R})$" such that either $\bar{f} \in \mathcal{C}^1$ or $\bar{f}$ is an autohomeomorphism of $\mathbb{R}$ with $\bar{f}^{-1} \in \mathcal{C}^1$.

(F5) For every Darboux function $g\colon \mathbb{R} \to \mathbb{R}$ there exists a continuous nowhere constant function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + g$ is Darboux.

(F6) There is a family $\mathcal{H}$ of ω_1 pairwise disjoint perfect subsets of $\mathbb{R}$ such that $H = \bigcup \mathcal{H}$ is a Hamel basis, that is, a linear basis of $\mathbb{R}$ over $\mathbb{Q}$. In particular,

- there is a non-measurable subset X of $\mathbb{R}$ without the Baire property which is $\mathcal{N} \cap \mathcal{M}$-rigid, that is, such that $X \triangle (r + X) \in \mathcal{N} \cap \mathcal{M}$ for every $r \in \mathbb{R}$,

- there is a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $h \in \mathbb{R}$ the difference function $\Delta_h(x) = f(x+h) - f(x)$ is Borel, but for every $\alpha < \omega_1$ there is an $h \in \mathbb{R}$ such that Δ_h is not of Borel class α.

(F7) There exists a discontinuous, almost continuous, and additive function $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is of measure zero.

(F8) There exists a Hamel basis H such that $E^+(H)$ has measure zero, where $E^+(A)$ is a linear combination of $A \subset \mathbb{R}$ with non-negative rational coefficients.

(F9) For a Polish space X and uniformly bounded sequence $\langle f_n\colon X \to \mathbb{R}\rangle_{n<\omega}$ of Borel measurable functions there are the sequences: $\langle P_\xi\colon \xi < \omega_1\rangle$ of compact subsets of X and $\langle W_\xi \in [\omega]^\omega\colon \xi < \omega_1\rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:

$\langle f_n \restriction P_\xi\rangle_{n\in W_\xi}$ is a monotone uniformly convergent sequence of uniformly continuous functions.

(F10) Let X be an arbitrary set and $f_n\colon X \to \mathbb{R}$ be a sequence of functions such that the set $\{f_n(x)\colon n < \omega\}$ is bounded for every $x \in X$. Then there are the sequences: $\langle P_\xi\colon \xi < \omega_1\rangle$ of subsets of X and $\langle W_\xi \in [\omega]^\omega\colon \xi < \omega_1\rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:

$\langle f_n \restriction P_\xi\rangle_{n\in W_\xi}$ is monotone and uniformly convergent.

Combinatorial Cardinal Characteristics

(C1) $\mathrm{cof}(\mathcal{N}) = \omega_1$, i.e., the cofinality of the measure ideal $\mathcal{N}$ is ω_1. In particular

- $\mathfrak{c} > \omega_1$ and there exists a Boolean algebra B of cardinality ω_1 which is not a union of strictly increasing ω-sequence of subalgebras of B.

(C2) There exists a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 which is simultaneously independent and splitting. In particular, $\mathfrak{i} = \mathfrak{s} = \omega_1$.

(C3) There exists a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 which is simultaneously maximal almost disjoint and reaping. In particular, $\mathfrak{a} = \mathfrak{r} = \omega_1$.

(C4) $\mathfrak{u} = \mathfrak{r}_\sigma = \omega_1$, where $\mathfrak{u}$ is the smallest cardinality of the base for a non-principal ultrafilter on ω.

(C5) $\mathrm{add}(s_0) = \omega_1$, where s_0 is the Marczewski's ideal.

(C6) $\mathrm{cov}(s_0) = \mathfrak{c}$

(C7) $\mathfrak{c} > \omega_1$ and for every Polish space there exists a partition of X into ω_1 *disjoint* closed nowhere dense measure zero sets.

Small Sets

(S1) Every perfectly meager set $S \subset \mathbb{R}$ has cardinality less than $\mathfrak{c}$.

(S2) Every universally null set $S \subset \mathbb{R}$ has cardinality less than $\mathfrak{c}$.

(S3) (Nowik) Every uniformly completely Ramsey null set $S \subset [\omega]^\omega$ has cardinality less than $\mathfrak{c}$.

(S4) There exists an uncountable γ-set. It can be ensured that it is strongly meager, or that it is not strongly meager.

$\beta\,\mathbb{N}$ and $\beta\,\mathbb{Q}$

(β1) There exist 2^{ω_1}-many distinct selective ultrafilter on ω.

(β2) Every selective filter on ω can be extended to a selective ultrafilter.

(β3) Every selective ultrafilter on ω is generated by ω_1-many sets.

(β4) There exist 2^{ω_1}-many distinct non-selective P-points.

(β5) There exists a non-principal ultrafilter on $\mathbb{Q}$ which is crowded, that is, it is generated by (relatively) closed sets without isolated points.

Other

- TOTAL FAILURE OF MARTIN'S AXIOM: $\mathfrak{c} > \omega_1$ and for every non-trivial ccc forcing $\mathbb{P}$ there exists ω_1-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.