## Axiom $\mathrm{CPA}_{\text {cube }}$ and its consequences: properties (A)-(E)

For a Polish space $X$ we will consider $\operatorname{Perf}(X)$, the family of all subsets of $X$ homeomorphic to the Cantor set $\mathfrak{C}$, as ordered by inclusion. Thus, a family $\mathcal{E} \subset \operatorname{Perf}(X)$ is dense in $\operatorname{Perf}(X)$ provided for every $P \in \operatorname{Perf}(X)$ there exists a $Q \in \mathcal{E}$ such that $Q \subset P$.

All different versions of our axiom will be more or less of the form:
If $\mathcal{E} \subset \operatorname{Perf}(X)$ is appropriately dense in $\operatorname{Perf}(X)$, then some portion $\mathcal{E}_{0}$ of $\mathcal{E}$ covers almost all of $X$ in a sense that $\left|X \backslash \bigcup \mathcal{E}_{0}\right|<\mathfrak{c}$.

If the word "appropriately" in the above is ignored, then it implies the following statement:

Naïve-CPA: If $\mathcal{E}$ is dense in $\operatorname{Perf}(X)$, then $|X \backslash \bigcup \mathcal{E}|<\mathfrak{c}$.
It is a very good candidate for our axiom in the sense that it implies all the properties we are interested in. It has, however, one major flaw - it is false! This is the case since $S \subset X \backslash \bigcup \mathcal{E}$ for some dense set $\mathcal{E}$ in $\operatorname{Perf}(X)$ provided:

For each $P \in \operatorname{Perf}(X)$ there is a $Q \in \operatorname{Perf}(X)$ such that $Q \subset P \backslash S$.
This means that the family $\mathcal{G}$ of all sets of the form $X \backslash \bigcup \mathcal{E}$, where $\mathcal{E}$ is dense in $\operatorname{Perf}(X)$, coincides with the $\sigma$-ideal $s_{0}$ of Marczewski's sets, since $\mathcal{G}$ is clearly hereditary. Thus we have

$$
\begin{equation*}
s_{0}=\{X \backslash \bigcup \mathcal{E}: \mathcal{E} \text { is dense in } \operatorname{Perf}(X)\} \tag{1.1}
\end{equation*}
$$

However, it is well known (see, e.g., [96, thm. 5.10]) that there are $s_{0}$-sets of cardinality $\mathfrak{c}$. Thus, our Naïve-CPA "axiom" cannot be consistent with ZFC.

In order to formulate the real axiom $\mathrm{CPA}_{\text {cube }}$, we need the following terminology and notation. A subset $C$ of a product $\mathfrak{C}^{\eta}$ of the Cantor set is
said to be a perfect cube if $C=\prod_{n \in \eta} C_{n}$, where $C_{n} \in \operatorname{Perf}(\mathfrak{C})$ for each $n$. For a fixed Polish space $X$ let $\mathcal{F}_{\text {cube }}$ stand for the family of all continuous injections from perfect cubes $C \subset \mathfrak{C}^{\omega}$ onto perfect subsets of $X$. Each such injection $f$ is called a cube in $X$ and is considered as a coordinate system imposed on $P=\operatorname{range}(f) .{ }^{1}$ We will usually abuse this terminology and refer to $P$ itself as a cube (in $X$ ) and to $f$ as a witness function for $P$. A function $g \in \mathcal{F}_{\text {cube }}$ is a subcube of $f$ provided $g \subset f$. In the above spirit we call $Q=\operatorname{range}(g)$ a subcube of a cube $P$. Thus, when we say that $Q$ is a subcube of a cube $P \in \operatorname{Perf}(X)$ we mean that $Q=f[C]$, where $f$ is a witness function for $P$ and $C \subset \operatorname{dom}(f) \subset \mathfrak{C}^{\omega}$ is a perfect cube. Here and in what follows, the symbol $\operatorname{dom}(f)$ stands for the domain of $f$.

We say that a family $\mathcal{E} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {cube-dense (or cube-dense) in }}$ $\operatorname{Perf}(X)$ provided every cube $P \in \operatorname{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {cube }}$-dense provided

$$
\begin{equation*}
\forall f \in \mathcal{F}_{\text {cube }} \exists g \in \mathcal{F}_{\text {cube }}(g \subset f \& \operatorname{range}(g) \in \mathcal{E}) \tag{1.2}
\end{equation*}
$$

It is easy to see that the notion of $\mathcal{F}_{\text {cube }}$-density is a generalization of the notion of density as defined in the first paragraph of this chapter:

$$
\begin{equation*}
\text { If } \mathcal{E} \text { is } \mathcal{F}_{\text {cube }} \text {-dense in } \operatorname{Perf}(X) \text {, then } \mathcal{E} \text { is dense in } \operatorname{Perf}(X) . \tag{1.3}
\end{equation*}
$$

On the other hand, the converse implication is not true, as shown by the following simple example.

Example 1.0.1 Let $X=\mathfrak{C} \times \mathfrak{C}$ and let $\mathcal{E}$ be the family of all $P \in \operatorname{Perf}(X)$ such that either

- all vertical sections $P_{x}=\{y \in \mathfrak{C}:\langle x, y\rangle \in P\}$ of $P$ are countable, or
- all horizontal sections $P^{y}=\{x \in \mathfrak{C}:\langle x, y\rangle \in P\}$ of $P$ are countable.

Then $\mathcal{E}$ is dense in $\operatorname{Perf}(X)$, but it is not $\mathcal{F}_{\text {cube-dense in }} \operatorname{Perf}(X)$.
Proof. To see that $\mathcal{E}$ is dense $\operatorname{in} \operatorname{Perf}(X)$, let $R \in \operatorname{Perf}(X)$. We need to find a $P \subset R$ with $P \in \mathcal{E}$. If all vertical sections of $R$ are countable, then $P=R \in \mathcal{E}$. Otherwise, there exists an $x$ such that $R_{x}$ is uncountable. Then there exists a perfect subset $P$ of $\{x\} \times R_{x} \subset R$ and clearly $P \in \mathcal{E}$.

To see that $\mathcal{E}$ is not $\mathcal{F}_{\text {cube }}$-dense in $\operatorname{Perf}(X)$, it is enough to notice that $P=X=\mathfrak{C} \times \mathfrak{C}$ considered as a cube, where the second coordinate is identified with $\mathfrak{C}^{\omega \backslash\{0\}}$, has no subcube in $\mathcal{E}$. More formally, let $h$ be a homeomorphism from $\mathfrak{C}$ onto $\mathfrak{C}^{\omega} \backslash\{0\}$, let $g: \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}^{\omega}=\mathfrak{C} \times \mathfrak{C}^{\omega \backslash\{0\}}$ be given by $g(x, y)=\langle x, h(y)\rangle$, and let $f=g^{-1}: \mathfrak{C}^{\omega} \rightarrow \mathfrak{C} \times \mathfrak{C}$ be the coordinate
1 In a language of forcing, a coordinate function $f$ is simply a nice name for an element from $X$.
function making $\mathfrak{C} \times \mathfrak{C}=$ range $(f)$ a cube. Then range $(f)$ does not contain a subcube from $\mathcal{E}$.

With these notions in hand we are ready to formulate our axiom $\mathrm{CPA}_{\text {cube }}$. For a Polish space $X$ let
$\mathrm{CPA}_{\text {cube }}[X]: \mathfrak{c}=\omega_{2}$, and for every $\mathcal{F}_{\text {cube-dense family }} \mathcal{E} \subset \operatorname{Perf}(X)$ there is an $\mathcal{E}_{0} \subset \mathcal{E}$ such that $\left|\mathcal{E}_{0}\right| \leq \omega_{1}$ and $\left|X \backslash \bigcup \mathcal{E}_{0}\right| \leq \omega_{1}$.

Then
$\mathrm{CPA}_{\text {cube }}: \mathrm{CPA}_{\text {cube }}[X]$ for every Polish space $X$.
We will show in Remark 1.8.3 that both these versions of the axiom are equivalent, that is, that $\mathrm{CPA}_{\text {cube }}[X]$ is equivalent to $\mathrm{CPA}_{\text {cube }}[Y]$ for arbitrary Polish spaces $X$ and $Y$.

The proof that $\mathrm{CPA}_{\text {cube }}$ is consistent with ZFC (it holds in the iterated perfect set model) will be presented in the next chapters. In the remainder of this chapter we will take a closer look at $\mathrm{CPA}_{\text {cube }}$ and its consequences.

It is also worth noticing that, in order to check that $\mathcal{E}$ is $\mathcal{F}_{\text {cube }}$-dense, it is enough to consider in condition (1.2) only functions $f$ defined on the entire space $\mathfrak{C}^{\omega}$, that is:

Fact 1.0.2 $\mathcal{E} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {cube }}$-dense if and only if

$$
\begin{equation*}
\forall f \in \mathcal{F}_{\text {cube }}, \operatorname{dom}(f)=\mathfrak{C}^{\omega}, \exists g \in \mathcal{F}_{\text {cube }}(g \subset f \& \operatorname{range}(g) \in \mathcal{E}) \tag{1.4}
\end{equation*}
$$

Proof. To see this, let $\Phi$ be the family of all bijections $h=\left\langle h_{n}\right\rangle_{n<\omega}$ between perfect subcubes $\prod_{n \in \omega} D_{n}$ and $\prod_{n \in \omega} C_{n}$ of $\mathfrak{C}^{\omega}$ such that each $h_{n}$ is a homeomorphism between $D_{n}$ and $C_{n}$. Then

$$
f \circ h \in \mathcal{F}_{\text {cube }} \text { for every } f \in \mathcal{F}_{\text {cube }} \text { and } h \in \Phi \text { with range }(h) \subset \operatorname{dom}(f)
$$

Now take an arbitrary $f: C \rightarrow X$ from $\mathcal{F}_{\text {cube }}$ and choose an $h \in \Phi$ mapping $\mathfrak{C}^{\omega}$ onto $C$. Then $\hat{f}=f \circ h \in \mathcal{F}_{\text {cube }}$ maps $\mathfrak{C}^{\omega}$ into $X$, and, using (1.4), we can find a $\hat{g} \in \mathcal{F}_{\text {cube }}$ such that $\hat{g} \subset \hat{f}$ and range $(\hat{g}) \in \mathcal{E}$. Then $g=f \upharpoonright h[\operatorname{dom}(\hat{g})]$ satisfies (1.2).

Next, let us consider ${ }^{1}$

$$
\begin{align*}
s_{0}^{\text {cube }} & =\left\{X \backslash \bigcup \mathcal{E}: \mathcal{E} \text { is } \mathcal{F}_{\text {cube }}-\text { dense in } \operatorname{Perf}(X)\right\}  \tag{1.5}\\
& =\{S \subset X: \forall \text { cube } P \in \operatorname{Perf}(X) \exists \operatorname{subcube} Q \subset P \backslash S\}
\end{align*}
$$

1 The second equation follows immediately from the fact that if $\mathcal{E}$ is $\mathcal{F}_{\text {cube }}$-dense and $Y \subset X \backslash \bigcup \mathcal{E}$, then $Y=X \backslash \bigcup \mathcal{E}^{\prime}$ for some $\mathcal{F}_{\text {cube }}$-dense $\mathcal{E}^{\prime}$. To see this, for every $x \in X$ choose $T_{x} \in \operatorname{Perf}(X)$ such that $T_{x} \subset\{x\} \cup \bigcup \mathcal{E}$ and note that $\mathcal{E}^{\prime}=\mathcal{E} \cup\left\{T_{x}: x \in X \backslash Y\right\}$ is as desired.

It can be easily shown, in ZFC, that $s_{0}^{\text {cube }}$ forms a $\sigma$-ideal. However, we will not use this fact in this text in that general form. This is the case, since we will usually assume that $\mathrm{CPA}_{\text {cube }}$ holds while $\mathrm{CPA}_{\text {cube }}$ implies the following stronger fact.

Proposition 1.0.3 If $\mathrm{CPA}_{\text {cube }}$ holds, then $s_{0}^{\text {cube }}=[X] \leq \omega_{1}$.
Proof. It is obvious that $\mathrm{CPA}_{\text {cube }}$ implies $s_{0}^{\text {cube }} \subset[X]^{<\mathbf{c}}$. The other inclusion is always true, and it follows from the following simple fact.

Fact 1.0.4 $[X]^{<\mathfrak{c}} \subset s_{0}^{\text {cube }} \subset s_{0}$ for every Polish space $X$.
Proof. Choose $S \in[X]^{<\mathfrak{c}}$. In order to see that $S \in s_{0}^{\text {cube }}$, note that the family $\mathcal{E}=\{P \in \operatorname{Perf}(X): P \cap S=\emptyset\}$ is $\mathcal{F}_{\text {cube-dense }}$ in $\operatorname{Perf}(X)$. Indeed, if function $f: \mathfrak{C}^{\omega} \rightarrow X$ is from $\mathcal{F}_{\text {cube }}$, then there is a perfect subset $P_{0}$ of $\mathfrak{C}$ that is disjoint with the projection $\pi_{0}\left(f^{-1}(S)\right)$ of $f^{-1}(S)$ into the first coordinate. Then $f\left[\prod_{i<\omega} P_{i}\right] \cap S=\emptyset$, where $P_{i}=\mathfrak{C}$ for all $0<i<\omega$. Therefore, $f\left[\prod_{i<\omega} P_{i}\right] \in \mathcal{E}$. Thus, $X \backslash \bigcup \mathcal{E} \in s_{0}^{\text {cube }}$. Since clearly $S \subset X \backslash \bigcup \mathcal{E}$, we get $S \in s_{0}^{\text {cube }}$.

The inclusion $s_{0}^{\text {cube }} \subset s_{0}$ follows immediately from (1.1), (1.5), and (1.3).

### 1.1 Perfectly meager sets, universally null sets, and continuous images of sets of cardinality continuum

The results presented in this section come from K. Ciesielski and J. Pawlikowski [39]. An important quality of the ideal $s_{0}^{\text {cube }}$, and so the power of the assumption $s_{0}^{\text {cube }}=[X]^{<\mathbf{c}}$, is well depicted by the following fact.

Proposition 1.1.1 If $X$ is a Polish space and $S \subset X$ does not belong to $s_{0}^{\text {cube }}$, then there exist a $T \in[S]^{\text {c }}$ and a uniformly continuous function $h$ from $T$ onto $\mathfrak{C}$.

Proof. Take an $S$ as above and let $f: \mathfrak{C}^{\omega} \rightarrow X$ be a continuous injection such that $f[C] \cap S \neq \emptyset$ for every perfect cube $C$. Let $g: \mathfrak{C} \rightarrow \mathfrak{C}$ be a continuous function such that $g^{-1}(y)$ is perfect for every $y \in \mathfrak{C}$. Then clearly $h_{0}=g \circ \pi_{0} \circ f^{-1}: f\left[\mathfrak{C}^{\omega}\right] \rightarrow \mathfrak{C}$ is uniformly continuous. Moreover, if $T=S \cap f\left[\mathfrak{C}^{\omega}\right]$, then $h_{0}[T]=\mathfrak{C}$ since

$$
T \cap h_{0}^{-1}(y)=T \cap f\left[\pi_{0}^{-1}\left(g^{-1}(y)\right)\right]=S \cap f\left[g^{-1}(y) \times \mathfrak{C} \times \mathfrak{C} \times \cdots\right] \neq \emptyset
$$

for every $y \in \mathfrak{C}$.

Corollary 1.1.2 Assume $s_{0}^{\text {cube }}=[X]^{<\mathfrak{c}}$ for a Polish space $X$. If $S \subset X$ has cardinality $\mathfrak{c}$, then there is a uniformly continuous function $f: X \rightarrow[0,1]$ such that $f[S]=[0,1]$. In particular, $\mathrm{CPA}_{\text {cube }}$ implies property $(\mathrm{A})$.

Proof. If $S$ is as above, then, by $\mathrm{CPA}_{\text {cube }}, S \notin s_{0}^{\text {cube }}$. Thus, by Proposition 1.1.1 there exists a uniformly continuous function $h$ from a subset of $S$ onto $\mathfrak{C}$. Consider $\mathfrak{C}$ as a subset of $[0,1]$ and let $\hat{h}: X \rightarrow[0,1]$ be a uniformly continuous extension of $h$. If $g:[0,1] \rightarrow[0,1]$ is continuous and such that $g[\mathfrak{C}]=[0,1]$, then $f=g \circ \hat{h}$ is as desired.

For more on property (A) see also Corollary 3.3.5.
It is worth noticing here that the function $f$ in Corollary 1.1.2 cannot be required to be either monotone or in the class " $D$ " of all functions having a finite or infinite derivative at every point. This follows immediately from the following proposition, since each function that is either monotone or " $D^{1}$ " belongs to the Banach class

$$
\left(T_{2}\right)=\left\{f \in \mathcal{C}(\mathbb{R}):\left\{y \in \mathbb{R}:\left|f^{-1}(y)\right|>\omega\right\} \in \mathcal{N}\right\}
$$

(See [58] or [114, p. 278].)
Proposition 1.1.3 There is, in $Z F C$, an $S \in[\mathbb{R}]^{\mathfrak{c}}$ such that $[0,1] \not \subset f[S]$ for every $f \in\left(T_{2}\right)$.

Proof. Let $\left\{f_{\xi}: \xi<\mathfrak{c}\right\}$ be an enumeration of all functions from $\left(T_{2}\right)$ whose range contains $[0,1]$. Construct by induction a sequence $\left\langle\left\langle s_{\xi}, y_{\xi}\right\rangle: \xi<\mathfrak{c}\right\rangle$ such that, for every $\xi<\mathfrak{c}$,
(i) $y_{\xi} \in[0,1] \backslash f_{\xi}\left[\left\{s_{\zeta}: \zeta<\xi\right\}\right]$ and $\left|f_{\xi}^{-1}\left(y_{\xi}\right)\right| \leq \omega$.
(ii) $s_{\xi} \in \mathbb{R} \backslash\left(\left\{s_{\zeta}: \zeta<\xi\right\} \cup \bigcup_{\zeta \leq \xi} f_{\zeta}^{-1}\left(y_{\zeta}\right)\right)$.

Then the set $S=\left\{s_{\xi}: \xi<\mathfrak{c}\right\}$ is as required since $y_{\xi} \in[0,1] \backslash f_{\xi}[S]$ for every $\xi<\mathrm{c}$.

Theorem 1.1.4 If $S \subset \mathbb{R}$ is either perfectly meager or universally null, then $S \in s_{0}^{\text {cube }}$. In particular,

$$
\mathrm{CPA}_{\text {cube }} \Longrightarrow " s_{0}^{\text {cube }}=[\mathbb{R}]^{<\mathfrak{c} "} \Longrightarrow "(\mathrm{~B}) \&(\mathrm{C}) . "
$$

Proof. Take an $S \subset \mathbb{R}$ that is either perfectly meager or universally null and let $f: \mathfrak{C}^{\omega} \rightarrow \mathbb{R}$ be a continuous injection. Then $S \cap f\left[\mathfrak{C}^{\omega}\right]$ is either meager or null in $f\left[\mathfrak{C}^{\omega}\right]$. Thus $G=\mathfrak{C}^{\omega} \backslash f^{-1}(S)$ is either comeager or of full measure in $\mathfrak{C}^{\omega}$. Hence the theorem follows immediately from the following claim, which will be used many times in the sequel.

Claim 1.1.5 Consider $\mathfrak{C}^{\omega}$ with its usual topology and its usual product measure. If $G$ is a Borel subset of $\mathfrak{C}^{\omega}$ that is either of the second category or of positive measure, then $G$ contains a perfect cube $\prod_{i<\omega} P_{i}$.

In particular, if $\mathcal{G}$ is a countable cover of $\mathfrak{C}^{\omega}$ formed by either measurable sets or by sets with the Baire property, then there is a $G \in \mathcal{G}$ that contains a perfect cube.

The measure version of the claim is a variant the following theorem:
(m) For every full measure subset $H$ of $[0,1] \times[0,1]$ there are a perfect set $P \subset[0,1]$ and a positive inner measure subset $\hat{H}$ of $[0,1]$ such that $P \times \hat{H} \subset H$.

This was proved by H.G. Eggleston [52] and, independently, by M.L. Brodskií [13]. The category version of the claim is a consequence of the category version of ( m ):
(c) For every Polish space $X$ and every comeager subset $G$ of $X \times X$ there are a perfect set $P \subset X$ and a comeager subset $\hat{G}$ of $X$ such that $P \times \hat{G} \subset G$.

This well-known result can be found in [74, exercise 19.3]. (Its version for $\mathbb{R}^{2}$ is also proved, for example, in [45, condition ( $\star$ ), p. 416].) For completeness, we will show here in detail how to deduce the claim from (m) and (c).

We will start the argument with a simple fact, in which we will use the following notations. If $X$ is a Polish space endowed with a Borel measure, then $\psi_{0}(X)$ will stand for the sentence
$\psi_{0}(X)$ : For every full measure subset $H$ of $X \times X$ there are a perfect set $P \subset X$ and a positive inner measure subset $\hat{H}$ of $X$ such that $P \times \hat{H} \subset H$.

Thus $\psi_{0}([0,1])$ is a restatement of $(\mathrm{m})$. We will also use the following seemingly stronger variants of $\psi_{0}(X)$.
$\psi_{1}(X)$ : For every full measure subset $H$ of $X \times X$ there are a perfect set $P \subset X$ and a subset $\hat{H}$ of $X$ of full measure such that $P \times \hat{H} \subset H$.
$\psi_{2}(X)$ : For a subset $H$ of $X \times X$ of positive inner measure there are a perfect set $P \subset X$ and a positive inner measure subset $\hat{H}$ of $X$ such that $P \times \hat{H} \subset H$.

Fact 1.1.6 Let $n=1,2,3, \ldots$..
(i) If $E$ is a subset of $\mathbb{R}^{n}$ of a positive Lebesgue measure, then the set $\mathbb{Q}^{n}+E=\bigcup_{q \in \mathbb{Q}^{n}}(q+E)$ has a full measure.
(ii) $\psi_{k}(X)$ holds for all $k<3$ and $X \in\{[0,1],(0,1), \mathbb{R}, \mathfrak{C}\}$.

Proof. (i) Let $\lambda$ be the Lebesgue measure on $\mathbb{R}^{n}$, and for $\varepsilon>0$ and $x \in \mathbb{R}^{n}$ let $B(x, \varepsilon)$ be an open ball in $\mathbb{R}^{n}$ of radius $\varepsilon$ centered at $x$. By way of contradiction, assume that there exists a positive measure set $A \subset \mathbb{R}^{n}$ disjoint with $\mathbb{Q}^{n}+E$. Let $a \in A$ and $x \in E$ be the Lebesgue density points of $A$ and $X$, respectively. Take an $\varepsilon>0$ such that $\lambda(A \cap B(a, \varepsilon))>$ $\left(1-4^{-n}\right) \lambda(B(a, \varepsilon))$ and $\lambda(E \cap B(x, \varepsilon))>\left(1-4^{-n}\right) \lambda(B(x, \varepsilon))$. Now, if $q \in \mathbb{Q}^{n}$ is such that $q+x \in B(a, \varepsilon / 2)$, then $A \cap(q+E) \cap B(a, \varepsilon / 2) \neq \emptyset$ since $B(a, \varepsilon / 2) \subset B(a, \varepsilon) \cap B(q+x, \varepsilon)$, and thus $\lambda(A \cap(q+E) \cap B(a, \varepsilon / 2))>$ $\lambda(B(a, \varepsilon / 2))-2 \cdot 4^{-n} \lambda(B(a, \varepsilon)) \geq 0$. Hence $A \cap\left(\mathbb{Q}^{n}+E\right) \neq \emptyset$, contradicting the choice of $A$.
(ii) First note that $\psi_{k}(\mathbb{R}) \Leftrightarrow \psi_{k}((0,1)) \Leftrightarrow \psi_{k}([0,1]) \Leftrightarrow \psi_{k}(\mathfrak{C})$ for every $k<3$. This is justified by the fact that, for the mappings $f:(0,1) \rightarrow \mathbb{R}$ given by $f(x)=\cot (x \pi)$, the identity mapping $i d:(0,1) \rightarrow[0,1]$, and a function $d: \mathfrak{C} \rightarrow[0,1]$ given by $d(x)=\sum_{i<\omega} \frac{x(i)}{2^{i+1}}$, the image and the preimage of a measure zero (respectively, full measure) set is of measure zero (respectively, of full measure).

Since, by $(\mathrm{m}), \psi_{0}([0,1])$ is true, we also have that $\psi_{0}(X)$ also holds for $X \in\{(0,1), \mathbb{R}, \mathfrak{C}\}$. To finish the proof it is enough to show that $\psi_{0}(\mathbb{R})$ implies $\psi_{1}(\mathbb{R})$ and $\psi_{2}(\mathbb{R})$.

To prove $\psi_{1}(\mathbb{R})$, let $H$ be a full measure subset of $\mathbb{R} \times \mathbb{R}$ and let us define $H_{0}=\bigcap_{q \in \mathbb{Q}}(\langle 0, q\rangle+H)$. Then $H_{0}$ is still of full measure, so, by $\psi_{0}(\mathbb{R})$, there are perfect set $P \subset \mathbb{R}$ and a positive inner measure subset $\hat{H}_{0}$ of $\mathbb{R}$ such that $P \times \hat{H}_{0} \subset H_{0}$. Thus, $P \times\left(q+\hat{H}_{0}\right) \subset\langle 0, q\rangle+H_{0}=H_{0}$ for every $q \in \mathbb{Q}$. Let $\hat{H}=\bigcup_{q \in \mathbb{Q}}\left(q+\hat{H}_{0}\right)$. Then $P \times \hat{H} \subset H_{0} \subset H$, and, by (i), $\hat{H}$ has full measure. So, $\psi_{1}(\mathbb{R})$ is proved.

To prove $\psi_{2}(\mathbb{R})$, let $H \subset \mathbb{R} \times \mathbb{R}$ be of positive inner measure. Decreasing $H$, if necessary, we can assume that $H$ is compact. Let $H_{0}=\mathbb{Q}^{2}+H$. Then, by (i), $H_{0}$ is of full measure, and so, by $\psi_{0}(\mathbb{R})$, there are a perfect set $P_{0} \subset \mathbb{R}$ and a positive inner measure subset $\hat{H}_{0}$ of $\mathbb{R}$ such that $P_{0} \times \hat{H}_{0} \subset H_{0}$. Once again, decreasing $P_{0}$ and $\hat{H}_{0}$ if necessary, we can assume that they are homeomorphic to $\mathfrak{C}$ and that no relatively open subset of $\hat{H}_{0}$ has measure zero. Since $P_{0} \times \hat{H}_{0} \subset \bigcup_{q \in \mathbb{Q}^{2}}(q+H)$ is covered by countably many compact sets $\left(P_{0} \times \hat{H}_{0}\right) \cap(q+H)$ with $q \in \mathbb{Q}^{2}$, there is a $q=\left\langle q_{0}, q_{1}\right\rangle \in \mathbb{Q}^{2}$ such that $\left(P_{0} \times \hat{H}_{0}\right) \cap(q+H)$ has a nonempty interior in $P_{0} \times \hat{H}_{0}$. Let $U$ and $V$ be nonempty clopen (i.e., simultaneously closed and open) subsets of $P_{0}$ and
$\hat{H}_{0}$, respectively, such that $U \times V \subset\left(P_{0} \times \hat{H}_{0}\right) \cap(q+H) \subset\left\langle q_{0}, q_{1}\right\rangle+H$. Then $U$ and $V$ are perfect and $V$ has positive measure. Let $P=-q_{0}+U$ and $\hat{H}=-q_{1}+V$. Then $P \times \hat{H}=\left(-q_{0}+U\right) \times\left(-q_{1}+V\right)=-\left\langle q_{0}, q_{1}\right\rangle+(U \times V) \subset$ $H$, and so $\psi_{2}(\mathbb{R})$ holds.

Proof of Claim 1.1.5. Since the natural homeomorphism between $\mathfrak{C}$ and $\mathfrak{C}^{\omega \backslash\{0\}}$ preserves product measure, we can identify $\mathfrak{C}^{\omega}=\mathfrak{C} \times \mathfrak{C}^{\omega} \backslash\{0\}$ with $\mathfrak{C} \times \mathfrak{C}$ considered with its usual topology and its usual product measure. With this identification, the result follows easily, by induction on coordinates, from the following fact:
(•) For every Borel subset $H$ of $\mathfrak{C} \times \mathfrak{C}$ that is of the second category (of positive measure) there are a perfect set $P \subset \mathfrak{C}$ and a second category (positive measure) subset $\hat{H}$ of $\mathfrak{C}$ such that $P \times \hat{H} \subset H$.

The measure version of $(\bullet)$ is a restatement of $\psi_{2}(\mathfrak{C})$, which was proved in Fact 1.1.6(ii). To see the category version of $(\bullet)$, let $H$ be a Borel subset of $\mathfrak{C} \times \mathfrak{C}$ of the second category. Then there are clopen subsets $U$ and $V$ of $\mathfrak{C}$ such that $H_{0}=H \cap(U \times V)$ is comeager in $U \times V$. Since $U$ and $V$ are homeomorphic to $\mathfrak{C}$, we can apply (c) to $H_{0}$ and $U \times V$ to find a perfect set $P \subset U$ and a comeager Borel subset $\hat{H}$ of $V$ such that $P \times \hat{H} \subset H_{0} \subset H$, finishing the proof.

We will finish this section with the following consequence of $\mathrm{CPA}_{\text {cube }}$ that follows easily from Claim 1.1.5. In what follows we will use the following notation: $\Sigma_{1}^{1}$ will stand for the class of analytic sets, that is, continuous images of Borel sets; $\Pi_{1}^{1}$ will stand for the class of coanalytic sets, the complements of analytic sets; and $\Sigma_{2}^{1}$ will stand for continuous images of coanalytic sets, and $\Pi_{2}^{1}$ for the class of all complements of $\Sigma_{2}^{1}$ sets. For the argument that follows we also need to recall a theorem of W. Sierpiński that every $\Sigma_{2}^{1}$ set is the union of $\omega_{1}$ Borel sets. (See, e.g., [74, p. 324].)

Fact 1.1.7 If $\mathrm{CPA}_{\text {cube }}$ holds, then for every $\Sigma_{2}^{1}$ subset $B$ of a Polish space $X$ there exists a family $\mathcal{P}$ of $\omega_{1}$ many compact sets such that $B=\bigcup \mathcal{P}$.

Proof. Since every $\Sigma_{2}^{1}$ set is a union of $\omega_{1}$ Borel sets, we can assume that $B$ is Borel. Let $\mathcal{E}$ be the family of all $P \in \operatorname{Perf}(X)$ such that either $P \subset B$ or $P \cap B=\emptyset$. We claim that $\mathcal{E}$ is $\mathcal{F}_{\text {cube-dense. Indeed, if } f: \mathfrak{C}^{\omega} \rightarrow X \text { is }}$ a continuous injection, then $f^{-1}(B)$ is Borel in $\mathfrak{C}^{\omega}$. Thus, there exists a basic open set $U$ in $\mathfrak{C}^{\omega}$, which is homeomorphic to $\mathfrak{C}^{\omega}$, such that either $U \cap f^{-1}(B)$ or $U \backslash f^{-1}(B)$ is comeager in $U$. Apply Claim 1.1.5 to this
comeager set to find a perfect cube $P$ contained in it. Then $f[P] \in \mathcal{E}$ is a subcube of range $(f)$. So, $\mathcal{E}$ is $\mathcal{F}_{\text {cube }}$-dense.

By $\mathrm{CPA}_{\text {cube }}$, there is an $\mathcal{E}_{0} \subset \mathcal{E}$ such that $\left|\mathcal{E}_{0}\right| \leq \omega_{1}$ and $\left|X \backslash \bigcup \mathcal{E}_{0}\right| \leq \omega_{1}$. Let $\mathcal{P}_{0}=\left\{P \in \mathcal{E}_{0}: P \subset B\right\}$ and $\mathcal{P}=\mathcal{P}_{0} \cup\left\{\{x\}: x \in B \backslash \bigcup \mathcal{E}_{0}\right\}$. Then $\mathcal{P}$ is as desired.

### 1.2 Uniformly completely Ramsey null sets

Uniformly completely Ramsey null sets are small subsets of $[\omega]^{\omega}$ that are related to the Ramsey property. The notion has been formally defined by U. Darji [47], though it was already studied by F. Galvin and K. Prikry in [63]. Instead of using the original definition for this class, we will use its characterization due to A. Nowik [107], in which we consider $\mathcal{P}(\omega)$ as a Polish space by identifying it with $2^{\omega}$ via the characteristic functions.

Proposition 1.2.1 (A. Nowik [107]) A subset $X$ of $[\omega]^{\omega}$ is uniformly completely Ramsey null if and only if for every continuous function $G: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ there exists an $A \in[\omega]^{\omega}$ such that $|G[\mathcal{P}(A)] \cap X| \leq \omega$.

Recently A. Nowik [108] proved that under $\mathrm{CPA}_{\text {cube }}$ every uniformly completely Ramsey null set has cardinality less than continuum. This answered a question of U . Darji, who asked whether there is a ZFC example of a uniformly completely Ramsey null set of cardinality continuum. Since Nowik's argument is typical for the use of $\mathrm{CPA}_{\text {cube }}$, we reproduce it here, with the author's approval.

Theorem 1.2.2 (A. Nowik [108]) If $X \in[\omega]^{\omega}$ is uniformly completely Ramsey null, then $X \in s_{0}^{\text {cube }}$.

Proof. Let $f: \mathfrak{C}^{\omega} \rightarrow \mathcal{P}(\omega)$ be a continuous injection. We need to find a perfect cube $C \subset \mathfrak{C}^{\omega}$ such that $f[C] \cap X=\emptyset$.

Let $\langle\cdot, \cdot\rangle: \omega \times \omega \rightarrow \omega$ be a bijection and define a function $F: \mathcal{P}(\omega) \rightarrow \mathfrak{C}^{\omega}$ by $F(A)(n)=\chi_{\left\{a_{\langle k, n\rangle}: k<\omega\right\}}$, where $\left\{a_{0}, a_{1}, \ldots\right\}$ is an increasing enumeration of $A$. It is easy to see that $F$ is a continuous injection. Therefore, the function $G=f \circ F: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ is continuous and so, by Proposition 1.2.1, there exists an $A \in[\omega]^{\omega}$ such that $|(f \circ F)[\mathcal{P}(A)] \cap X| \leq \omega$.

Let $A=\left\{a_{0}, a_{1}, a_{2}, \ldots\right\}$ be an increasing enumeration of elements of $A$ and define a function $\Xi: \mathfrak{C}^{\omega} \rightarrow \mathcal{P}(A)$ by

$$
\Xi(x)=\left\{a_{2\langle k, n\rangle}: x(n)(k)=0\right\} \cup\left\{a_{2\langle k, n\rangle+1}: x(n)(k)=1\right\} .
$$

We claim that:
$(*) F\left[\Xi\left[\mathfrak{C}^{\omega}\right]\right]$ is a perfect cube in $\mathfrak{C}^{\omega}$.
To see this, for every $k, n<\omega$, let $E_{k, n}=\left\{a_{2\langle k, n\rangle}, a_{2\langle k, n\rangle+1}\right\}$ and put

$$
D_{n}=\left\{x \in \mathfrak{C}: x^{-1}(1) \subseteq \bigcup_{k \in \omega} E_{k, n} \&(\forall k \in \omega)\left|E_{k, n} \cap x^{-1}(1)\right|=1\right\} .
$$

It is easy to see that each $D_{n}$ is perfect in $\mathfrak{C}$. We will show that

$$
F\left[\Xi\left[\mathfrak{C}^{\omega}\right]\right]=\prod_{n \in \omega} D_{n}
$$

So, let $x \in \mathfrak{C}^{\omega}$. To see that $F(\Xi(x)) \in \prod_{n \in \omega} D_{n}$, first notice that, if $\left\{b_{0}, b_{1}, \ldots\right\}$ is an increasing enumeration of $\Xi(x)$, then $b_{i} \in\left\{a_{2 i}, a_{2 i+1}\right\}$ for every $i<\omega$. Therefore $b_{\langle k, n\rangle} \in E_{k, n}$ for every $k, n<\omega$. In particular, $F(\Xi(x))(n)^{-1}(1)=\left\{b_{\langle k, n\rangle}: k<\omega\right\} \in D_{n}$ for every $n<\omega$.

To see the other inclusion, take $\left\langle x_{n}: n<\omega\right\rangle \in \prod_{n \in \omega} D_{n}$ and define $B=\bigcup_{n<\omega}\left(x_{n}\right)^{-1}(1)$. Then $F(B)=\left\langle x_{n}: n<\omega\right\rangle$ and $\left|B \cap E_{k, n}\right|=1$ for every $k, n<\omega$. Let $x \in \mathfrak{C}^{\omega}$ be such that $x(n)(k)=0$ if and only if $a_{2\langle k, n\rangle} \in B$. Then $\Xi(x)=B$ and so $\left\langle x_{n}: n<\omega\right\rangle=F(\Xi(x)) \in \prod_{n \in \omega} D_{n}$, finishing the proof of $(*)$.

Now, $D=F\left[\Xi\left[\mathfrak{C}^{\omega}\right]\right] \subset \mathfrak{C}^{\omega}$ is a perfect cube and $|f[D] \cap X| \leq \omega$, since $f[D]=f\left[F\left[\Xi\left[\mathfrak{C}^{\omega}\right]\right]\right] \subset f[F[\mathcal{P}(A)]]=(f \circ F)[\mathcal{P}(A)]$. Since $D$ can be partitioned into continuum many disjoint perfect cubes, for some member of the partition, say $C$, we will have $f[C] \cap X=\emptyset$.

Corollary 1.2.3 (A. Nowik [108]) $\mathrm{CPA}_{\text {cube }}$ implies that every uniformly completely Ramsey null set has cardinality less than continuum.

To discuss another application of $\mathrm{CPA}_{\text {cube }}$, let us consider the following covering number connected to a theorem of H. Blumberg (see Section 1.7) and studied by F. Jordan in [70]. Here $\mathcal{B}_{1}$ stands for the class of all Baire class 1 functions $f: \mathbb{R} \rightarrow \mathbb{R}$.

- $\operatorname{cov}\left(\mathcal{B}_{1}, \operatorname{Perf}(\mathbb{R})\right)$ is the smallest cardinality of $F \subset \mathcal{B}_{1}$ such that for each $P \in \operatorname{Perf}(\mathbb{R})$ there is an $f \in F$ with $f \upharpoonright P$ not continuous.

Jordan also proves $\left[70\right.$, thm. $7(\mathrm{a})$ ] that $\operatorname{cov}\left(\mathcal{B}_{1}, \operatorname{Perf}(\mathbb{R})\right)$ is equal to the covering number of the space $\operatorname{Perf}(\mathbb{R})$ (considered with the Hausdorff metric) by the elements of some $\sigma$-ideal $\mathcal{Z}_{p}$ and notices [70, thm. 17(a)] that every compact set $C \in \operatorname{Perf}(\mathbb{R})$ contains a dense $G_{\delta}$ subset that belongs to $\mathcal{Z}_{p}$. So, by Claim 1.1.5, the elements of $\operatorname{Perf}(\mathbb{R}) \cap \mathcal{Z}_{p}$ are $\mathcal{F}_{\text {cube }}$-dense in $\operatorname{Perf}(\mathbb{R})$. Thus

Corollary 1.2.4 $\mathrm{CPA}_{\text {cube }}$ implies that $\operatorname{cov}\left(\mathcal{B}_{1}, \operatorname{Perf}(\mathbb{R})\right)=\operatorname{cov}\left(\mathcal{Z}_{p}\right)=\omega_{1}$.

$$
1.3 \operatorname{cof}(\mathcal{N})=\omega_{1}
$$

$$
1.3 \operatorname{cof}(\mathcal{N})=\omega_{1}
$$

Next, following the argument of K. Ciesielski and J. Pawlikowski from [39], we show that $\mathrm{CPA}_{\text {cube }}$ implies that $\operatorname{cof}(\mathcal{N})=\omega_{1}$. So, under $\mathrm{CPA}_{\text {cube }}$, all cardinals from Cichońs diagram (see, e.g., [4]) are equal to $\omega_{1}$.

Let $\mathcal{C}_{H}$ be the family of all subsets $\prod_{n<\omega} T_{n}$ of $\omega^{\omega}$ such that $T_{n} \in[\omega] \leq n+1$ for all $n<\omega$. We will use the following characterization.

## Proposition 1.3.1 (T. Bartoszyński [4, thm. 2.3.9])

$$
\operatorname{cof}(\mathcal{N})=\min \left\{|\mathcal{F}|: \mathcal{F} \subset \mathcal{C}_{H} \& \bigcup \mathcal{F}=\omega^{\omega}\right\}
$$

Lemma 1.3.2 The family $\mathcal{C}_{H}^{*}=\left\{X \subset \omega^{\omega}: X \subset T\right.$ for some $\left.T \in \mathcal{C}_{H}\right\}$ is $\mathcal{F}_{\text {cube }}$-dense in $\operatorname{Perf}\left(\omega^{\omega}\right)$.

Proof. Let $f: \mathfrak{C}^{\omega} \rightarrow \omega^{\omega}$ be a continuous function. By (1.4) it is enough to find a perfect cube $C$ in $\mathfrak{C}^{\omega}$ such that $f[C] \in \mathcal{C}_{H}^{*}$.

Construct, by induction on $n<\omega$, the families $\left\{E_{s}^{i}: s \in 2^{n} \& i<\omega\right\}$ of nonempty clopen subsets of $\mathfrak{C}$ such that, for every $n<\omega$ and $s, t \in 2^{n}$,
(i) $E_{s}^{i}=E_{t}^{i}$ for every $n \leq i<\omega$;
(ii) $E_{s^{\wedge} 0}^{i}$ and $E_{s^{\wedge} 1}^{i}$ are disjoint subsets of $E_{s}^{i}$ for every $i<n+1$;
(iii) for every $\left\langle s_{i} \in 2^{n}: i<\omega\right\rangle$

$$
f\left(x_{0}\right) \upharpoonright 2^{(n+1)^{2}}=f\left(x_{1}\right) \upharpoonright 2^{(n+1)^{2}} \quad \text { for every } \quad x_{0}, x_{1} \in \prod_{i<\omega} E_{s_{i}}
$$

For each $i<\omega$ the fusion of $\left\{E_{s}^{i}: s \in 2^{<\omega}\right\}$ will give us the $i$-th coordinate set of the desired perfect cube $C$.

Condition (iii) can be ensured by the uniform continuity of $f$. Indeed, let $\delta>0$ be such that $f\left(x_{0}\right) \upharpoonright 2^{(n+1)^{2}}=f\left(x_{1}\right) \upharpoonright 2^{(n+1)^{2}}$ for every $x_{0}, x_{1} \in \mathfrak{C}^{\omega}$ of distance less than $\delta$. Then it is enough to choose $\left\{E_{s}^{i}: s \in 2^{n} \& i<\omega\right\}$ such that (i) and (ii) are satisfied and every set $\prod_{i<\omega} E_{s_{i}}$ from (iii) has diameter less than $\delta$. This finishes the construction.

Next, for every $i, n<\omega$, let $E_{n}^{i}=\bigcup\left\{E_{s}^{i}: s \in 2^{n}\right\}$ and $E_{n}=\prod_{i<\omega} E_{n}^{i}$. Then $C=\bigcap_{n<\omega} E_{n}=\prod_{i<\omega}\left(\bigcap_{n<\omega} E_{n}^{i}\right)$ is a perfect cube in $\mathfrak{C}^{\omega}$, since $\bigcap_{n<\omega} E_{n}^{i} \in \operatorname{Perf}(\mathfrak{C})$ for every $i<\omega$. Thus, to finish the proof it is enough to show that $f[C] \in \mathcal{C}_{H}^{*}$.

So, for every $k<\omega$, let $n<\omega$ be such that $2^{n^{2}} \leq k+1<2^{(n+1)^{2}}$, put $T_{k}=\left\{f(x)(k): x \in E_{n}\right\}=\left\{f(x)(k): x \in \prod_{i<\omega} E_{s_{i}}\right.$ for some $\left.\left\langle s_{i} \in 2^{n}: i<\omega\right\rangle\right\}$, and notice that $T_{k}$ has at most $2^{n^{2}} \leq k+1$ elements. Indeed, by (iii), the
set $\left\{f(x)(k): x \in \prod_{i<\omega} E_{s_{i}}\right\}$ is a singleton for every $\left\langle s_{i} \in 2^{n}: i<\omega\right\rangle$ while (i) implies that $\left\{\prod_{i<\omega} E_{s_{i}}:\left\langle s_{i} \in 2^{n}: i<\omega\right\rangle\right\}$ has $2^{n^{2}}$ elements. Therefore $\prod_{k<\omega} T_{k} \in \mathcal{C}_{H}$.

To finish the proof it is enough to notice that $f[C] \subset \prod_{k<\omega} T_{k}$.

Corollary 1.3.3 If $\mathrm{CPA}_{\text {cube }}$ holds, then $\operatorname{cof}(\mathcal{N})=\omega_{1}$.
Proof. By $\mathrm{CPA}_{\text {cube }}$ and Lemma 1.3.2, there exists an $\mathcal{F} \in\left[\mathcal{C}_{H}\right]^{\leq \omega_{1}}$ such that $\left|\omega^{\omega} \backslash \bigcup \mathcal{F}\right| \leq \omega_{1}$. This and Proposition 1.3.1 imply $\operatorname{cof}(\mathcal{N})=\omega_{1}$.

### 1.4 Total failure of Martin's axiom

In this section we prove that $\mathrm{CPA}_{\text {cube }}$ implies the total failure of Martin's axiom, that is, the property that:

For every nontrivial ccc forcing $\mathbb{P}$ there exists $\omega_{1}$ many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

The consistency of this fact with $\mathfrak{c}>\omega_{1}$ was first proved by J. Baumgartner [6] in a model obtained by adding Sacks reals side by side. The topological and Boolean algebraic formulations of the theorem follow immediately from the following proposition. The proof presented below comes from K. Ciesielski and J. Pawlikowski [39].

Proposition 1.4.1 The following conditions are equivalent.
(a) For every nontrivial ccc forcing $\mathbb{P}$ there exists $\omega_{1}$ many dense sets in $\mathbb{P}$ such that no filter intersects all of them.
(b) Every compact ccc topological space without isolated points is a union of $\omega_{1}$ nowhere dense sets.
(c) For every atomless ccc complete Boolean algebra $B$ there exist $\omega_{1}$ many dense sets in $B$ such that no filter intersects all of them.
(d) For every atomless ccc complete Boolean algebra $B$ there exist $\omega_{1}$ many maximal antichains in $B$ such that no filter intersects all of them.
(e) For every countably generated atomless ccc complete Boolean algebra $B$ there exists $\omega_{1}$ many maximal antichains in $B$ such that no filter intersects all of them.

Proof. The equivalence of conditions (a), (b), (c), and (d) is well known. In particular, equivalences (a)-(c) are explicitly given in [6, thm. 0.1].

Clearly (d) implies (e). The remaining implication, $(\mathrm{e}) \Longrightarrow(\mathrm{d})$, is a version of the theorem from [89, p. 158]. However, it is expressed there in slightly different language, so we include its proof here.
Let $\langle B, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ be an atomless ccc complete Boolean algebra. For every $\sigma \in 2^{<\omega_{1}}$ define, by induction on the length $\operatorname{dom}(\sigma)$ of a sequence $\sigma$, a $b_{\sigma} \in B$ such that the following conditions are satisfied:

- $b_{\emptyset}=1$.
- $b_{\sigma}$ is a disjoint union of $b_{\sigma^{\wedge} 0}$ and $b_{\sigma^{\wedge} 1}$.
- If $b_{\sigma}>\mathbf{0}$, then $b_{\sigma^{\wedge} 0}>\mathbf{0}$ and $b_{\sigma^{\wedge} 1}>\mathbf{0}$.
- If $\lambda=\operatorname{dom}(\sigma)$ is a limit ordinal, then $b_{\sigma}=\bigwedge_{\xi<\lambda} b_{\sigma \upharpoonright \xi}$.

Let $T=\left\{s \in 2^{<\omega_{1}}: b_{s}>\mathbf{0}\right\}$. Then $T$ is a subtree of $2^{<\omega_{1}}$; its levels determine antichains in $B$, so they are countable.

First assume that $T$ has a countable height. Then $T$ itself is countable. Let $B_{0}$ be the smallest complete subalgebra of $B$ containing $\left\{b_{\sigma}: \sigma \in T\right\}$ and notice that $B_{0}$ is atomless. Indeed, if there were an atom $a$ in $B_{0}$, then $S=\left\{\sigma \in T: a \leq b_{\sigma}\right\}$ would be a branch in $T$ so that $\delta=\bigcup S$ would belong to $2^{<\omega_{1}}$. Since $b_{\delta} \geq a>\mathbf{0}$, we would also have $\delta \in T$. But then $a \leq b_{\delta}=b_{\delta^{\wedge} 0} \vee b_{\delta^{\wedge} 1}$, so that either $\delta^{\wedge} 0$ or $\delta^{\wedge} 1$ belongs to $S$, which is impossible.

Thus, $B_{0}$ is a complete, countably generated, atomless subalgebra of $B$. So, by (e), there exists a family $\mathcal{A}$ of $\omega_{1}$ many maximal antichains in $B_{0}$ with no filter in $B_{0}$ intersecting all of them. But then each $A \in \mathcal{A}$ is also a maximal antichain in $B$ and no filter in $B$ would intersect all of them. So, we have (d).

Next, assume that $T$ has height $\omega_{1}$ and for every $\alpha<\omega_{1}$ let

$$
T_{\alpha}=\{\sigma \in T: \operatorname{dom}(\sigma)=\alpha\}
$$

be the $\alpha$-th level of $T$. Also let $b_{\alpha}=\bigvee_{\sigma \in T_{\alpha}} b_{\sigma}$. Notice that $b_{\alpha}=b_{\alpha+1}$ for every $\alpha<\omega_{1}$. On the other hand, it may happen that $b_{\lambda}>\bigwedge_{\alpha<\lambda} b_{\alpha}$ for some limit $\lambda<\omega_{1}$; however, this may happen only countably many times, since $B$ is ccc. Thus, there is an $\alpha<\omega_{1}$ such that $b_{\beta}=b_{\alpha}$ for every $\alpha<\beta<\omega_{1}$.

Now, let $B_{0}$ be the smallest complete subalgebra of $B$ below $\mathbf{1} \backslash b_{\alpha}$ containing $\left\{b_{\sigma} \backslash b_{\alpha}: \sigma \in T\right\}$. Then $B_{0}$ is countably generated and, as before, it can be shown that $B_{0}$ is atomless. Thus, there exists a family $\mathcal{A}_{0}$ of $\omega_{1}$ many maximal antichains in $B_{0}$ with no filter in $B_{0}$ intersecting all of them. Then no filter in $B$ containing $\mathbf{1} \backslash b_{\alpha}$ intersects every $A \in \mathcal{A}_{0}$. But for every $\alpha<\beta<\omega_{1}$ the set $A^{\beta}=\left\{b_{\sigma}: \sigma \in T_{\beta}\right\}$ is a maximal antichain in $B$ below $b_{\alpha}$. Therefore, $\mathcal{A}_{1}=\left\{A^{\beta}: \alpha<\beta<\omega_{1}\right\}$ is an uncountable
family of maximal antichains in $B$ below $b_{\alpha}$ with no filter in $B$ containing $b_{\alpha}$ intersecting every $A \in \mathcal{A}_{1}$. Then it is easy to see that the family $\mathcal{A}=\left\{A_{0} \cup A_{1}: a_{0} \in \mathcal{A}_{0} \& A_{1} \in \mathcal{A}_{1}\right\}$ is a family of $\omega_{1}$ many maximal antichains in $B$ with no filter in $B$ intersecting all of them. This proves condition (d).

Theorem 1.4.2 $\mathrm{CPA}_{\text {cube }}$ implies the total failure of Martin's axiom.
Proof. Let $\mathcal{A}$ be a countably generated, atomless, ccc complete Boolean algebra and let $\left\{A_{n}: n<\omega\right\}$ generate $\mathcal{A}$. By Proposition 1.4.1 it is enough to show that $\mathcal{A}$ contains $\omega_{1}$ many maximal antichains such that no filter in $\mathcal{A}$ intersects all of them.

Next let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of a space $\mathfrak{C}=2^{\omega}$. Recall that it is a free countably generated $\sigma$-algebra, with the free generators $B_{i}=\{s \in \mathfrak{C}: s(i)=0\}$. Define $h_{0}:\left\{B_{n}: n<\omega\right\} \rightarrow\left\{A_{n}: n<\omega\right\}$ by $h_{0}\left(B_{n}\right)=A_{n}$ for all $n<\omega$. Then $h_{0}$ can be uniquely extended to a $\sigma$ homomorphism $h: \mathcal{B} \rightarrow \mathcal{A}$ between $\sigma$-algebras $\mathcal{B}$ and $\mathcal{A}$. (See, e.g., [121, 34.1 p. 117].) Let $\mathcal{I}=\{B \in \mathcal{B}: h[B]=\mathbf{0}\}$. Then $\mathcal{I}$ is a $\sigma$-ideal in $\mathcal{B}$ and the quotient algebra $\mathcal{B} / \mathcal{I}$ is isomorphic to $\mathcal{A}$. (Compare also Loomis and Sikorski's theorem in [121, p. 117] or [86].) In particular, $\mathcal{I}$ contains all singletons and is ccc, since $\mathcal{A}$ is atomless and ccc.
It follows that we need only to consider complete Boolean algebras of the form $\mathcal{B} / \mathcal{I}$, where $\mathcal{I}$ is some ccc $\sigma$-ideal of Borel sets containing all singletons. To prove that such an algebra has $\omega_{1}$ maximal antichains as desired, it is enough to prove that:
$(*) \mathfrak{C}$ is a union of $\omega_{1}$ perfect sets $\left\{N_{\xi}: \xi<\omega_{1}\right\}$ that belong to $\mathcal{I}$.
Indeed, assume that $(*)$ holds and for every $\xi<\omega_{1}$ let $\mathcal{D}_{\xi}^{*}$ be a family of all $B \in \mathcal{B} \backslash \mathcal{I}$ with closures $\operatorname{cl}(B)$ disjoint from $N_{\xi}$. Then $\mathcal{D}_{\xi}=\left\{B / \mathcal{I}: B \in \mathcal{D}_{\xi}^{*}\right\}$ is dense in $\mathcal{B} / \mathcal{I}$, since $\mathfrak{C} \backslash N_{\xi}$ is $\sigma$-compact and $\mathcal{B} / \mathcal{I}$ is a $\sigma$-algebra. Let $\mathcal{A}_{\xi}^{*} \subset \mathcal{D}_{\xi}^{*}$ be such that $\mathcal{A}_{\xi}=\left\{B / \mathcal{I}: B \in \mathcal{A}_{\xi}^{*}\right\}$ is a maximal antichain in $\mathcal{B} / \mathcal{I}$. It is enough to show that no filter intersects all $\mathcal{A}_{\xi}$ 's. But if there were a filter $\mathcal{F}$ in $\mathcal{B} / \mathcal{I}$ intersecting all $\mathcal{A}_{\xi}$ 's, then for every $\xi<\omega_{1}$ there would exist a $B_{\xi} \in \mathcal{A}_{\xi}^{*}$ with $B_{\xi} / \mathcal{I} \in \mathcal{F} \cap \mathcal{A}_{\xi}$. Thus, the set $\bigcap_{\xi<\omega_{1}} \operatorname{cl}\left(B_{\xi}\right)$ would be nonempty, despite the fact that it is disjoint from $\bigcup_{\xi<\omega_{1}} N_{\xi}=\mathfrak{C}$.

To finish the proof it is enough to show that $(*)$ follows from $\mathrm{CPA}_{\text {cube }}$. But this follows immediately from the fact that any cube $P$ in $\mathfrak{C}$ contains a subcube $Q \in \mathcal{I}$ as any cube $P$ can be partitioned into $\mathfrak{c}$ many disjoint subcubes, and, by the ccc property of $\mathcal{I}$, only countably many of them can be outside $\mathcal{I}$.

### 1.5 Selective ultrafilters and the reaping numbers $\mathfrak{r}$ and $\mathfrak{r}_{\sigma}$

In this section, which is based in part on a paper by K. Ciesielski and J. Pawlikowski [37], we will show that $\mathrm{CPA}_{\text {cube }}$ implies that every selective ultrafilter is generated by $\omega_{1}$ sets and that the reaping number $\mathfrak{r}$ is equal to $\omega_{1}$. The actual construction of a selective ultrafilter will require a stronger version of the axiom and will be done in Theorem 5.3.3.

We will use here the terminology introduced in the Preliminaries chapter. In particular, recall that the ideal $[\omega]^{<\omega}$ of finite subsets of $\omega$ is semiselective.

The most important combinatorial fact for us concerning semiselective ideals is the following property. (See Theorem 2.1 and Remark 4.1 in [55].) This is a generalization of a theorem of R. Laver [85], who proved this fact for the ideal $\mathcal{I}=[\omega]^{<\omega}$.

Proposition 1.5.1 (I. Farah [55]) Let $\mathcal{I}$ be a semiselective ideal on $\omega$. For every analytic set $S \subset \mathfrak{C}^{\omega} \times[\omega]^{\omega}$ and every $A \in \mathcal{I}^{+}$there exist a $B \in \mathcal{I}^{+} \cap \mathcal{P}(A)$ and a perfect cube $C$ in $\mathfrak{C}^{\omega}$ such that $C \times[B]^{\omega}$ is either contained in or disjoint with $S$.

With this fact in hand we can prove the following theorem.

Theorem 1.5.2 Assume that $\mathrm{CPA}_{\text {cube }}$ holds. If $\mathcal{I}$ is a semiselective ideal, then there is a family $\mathcal{W} \subset \mathcal{I}^{+},|\mathcal{W}| \leq \omega_{1}$, such that for every analytic set $A \subset[\omega]^{\omega}$ there is a $W \in \mathcal{W}$ for which either $[W]^{\omega} \subset A$ or $[W]^{\omega} \cap A=\emptyset$.

Proof. Let $S \subset \mathfrak{C} \times[\omega]^{\omega}$ be a universal analytic set, that is, such that the family $\left\{S_{x}: x \in \mathfrak{C}\right\}$ (where $S_{x}=\left\{y \in[\omega]^{\omega}:\langle x, y\rangle \in S\right\}$ ) contains all analytic subsets of $[\omega]^{\omega}$. (See, e.g., [69, lem. 39.4].) In fact, we will take $S$ such that, for any analytic set $A$ in $[\omega]^{\omega}$,

$$
\begin{equation*}
\left|\left\{x \in \mathfrak{C}: S_{x}=A\right\}\right|=\mathfrak{c} \tag{1.6}
\end{equation*}
$$

(If $U \subset \mathfrak{C} \times[\omega]^{\omega}$ is a universal analytic set, then $S=\mathfrak{C} \times U \subset \mathfrak{C} \times \mathfrak{C} \times[\omega]^{\omega}$ satisfies (1.6), where we identify $\mathfrak{C} \times \mathfrak{C}$ with $\mathfrak{C}$.) For this particular set $S$ consider the family $\mathcal{E}$ of all $Q \in \operatorname{Perf}(\mathfrak{C})$ for which there exists a $W_{Q} \in \mathcal{I}^{+}$ such that

$$
\begin{equation*}
Q \times\left[W_{Q}\right]^{\omega} \text { is either contained in or disjoint from } S \tag{1.7}
\end{equation*}
$$

Note that, by Proposition 1.5.1, the family $\mathcal{E}$ is $\mathcal{F}_{\text {cube }}$-dense in $\operatorname{Perf}(\mathfrak{C})$. So, by $\mathrm{CPA}_{\text {cube }}$, there exists an $\mathcal{E}_{0} \subset \mathcal{E},\left|\mathcal{E}_{0}\right| \leq \omega_{1}$, such that $\left|\mathfrak{C} \backslash \bigcup \mathcal{E}_{0}\right|<\mathfrak{c}$. Let

$$
\mathcal{W}=\left\{W_{Q}: Q \in \mathcal{E}_{0}\right\}
$$

It is enough to see that this $\mathcal{W}$ is as required.
Clearly $|\mathcal{W}| \leq \omega_{1}$. Also, by (1.6), for an analytic set $A \subset[\omega]^{\omega}$ there exist a $Q \in \mathcal{E}_{0}$ and an $x \in Q$ such that $A=S_{x}$. So, by (1.7), $\{x\} \times\left[W_{Q}\right]^{\omega}$ is either contained in or disjoint from $\{x\} \times S_{x}=\{x\} \times A$.

Recall (see, e.g., [4] or [128]) that a family $\mathcal{W} \subset[\omega]^{\omega}$ is a reaping family provided

$$
\forall A \in[\omega]^{\omega} \exists W \in \mathcal{W}(W \subset A \text { or } W \subset \omega \backslash A)
$$

The reaping (or refinement) number $\mathfrak{r}$ is defined as the minimum cardinality of a reaping family. Also, a number $\mathfrak{r}_{\sigma}$ is defined as the smallest cardinality of a family $\mathcal{W} \subset[\omega]^{\omega}$ such that for every sequence $\left\langle A_{n} \in[\omega]^{\omega}: n<\omega\right\rangle$ there exists a $W \in \mathcal{W}$ such that for every $n<\omega$ either $W \subseteq^{*} A_{n}$ or $W \subseteq^{*} \omega \backslash A_{n}$. (See [18] or [128].) Clearly $\mathfrak{r} \leq \mathfrak{r}_{\sigma}$.

Corollary 1.5.3 If $\mathrm{CPA}_{\text {cube }}$ holds, then for every semiselective ideal $\mathcal{I}$ there exists a family $\mathcal{W} \subset \mathcal{I}^{+},|\mathcal{W}| \leq \omega_{1}$, such that for every $A \in[\omega]^{\omega}$ there is a $W \in \mathcal{W}$ for which either $W \subseteq^{*} A$ or $W \subseteq^{*} \omega \backslash A$. In particular, $\mathrm{CPA}_{\text {cube }}$ implies that $\mathfrak{r}=\omega_{1}<\mathfrak{c}$.

Proof. The family $\mathcal{W}$ from Theorem 1.5.2 works: since $[A]^{\omega}$ is analytic in $[\omega]^{\omega}$, there exists a $W \in \mathcal{W}$ such that either $[W]^{\omega} \subset[A]^{\omega}$ or $[W]^{\omega} \cap[A]^{\omega}=\emptyset$.

Note also that $\mathrm{CPA}_{\text {cube }}$ implies the second part of property (E).

Corollary 1.5.4 If $\mathrm{CPA}_{\text {cube }}$ holds, then every selective ultrafilter $\mathcal{F}$ on $\omega$ is generated by a family of size $\omega_{1}<\mathfrak{c}$.

Proof. If $\mathcal{F}$ is a selective ultrafilter on $\omega$, then $\mathcal{I}=\mathcal{P}(\omega) \backslash \mathcal{F}$ is a selective ideal and $\mathcal{I}^{+}=\mathcal{F}$. Let $\mathcal{W} \subset \mathcal{I}^{+}=\mathcal{F}$ be as in Corollary 1.5.3. Then $\mathcal{W}$ generates $\mathcal{F}$.

Indeed, if $A \in \mathcal{F}$, then there exists a $W \in \mathcal{W}$ such that either $W \subset A$ or $W \subset \omega \backslash A$. But it is impossible that $W \subset \omega \backslash A$, since then we would have $\emptyset=A \cap W \in \mathcal{F}$.

As mentioned above, in Theorem 5.3.3 we will prove that some version of our axiom implies that there exists a selective ultrafilter on $\omega$. In particular, the assumptions of the next corollary are implied by such a version of our axiom.

Corollary 1.5.5 If $\mathrm{CPA}_{\text {cube }}$ holds and there exists a selective ultrafilter $\mathcal{F}$ on $\omega$, then $\mathfrak{r}_{\sigma}=\omega_{1}<\mathfrak{c}$.

Proof. Let $\mathcal{W} \in[\mathcal{F}]^{\leq \omega_{1}}$ be a generating family for $\mathcal{F}$. We will show that it justifies $\mathfrak{r}_{\sigma}=\omega_{1}$. Indeed, take a sequence $\left\langle A_{n} \in[\omega]^{\omega}: n<\omega\right\rangle$. For every $n<\omega$, let $A_{n}^{*}$ belong to $\mathcal{F} \cap\left\{A_{n}, \omega \backslash A_{n}\right\}$. Since $\mathcal{F}$ is selective, there exists an $A \in \mathcal{F}$ such that $A \subseteq^{*} A_{n}^{*}$ for every $n<\omega$. Let $W \in \mathcal{W}$ be such that $W \subset A$. Then for every $n<\omega$ either $W \subseteq^{*} A_{n}$ or $W \subseteq^{*} \omega \backslash A_{n}$.

We are particularly interested in the number $\mathfrak{r}_{\sigma}$ since it is related to different variants of sets of uniqueness coming from harmonic analysis, as described in the survey paper [18]. In particular, from [18, thm. 12.6] it follows that an appropriate version of our axiom implies that all covering numbers described in the paper are equal to $\omega_{1}$.

### 1.6 On the convergence of subsequences of real-valued functions

This section can be viewed as an extension of the discussion of Egorov's theorem presented in [77, chapter 9]. In 1932, S. Mazurkiewicz [91] proved the following variant of Egorov's theorem, where a sequence $\left\langle f_{n}\right\rangle_{n<\omega}$ of real-valued functions is uniformly bounded provided there exists an $r \in \mathbb{R}$ such that range $\left(f_{n}\right) \subset[-r, r]$ for every $n$.
Mazurkiewicz' Theorem For every uniformly bounded sequence $\left\langle f_{n}\right\rangle_{n<\omega}$ of real-valued continuous functions defined on a Polish space $X$ there exists a subsequence that is uniformly convergent on some perfect set $P$.

The proof of the next theorem comes from the paper [39] of K. Ciesielski and J. Pawlikowski .

Theorem 1.6.1 If $\mathrm{CPA}_{\text {cube }}$ holds, then for every Polish space $X$ and every uniformly bounded sequence $\left\langle f_{n}: X \rightarrow \mathbb{R}\right\rangle_{n<\omega}$ of Borel measurable functions there are two sequences, $\left\langle P_{\xi}: \xi<\omega_{1}\right\rangle$ of compact subsets of $X$ and $\left\langle W_{\xi} \in[\omega]^{\omega}: \xi<\omega_{1}\right\rangle$, such that $X=\bigcup_{\xi<\omega_{1}} P_{\xi}$ and for every $\xi<\omega_{1}$ :
$\left\langle f_{n} \upharpoonright P_{\xi}\right\rangle_{n \in W_{\xi}}$ is a monotone uniformly convergent sequence of uniformly continuous functions.

Proof. We first note that the family $\mathcal{E}$ of all $P \in \operatorname{Perf}(X)$ for which there exists a $W \in[\omega]^{\omega}$ such that
the sequence $\left\langle f_{n} \upharpoonright P\right\rangle_{n \in W}$ is monotone and uniformly convergent
is $\mathcal{F}_{\text {cube }}$-dense in $\operatorname{Perf}(X)$.
Indeed, let $g \in \mathcal{F}_{\text {cube }}, g: \mathfrak{C}^{\omega} \rightarrow X$ and consider the functions $h_{n}=f_{n} \circ g$. Since $h=\left\langle h_{n}: n<\omega\right\rangle: \mathfrak{C}^{\omega} \rightarrow \mathbb{R}^{\omega}$ is Borel measurable, there is a dense $G_{\delta}$ subset $G$ of $\mathfrak{C}^{\omega}$ such that $h \upharpoonright G$ is continuous. So, we can find a perfect

## Games and axiom $\mathrm{CPA}_{\text {cube }}^{\text {game }}$

Before we get to the formulation of our next version of the axiom, it would be good to note that in many applications we would prefer to have a full covering of a Polish space $X$ rather that the almost covering as claimed by $\mathrm{CPA}_{\text {cube }}$. To get better access to the missing singletons ${ }^{1}$ we will extend the notion of a cube by also allowing the "constant cubes:" A family $\mathcal{C}_{\text {cube }}(X)$ of constant cubes is defined as the family of all constant functions from a perfect cube $C \subset \mathfrak{C}^{\omega}$ into $X$. We also define $\mathcal{F}_{\text {cube }}^{*}(X)$ as

$$
\begin{equation*}
\mathcal{F}_{\text {cube }}^{*}=\mathcal{F}_{\text {cube }} \cup \mathcal{C}_{\text {cube }} . \tag{2.1}
\end{equation*}
$$

Thus, $\mathcal{F}_{\text {cube }}^{*}$ is the family of all continuous functions from a perfect cube $C \subset \mathfrak{C}^{\omega}$ into $X$ that are either one to one or constant. Now the range of every $f \in \mathcal{F}_{\text {cube }}^{*}$ belongs to the family $\operatorname{Perf}^{*}(X)$ of all sets $P$ such that either $P \in \operatorname{Perf}(X)$ or $P$ is a singleton. The terms " $P \in \operatorname{Perf}^{*}(X)$ is a cube" and " $Q$ is a subcube of a cube $P \in \operatorname{Perf}^{*}(X)$ " are defined in a natural way.

Consider also the following game $\operatorname{GAME}_{\text {cube }}(X)$ of length $\omega_{1}$. The game has two players, Player I and Player II. At each stage $\xi<\omega_{1}$ of the game Player I can play an arbitrary cube $P_{\xi} \in \operatorname{Perf}^{*}(X)$ and Player II must respond with a subcube $Q_{\xi}$ of $P_{\xi}$. The game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ is won by Player I provided

$$
\bigcup_{\xi<\omega_{1}} Q_{\xi}=X
$$

otherwise, the game is won by Player II.
By a strategy for Player II we will consider any function $S$ such that $S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)$ is a subcube of $P_{\xi}$, where $\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle$ is any

1 The logic for accessing the singletons in such a strange is justified by the versions of the axiom that will be presented in Chapter 6.
partial game. (We abuse here slightly the notation, since the function $S$ also depends on the implicitly given coordinate functions $f_{\eta}: \mathbb{C}^{\omega} \rightarrow P_{\eta}$, making each $P_{\eta}$ a cube.) A game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ is played according to a strategy $S$ for Player II provided $Q_{\xi}=S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)$ for every $\xi<\omega_{1}$. A strategy $S$ for Player II is a winning strategy for Player II provided Player II wins any game played according to the strategy $S$.
Here is our new version of the axiom.
$\mathrm{CPA}_{\text {cube }}^{\text {game }}: \mathfrak{c}=\omega_{2}$, and for any Polish space $X$ Player II has no winning strategy in the game $\operatorname{GAME}_{\text {cube }}(X)$.

Notice that
Proposition 2.0.1 Axiom $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies $\mathrm{CPA}_{\text {cube }}$.
Proof. Let $\mathcal{E} \subset \operatorname{Perf}(X)$ be an $\mathcal{F}_{\text {cube- }}$ dense family. Thus, for every cube $P \in \operatorname{Perf}(X)$ there exists a subcube $s(P) \in \mathcal{E}$ of $P$. Now, for a singleton $P \in \operatorname{Perf}^{*}(X)$, put $s(P)=P$ and consider the following strategy (in fact, it is a tactic) $S$ for Player II:

$$
S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)=s\left(P_{\xi}\right) .
$$

By $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ it is not a winning strategy for Player II. So there exists a game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ in which $Q_{\xi}=s\left(P_{\xi}\right)$ for every $\xi<\omega_{1}$ and Player II loses, that is, $X=\bigcup_{\xi<\omega_{1}} Q_{\xi}$. Now, let $\mathcal{E}_{0}=\left\{Q_{\xi}: \xi<\omega_{1} \& Q_{\xi} \in \operatorname{Perf}(X)\right\}$. Then $\left|X \backslash \bigcup \mathcal{E}_{0}\right| \leq \omega_{1}$, so $\mathrm{CPA}_{\text {cube }}$ is justified.

## $2.1 \mathrm{CPA}_{\text {cube }}^{\text {game }}$ and disjoint coverings

The results presented in this section come from a paper [39] of K. Ciesielski and J. Pawlikowski.

Theorem 2.1.1 Assume that $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ holds and let $X$ be a Polish space. If $\mathcal{D} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {cube-dense }}$ and it is closed under perfect subsets, then there exists a partition of $X$ into $\omega_{1}$ disjoint sets from $\mathcal{D} \cup\{\{x\}: x \in X\}$.

In the proof we will use the following easy lemma.
Lemma 2.1.2 Let $\mathcal{P}=\left\{P_{i}: i<\omega\right\} \subset \operatorname{Perf}^{*}(X)$, where $X$ is a Polish space. For every cube $P \in \operatorname{Perf}(X)$ there exists a subcube $Q$ of $P$ such that either $Q \cap \bigcup_{i<\omega} P_{i}=\emptyset$ or $Q \subset P_{i}$ for some $i<\omega$.

Proof. Let $f \in \mathcal{F}_{\text {cube }}$ be such that $f\left[\mathfrak{C}^{\omega}\right]=P$.
If $P \cap \bigcup_{i<\omega} P_{i}$ is meager in $P$, then, by Claim 1.1.5, we can find a subcube $Q$ of $P$ such that $Q \subset P \backslash \bigcup_{i<\omega} P_{i}$.

If $P \cap \bigcup_{i<\omega} P_{i}$ is not meager in $P$, then there exists an $i<\omega$ such that $P \cap P_{i}$ has a nonempty interior in $P$. Thus, there exists a basic clopen set $C$ in $\mathfrak{C}^{\omega}$, which is a perfect cube, such that $f[C] \subset P_{i}$. So, $Q=f[C]$ is a desired subcube of $P$.

Proof of Theorem 2.1.1. For a cube $P \in \operatorname{Perf}(X)$ and a countable family $\mathcal{P} \subset \operatorname{Perf}^{*}(X)$, let $D(P) \in \mathcal{D}$ be a subcube of $P$ and $Q(\mathcal{P}, P) \in \mathcal{D}$ be as in Lemma 2.1.2 used with $D(P)$ in place of $P$. For a singleton $P \in \operatorname{Perf}^{*}(X)$ we just put $Q(\mathcal{P}, P)=P$.

Consider the following strategy $S$ for Player II:

$$
S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)=Q\left(\left\{Q_{\eta}: \eta<\xi\right\}, P_{\xi}\right)
$$

By $\mathrm{CPA}_{\text {cube }}^{\text {game }}$, strategy $S$ is not a winning strategy for Player II. So there exists a game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ played according to $S$ in which Player II looses, that is, $X=\bigcup_{\xi<\omega_{1}} Q_{\xi}$.

Notice that for every $\xi<\omega_{1}$ either $Q_{\xi} \cap \bigcup_{\eta<\xi} Q_{\eta}=\emptyset$ or there is an $\eta<\omega_{1}$ such that $Q_{\xi} \subset Q_{\eta}$. Let

$$
\mathcal{F}=\left\{Q_{\xi}: \xi<\omega_{1} \& Q_{\xi} \cap \bigcup_{\eta<\xi} Q_{\eta}=\emptyset\right\}
$$

Then $\mathcal{F}$ is as desired.
Since a family of all measure zero perfect subsets of $\mathbb{R}^{n}$ is $\mathcal{F}_{\text {cube }}$-dense, we get the following corollary.

Corollary 2.1.3 $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies that there exists a partition of $\mathbb{R}^{n}$ into $\omega_{1}$ many disjoint closed nowhere dense measure zero sets.

Note that the conclusion of Corollary 2.1.3 does not follow from the fact that $\mathbb{R}^{n}$ can be covered by $\omega_{1}$ perfect measure zero subsets. (See A. Miller [94, thm. 6].)

The next corollary is a generalization of Fact 1.1.7.
Corollary 2.1.4 $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies that for every Borel subset $B$ of a Polish space $X$ there exists a family $\mathcal{P}$ of $\omega_{1}$ many disjoint compact sets such that $B=\bigcup \mathcal{P}$.

Proof. This follows immediately from Theorem 2.1.1 and the fact that the family $\mathcal{E}=\{P \in \operatorname{Perf}(X): P \subset B$ or $P \cap B=\emptyset\}$ is $\mathcal{F}_{\text {cube-dense }}$.

2 Games and axiom $\mathrm{CPA}_{\text {cube }}^{\text {game }}$

### 2.2 MAD families and the numbers $\mathfrak{a}$ and $\mathfrak{r}$

Recall that a family $\mathcal{A} \subset[\omega]^{\omega}$ is almost disjoint provided $|A \cap B|<\omega$ and it is maximal almost disjoint, MAD, provided it is not a proper subfamily of any other almost disjoint family. The cardinal number $\mathfrak{a}$ is defined as follows:

$$
\mathfrak{a}=\min \{|\mathcal{A}|: \mathcal{A} \text { is infinite and } \operatorname{MAD}\}
$$

The fact that $\mathfrak{a}=\omega_{1}$ holds in the iterated perfect set model was apparently first noticed by Otmar Spinas (see A. Blass [9, sec. 11.5]), though it seems that the proof of this result was never provided. The argument presented below comes from K. Ciesielski and J. Pawlikowski [37].

Theorem 2.2.1 $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies that $\mathfrak{a}=\omega_{1}$.
Our proof of Theorem 2.2.1 is based on the following lemma.
Lemma 2.2.2 For every countable infinite family $\mathcal{W} \subset[\omega]^{\omega}$ of almost disjoint sets and a cube $P \in \operatorname{Perf}\left([\omega]^{\omega}\right)$ there exist a $W \in[\omega]^{\omega}$ and a subcube $Q$ of $P$ such that $\mathcal{W} \cup\{W\}$ is almost disjoint but $\mathcal{W} \cup\{W, x\}$ is not almost disjoint for every $x \in Q$.

Proof. Let $\mathcal{W}=\left\{W_{i}: i<\omega\right\}$. For every $i<\omega$ choose sets $V_{i} \subset W_{i}$ such that the $V_{i}$ 's are pairwise disjoint and each $W_{i} \backslash V_{i}$ is finite, but $V_{\omega}=$ $\omega \backslash \bigcup_{i<\omega} V_{i}$ is infinite. (For example, for every $i<\omega$, put $V_{i}^{*}=W_{i} \backslash \bigcup_{j<i} W_{j}$ and $V_{i}=V_{i}^{*} \backslash\left\{\min V_{i}^{*}\right\}$.) Let

$$
B=\left\{x \in P:(\forall i \leq \omega)\left|x \cap V_{i}\right|<\omega\right\}
$$

and notice that $B=\bigcap_{i \leq \omega} \bigcup_{a \in\left[V_{i}\right]<\omega}\left\{x \in P: x \cap V_{i}=a\right\}$ is a Borel subset of $P$, since each set $\left\{x \in P: x \cap V_{i}=a\right\}$ is closed. Since either $B$ or $P \backslash B$ must be of the second category in $P$, by Claim 1.1.5 there is a subcube $P^{*}$ of $P$ such that either $P^{*} \subset B$ or $P^{*} \cap B=\emptyset$.

If $P^{*} \cap B=\emptyset$, then $W=V_{\omega}$ and $Q=P^{*}$ satisfy the conclusion of the lemma. So, suppose that $P^{*} \subset B$. Let $h: \mathfrak{C}^{\omega} \rightarrow P^{*}, h \in \mathcal{F}_{\text {cube }}$, be a coordinate function making $P^{*}$ a cube, let $\lambda$ be the standard product probability measure on $\mathfrak{C}^{\omega}$, and define a Borel measure $\mu$ on $P^{*}$ by a formula $\mu(B)=\lambda\left(h^{-1}(B)\right)$.

For $i, n<\omega$ let

$$
P_{i}^{n}=\left\{x \in P^{*}: x \cap V_{i} \subset n\right\}
$$

Then all the sets $P_{i}^{n}=\bigcup_{a \subset n}\left\{x \in P^{*}: x \cap V_{i}=a\right\}$ are Borel (since each of the sets $\left\{x \in P^{*}: x \cap V_{i}=a\right\}$ is closed) and $P^{*}=\bigcup_{n<\omega} P_{i}^{n}$ for every
$i<\omega$. Thus for each $i<\omega$ there exists an $n(i)<\omega$ such that

$$
\mu\left(P_{i}^{n(i)}\right)>1-2^{-i}
$$

Then the set $T=\bigcup_{j<\omega} \bigcap_{j<i<\omega} P_{i}^{n(i)}$ has a $\mu$-measure 1 so, by Claim 1.1.5, there is a subcube $Q$ of $P^{*}$ that is a subset of $T$. Let

$$
W=\bigcup_{i<\omega}\left[V_{i} \cap n(i)\right]
$$

We claim that $W$ and $Q$ satisfy the lemma.
It is obvious that $W$ is almost disjoint with each $W_{i}$. So, fix an $x \in Q$. To finish the proof it is enough to show that

$$
x \subseteq^{*} W
$$

But $x \in Q \subset \bigcup_{j<\omega} \bigcap_{j<i<\omega} P_{i}^{n(i)}$. Thus, there exists a $j<\omega$ such that $x \in \bigcap_{j<i<\omega} P_{i}^{n(i)}$. So,

$$
x \cap \bigcup_{j<i<\omega} V_{i}=\bigcup_{j<i<\omega}\left(x \cap V_{i}\right) \subset \bigcup_{j<i<\omega}\left(V_{i} \cap n(i)\right) \subset W
$$

and the set $x \backslash W \subset x \cap\left(V_{\omega} \cup \bigcup_{i \leq j} V_{i}\right)=\left(x \cap V_{\omega}\right) \cup \bigcup_{i \leq j}\left(x \cap V_{i}\right)$ is finite, as $x \in Q \subset P^{*} \subset B$.

Proof of Theorem 2.2.1. For a countably infinite almost disjoint family $\mathcal{W} \subset[\omega]^{\omega}$ and a cube $P \in \operatorname{Perf}\left([\omega]^{\omega}\right)$, let $W(\mathcal{W}, P) \in[\omega]^{\omega}$ and a subcube $Q(\mathcal{W}, P)$ of $P$ be as in Lemma 2.2.2. For $P=\{x\} \in \operatorname{Perf}^{*}\left([\omega]^{\omega}\right)$, we put $Q(\mathcal{W}, P)=P$ and define $W(\mathcal{W}, P)$ as some arbitrary $W$ almost disjoint with each set from $\mathcal{W}$ and such that $A \cap x$ is infinite for some $A \in \mathcal{W} \cup\{W\}$. (If $|x \cap V|<\omega$ for every $V \in \mathcal{W}$, we put $W=x$; otherwise, $W$ is chosen as an arbitrary set almost disjoint with each set from $\mathcal{W}$.)

Let $\mathcal{A}_{0} \subset[\omega]^{\omega}$ be an arbitrary infinite almost disjoint family and consider the following strategy $S$ for Player II:

$$
S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)=Q\left(\mathcal{A}_{0} \cup\left\{W_{\eta}: \eta<\xi\right\}, P_{\xi}\right)
$$

where the $W_{\eta}$ 's are defined inductively by $W_{\eta}=W\left(\mathcal{A}_{0} \cup\left\{W_{\zeta}: \zeta<\eta\right\}, P_{\eta}\right)$. In other words, Player II remembers (recovers) the sets $W_{\eta}$ associated with the sets $P_{\eta}$ played so far, and he uses them (and Lemma 2.2.2) to get the next answer, $Q_{\xi}=Q\left(\mathcal{A}_{0} \cup\left\{W_{\eta}: \eta<\xi\right\}, P_{\xi}\right)$, while remembering (or recovering each time) the set $W_{\xi}=W\left(\mathcal{A}_{0} \cup\left\{W_{\eta}: \eta<\xi\right\}, P_{\xi}\right)$.

By $\mathrm{CPA}_{\text {cube }}^{\text {game }}$, strategy $S$ is not a winning strategy for Player II. So there exists a game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ played according to $S$ in which Player II loses, that is, $[\omega]^{\omega}=\bigcup_{\xi<\omega_{1}} Q_{\xi}$.

Now, notice that the family $\mathcal{A}=\mathcal{A}_{0} \cup\left\{W_{\xi}: \xi<\omega_{1}\right\}$ is a MAD family. It is clear that $\mathcal{A}$ is almost disjoint, since every set $W_{\xi}$ was chosen as almost disjoint with every set from $\mathcal{A}_{0} \cup\left\{W_{\zeta}: \zeta<\xi\right\}$. To see that $\mathcal{A}$ is maximal it is enough to note that every $x \in[\omega]^{\omega}$ belongs to a $Q_{\xi}$ for some $\xi<\omega_{1}$, and so there is an $A \in \mathcal{A}_{0} \cup\left\{W_{\eta}: \eta \leq \xi\right\}$ such that $A \cap x$ is infinite.

By Theorem 2.2.1 we see that $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies the existence of a MAD family of size $\omega_{1}$. Next we will show that such a family can be simultaneously a reaping family. This result is similar in flavor to that from Theorem 5.4.9.

Theorem 2.2.3 $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies that there exists a family $\mathcal{F} \subset[\omega]^{\omega}$ of cardinality $\omega_{1}$ that is simultaneously $M A D$ and reaping.

Proof. The proof is just a slight modification of that for Theorem 2.2.1.
For a countable infinite almost disjoint family $\mathcal{W} \subset[\omega]^{\omega}$ and a cube $P \in \operatorname{Perf}\left([\omega]^{\omega}\right)$, let $W_{0} \in[\omega]^{\omega}$ and a subcube $Q_{0}$ of $P$ be as in Lemma 2.2.2. Let $A \in[\omega]^{\omega}$ be almost disjoint with every set from $\mathcal{W} \cup\left\{W_{0}\right\}$. By Laver's theorem [85] we can also find a subcube $Q_{1}$ of $Q_{0}$ and a $W_{1} \in[A]^{\omega}$ such that

- either $W_{1} \cap x=\emptyset$ for every $x \in Q_{1}$,
- or else $W_{1} \subset x$ for every $x \in Q_{1}$.

Let $Q(\mathcal{W}, P)=Q_{1}$ and $\mathcal{W}(\mathcal{W}, P)=\left\{W_{0}, W_{1}\right\}$. If $P \in \operatorname{Perf}^{*}\left([\omega]^{\omega}\right)$ is a singleton, then we put $Q(\mathcal{W}, P)=P$ and we can easily find $W_{0}$ and $W_{1}$ satisfying the above conditions.

Let $\mathcal{A}_{0} \subset[\omega]^{\omega}$ be an arbitrary infinite almost disjoint family and consider the following strategy $S$ for Player II:

$$
S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)=Q\left(\mathcal{A}_{0} \cup \bigcup\left\{\mathcal{W}_{\eta}: \eta<\xi\right\}, P_{\xi}\right)
$$

where the $\mathcal{W}_{\eta}$ 's are defined as $\mathcal{W}_{\eta}=\mathcal{W}\left(\mathcal{A}_{0} \cup \bigcup\left\{\mathcal{W}_{\eta}: \eta<\xi\right\}, P_{\eta}\right)$.
By $\mathrm{CPA}_{\text {cube }}^{\text {game }}$, strategy $S$ is not a winning strategy for Player II. So there exists a game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ played according to the strategy $S$ in which Player II loses, that is, $[\omega]^{\omega}=\bigcup_{\xi<\omega_{1}} Q_{\xi}$. Then the family $\mathcal{F}=\mathcal{A}_{0} \cup \bigcup\left\{\mathcal{W}_{\xi}: \xi<\omega_{1}\right\}$ is MAD and reaping.

### 2.3 Uncountable $\gamma$-sets and strongly meager sets

In this section we will prove that $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ implies the existence of an uncountable $\gamma$-set. We will also show that such a set can be, but need

## 3

## Prisms and axioms $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ and $\mathrm{CPA}_{\text {prism }}$

The axioms $\mathrm{CPA}_{\text {cube }}^{\text {game }}$ and $\mathrm{CPA}_{\text {cube }}$ deal with the notion of $\mathcal{F}_{\text {cube }}$-density, where $\mathcal{F}_{\text {cube }}$ is the family of all injections $f: C \rightarrow X$ with $C$ being a perfect cube in $\mathfrak{C}^{\omega}$. In the applications of these axioms we were using the facts that different subfamilies of $\operatorname{Perf}(X)$ are $\mathcal{F}_{\text {cube-dense. Unfortunately, in }}$ many cases, the notion of $\mathcal{F}_{\text {cube }}$-density is too weak to do the job - in the applications that follow, the families $\mathcal{E} \subset \operatorname{Perf}(X)$ will not be $\mathcal{F}_{\text {cube-dense }}$, but they will be dense in a weaker sense defined below. Luckily, this weaker notion of density still leads to consistent axioms.

To define this weaker notion of density, let us first take another look at the notion of "cube." Let $A$ be a nonempty countable set of ordinal numbers. The notion of a perfect cube in $\mathfrak{C}^{A}$ can be defined the same way as it was done for $\mathfrak{C}^{\omega}$. However, it will be more convenient for us to define it as follows. Let $\Phi_{\text {cube }}(A)$ be the family of all continuous injections $f: \mathfrak{C}^{A} \rightarrow \mathfrak{C}^{A}$ such that

$$
f(x)(\alpha)=f(y)(\alpha) \text { for all } \alpha \in A \text { and } x, y \in \mathfrak{C}^{A} \text { with } x(\alpha)=y(\alpha) .
$$

In other words, $\Phi_{\text {cube }}(A)$ is the family of all functions of the form $f=$ $\left\langle f_{\alpha}\right\rangle_{\alpha \in A}$, where each $f_{\alpha}$ is an injection from $\mathfrak{C}$ into $\mathfrak{C}$. Then the family of all perfect cubes in $\mathfrak{C}^{A}$ for an appropriate $A$ is equal to

$$
\operatorname{CUBE}(A)=\left\{\operatorname{range}(f): f \in \Phi_{\text {cube }}(A)\right\},
$$

while the family $\mathcal{F}_{\text {cube }}$ defined in the first chapter consists of all continuous injections $f: C \rightarrow X$ with $C \in \operatorname{CUBE}(\omega)$.

In the definitions that follow, the notion of a "cube" will be replaced by that of a "prism." So, let $\Phi_{\text {prism }}(A)$ be the family of all continuous injections $f: \mathfrak{C}^{A} \rightarrow \mathfrak{C}^{A}$ with the property that

$$
\begin{equation*}
f(x) \upharpoonright \alpha=f(y) \upharpoonright \alpha \Leftrightarrow x \upharpoonright \alpha=y \upharpoonright \alpha \text { for all } \alpha \in A \text { and } x, y \in \mathfrak{C}^{A} \tag{3.1}
\end{equation*}
$$

or, equivalently, such that, for every $\alpha \in A$,

$$
f \upharpoonright \alpha \stackrel{\text { def }}{=}\{\langle x \upharpoonright \alpha, y \upharpoonright \alpha\rangle:\langle x, y\rangle \in f\}
$$

is a one to one function from $\mathfrak{C}^{A \cap \alpha}$ into $\mathfrak{C}^{A \cap \alpha}$. For example, if $A=$ $\{0,1,2\}$, then function $f$ belongs to $\Phi_{\text {prism }}(A)$ provided there exist continuous functions $f_{0}: \mathfrak{C} \rightarrow \mathfrak{C}, f_{1}: \mathfrak{C}^{2} \rightarrow \mathfrak{C}$, and $f_{2}: \mathfrak{C}^{3} \rightarrow \mathfrak{C}$ such that $f\left(x_{0}, x_{1}, x_{2}\right)=\left\langle f_{0}\left(x_{0}\right), f_{1}\left(x_{0}, x_{1}\right), f_{2}\left(x_{0}, x_{1}, x_{2}\right)\right\rangle$ for all $x_{0}, x_{1}, x_{2} \in \mathfrak{C}$ and maps $f_{0},\left\langle f_{0}, f_{1}\right\rangle$, and $f$ are one to one. Functions $f$ from $\Phi_{\text {prism }}(A)$ were first introduced, in a more general setting, in [73], where they are called projection-keeping homeomorphisms. Note that

$$
\begin{equation*}
\Phi_{\mathrm{prism}}(A) \text { is closed under compositions } \tag{3.2}
\end{equation*}
$$

and tha, t for every ordinal number $\alpha>0$,

$$
\begin{equation*}
\text { if } f \in \Phi_{\text {prism }}(A) \text {, then } f \upharpoonright \alpha \in \Phi_{\text {prism }}(A \cap \alpha) \text {. } \tag{3.3}
\end{equation*}
$$

Let

$$
\mathbb{P}_{A}=\left\{\operatorname{range}(f): f \in \Phi_{\text {prism }}(A)\right\}
$$

We will write $\Phi_{\text {prism }}$ for $\bigcup_{0<\alpha<\omega_{1}} \Phi_{\text {prism }}(\alpha)$ and define

$$
\mathbb{P}_{\omega_{1}} \stackrel{\text { def }}{=} \bigcup_{0<\alpha<\omega_{1}} \mathbb{P}_{\alpha}=\left\{\operatorname{range}(f): f \in \Phi_{\text {prism }}\right\} .
$$

Following [73], we will refer to elements of $\mathbb{P}_{\omega_{1}}$ as iterated perfect sets. (In [131], the elements of $\mathbb{P}_{\alpha}$ are called $I$-perfect, where $I$ is the ideal of countable sets.)

Let $\mathcal{F}_{\text {prism }}(X)$ (or just $\mathcal{F}_{\text {prism }}$, if $X$ is clear from the context) be the family of all continuous injections $f: E \rightarrow X$, where $E \in \mathbb{P}_{\omega_{1}}$ and $X$ is a fixed Polish space. We adopt the shortcuts similar to those for cubes. Thus, we say that $P \in \operatorname{Perf}(X)$ is a prism if we consider it with an (implicitly given) witness function $f \in \mathcal{F}_{\text {prism }}$ onto $P$. Then $Q$ is a subprism of a prism $P$ provided $Q=f[E]$, where $E \in \mathbb{P}_{\alpha}$ and $E \subset \operatorname{dom}(f)$. Also, singletons $\{x\}$ in $X$ will be identified with constant functions from $E \in \mathbb{P}_{\omega_{1}}$ to $\{x\}$, and these functions will be considered as elements of $\mathcal{C}_{\text {prism }} \subset \mathcal{F}_{\text {prism }}^{*}$, similarly as in (2.1).

Following the schema presented in (1.9), we say that a family $\mathcal{E} \subset$ $\operatorname{Perf}(X)$ is $\mathcal{F}_{\text {prism }}$-dense provided

$$
\forall f \in \mathcal{F}_{\text {prism }} \exists g \in \mathcal{F}_{\text {prism }}(g \subset f \& \operatorname{range}(g) \in \mathcal{E})
$$

Similarly as in Fact 1.0.2, using (3.2) we can also prove that:

Fact 3.0.1 $\mathcal{E} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {prism }}$-dense if and only if

$$
\forall \alpha<\omega_{1} \forall f \in \mathcal{F}_{\text {prism }}, \operatorname{dom}(f)=\mathfrak{C}^{\alpha} \exists g \in \mathcal{F}_{\text {prism }}(g \subset f \& \operatorname{range}(g) \in \mathcal{E})
$$

Notice also that $\Phi_{\text {cube }}(A) \subset \Phi_{\text {prism }}(A)$, so every cube is also a prism. From this and Fact 3.0.1 (see also Fact 1.8.5) it is also easy to see that
if $\mathcal{E} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {cube }}$-dense, then $\mathcal{E}$ is also $\mathcal{F}_{\text {prism }}$-dense.
The converse of (3.4), however, is false. (See Remark 3.2.6.)
Now we are ready to state the next version of our axiom, in which the game $\operatorname{GAME}_{\text {prism }}(X)$ is an obvious generalization of $\operatorname{GAME}_{\text {cube }}(X)$.
$\mathrm{CPA}_{\text {prism }}^{\text {game }}: \mathfrak{c}=\omega_{2}$, and for any Polish space $X$ Player II has no winning strategy in the game $\mathrm{GAME}_{\text {prism }}(X)$.

Remark 3.0.2 In order to apply $\mathrm{CPA}_{\text {prism }}^{\text {game }}$, we will always construct some strategy $S$ for Player II and then use the axiom to conclude that, since $S$ is not winning, there exists a game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ played according to $S$ in which Player I wins. But in any such game, for every $\xi<\omega_{1}$, we have

$$
\begin{equation*}
Q_{\xi}=S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right) \tag{3.5}
\end{equation*}
$$

Thus, in order to construct a meaningful Player II strategy $S$, for each sequence $\left\langle P_{\xi} \in \operatorname{Perf}^{*}(X): \xi<\omega_{1}\right\rangle$ we will be defining by induction a sequence $\left\langle Q_{\xi}: \xi<\omega_{1}\right\rangle$ such that each $Q_{\xi}$ is a subprism of $P_{\xi}$ and the definition of $Q_{\xi}$ may depend only on $\left\langle P_{\eta}: \eta \leq \xi\right\rangle$, that is, it cannot depend on any $P_{\eta}$ with $\xi<\eta<\omega_{1}$. If such a sequence $\left\langle Q_{\xi}: \xi<\omega_{1}\right\rangle$ is defined, then each $Q_{\xi}$ can be expressed as in (3.5), where $S$ is the function representing our inductive construction.

If we proceed as described above, then we say that the strategy $S$ is associated with our inductive construction. (Such an $S$ is not defined yet on all required sequences, but it is defined on all sequences relevant for us. So, we will be assuming that, on the other sequences, it is defined in some fixed, trivial way.)

Notice that if a prism $P \in \operatorname{Perf}(X)$ is considered with a witness function $f \in \mathcal{F}_{\text {prism }}$ from $\mathfrak{C}^{\alpha}$ onto $P$, then $P$ is also a cube and any subcube of $P$ is also a subprism of $P$. Thus, any Player II strategy in a game $\operatorname{GAME}_{\text {cube }}(X)$ can be translated to a strategy in a game $\mathrm{GAME}_{\text {prism }}(X)$. (You need to identify appropriately $\mathfrak{C}^{\alpha}$ with $\mathfrak{C}^{\omega}$ : First you identify $\mathfrak{C}^{\alpha}$ with the product $\mathfrak{C}^{\omega} \times \mathfrak{C}^{\alpha} \backslash\{0\}$, which is important for a finite $\alpha$, and then this second space is identified with $\mathfrak{C}^{\omega}$ coordinatewise.) In particular, $\mathrm{CPA}_{\mathrm{prism}}^{\text {game }}$
implies $\mathrm{CPA}_{\text {cube }}^{\text {game }}$. In addition, essentially the same argument as was used for Proposition 2.0.1 also gives the following.

Proposition 3.0.3 Axiom $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ implies the following prism version of the axiom $\mathrm{CPA}_{\text {cube }}$ :
$\mathrm{CPA}_{\text {prism }}: \mathfrak{c}=\omega_{2}$, and for every Polish space $X$ and every $\mathcal{F}_{\text {prism-dense }}$ family $\mathcal{E} \subset \operatorname{Perf}(X)$ there is an $\mathcal{E}_{0} \subset \mathcal{E}$ such that $\left|\mathcal{E}_{0}\right| \leq \omega_{1}$ and $\left|X \backslash \bigcup \mathcal{E}_{0}\right| \leq \omega_{1}$.

By (3.4) it is also obvious that $\mathrm{CPA}_{\text {prism }}$ implies $\mathrm{CPA}_{\text {cube }}$. All these implications can be summarized by a graph.


We will prove the consistency of $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ in Chapter 7. For the remainder of this chapter we will concentrate on some basic consequences of $\mathrm{CPA}_{\text {prism }}$. Most of the applications of the axioms $\mathrm{CPA}_{\text {prism }}$ and $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ will be presented in the following two chapters. We finish this section with few simple but important general remarks.

Although we will not use this, it is illuminating to note that every iterated perfect set $E \in \mathbb{P}_{\alpha}$ comes with a canonical projection-keeping homeomorphism $f \in \Phi_{\text {prism }}(\alpha)$ for which range $(f)=E$. To see it, first note that for every $T \in \operatorname{Perf}(\mathfrak{C})$ there is a canonical homeomorphism $h_{T}$ from $T$ onto $\mathfrak{C}$ defined by $h_{T}(t)(i)=t\left(m_{T}^{i}\right)$, where $m_{T}^{i}$ is the $i$-th forking place of $t$ in $T$, that is,

$$
m_{T}^{i}=\min \left\{k<\omega: k>m_{T}^{i-1} \&(\exists s \in T) s \upharpoonright k=t \upharpoonright k \& s(k) \neq t(k)\right\}
$$

Then for $x \in E, \beta<\alpha$, and $i<\omega$ we define

$$
f^{-1}(x)(\beta)(i)=h_{\{y(\beta): y \in E \& x \upharpoonright \beta=y \upharpoonright \beta\}}(x(\beta))(i) .
$$

In other wards, we obtain the value of $f^{-1}(x)$ by removing from each 0 1 sequence $x(\beta)$ its subsequences, where $x(\beta)$ does not branch. It is not difficult to see that such a defined $f$ indeed belongs to $\Phi_{\text {prism }}(\alpha)$ and that range $(f)=E$.

Note also that for every $0<\alpha<\omega_{1}$

$$
\begin{equation*}
\text { if } f \in \Phi_{\text {prism }}(\alpha) \text { and } P \in \mathbb{P}_{\alpha} \text {, then } f[P] \in \mathbb{P}_{\alpha} \tag{3.6}
\end{equation*}
$$

Indeed, if $P=g\left[\mathfrak{C}^{\alpha}\right]$ for some $g \in \Phi_{\text {prism }}(\alpha)$, then, by condition (3.2), we have $f[P]=f\left[g\left[\mathfrak{C}^{\alpha}\right]\right]=(f \circ g)\left[\mathfrak{C}^{\alpha}\right] \in \mathbb{P}_{\alpha}$.

In what follows, for a fixed $0<\alpha<\omega_{1}$ and $0<\beta \leq \alpha$, the symbol $\pi_{\beta}$ will stand for the projection from $\mathfrak{C}^{\alpha}$ onto $\mathfrak{C}^{\beta}$, that is, $\mathfrak{C}^{\alpha} \ni x \stackrel{\pi_{\beta}}{\mapsto} x \upharpoonright \beta \in \mathfrak{C}^{\beta}$. We will always consider $\mathfrak{C}^{\alpha}$ with the following metric $\rho$ : Fix an enumeration $\left\{\left\langle\beta_{k}, n_{k}\right\rangle: k<\omega\right\}$ of $\alpha \times \omega$ and for distinct $x, y \in \mathfrak{C}^{\alpha}$ define

$$
\begin{equation*}
\rho(x, y)=2^{-\min \left\{k<\omega: x\left(\beta_{k}\right)\left(n_{k}\right) \neq y\left(\beta_{k}\right)\left(n_{k}\right)\right\}} \tag{3.7}
\end{equation*}
$$

The open ball in $\mathfrak{C}^{\alpha}$ with a center at $z \in \mathfrak{C}^{\alpha}$ and radius $\varepsilon>0$ will be denoted by $B_{\alpha}(z, \varepsilon)$. Notice that in this metric any two open balls are either disjoint or one is a subset of the other. Also, for every $\gamma<\alpha$ and $\varepsilon>0$

$$
\begin{equation*}
\pi_{\gamma}\left[B_{\alpha}(x, \varepsilon)\right]=B_{\gamma}[(x \upharpoonright \gamma, \varepsilon)] \quad \text { for every } x \in \mathfrak{C}^{\alpha} \tag{3.8}
\end{equation*}
$$

It is also easy to see that any $B_{\alpha}(z, \varepsilon)$ is a clopen set, and, in fact, it is a perfect cube in $\mathfrak{C}^{\alpha}$, so it belongs to $\mathbb{P}_{\alpha}$. In fact, more can be said:

$$
\begin{equation*}
\text { If } \mathcal{B}_{\alpha} \stackrel{\text { def }}{=}\left\{B \subset \mathfrak{C}^{\alpha}: B \text { is clopen in } \mathfrak{C}^{\alpha}\right\}, \text { then } \mathcal{B}_{\alpha} \subset \mathbb{P}_{\alpha} \tag{3.9}
\end{equation*}
$$

This is the case, since any clopen $E$ in $\mathfrak{C}^{\alpha}$ is a finite union of disjoint open balls, each of which belongs to $\mathbb{P}_{\alpha}$, and it is easy to see that $\mathbb{P}_{\alpha}$ is closed under finite unions of open balls.

From this we conclude immediately that

$$
\begin{equation*}
\text { a clopen subset of } E \in \mathbb{P}_{\alpha} \text { belongs to } \mathbb{P}_{\alpha} \tag{3.10}
\end{equation*}
$$

and

> a clopen subset of a prism is its subprism,
while (3.3) implies

$$
\begin{equation*}
\pi_{\beta}[E] \in \mathbb{P}_{\beta} \text { for every } 0<\beta<\alpha<\omega_{1} \text { and } E \in \mathbb{P}_{\alpha} \tag{3.12}
\end{equation*}
$$

Notice also that if $P \in \mathbb{P}_{\alpha}$ and $0<\beta<\alpha$, then

$$
\begin{equation*}
P \cap \pi_{\beta}^{-1}\left(P^{\prime}\right) \in \mathbb{P}_{\alpha} \quad \text { for every } P^{\prime} \in \mathbb{P}_{\beta} \text { with } P^{\prime} \subset \pi_{\beta}[P] \tag{3.13}
\end{equation*}
$$

Indeed, let $f \in \Phi_{\text {prism }}(\beta)$ and $g \in \Phi_{\text {prism }}(\alpha)$ be such that $f\left[\mathfrak{C}^{\beta}\right]=P^{\prime}$ and $g\left[\mathfrak{C}^{\alpha}\right]=P$. Let $Q=(g \upharpoonright \beta)^{-1}\left[P^{\prime}\right]=(g \upharpoonright \beta)^{-1} \circ f\left[\mathfrak{C}^{\beta}\right]$. Then, $Q \in \mathbb{P}_{\beta}$ since, by $(3.3),(g \upharpoonright \beta)^{-1} \circ f \in \Phi_{\text {prism }}(\beta)$. Thus $\pi_{\beta}^{-1}(Q)$ belongs to $\mathbb{P}_{\alpha}$ and $P \cap \pi_{\beta}^{-1}\left(P^{\prime}\right)=g\left[\pi_{\beta}^{-1}(Q)\right] \in \mathbb{P}_{\alpha}$.

### 3.1 Fusion for prisms

One of the main technical tools used to prove that a family of perfect sets is dense is the so-called fusion lemma. It says that, for an appropriately chosen decreasing sequence $\left\{P_{n}: n<\omega\right\}$ of perfect sets, its intersection $P=\bigcap_{n<\omega} P_{n}$, called the fusion, is still a perfect set. The simple structure of perfect cubes makes it quite easy to formulate a "cube fusion lemma" in which the fusion set $P$ is also a perfect cube. However, so far we have not had any need for such a lemma (at least in an explicit form), since its use was always hidden in the proofs of the results we quoted, like Claim 1.1.5 or Proposition 1.5.1. On the other hand, the new and more complicated structure of prisms does not leave us the option of avoiding fusion arguments any longer - we have to face it up front.
For a fixed $0<\alpha<\omega_{1}$, let $\left\{\left\langle\beta_{k}, n_{k}\right\rangle: k<\omega\right\}$ be the enumeration of $\alpha \times \omega$ used in the definition (3.7) of the metric $\rho$ and let

$$
\begin{equation*}
A_{k}=\left\{\left\langle\beta_{i}, n_{i}\right\rangle: i<k\right\} \quad \text { for every } k<\omega . \tag{3.14}
\end{equation*}
$$

Lemma 3.1.1 (Fusion Sequence) Let $0<\alpha<\omega_{1}$, and for $k<\omega$ let $\mathcal{E}_{k}=\left\{E_{s}: s \in 2^{A_{k}}\right\}$ be a family of closed subsets of $\mathfrak{C}^{\alpha}$. Assume that for every $k<\omega, s, t \in 2^{A_{k}}$, and $\beta<\alpha$ we have:
(i) The diameter of $E_{s}$ goes to 0 as the length of $s$ goes to $\infty$.
(ii) If $i<k$, then $E_{s} \subset E_{s\left\lceil A_{i}\right.}$.
(ag) (agreement) If $s \upharpoonright(\beta \times \omega)=t \upharpoonright(\beta \times \omega)$, then $\pi_{\beta}\left[E_{s}\right]=\pi_{\beta}\left[E_{t}\right]$.
(sp) (split) If $s \upharpoonright(\beta \times \omega) \neq t \upharpoonright(\beta \times \omega)$, then $\pi_{\beta}\left[E_{s}\right] \cap \pi_{\beta}\left[E_{t}\right]=\emptyset$.
Then $Q=\bigcap_{k<\omega} \cup \mathcal{E}_{k}$ belongs to $\mathbb{P}_{\alpha}$.
Proof. For $x \in \mathfrak{C}^{\alpha}$, let $\bar{x} \in 2^{\alpha \times \omega}$ be defined by $\bar{x}(\beta, n)=x(\beta)(n)$.
First note that, by conditions (i) and (sp), for every $k<\omega$ the sets in $\mathcal{E}_{k}$ are pairwise disjoint. Thus, taking into account (ii), the function $h: \mathfrak{C}^{\alpha} \rightarrow \mathfrak{C}^{\alpha}$ defined by

$$
h(x)=r \quad \Longleftrightarrow \quad\{r\}=\bigcap_{k<\omega} E_{\bar{x} \mid A_{k}}
$$

is well defined and is one to one. It is also easy to see that $h$ is continuous and that $Q=h\left[\mathfrak{C}^{\alpha}\right]$. Thus, we need to prove only that $h$ is projectionkeeping.
To show this, fix $\beta<\alpha$, put $S=\bigcup_{i<\omega} 2^{A_{i}}$, and notice that, by (i) and (ag), for every $x \in \mathfrak{C}^{\alpha}$ we have

$$
\begin{aligned}
\{h(x) \upharpoonright \beta\} & =\pi_{\beta}\left[\bigcap\left\{E_{\bar{x} \upharpoonright A_{k}}: k<\omega\right\}\right] \\
& =\bigcap\left\{\pi_{\beta}\left[E_{\bar{x} \upharpoonright A_{k}}\right]: k<\omega\right\} \\
& =\bigcap\left\{\pi_{\beta}\left[E_{s}\right]: s \in S \& s \subset \bar{x}\right\} \\
& =\bigcap\left\{\pi_{\beta}\left[E_{s}\right]: s \in S \& s \upharpoonright(\beta \times \omega) \subset \bar{x}\right\}
\end{aligned}
$$

Now, if $x \upharpoonright \beta=y \upharpoonright \beta$, then for every $s \in S$

$$
s \upharpoonright(\beta \times \omega) \subset \bar{x} \Leftrightarrow s \upharpoonright(\beta \times \omega) \subset \bar{y}
$$

so $h(x) \upharpoonright \beta=h(y) \upharpoonright \beta$.
On the other hand, if $x \upharpoonright \beta \neq y \upharpoonright \beta$, then there is a $k<\omega$ large enough such that for $s=\bar{x} \upharpoonright A_{k}$ and $t=\bar{y} \upharpoonright A_{k}$ we have $s \upharpoonright(\beta \times \omega) \neq t \upharpoonright(\beta \times \omega)$. But then $\{h(x) \upharpoonright \beta\}$ and $\{h(y) \upharpoonright \beta\}$ are subsets of $\pi_{\beta}\left[E_{s}\right]$ and $\pi_{\beta}\left[E_{t}\right]$, respectively, which, by ( sp ), are disjoint. So, $h(x) \upharpoonright \beta \neq h(y) \upharpoonright \beta$.

In most of our applications the task of constructing sequences $\left\langle\mathcal{E}_{k}: k<\omega\right\rangle$ satisfying the specific conditions (ag) and (sp) can be reduced to checking some simple density properties listed in our next lemma. In its statement we consider $\mathbb{P}_{\alpha}$ as ordered by inclusion and use the standard terminology from the theory of partially ordered sets: $D \subset \mathbb{P}_{\alpha}$ is dense provided for every $E \in \mathbb{P}_{\alpha}$ there is an $E^{\prime} \in D$ with $E^{\prime} \subset E$; it is open provided for every $E \in D$ if $E^{\prime} \in \mathbb{P}_{\alpha}$ and $E^{\prime} \subset E$, then $E^{\prime} \in D$. Moreover, for a family $\mathcal{E}$ of pairwise disjoint subsets of $\mathbb{P}_{\alpha}$, we say that $\mathcal{E}^{\prime} \subset \mathbb{P}_{\alpha}$ is a refinement of $\mathcal{E}$ provided $\mathcal{E}^{\prime}=\left\{P_{E}: E \in \mathcal{E}\right\}$, where $P_{E} \subset E$ for all $E \in \mathcal{E}$.

Lemma 3.1.2 Let $0<\alpha<\omega_{1}$ and $k<\omega$. If $\mathcal{E}_{k}=\left\{E_{s} \in \mathbb{P}_{\alpha}: s \in 2^{A_{k}}\right\}$ satisfies (ag) and (sp), then:
(A) There exists an $\mathcal{E}_{k+1}=\left\{E_{s} \in \mathbb{P}_{\alpha}: s \in 2^{A_{k+1}}\right\}$ of sets of diameter less than $2^{-(k+1)}$ such that (ii), (ag), and (sp) hold for all $s, t \in 2^{A_{k+1}}$ and $r \in 2^{A_{k}}$.

Moreover, if $\mathcal{D} \subset\left[\mathbb{P}_{\alpha}\right]^{<\omega}$ is a family of pairwise disjoint sets such that $\emptyset \in \mathcal{D}, \mathcal{D}$ is closed under refinements, and
$(\dagger)$ for every $\mathcal{E} \in \mathcal{D}$ and $E \in \mathbb{P}_{\alpha}$ which is disjoint with $\bigcup \mathcal{E}$ there exists an $E^{\prime} \in \mathbb{P}_{\alpha} \cap \mathcal{P}(E)$ such that $\left\{E^{\prime}\right\} \cup \mathcal{E} \in \mathcal{D}$,
then
(B) there exists a refinement $\mathcal{E}_{k}^{\prime} \in \mathcal{D}$ of $\mathcal{E}_{k}$ satisfying (ag) and (sp);
(C) there exists an $\mathcal{E}_{k+1}$ as in (A) such that $\mathcal{E}_{k+1} \in \mathcal{D}$.

Proof. For $s \in 2^{A_{k}}$ and $j<2$, let $s^{\wedge} j$ stand for $s \cup\left\{\left\langle\left\langle\beta_{k}, n_{k}\right\rangle, j\right\rangle\right\} \in 2^{A_{k+1}}$.
Let $\left\{s_{i}: i<2^{k+1}\right\}$ be an enumeration of $2^{A_{k+1}}$. By induction on $i<2^{k+1}$, we will construct a sequence $\left\langle x_{s_{i}} \in \mathfrak{C}^{\alpha}: i<2^{k+1}\right\rangle$ such that for every $i<2^{k+1}$
(a) $x_{s_{i}} \in E_{s_{i}\left\lceil A_{k}\right.}$,
(b) for every $m<i$, if $\beta=\max \left\{\bar{\beta}: s_{i} \upharpoonright(\bar{\beta} \times \omega)=s_{m} \upharpoonright(\bar{\beta} \times \omega)\right\}$, then

$$
x_{s_{i}} \upharpoonright \beta=x_{s_{m}} \upharpoonright \beta \text { and } x_{s_{i}}(\beta) \neq x_{s_{m}}(\beta)
$$

The point $x_{s_{0}}$ is chosen arbitrarily from $E_{s_{0} \upharpoonright A_{k}}$. To make an inductive step, if for some $0<i \leq 2^{k+1}$ points $\left\{x_{s_{m}}: m<i\right\}$ are already constructed, choose an $\bar{m}<i$ for which $\beta$ as in (b) is maximal. Notice that by the inductive assumption and the condition (ag) we have $x_{s_{\bar{m}}} \upharpoonright \beta \in \pi_{\beta}\left[E_{s_{\bar{m}}\left\lceil A_{k}\right.}\right]=\pi_{\beta}\left[E_{s_{i}\left\lceil A_{k}\right.}\right]$. So we can choose an $x_{s_{i}} \in E_{s_{i}\left\lceil A_{k}\right.}$ extending $x_{s_{\bar{m}}} \upharpoonright \beta$ and such that $x_{s_{i}}(\beta) \neq x_{s_{m}}(\beta)$ for all $m<i$. It is easy to see that such an $x_{s_{i}}$ satisfies (a) and condition (b) for $m=\bar{m}$. For other $m<i$, condition (b) follows from the maximality of $\beta$ and the assumption that $\mathcal{E}_{k}$ satisfies (ag) and ( $\mathrm{sp)}$.

Conditions (a) and (b) imply that $\mathcal{E}_{k+1}^{\prime}=\left\{\left\{x_{s}\right\}: s \in 2^{A_{k+1}}\right\}$ satisfy condition (A) except for being a subset of $\mathbb{P}_{\alpha}$. Let $\varepsilon \in\left(0,2^{-(k+1)}\right)$ be small enough that for every $m<i<2^{k+1}$ and $\beta$ as in (b) we have $\pi_{\beta+1}\left[B_{\alpha}\left(x_{s_{i}}, \varepsilon\right)\right] \cap \pi_{\beta+1}\left[B_{\alpha}\left(x_{s_{m}}, \varepsilon\right)\right]=\emptyset$. For $s \in 2^{A_{k}}$ and $j<2$, define

$$
E_{s^{\wedge} j}=E_{s} \cap B_{\alpha}\left(x_{s^{\wedge} j}, \varepsilon\right) .
$$

Then $\mathcal{E}_{k+1}=\left\{E_{s}: s \in 2^{A_{k+1}}\right\}$ is a subset of $\mathbb{P}_{\alpha}$ by (3.10). Condition (ii) is clear from the construction, while (ag) for $\mathcal{E}_{k+1}$ follows from (b) and (3.8). Property (sp) holds by (b) and the choice of $\varepsilon$, since ( sp ) was true for $\mathcal{E}_{k+1}^{\prime}$. We have completed the proof of (A).

To prove condition (B), fix an enumeration $\left\{s_{i}: i<2^{k}\right\}$ of $2^{A_{k}}$ and define $\gamma=\max \left\{\beta_{0}, \ldots, \beta_{k}\right\}<\alpha$. Also, for $i, m<2^{k}$, put $E_{s_{i}}^{-1}=E_{s_{i}}$ and

$$
\beta_{i}^{m}=\max \left\{\beta \leq \gamma: s_{i} \upharpoonright(\beta \times \omega)=s_{m} \upharpoonright(\beta \times \omega)\right\}
$$

By induction we will construct the sequences $\left\langle\left\{E_{s_{i}}^{m} \in \mathbb{P}_{\alpha}: i<2^{k}\right\}: m<2^{k}\right\rangle$ and $\left\langle P_{m} \in \mathbb{P}_{\alpha}: m<2^{k}\right\rangle$ such that, for every $j, m<2^{k}$,
(a) $\mathcal{E}^{m}=\left\{E_{s_{i}}^{m}: i<2^{k}\right\}$ satisfies (ag);
(b) $E_{s_{j}}^{m} \subset E_{s_{j}}^{m-1}$ and if $x \in E_{s_{j}}^{m-1}$ and $\pi_{\gamma}(x) \in \pi_{\gamma}\left[E_{s_{j}}^{m}\right]$, then $x \in E_{s_{j}}^{m}$;
(c) $\pi_{\gamma}\left[P_{m}\right]=\pi_{\gamma}\left[E_{s_{m}}^{m}\right]$;
(d) $P_{m} \subset E_{s_{m}}^{m-1}$ and $\left\{P_{i}: i \leq m\right\} \in \mathcal{D}$.

So, assume that for some $m<2^{k}$ the sequence $\left\langle P_{i}: i<m\right\rangle$ and the family $\mathcal{E}^{m-1}$ satisfying (ag) are already constructed. Notice that, by (b), sets in $\mathcal{E}^{m-1}$ are pairwise disjoint, since this was the case for $\mathcal{E}^{-1}=\mathcal{E}_{k}$. Thus, by condition ( $\dagger$ ) applied to the family $\mathcal{E}=\left\{P_{i}: i<m\right\}$, we can choose a $P_{m} \in$ $\mathbb{P}_{\alpha} \cap \mathcal{P}\left(E_{s_{m}}^{m-1}\right)$ such that $\left\{P_{m}\right\} \cup\left\{P_{i}: i<m\right\} \in \mathcal{D}$. This guarantees (d).

Next, for $i<2^{k}$ define

$$
E_{s_{i}}^{m}=E_{s_{i}}^{m-1} \cap \pi_{\beta_{i}^{m}}^{-1}\left(\pi_{\beta_{i}^{m}}\left[P_{m}\right]\right)=\left\{x \in E_{s_{i}}^{m-1}: x \upharpoonright \beta_{i}^{m} \in \pi_{\beta_{i}^{m}}\left[P_{m}\right]\right\}
$$

and notice that $\pi_{\beta_{i}^{m}}\left[P_{m}\right] \subset \pi_{\beta_{i}^{m}}\left[E_{s_{m}}^{m-1}\right]=\pi_{\beta_{i}^{m}}\left[E_{s_{i}}^{m-1}\right]$. So, by (3.13), $E_{s_{i}}^{m} \in \mathbb{P}_{\alpha}$. Also, the definition ensures (b) since $\beta_{i}^{m} \leq \gamma$.
Note that, by the inductive assumption (a), for all $i<2^{k}$ we have

$$
\pi_{\beta_{i}^{m}}\left[E_{s_{i}}^{m}\right]=\pi_{\beta_{i}^{m}}\left[E_{s_{i}}^{m-1}\right] \cap \pi_{\beta_{i}^{m}}\left[P_{m}\right]=\pi_{\beta_{i}^{m}}\left[E_{s_{m}}^{m-1}\right] \cap \pi_{\beta_{i}^{m}}\left[P_{m}\right]=\pi_{\beta_{i}^{m}}\left[P_{m}\right]
$$

Since $\beta_{m}^{m}=\gamma$, this implies (c). To prove (a), pick $\beta<\alpha$ and different $i, j<2^{k}$ such that $s_{i} \upharpoonright(\beta \times \omega)=s_{j} \upharpoonright(\beta \times \omega)$. If $\beta \leq \beta_{i}^{m}$, then also $\beta \leq \beta_{j}^{m}$ and $\pi_{\beta}\left[E_{s_{i}}^{m}\right]=\pi_{\beta}\left[P_{m}\right]=\pi_{\beta}\left[E_{s_{j}}^{m}\right]$. So, assume that $\beta>\beta_{i}^{m}$ and $\beta>\beta_{j}^{m}$. Then $\beta_{i}^{m}=\beta_{j}^{m}$ and

$$
\begin{aligned}
\pi_{\beta}\left[E_{s_{i}}^{m}\right] & =\left\{\pi_{\beta}(x): x \in E_{s_{i}}^{m-1} \& x \upharpoonright \beta_{i}^{m} \in \pi_{\beta_{i}^{m}}\left[P_{m}\right]\right\} \\
& =\left\{\pi_{\beta}(x): x \in E_{s_{j}}^{m-1} \& x \upharpoonright \beta_{j}^{m} \in \pi_{\beta_{j}^{m}}\left[P_{m}\right]\right\} \\
& =\pi_{\beta}\left[E_{s_{j}}^{m}\right]
\end{aligned}
$$

So $\mathcal{E}^{m}$ satisfies (a). This finishes the construction.
Notice that by the maximality of $\gamma$ and properties (a) and (c), the family $\mathcal{E}_{k}^{\prime}=\left\{P_{m}: m<2^{k}\right\}$ satisfies (ag). Since it is a refinement of $\mathcal{E}_{k}$, it also satisfies ( sp ). So (B) is proved.

To find $\mathcal{E}_{k+1}$ as in (C), first take an $\mathcal{E}_{k+1}^{\prime}$ satisfying (A) and then use (B) to find its refinement $\mathcal{E}_{k+1} \in \mathcal{D}$ satisfying (ag) and (sp).

One of the most important consequences of Lemma 3.1.2 is the following.
Corollary 3.1.3 Let $0<\alpha<\omega_{1}$ and let $\left\{D_{k}: k<\omega\right\}$ be a collection of dense open subsets of $\mathbb{P}_{\alpha}$. If for every $k<\omega$

$$
D_{k}^{*}=\left\{\bigcup \mathcal{D}: \mathcal{D} \in\left[D_{k}\right]^{<\omega} \text { and the sets in } \mathcal{D} \text { are pairwise disjoint }\right\}
$$

then $\bar{D}=\bigcap_{k<\omega} D_{k}^{*}$ is open and dense in $\mathbb{P}_{\alpha}$.
Proof. It is clear that $\bar{D}$ is open. To see its density, notice that the families

$$
\mathcal{D}_{k}=\left\{\mathcal{D} \in\left[D_{k}\right]^{<\omega} \text { and sets in } \mathcal{D} \text { are pairwise disjoint }\right\}
$$

satisfy condition $(\dagger)$. Let $E \in \mathbb{P}_{\alpha}$, choose an $E_{\emptyset} \in D_{0} \subset D_{0}^{*}$ below $E$, and put $\mathcal{E}_{0}=\left\{E_{\emptyset}\right\}$. Applying (C) from Lemma 3.1.2 by induction we can define families $\mathcal{E}_{k} \in \mathcal{D}_{k}, k<\omega$, such that conditions (i), (ii), (ag), and (sp) from Lemma 3.1.1 are satisfied. But then $Q=\bigcap_{k<\omega} \cup \mathcal{E}_{k} \subset E$ belongs to $\bar{D}$.

### 3.2 On $\mathcal{F}$-independent prisms

The following variant of the Kuratowski-Ulam theorem will be useful in what follows.

Lemma 3.2.1 Let $X$ be a Polish space and consider $X^{T}$ with the product topology, where $T \neq \emptyset$ is an arbitrary set. Fix at most countable family $\mathcal{K}$ of sets $K \subsetneq T$. Then for every comeager set $H \subset X^{T}$ there exists a comeager set $G \subset H$ such that for every $x \in G$ and $K \in \mathcal{K}$ the set

$$
G_{x \upharpoonright K}=\left\{y \in X^{T \backslash K}:(x \upharpoonright K) \cup y \in G\right\}
$$

is comeager in $X^{T \backslash K}$.
Proof. Let $\left\{K_{i}: i<\omega\right\}$ be an enumeration of $\mathcal{K}$ with infinite repetitions. We construct, by induction on $i<\omega$, a decreasing sequence $\left\langle G_{i}: i<\omega\right\rangle$ of comeager subsets of $H$ such that for every $i<\omega$ :
(i) The set $\left(G_{i}\right)_{x \upharpoonright K_{i}}$ is comeager in $X^{T \backslash K_{i}}$ for every $x \in G_{i}$.

Put $G_{-1}=H$ and assume that for some $i<\omega$ the comeager set $G_{i-1}$ is already constructed. To define $G_{i}$ identify $X^{T}$ with $X^{K_{i}} \times X^{T \backslash K_{i}}$. Then, by the Kuratowski-Ulam theorem, the set

$$
A=\left\{y \in X^{K_{i}}:\left(G_{i-1}\right)_{y} \text { is comeager in } X^{T \backslash K_{i}}\right\}
$$

is comeager in $X^{K_{i}}$. Put $G_{i}=G_{i-1} \cap\left(A \times X^{T \backslash K_{i}}\right)$.
Clearly $G_{i} \subset G_{i-1}$ is comeager in $X^{T}$. If $x \in G_{i}$, then $x \upharpoonright K_{i} \in A$ and so $\left(G_{i}\right)_{x \upharpoonright K_{i}}=\left(G_{i-1}\right)_{x \upharpoonright K_{i}}$ is comeager in $X^{T \backslash K_{i}}$. So, (i) holds. This completes the definition of the sequence $\left\langle G_{i}: i<\omega\right\rangle$.

Let $G=\bigcap_{i<\omega} G_{i}$. Clearly $G \subset H$ is comeager in $X^{T}$. To see the additional part, take a $K \in \mathcal{K}$. Since $G=\bigcap\left\{G_{i}: i<\omega \& K_{i}=K\right\}$, for every $x \in G$ the set

$$
G_{x \upharpoonright K}=\bigcap\left\{\left(G_{i}\right)_{x \upharpoonright K_{i}}: i<\omega \& K_{i}=K\right\}
$$

is comeager in $X^{T \backslash K}$.

Applying Lemma 3.2 .1 to $X=\mathfrak{C}, T=\alpha$, and $\mathcal{K}=\alpha$ we immediately obtain the following corollary.

Corollary 3.2.2 Let $0<\alpha<\omega_{1}$. For every comeager set $H \subset \mathfrak{C}^{\alpha}$ there exists a comeager set $G \subset H$ such that for every $x \in G$ and $\beta<\alpha$ the set

$$
G_{x \upharpoonright \beta}=\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}:(x \upharpoonright \beta) \cup y \in G\right\}
$$

is comeager in $\mathfrak{C}^{\alpha \backslash \beta}$.
Let $X$ be a Polish space, $0<n<\omega$, and $F \subset X^{n}$ be an $n$-ary relation. We say that a set $S \subset X$ is $F$-independent provided $F(x(0), \ldots, x(n-1))$ does not hold for any one to one $x: n \rightarrow S$. For a family $\mathcal{F}$ of finitary relations on $X$ (i.e., relations $F \subset X^{n}$, where $0<n<\omega$ ) we say that $S \subset X$ is $\mathcal{F}$-independent provided $S$ is $F$-independent for every $F \in \mathcal{F}$. We will use the term unary relation for any 1-ary relation.

Proposition 3.2.3 Let $0<\alpha<\omega_{1}$ and $\mathcal{F}$ be a countable family of closed finitary relations on $\mathfrak{C}^{\alpha}$. Assume that every unary relation in $\mathcal{F}$ is nowhere dense in $\mathfrak{C}^{\alpha}$ and that for every $F \in \mathcal{F}$ there exists a comeager subset $G_{F}$ of $\mathfrak{C}^{\alpha}$ such that:
(ex) For every $F$-independent finite set $S \subset G_{F}, x \in S$, and $\beta<\alpha$ the set

$$
\left\{z \in \mathfrak{C}^{\alpha \backslash \beta}: S \cup\{z \cup x \upharpoonright \beta\} \subset G_{F} \text { is } F \text {-independent }\right\}
$$

is dense in $\mathfrak{C}^{\alpha \backslash \beta}$.
Then there is an $E \in \mathbb{P}_{\alpha}$ that is $\mathcal{F}$-independent.
Note that, without the assumption that the unary relations in $\mathcal{F}$ are nowhere dense the proposition is false: The unary relation $F=\mathfrak{C}^{\alpha}$ satisfies the condition (ex) (with $G_{F}=\mathfrak{C}^{\alpha}$ ) and no nonempty set is $F$-independent. On the other hand, for any $n$-ary relation $F \in \mathcal{F}$ with $n>1$, condition (ex) implies that $F$ is nowhere dense in $\left(\mathfrak{C}^{\alpha}\right)^{n}$. However, not every nowhere dense binary relation satisfies (ex). For example, $F=\{\langle x, y\rangle: x(0)=y(0)\}$ is nowhere dense and it does not satisfy (ex) if $\alpha>1$.
Proof. First notice that, applying Corollary 3.2.2, if necessary, we can assume that for every $F \in \mathcal{F}, x \in G_{F}$, and $\beta<\alpha$ the set $\left(G_{F}\right)_{x \upharpoonright \beta}$ is comeager in $\mathfrak{C}^{\alpha \backslash \beta}$. But this implies that each set from the condition (ex) is comeager in $\mathfrak{C}^{\alpha \backslash \beta}$ since it is an intersection of $\left(G_{F}\right)_{x \upharpoonright \beta}$ and an open set $\left\{z \in \mathfrak{C}^{\alpha \backslash \beta}: S \cup\{z \cup x \upharpoonright \beta\}\right.$ is $F$-independent $\}$. In particular, if we put $G=\bigcap_{F \in \mathcal{F}} G_{F}$, then $G$ is comeager in $\mathfrak{C}^{\alpha}$ and it is easy to see that it satisfies the following condition.
(EX) For every $\mathcal{F}$-independent finite set $S \subset G, x \in S$, and $\beta<\alpha$ the set

$$
\left\{z \in \mathfrak{C}^{\alpha \backslash \beta}: S \cup\{z \cup x \upharpoonright \beta\} \subset G \text { is } \mathcal{F} \text {-independent }\right\}
$$

is dense in $\mathfrak{C}^{\alpha \backslash \beta}$.
Let $\left\{F_{k}: k<\omega\right\}$ be an enumeration of $\mathcal{F}$ with infinite repetitions. Also, for $k<\omega$, let $A_{k}=\left\{\left\langle\beta_{i}, n_{i}\right\rangle: i<k\right\}$ be as in condition (3.14). By induction on $k<\omega$ we will construct two sequences: $\left\langle\varepsilon_{k}>0: k<\omega\right\rangle$ converging to 0 and $\left\langle\left\{x_{s} \in G: s \in 2^{A_{k}}\right\}: k<\omega\right\rangle$ of $\mathcal{F}$-independent sets such that for every $\beta<\alpha, k<\omega$, and $s, t \in 2^{A_{k}}$ :
(a) $x_{s} \upharpoonright \beta=x_{t} \upharpoonright \beta$ if and only if $s \upharpoonright \beta \times \omega=t \upharpoonright \beta \times \omega$.
(b) If $E_{s}=B_{\alpha}\left(x_{s}, \varepsilon_{k}\right)$ and $\mathcal{E}_{k}=\left\{E_{s}: s \in 2^{A_{k}}\right\}$, then the $\mathcal{E}_{k}$ 's satisfy (ii), $(\mathrm{ag})$, and ( sp ) from Lemma 3.1.1.
(c) If $F_{k}$ is an $n$-ary relation, then $F_{k}\left(z_{0}, \ldots, z_{n-1}\right)$ does not hold provided each $z_{i}$ is chosen from a different ball from $\mathcal{E}_{k}$.

Before we construct such sequences, let us first note that $E=\bigcap_{k<\omega} \bigcup \mathcal{E}_{k}$ is as desired. Indeed, $E \in \mathbb{P}_{\alpha}$ by Lemma 3.1.1. To see that $E$ is $\mathcal{F}$-independent, pick an $n$-ary relation $F \in \mathcal{F},\left\{z_{0}, \ldots, z_{n-1}\right\} \in[E]^{n}$, and find a $k<\omega$ with $F_{k}=F$ that is big enough so that $\varepsilon_{k}$ is smaller than the distance between $z_{i}$ and $z_{j}$ for all $i<j<n$. Then the $z_{i}$ 's must belong to distinct elements of $\mathcal{E}_{k}$; so, by (c), $F\left(z_{0}, \ldots, z_{n-1}\right)$ does not hold.

For $k=0$ we pick an arbitrary $\mathcal{F}$-independent $x_{\emptyset} \in G$ by choosing an arbitrary element of $G$ that does not belong to any nowhere dense unary relation from $\mathcal{F}$. Also, we choose an $\varepsilon_{0} \in(0,1]$ ensuring (c), which can be done since $F_{0}$ is closed. (This is a nontrivial requirement only when $F_{0}$ is a unary relation.) Clearly (a)-(c) are satisfied.

Assume that for some $k<\omega$ the construction is done up to the level $k$. For $s \in 2^{A_{k}}$ and $j<2$, let $s^{\wedge} j=s \cup\left\{\left\langle\left\langle\beta_{k}, n_{k}\right\rangle, j\right\rangle\right\} \in 2^{A_{k+1}}$ and define $x_{s^{\wedge} 0}=x_{s}$. Let $\left\{s_{i}: i<2^{k}\right\}$ be an enumeration of $2^{A_{k}}$ and put $S=\left\{x_{s^{\wedge} 0}: s \in 2^{A_{k}}\right\}$. Points $x_{s_{i}{ }^{\wedge} 1} \in G \cap E_{s_{i}}$ will be chosen by induction on $i \leq 2^{k}$ such that the set $S_{i}=S \cup\left\{x_{s_{j}{ }^{\wedge} 1}: j<i\right\}$ is $\mathcal{F}$-independent and condition (a) is satisfied for the elements of $S_{i}$. Clearly, by the inductive assumption, (a) is satisfied for the elements of $S_{0}=S$. So, assume that for some $i \leq 2^{k}$ the set $S_{i}$ is already constructed. We need to find an appropriate $x_{s_{i}{ }^{\wedge} 1} \in G \cap E_{s_{i}}$. Let $\beta<\alpha$ be maximal such that there is an $s \in\left\{s^{\wedge} 0: s \in 2^{A_{k}}\right\} \cup\left\{s_{j}{ }^{\wedge} 1: j<i\right\}$ with $s \upharpoonright \beta \times \omega=\left(s_{i}{ }^{\wedge} 1\right) \upharpoonright \beta \times \omega$ and let $x=x_{s} \upharpoonright \beta$. We will choose $x_{s_{i}{ }^{\wedge} 1}$ extending $x$ and such that $x_{s_{i}{ }^{\wedge} 1}(\beta) \neq x_{t}(\beta)$ for all $x_{t} \in S_{i}$. Notice that this will ensure that condition (a) is satisfied for the elements of $S_{i+1}$. Surprisingly, a more difficult condition to ensure
will be that $x_{s_{i}{ }^{\wedge} 1} \in E_{s_{i}}=B_{\alpha}\left(x_{s_{i}{ }^{\wedge} 0}, \varepsilon_{k}\right)$, since at the first glance it is not even obvious that

$$
\begin{equation*}
B_{\alpha}\left(x_{s_{i} \wedge 0}, \varepsilon_{k}\right) \text { contains an extension of } x \tag{3.15}
\end{equation*}
$$

To argue for this, first notice that maximality of $\beta$ ensures that $\beta \geq \beta_{k}$, since $s_{i}{ }^{\wedge} 0 \in S_{i}$ and $\left(s_{i}{ }^{\wedge}\right) \upharpoonright \beta_{k} \times \omega=\left(s_{i}{ }^{\wedge} 1\right) \upharpoonright \beta_{k} \times \omega$. If $\beta=\beta_{k}$, we have $x=x_{s_{i} \wedge} \upharpoonright \upharpoonright \beta$ and (3.15) is obvious. So, assume that $\beta>\beta_{k}$. Then there is a $j<i$ such that $s=s_{j}{ }^{\wedge} 1$. We also have $s_{j} \upharpoonright \beta \times \omega=s_{i} \upharpoonright \beta \times \omega$; so, by the inductive assumption, $x_{s_{j}} \upharpoonright \beta=x_{s_{i}} \upharpoonright \beta$.
Now, let $n<\omega$ be the smallest such that $2^{-n}<\varepsilon_{k}$. Then, by the definition of the metric on $\mathfrak{C}^{\alpha}$, the fact that $x_{s}=x_{s_{j}{ }^{\wedge}} \in E_{s_{j}}=B_{\alpha}\left(x_{s_{j}}, \varepsilon_{k}\right)$ means that $x_{s}(\gamma)(m)=x_{s_{j}}(\gamma)(m)$ for every $\langle\gamma, m\rangle \in A_{n}$. Therefore, we have $x(\gamma)(m)=x_{s}(\gamma)(m)=x_{s_{j}}(\gamma)(m)=x_{s_{i}}(\gamma)(m)$ for every $\langle\gamma, m\rangle \in A_{n}$ with $\gamma<\beta$. Thus, we can extend $x$ to an element $y \in \mathfrak{C}^{\alpha}$ for which $y(\gamma)(m)=x_{s_{i}}(\gamma)(m)$ for every $\langle\gamma, m\rangle \in A_{n}$. But this $y$ witnesses (3.15).

To finish the construction of $x_{s_{i}{ }^{\wedge} 1}$, notice that by (3.15) we can find an open ball $B$ in $\mathfrak{C}^{\alpha \backslash \beta}$ such that $\{x\} \times B \subset B_{\alpha}\left(x_{s_{i}{ }^{\wedge} 0}, \varepsilon_{k}\right)$. Decreasing $B$, if necessary, we can also insure that $y(\beta) \neq x_{t}(\beta)$ for every $t \in S_{i}$ and $y \in\{x\} \times B$. By condition (EX) we can find a $z \in B$ such that $S_{i} \cup\{x \cup z\} \subset G$ is $\mathcal{F}$-independent. We put $x_{s_{i}{ }^{\wedge} 1}=x \cup z$.

Thus, we constructed an $\mathcal{F}$-independent set $\left\{x_{s^{\wedge} j}: s \in 2^{A_{k}} \& j<2\right\} \subset G$ satisfying (a) and such that $x_{s^{\wedge} 0}, x_{s^{\wedge} 1} \in E_{s}$ for every $s \in 2^{A_{k}}$. To finish the construction insuring (a)-(c) we need to choose an $\varepsilon_{k+1} \leq 2^{-(k+1)}$ small enough to guarantee the following properties.

- $E_{s^{\wedge} j}=B_{\alpha}\left(x_{s^{\wedge} 0}, \varepsilon_{k_{1}}\right) \subset E_{s}$ for every $s \in 2^{A_{k}}$ and $j<2$. This will ensure condition (ii).
- Condition (sp) holds. This can be done, since (a) is satisfied.
- Condition (c) is satisfied. This can be done since $\left\{x_{s}: s \in 2^{A_{k+1}}\right\}$ is $\mathcal{F}$-independent and $F_{k+1}$ is a closed relation.

Note that (ag) is guaranteed by (a) and our definition of the $E_{s}$ 's. This finishes the proof of Proposition 3.2.3.

It worth mentioning that Proposition 3.2.3 can be viewed as a generalization of J. Mycielski's theorem [101]. (Compare also [102].) Also, in the case when $\mathcal{F}$ consists of one binary relation $R$, a slight modification of the argument for Proposition 3.2.3 gives us the following.

Proposition 3.2.4 Let $0<\alpha<\omega_{1}$ and let $R$ be a closed binary relation on $\mathfrak{C}^{\alpha}$. Assume that there exists a comeager subset $G$ of $\mathfrak{C}^{\alpha}$ such that:
(bin) For every $x \in G, \varepsilon>0$, and $\beta<\alpha$ there is a $y \in G \cap B_{\alpha}(x, \varepsilon) \backslash\{x\}$ such that $y \upharpoonright \beta=x \upharpoonright \beta$ and $\{x, y\}$ is $R$-independent.

Then there is an $E \in \mathbb{P}_{\alpha}$ that is $R$-independent.
Proof. Let $\mathcal{F}=\{R\}$. We will just indicate the modifications needed in the proof of Proposition 3.2.3 to obtain the current result. Thus, we construct the sequences $\left\langle\varepsilon_{k}>0: k<\omega\right\rangle$ and $\left\langle\left\{x_{s} \in G: s \in 2^{A_{k}}\right\}: k<\omega\right\rangle$ subject to the same requirements. As before, this will give us the desired $E \in \mathbb{P}_{\alpha}$.

First notice that, by Corollary 3.2.2, decreasing $G$ if necessary, we can assume that

$$
G_{x \upharpoonright \beta}=\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}:(x \upharpoonright \beta) \cup y \in G\right\}
$$

is comeager for every $x \in G$ and $\beta<\alpha$.
The choice of $x_{\emptyset}$ is trivial, since any point from $G$ together with any $\varepsilon_{0}$ satisfy the requirements. To make an inductive step assume that for some $k<\omega$ the construction is done up to the level $k$. As before, for $s \in 2^{A_{k}}$ we put $x_{s^{\wedge} 0}=x_{s}$, fix an enumeration $\left\{s_{i}: i<2^{k}\right\}$ of $2^{A_{k}}$, and put $S=\left\{x_{s^{\wedge} 0}: s \in 2^{A_{k}}\right\}$. Before we choose points $x_{s_{i}{ }^{\wedge} 1}$ we need to make some preparations.

Let $\bar{\beta}=\max \left\{\beta_{i}: i \leq k\right\}<\alpha$ and notice that, by (bin), for every $i<2^{k}$ we can find a $y_{i} \in G \cap E_{s_{i}} \backslash\left\{x_{s_{i}}\right\}$ such that $y_{i} \upharpoonright \bar{\beta}=x_{s_{i}} \upharpoonright \bar{\beta}$ and $\left\{x_{s_{i}}, y_{i}\right\}$ is $R$-independent. Note that this, together with condition (c) for the step $k$, implies that the set $T=S \cup\left\{y_{i}: i<2^{k}\right\}$ is $R$-independent. Now, by the closure assumption about $R$, we can find an $\varepsilon \in\left(0, \varepsilon_{k}\right]$ such that the balls $\left\{B_{\alpha}(t, \varepsilon): t \in T\right\}$ are pairwise disjoint and any selector from this family is $R$-independent. Note also that $B_{\alpha}\left(x_{s_{i}}, \varepsilon\right) \cup B_{\alpha}\left(y_{i}, \varepsilon\right) \subset E_{s_{i}}$ for every $i<2^{k}$ as $\varepsilon \leq \varepsilon_{k}$. Since we will choose $x_{s_{i}{ }^{\wedge} 1} \in B_{\alpha}\left(y_{i}, \varepsilon\right) \cap G$, the resulting set $\left\{x_{s}: s \in 2^{A_{k+1}}\right\} \subset G$ will be $R$-independent. We just need to insure that it satisfies (a).

Points $x_{s_{i}{ }^{\wedge} 1}$ will be chosen by induction on $i<2^{k}$ such that the elements of the set $S_{i}=S \cup\left\{x_{s_{j}{ }^{\wedge} 1}: j<i\right\}$ satisfy condition (a). For this, we proceed precisely as in the proof of Proposition 3.2.3. We take a maximal $\beta<\alpha$ for which there exists an $s \in\left\{s^{\wedge} 0: s \in 2^{A_{k}}\right\} \cup\left\{s_{j}{ }^{\wedge} 1: j<i\right\}$ with $s \upharpoonright \beta \times \omega=\left(s_{i}{ }^{\wedge} 1\right) \upharpoonright \beta \times \omega$ and let $x=x_{s} \upharpoonright \beta$. We will choose $x_{s_{i}{ }^{\wedge} 1}$ extending $x$ and such that $x_{s_{i}{ }^{\wedge} 1}(\beta) \neq x_{t}(\beta)$ for all $x_{t} \in S_{i}$, ensuring that condition (a) is satisfied for the elements of $S_{i+1}$. For this we need to notice
that, similarly as for (3.15), we can prove that

$$
B_{\alpha}\left(y_{i}, \varepsilon\right) \text { contains an extension of } x
$$

The same argument works here, since we have shown there that $x(\gamma)(m)=x_{s}(\gamma)(m)=x_{s_{j}}(\gamma)(m)=x_{s_{i}}(\gamma)(m)$ for all appropriate pairs $\langle\gamma, m\rangle \in A_{n}$ with $\gamma<\beta$, while we have $\beta \leq \bar{\beta}$ and $y_{i} \upharpoonright \bar{\beta}=x_{s_{i}} \upharpoonright \bar{\beta}$. In other words, $x(\gamma)(m)=x_{t_{i}}(\gamma)(m)$ for all appropriate pairs $\langle\gamma, m\rangle \in A_{n}$ with $\gamma<\beta$, which gives us the above condition.

Now, since $G_{x}=G_{x_{s} \upharpoonright \beta}$ is comeager in $\mathfrak{C}^{\alpha \backslash \beta}$, there exists an

$$
x_{s_{i}{ }^{\wedge} 1} \in B_{\alpha}\left(y_{i}, \varepsilon\right) \cap\left(\{x\} \times\left\{z \in G_{x}: z(\beta) \neq x_{t}(\beta) \text { for all } x_{t} \in S_{i}\right\}\right) .
$$

It is easy to see that such an $x_{s_{i}{ }^{\wedge} 1}$ satisfies all the requirements.
The choice of an appropriate $\varepsilon_{k+1}$ is done as in Proposition 3.2.3.
In what follows we will also need the following fact, which is one of the most important properties of prisms (or, more precisely, iterated perfect sets) and distinguishes them from cubes. (Compare also [73, thm. 20].)

Lemma 3.2.5 For every $0<\alpha<\omega_{1}, E \in \mathbb{P}_{\alpha}$, a Polish space $X$, and a continuous function $f: E \rightarrow X$ there exists a $P \in \mathbb{P}_{\alpha}$ such that $P \subset E$ and either $f$ is constant on $P$ or else there exists a $0<\beta \leq \alpha$ such that $f \circ \pi_{\beta}^{-1}$ is a one to one function on $\pi_{\beta}[P] \in \mathbb{P}_{\beta}$.

Notice that the property that " $f \circ \pi_{\beta}^{-1}$ is a function on $\pi_{\beta}[P]$ " means simply that the value of $f \upharpoonright P$ at $x \in P$ depends only on $x \upharpoonright \beta$, the first $\beta$ coordinates of $x$. Also, we could eliminate from the lemma the case " $f$ is constant on $P$ " if we allow $\beta=0$, but then we could not claim that $\pi_{\beta}[P]=\{\emptyset\}$ belongs to $\mathbb{P}_{\beta}$ because $\mathbb{P}_{0}$ is not defined.

Proof. Since $E \in \mathbb{P}_{\alpha}$, there is a $g \in \Phi_{\text {prism }}(\alpha)$ mapping $\mathfrak{C}^{\alpha}$ onto $E$. Notice that it is enough to prove the lemma for $\bar{f}=f \circ g$ and $\bar{E}=\mathfrak{C}^{\alpha}$ : If $\bar{P}$ satisfies the lemma for this pair, then $P=g[\bar{P}]$ satisfies it for the original pair. Thus, without loss of generality we can assume that $E=\mathfrak{C}^{\alpha}$.

If there is a $P \in \mathbb{P}_{\alpha}$ on which $f$ is constant, then we are done. So, assume that this is not the case, that is, that

$$
\begin{equation*}
\text { there is no } P \in \mathbb{P}_{\alpha} \text { such that } f \text { is constant on } P \text {. } \tag{3.16}
\end{equation*}
$$

First we prove that under the additional assumption that
(•) $f \circ \pi_{\gamma}^{-1}$ is a function on $\pi_{\beta}[\hat{E}]$ for no $\gamma<\alpha$ and $\hat{E} \in \mathbb{P}_{\alpha}$
we can find a $P \in \mathbb{P}_{\alpha}$ on which $f$ is one to one.
To see this, for every $0<\gamma<\alpha$ consider the closed set

$$
F_{\gamma}=\left\{z \in \mathfrak{C}^{\gamma}: f \text { is constant on }\{z\} \times \mathfrak{C}^{\alpha \backslash \gamma}\right\}
$$

and notice that it must be nowhere dense: If it had contained a nonempty open ball $B$, then $P=B \times \mathfrak{C}^{\alpha \backslash \gamma} \in \mathbb{P}_{\alpha}$ would have contradicted ( $\bullet$ ). Thus, the set $F=\bigcup_{0<\gamma<\alpha} F_{\gamma} \times \mathfrak{C}^{\alpha \backslash \gamma}$ is meager, and so, by Corollary 3.2.2, we can find a comeager set $G \subset \mathfrak{C}^{\alpha} \backslash F$ such that for every $x \in G$ and $\gamma<\alpha$ the set $G_{x \upharpoonright \gamma}$ is comeager in $\mathfrak{C}^{\alpha \backslash \gamma}$. But this implies that
$f$ is not constant on $\{x \upharpoonright \gamma\} \times G_{x \upharpoonright \gamma}$ for any $\gamma<\alpha$ and $x \in G$,
since otherwise $f$ would be constant on the closure of $\{x \upharpoonright \gamma\} \times G_{x \upharpoonright \gamma}$, which is equal to $\{x \upharpoonright \gamma\} \times \mathfrak{C}^{\alpha \backslash \gamma}$, and $x$ would belong to $F$, contradicting $x \in G$. So the relation $R=\left\{\left\langle x_{0}, x_{1}\right\rangle: f\left(x_{0}\right)=f\left(x_{1}\right)\right\}$ and $G$ satisfy the condition (bin) from Proposition 3.2.4. Therefore, there exists an $R$-independent $P \in \mathbb{P}_{\alpha}$ and it is easy to see that $f$ is one to one on such a $P$. Thus (•) implies what we promised.
To prove the lemma in a general case, let $\beta \leq \alpha$ be the smallest ordinal such that $\hat{f}=f \circ \pi_{\beta}^{-1}$ is a function on $\pi_{\beta}[\hat{E}] \in \mathbb{P}_{\beta}$ for some $\hat{E} \in \mathbb{P}_{\alpha}$. Note that, by (3.16), $\beta>0$. Using the argument from the first paragraph of the proof we can assume that $\hat{E}=\mathfrak{C}^{\alpha}$. (Recall that the function $g$ is in this argument is projection-keeping.) Then, the minimality of $\beta$ implies that $\hat{f}$ satisfies ( $\bullet$ ). Thus, from what we have already proved, we can conclude that there is a $\hat{P} \in \mathbb{P}_{\beta}$ such that $\hat{f}$ is one to one on $\hat{P}$. Then $P=\hat{P} \times \mathfrak{C}^{\alpha \backslash \beta} \in \mathbb{P}_{\alpha}$ and $\beta$ are as desired.

Remark 3.2.6 Notice that Lemma 3.2 .5 is false if we replace $\mathbb{P}_{\alpha}$ with $\operatorname{CUBE}(\alpha)$. Indeed, this is obviously the case if we take $f: \mathfrak{C}^{2} \rightarrow \mathfrak{C}$ given by $f\left(x_{0}, x_{1}\right)=x_{1}$.

The result presented in the reminder of this section will be used only in Section 5.1. We say that an $n$-ary relation $F$ on a Polish space $X$ is symmetric provided for any sequence $\left\langle x_{i} \in X: i<n\right\rangle$ and any permutation $\pi$ of $n$

$$
F\left(x_{0}, \ldots, x_{n-1}\right) \text { holds if and only if } F\left(x_{\pi(0)}, \ldots, x_{\pi(n-1)}\right) \text { holds. }
$$

For such an $F$ and $A \subset X$ we put

$$
F * A=A \cup\left\{x \in X:\left(\exists a_{1}, \ldots, a_{n-1} \in A\right) \quad F\left(x, a_{1}, \ldots, a_{n-1}\right)\right\}
$$

If $F$ is unary relation, we interpret the above as $F * A=A \cup F$. If $\mathcal{F}$ is a family of symmetric finitary relations on $X$, then we put $\mathcal{F} * A=$ $\bigcup_{F \in \mathcal{F}} F * A$. Also, an $\mathcal{F}$-closure of $A$, denoted by $\mathrm{cl}_{\mathcal{F}}(A)$, is the least $B \subset X$ containing $A$ such that $\mathcal{F} * B=B$. Note that $\operatorname{cl}_{\mathcal{F}}(A)=\bigcup_{n<\omega} \mathcal{F}^{n} * A$, where $\mathcal{F}^{0} * A=A$ and $\mathcal{F}^{n+1} * A=\mathcal{F} *\left(\mathcal{F}^{n} * A\right)$. Thus, if $\mathcal{F}$ is a countable family of closed symmetric finitary relations, then $\operatorname{cl}_{\mathcal{F}}(A)$ is $F_{\sigma}$ in $X$ for a $\sigma$-compact $A \subset X$ since $F * K$ is closed for every $F \in \mathcal{F}$ and compact $K \subset X$.

We are most interested in these notions when we are concerned with either linear independence (over $\mathbb{Q}$ ) or algebraic independence in $\mathbb{R}$. In the first case, $\mathcal{F}$ is defined as the family of all relations $F_{w}$ of all $\left\langle x_{0}, \ldots, x_{n-1}\right\rangle$ for which

$$
w\left(x_{\pi(0)}, \ldots, x_{\pi(n-1)}\right)=0 \text { for some permutation } \pi \text { of } n
$$

where $w$ is a nonzero linear function with rational coefficients. In this case $\mathcal{F}$-independence stands for linear independence (over $\mathbb{Q}$ ) and $\operatorname{cl}_{\mathcal{F}}(A)$ is the linear span of $A$. When $\mathcal{F}$ is the family of all relations $F_{w}$, where $w$ spans over all nonzero polynomials with rational coefficients, then $\mathcal{F}$-independence stands for algebraic independence while $\operatorname{cl}_{\mathcal{F}}(A)$ is the algebraic closure of $\mathbb{Q}(A)$.

We will also need one more notion. For a family $\mathcal{F}$ of closed symmetric finitary relations on $X$ and an $M \subset X$ we define $\mathcal{F}_{M}$ as the collection of all possible projections of the relations from $\mathcal{F}$ along $M$. In other words, $\mathcal{F}_{M}$ is the collection of all (symmetric) relations

$$
\left\{\left\langle x_{0}, \ldots, x_{k-1}\right\rangle:\left(\exists a_{k}, \ldots, a_{n-1} \in M\right) F\left(x_{0}, \ldots, x_{k-1}, a_{k}, \ldots, a_{n-1}\right)\right\}
$$

where $F \in \mathcal{F}$ is an $n$-ary relation and $0<k \leq n$. Note that if $M$ is compact, then each relation in $\mathcal{F}_{M}$ is still closed and for every $A \subset X$ we have

$$
\begin{equation*}
\operatorname{cl}_{\mathcal{F}}(M \cup A)=\operatorname{cl}_{\mathcal{F}_{M}}(A) \tag{3.17}
\end{equation*}
$$

Also, if $M$ is $\mathcal{F}$-independent, then
$A \cup M$ is $\mathcal{F}$-independent provided $A$ is $\mathcal{F}_{M}$-independent.
The following lemma will be the crucial for our applications presented in Section 5.1. We will also present there, in Remark 5.1.6, an example showing that in the lemma we cannot require $R=Q$.

Lemma 3.2.7 Let $\mathcal{F}$ be an arbitrary family of closed symmetric finitary relations in a Polish space $X$. Then for every prism $P$ in $X$ there exists a subprism $Q$ of $P$ and a compact $\mathcal{F}$-independent set $R \subset P$ such that $Q \subset \operatorname{cl}_{\mathcal{F}}(R)$.

Proof. For $0<\alpha<\omega_{1}$, let $I_{\alpha}$ be the statement:
$I_{\alpha}$ : The lemma holds for any prism $P$ with witness function $f: \mathfrak{C}^{\alpha} \rightarrow P$.
We will prove $I_{\alpha}$ by induction on $\alpha$.
First notice that $I_{\alpha}$ implies the following:
$I_{\alpha}^{*}$ : For every $k<\omega$ and continuous functions $g_{0}, \ldots, g_{k}: \mathfrak{C}^{\alpha} \rightarrow X$, there exist an $E \in \mathbb{P}_{\alpha}$ and a compact $\mathcal{F}$-independent set $R \subset \bigcup_{i \leq k} g_{i}\left[\mathfrak{C}^{\alpha}\right]$ such that $\bigcup_{i \leq k} g_{i}[E] \subset \operatorname{cl}_{\mathcal{F}}(R)$.

To see that $I_{\alpha}^{*}$ holds true for $k=0$, for every $n$-ary relation $F \in \mathcal{F}$ define $F^{0}=\left\{\left\langle x_{0}, \ldots, x_{n-1}\right\rangle \in\left(\mathfrak{C}^{\alpha}\right)^{n}: F\left(g_{0}\left(x_{0}\right), \ldots, g_{0}\left(x_{n-1}\right)\right)\right\}$. By $I_{\alpha}$ applied to $\mathcal{F}_{0}=\left\{F^{0}: F \in \mathcal{F}\right\}$ we can find an $\mathcal{F}_{0}$-independent set $R_{0} \subset \mathfrak{C}^{\alpha}$ and an $E \in \mathbb{P}_{\alpha}$ such that $E \subset \operatorname{cl}_{\mathcal{F}_{0}}(R)$. But then $R=g_{0}\left[R_{0}\right]$ is compact $\mathcal{F}$-independent and $g_{0}[E] \subset \operatorname{cl}_{\mathcal{F}}\left(g_{0}\left[R_{0}\right]\right)=\operatorname{cl}_{\mathcal{F}}(R)$.

To make an inductive step assume that $I_{\alpha}^{*}$ holds for some $k<\omega$ and take continuous functions $g_{0}, \ldots, g_{k+1}: \mathfrak{C}^{\alpha} \rightarrow X$. By the inductive assumption we can find an $E_{0} \in \mathbb{P}_{\alpha}$ and a compact $\mathcal{F}$-independent set $R_{0} \subset \bigcup_{i \leq k} g_{i}\left[\mathfrak{C}^{\alpha}\right]$ such that $\bigcup_{i \leq k} g_{i}\left[E_{0}\right] \subset \operatorname{cl}_{\mathcal{F}}\left(R_{0}\right)$. Let $h \in \Phi_{\text {prism }}(\alpha)$ be a mapping from $\mathfrak{C}^{\alpha}$ onto $E_{0}$. Using the case $k=0$ to the function $g_{k+1} \circ h$ and the family $\mathcal{F}_{R_{0}}$ we can find an $E_{1} \in \mathbb{P}_{\alpha}$ and a compact $\mathcal{F}_{R_{0}}$-independent set $R_{1} \subset\left(g_{k+1} \circ h\right)\left[\mathfrak{C}^{\alpha}\right]$ such that $\left(g_{k+1} \circ h\right)\left[E_{1}\right] \subset \operatorname{cl}_{\mathcal{F}_{R_{0}}}\left(R_{1}\right)$. Then, by (3.18), we conclude that $R=R_{0} \cup R_{1}$ is $\mathcal{F}$-independent. Put $E=h\left[E_{1}\right] \in \mathbb{P}_{\alpha}$. Then, by (3.17), we have $g_{k+1}[E] \subset \operatorname{cl}_{\mathcal{F}_{R_{0}}}\left(R_{1}\right)=\operatorname{cl}_{\mathcal{F}}\left(R_{0} \cup R_{1}\right)=\operatorname{cl}_{\mathcal{F}}(R)$, while clearly $\bigcup_{i \leq k} g_{i}[E] \subset \bigcup_{i \leq k} g_{i}\left[E_{0}\right] \subset \operatorname{cl}_{\mathcal{F}}\left(R_{0}\right) \subset \operatorname{cl}_{\mathcal{F}}(R)$. Thus, $E$ and $R$ satisfy $I_{\alpha}^{*}$.

Now we are ready to prove $I_{\alpha}$. So, fix $0<\alpha<\omega_{1}$ and assume that $I_{\gamma}$ is true for all $0<\gamma<\alpha$. Let $P$ be a prism in $X$ with witness function $f: \mathfrak{C}^{\alpha} \rightarrow P$. We need to find an appropriate $Q$ and $R$.

Let $W$ be the set of all $\beta \leq \alpha$ for which there exists an $E \in \mathbb{P}_{\alpha}$ and an $F \in \mathcal{F}$ such that for every $z \in \pi_{\beta}[E]$ there is a finite set $R_{z} \subset P$ for which

$$
\begin{equation*}
f[\{x \in E: z \subset x\}] \subset F * R_{z} \tag{3.19}
\end{equation*}
$$

Notice that $W$ is nonempty since $\alpha \in W$. So $\beta=\min W$ is well defined. Let $E \in \mathbb{P}_{\alpha}$ be such that (3.19) holds for $\beta$. As usual, replacing $f$ with its
composition with an appropriate function from $\Phi_{\text {prism }}(\alpha)$, if necessary, we can assume that $E=\mathfrak{C}^{\alpha}$.

If $\beta=0$, then $f\left[\mathfrak{C}^{\alpha}\right] \subset \operatorname{cl}_{\mathcal{F}}\left(R_{0}\right)$ for some finite set $R_{0} \subset P$, and we can find an $\mathcal{F}$-independent finite $R \subset R_{0}$ with $f\left[\mathfrak{C}^{\alpha}\right] \subset \operatorname{cl}_{\mathcal{F}}(R)$. (Note that if $T$ is $\mathcal{F}$-independent and $x \in X \backslash \operatorname{cl}_{\mathcal{F}}(T)$, then $T \cup\{x\}$ is also $\mathcal{F}$-independent.) Thus, $Q=f\left[\mathfrak{C}^{\alpha}\right]$ and $R$ satisfy $I_{\alpha}$. So, for the rest of the proof we will assume that $\beta>0$.

Next, assume that $0<\beta<\alpha$. Let $\mathcal{B}_{\beta}$ be a countable basis of $\mathfrak{C}^{\alpha \backslash \beta}$ consisting of nonempty clopen sets and assume that $F$ satisfying (3.19) is ( $n+1$ )-ary. For every $B \in \mathcal{B}_{\beta}$ consider the set

$$
K_{B}=\left\{z \in \mathfrak{C}^{\beta}:\left(\exists\left\langle x_{1}, \ldots, x_{n}\right\rangle \in P^{n}\right)(\forall y \in B) F\left(f(z \cup y), x_{1}, \ldots, x_{n}\right)\right\}
$$

It is easy to see that each set $K_{B}$ is closed. Notice also that

$$
\begin{equation*}
\mathfrak{C}^{\beta}=\bigcup_{B \in \mathcal{B}_{\beta}} K_{B} \tag{3.20}
\end{equation*}
$$

To see this, fix a $z \in \mathfrak{C}^{\beta}$. By (3.19), there exists a finite set $S_{z} \subset \mathfrak{C}^{\alpha}$ such that $\mathfrak{C}^{\alpha \backslash \beta}=\bigcup_{x_{1}, \ldots, x_{n} \in f\left[S_{z}\right]}\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}: F\left(f(z \cup y), x_{1}, \ldots, x_{n}\right)\right\}$. Since each set $\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}: F\left(f(z \cup y), x_{1}, \ldots, x_{n}\right)\right\}$ is closed, one of them must contain a $B \in \mathcal{B}_{\beta}$, and so $z \in K_{B}$.

Thus, by (3.20), there exists a $B \in \mathcal{B}_{\beta}$ such that $K_{B}$ has a nonempty interior. In particular, there is a nonempty clopen set $U \subset K_{B}$. But then for every $z \in U$ there is a $g(z)=\left\langle g_{1}(z), \ldots, g_{n}(z)\right\rangle \in P^{n}$ such that $F\left(f(z \cup y), g_{1}(z), \ldots, g_{n}(z)\right)$ holds for every $y \in B$. Now

$$
T=\left\{\langle z, \bar{p}\rangle \in U \times P^{n}:(\forall y \in B) F(f(z \cup y), \bar{p})\right\}
$$

is a compact subset of $U \times P^{n}$ and $g$ constitutes a selector of $T$. Thus, we can choose $g$ to be Borel. In particular, there is a dense $G_{\delta}$ subset $W$ of $U$ such that $g \upharpoonright W$ is continuous. So, by Claim 1.1.5, we can find a perfect cube $C \subset W \subset \mathfrak{C}^{\beta}$. Now, identifying $C$ with $\mathfrak{C}^{\beta}$, we conclude that the functions $g_{1}, \ldots, g_{n}: \mathfrak{C}^{\beta} \rightarrow P$ are continuous and that the relation $F\left(f(z \cup y), g_{1}(z), \ldots, g_{n}(z)\right)$ holds for every $z \in \mathfrak{C}^{\beta}$ and $y \in B$.

Since, by the inductive hypothesis, $I_{\beta}$ is true, condition $I_{\beta}^{*}$ holds as well. Thus, there exist an $E \in \mathbb{P}_{\beta}$ and a compact $\mathcal{F}$-independent set $R \subset P$ such that $\bigcup_{i=1}^{n} g_{i}[E] \subset \operatorname{cl}_{\mathcal{F}}(R)$. Since $Q=f[E \times B]$ is a subprism of $P$, we just need to show that $Q \subset \operatorname{cl}_{\mathcal{F}}(R)$. To see this just note that for every $z \in E$ we have $f[\{z\} \times B] \subset F *\left\{g_{1}(z), \ldots, g_{n}(z)\right\} \subset \operatorname{cl}_{\mathcal{F}}\left(\bigcup_{i=1}^{n} g_{i}[E]\right) \subset \operatorname{cl}_{\mathcal{F}}(R)$. This finishes the proof of the case $0<\beta<\alpha$.

For the reminder of the proof we will assume that $\beta=\alpha$. This means that there is no $E \in \mathbb{P}_{\alpha}$ such that for some $F \in \mathcal{F}$ and $\beta<\alpha$

$$
\begin{equation*}
\left(\forall z \in \pi_{\beta}[E]\right)\left(\exists R_{z} \in[P]^{<\omega}\right) f[\{x \in E: z \subset x\}] \subset F * R_{z} \tag{3.21}
\end{equation*}
$$

For every $n$-ary $F \in \mathcal{F}$, let $F^{*}=\left\{\left\langle x_{0}, \ldots, x_{n-1}\right\rangle: F\left(f\left(x_{0}\right), \ldots, f\left(x_{n-1}\right)\right)\right\}$ and let $\mathcal{F}^{*}=\left\{F^{*}: F \in \mathcal{F}\right\}$. We will apply Proposition 3.2 .3 to find an $\mathcal{F}^{*}$-independent $E \in \mathbb{P}_{\alpha}$. Then $Q=f[E]$ is an $\mathcal{F}$-independent subprism of $P$ and together with $R=Q$ they satisfy the lemma.

To see that the assumptions of Proposition 3.2.3 are satisfied, first notice that unary relations in $\mathcal{F}^{*}$ are nowhere dense. Indeed, otherwise there is a unary relation $F^{*} \in \mathcal{F}^{*}$ and a nonempty clopen set $E \subset F^{*}$. But then $E$ contradicts (3.21), as $f[E] \subset F * \emptyset$. Thus, we just need to show that the condition (ex) is satisfied.

So, fix an $F \in \mathcal{F}$. For $0<\beta<\alpha$ and $B \in \mathcal{B}_{\beta}$ let

$$
K(B)=\left\{z \in \mathfrak{C}^{\beta}:\left(\exists R_{z} \in[P]^{<\omega}\right) f[\{z\} \times B] \subset F * R_{z}\right\}
$$

Clearly $K(B)$ is $F_{\sigma}$. Notice also that it is meager, since otherwise there would exist a nonempty clopen $U \subset K(B)$ and $E=U \times B$ would contradict (3.21). Thus, each set $K_{\beta}=\bigcup_{B \in \mathcal{B}_{\alpha}} K(B)$ is meager. Also, for every $z \in \mathfrak{C}^{\beta} \backslash K_{\beta}$ and for every finite $R \subset P$ the set $\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}: f(z \cup y) \notin F * R\right\}$ is dense and open. In particular, if $R$ is a finite $F$-independent subset of $P$, then

$$
\begin{equation*}
W_{R}=\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}: R \cup\{f(z \cup y)\} \text { is } F \text {-independent }\right\} \tag{3.22}
\end{equation*}
$$

is dense and open. Let

$$
H=\bigcap_{0<\beta<\alpha}\left(\left(\mathfrak{C}^{\beta} \backslash K_{\beta}\right) \times \mathfrak{C}^{\alpha \backslash \beta}\right)
$$

and notice that $H$ is comeager since each $K_{\beta}$ is meager in $\mathfrak{C}^{\beta}$. By Corollary 3.2 .2 we can find a comeager set $G \subset H$ such that

$$
G_{x \upharpoonright \beta}=\left\{y \in \mathfrak{C}^{\alpha \backslash \beta}:(x \upharpoonright \beta) \cup y \in G\right\}
$$

is comeager for every $x \in G$ and $\beta<\alpha$. To finish the proof it is enough to show that $G$ satisfies (ex) for $F^{*}$. So, take an $F^{*}$-independent finite set $S \subset G$, an $x \in S$, and a $\beta<\alpha$.

First let us assume that $\beta>0$. Then $x \in S \subset G \subset H$ implies that $z=x \upharpoonright \beta \in \mathfrak{C}^{\beta} \backslash K_{\beta}$. In particular, the set $W_{f[S]}$ from (3.22) is comeager, and so is $W_{f[S]} \cap G_{x \upharpoonright \beta}$. To get (ex) it is enough to notice that $W_{f[S]} \cap G_{x \upharpoonright \beta}$ is a subset of $\left\{y \in \mathfrak{C}^{\alpha} \backslash \beta: S \cup\{y \cup z\} \subset G\right.$ is $F^{*}$-independent $\}$.

Finally, assume that $\beta=0$. We need to show that the set

$$
\left\{y \in G: S \cup\{y\} \text { is } F^{*} \text {-independent }\right\}
$$

is dense. But this set must be comeager, since otherwise its complement would contain a nonempty clopen set $E$, which wold contradict (3.21) with $\beta=0$.

## 3.3 $\mathrm{CPA}_{\text {prism }}$, additivity of $s_{0}$, and more on (A)

The results presented in this section come from K. Ciesielski and J. Pawlikowski [43].

We will start with noticing that the axiom $\mathrm{CPA}_{\text {prism }}$ leads in a natural way to the following generalization of the ideal $s_{0}^{\text {cube }}$ :

$$
s_{0}^{\text {prism }}=\left\{X \backslash \bigcup \mathcal{E}: \mathcal{E} \text { is } \mathcal{F}_{\text {prism-dense in }} \operatorname{Perf}(X)\right\} .
$$

Similarly as for Proposition 1.0.3, it can be shown that
Proposition 3.3.1 If $\mathrm{CPA}_{\text {prism }}$ holds, then $s_{0}^{\text {prism }}=[X]^{\leq \omega_{1}}$.
It can be also shown, refining the argument for Fact 1.0.4, that
Fact 3.3.2 For a Polish space $X$ we have $[X]^{<\mathfrak{c}} \subset s_{0}^{\text {cube }} \subset s_{0}^{\text {prism }} \subset s_{0}$.
However, we will not use these facts in the rest of this text.
The next lemma and its corollaries represent a very useful application of Lemma 3.2.5. For a fixed Polish space $X$ and $0<\alpha<\omega_{1}$, let $\mathcal{F}^{\alpha}$ denote the family of all continuous injections from $\mathfrak{C}^{\alpha}$ into $X$. Note that if we consider $\mathcal{F}^{\alpha}$ with the topology of uniform convergence, then

$$
\begin{equation*}
\mathcal{F}^{\alpha} \text { is a Polish space. } \tag{3.23}
\end{equation*}
$$

To prove (3.23) it is enough to show that $\mathcal{F}^{\alpha}$ is a $G_{\delta}$ subset of the space $\mathcal{C}=\mathcal{C}\left(\mathfrak{C}^{\alpha}, X\right)$ of all continuous functions from $\mathfrak{C}^{\alpha}$ into $X$. But $\mathcal{F}^{\alpha}$ is the intersection of the open sets $G_{n}, n<\omega$, where the sets $G_{n}$ are constructed as follows. Fix a finite partition $\mathcal{P}_{n}$ of $\mathfrak{C}^{\alpha}$ into clopen sets each of the diameter less than $2^{-n}$, and let $\mathcal{H}_{n}$ be the family of all mappings $h$ from $\mathcal{P}_{n}$ into the topology of $X$ such that $h(P) \cap h\left(P^{\prime}\right)=\emptyset$ for distinct $P, P^{\prime} \in \mathcal{P}_{n}$. We put

$$
G_{n}=\bigcup_{h \in \mathcal{H}_{n}}\left\{f \in \mathcal{C}:\left(\forall P \in \mathcal{P}_{n}\right)(\forall x \in P) f(x) \in h(P)\right\}
$$

This completes the argument for (3.23).

Lemma 3.3.3 Let $X$ be a Polish space and $0<\alpha<\omega_{1}$. Then every function $f: \mathfrak{C}^{\beta} \rightarrow \mathcal{F}^{\alpha}$ from $\mathcal{F}_{\text {prism }}\left(\mathcal{F}^{\alpha}\right)$ has a restriction $f^{*} \in \mathcal{F}_{\text {prism }}\left(\mathcal{F}^{\alpha}\right)$ with the property that there exists an $\hat{f} \in \mathcal{F}_{\text {prism }}(X)$ defined on a subset of $\mathfrak{C}^{\beta+\alpha}$ such that
(a) $\hat{f}(s, t)=f^{*}(s)(t)$ for all $\langle s, t\rangle \in\left(\mathfrak{C}^{\beta} \times \mathfrak{C}^{\alpha}\right) \cap \operatorname{dom}(\hat{f})$, and
(b) for each $s \in \operatorname{dom}\left(f^{*}\right)$ function $\hat{f}(s, \cdot):\left\{t \in \mathfrak{C}^{\alpha}:\langle s, t\rangle \in \operatorname{dom}(\hat{f})\right\} \rightarrow X$ is a restriction of $f^{*}(s)$ and belongs to $\mathcal{F}_{\text {prism }}(X)$.

Proof. Let $f: \mathfrak{C}^{\beta} \rightarrow \mathcal{F}^{\alpha}, f \in \mathcal{F}_{\text {prism }}\left(\mathcal{F}^{\alpha}\right)$, and define a function $g$ from a set $\mathfrak{C}^{\beta} \times \mathfrak{C}^{\alpha}=\mathfrak{C}^{\beta+\alpha}$ into $X$ by $g(s, t)=f(s)(t)$ for $\langle s, t\rangle \in \mathfrak{C}^{\beta} \times \mathfrak{C}^{\alpha}$. It is easy to see that $g$ is continuous.

Apply Lemma 3.2 .5 to $E=\mathfrak{C}^{\beta+\alpha} \in \mathbb{P}_{\beta+\alpha}$ and to the function $g$ to find a $\gamma \leq \beta+\alpha$ and a subset $P \in \mathbb{P}_{\beta+\alpha}$ of $E$ such that $g \circ \pi_{\gamma}^{-1}$ is a function on $\pi_{\gamma}[P] \in \mathbb{P}_{\gamma}$ that is either one to one or constant. Let $f^{*}=f \upharpoonright \pi_{\beta}[P]$. We will show that it is as desired.

First note that

$$
\gamma=\beta+\alpha \text { and } g \text { is one to one on } P
$$

Indeed, if $z \in \operatorname{range}\left(f^{*}\right) \cap \mathcal{F}_{\text {prism }}(X)$ and $z=f^{*}(s)$, then for every different $t_{0}, t_{1} \in \mathfrak{C}^{\alpha}$ with $\left\langle s, t_{0}\right\rangle,\left\langle s, t_{1}\right\rangle \in P$ we have $g\left(s, t_{0}\right)=f(s)\left(t_{0}\right)=z\left(t_{0}\right) \neq$ $z\left(t_{1}\right)=g\left(s, t_{1}\right)$. So, $g$ cannot be constant and if $\gamma<\beta+\alpha$, then we can find $t_{0}$ and $t_{1}$ such that $\pi_{\gamma}\left(\left\langle s, t_{0}\right\rangle\right)=\pi_{\gamma}\left(\left\langle s, t_{1}\right\rangle\right)$, contradicting the above calculation.

It is easy to see that $\hat{f}=g \upharpoonright P$ is as desired.
Lemma 3.3.3 implies the following useful fact.
Proposition 3.3.4 $\mathrm{CPA}_{\text {prism }}$ implies that for every Polish space $X$ there exists a family $\mathcal{H}$ of continuous functions from compact subsets of $X$ onto $\mathfrak{C} \times \mathfrak{C}$ such that $|\mathcal{H}| \leq \omega_{1}$ and

- for every prism $P$ in $X$ there are $h \in \mathcal{H}$ and $c \in \mathfrak{C}$ such that $h^{-1}(\{c\} \times \mathfrak{C})$ and $h^{-1}(\langle c, d\rangle)$ are subprisms of $P$ for every $d \in \mathfrak{C}$.
In particular, $\mathcal{F}=\left\{h^{-1}(\{c\} \times \mathfrak{C}): h \in \mathcal{H} \& c \in \mathfrak{C}\right\}$ is $\mathcal{F}_{\text {prism-dense in } X}$.
Proof. Let $0<\alpha<\omega_{1}$. We will use the notation as in Lemma 3.3.3.
Since the family of all sets range $\left(f^{*}\right)$ is $\mathcal{F}_{\text {prism }}$-dense in $\mathcal{F}^{\alpha}$ by $\mathrm{CPA}_{\text {prism }}$, we can find a $\mathcal{G}_{\alpha}=\left\{f_{\xi}^{*}\right.$ : $\left.\xi<\omega_{1}\right\}$ such that $R_{\alpha}=\mathcal{F}^{\alpha} \backslash \bigcup_{\xi<\omega_{1}}$ range $\left(f_{\xi}^{*}\right)$ has cardinality less than or equal to $\omega_{1}$.

If $f^{*} \in \mathcal{G}_{\alpha}$, then $\hat{f}$ maps injectively a $P=P_{f} \in \mathbb{P}_{\beta+\alpha}$ onto $Q=Q_{f} \subset X$.

Moreover, for every $z \in \mathcal{F}^{\alpha} \backslash R_{\alpha}$ there are $f^{*} \in \mathcal{G}_{\alpha}$ and $s \in \operatorname{dom}\left(f^{*}\right)$ such that $z=f^{*}(s)$ and $\hat{f}(s, \cdot) \in \mathcal{F}_{\text {prism }}(X)$ is a restriction of $z$.

Now, let $H_{f} \in \Phi_{\text {prism }}(\beta+\alpha)$ be from $\mathfrak{C}^{\beta+\alpha}$ onto $P$ and consider the composition $\hat{f} \circ H_{f}: \mathfrak{C}^{\beta+\alpha} \rightarrow Q$. Then the functions $\left(\hat{f} \circ H_{f}\right)^{-1}: Q_{f} \rightarrow \mathfrak{C}^{\beta+\alpha}$ are our desired functions modulo some projections. More precisely, let $k_{0}: \mathfrak{C}^{\beta} \rightarrow \mathfrak{C}$ be a homeomorphism and choose a mapping $k_{1}: \mathfrak{C} \rightarrow \mathfrak{C}$ be such that $k_{1}^{-1}(c) \in \operatorname{Perf}(\mathfrak{C})$ for every $c \in \mathfrak{C}$. Define $h_{f}^{\alpha}: Q_{f} \rightarrow \mathfrak{C} \times \mathfrak{C}$ by

$$
h_{f}^{\alpha}(x)=\left\langle\left(k_{0} \circ \pi_{\beta}\right)\left(\left(\hat{f} \circ H_{f}\right)^{-1}(x)\right), k_{1}\left(\left[\left(\hat{f} \circ H_{f}\right)^{-1}(x)\right](\beta)\right)\right\rangle .
$$

Then the family $\mathcal{H}_{0}=\left\{h_{f}^{\alpha}: \alpha<\omega_{1} \& f^{*} \in \mathcal{G}_{\alpha}\right\}$ works for all functions not in $R=\bigcup_{0<\alpha<\omega_{1}} R_{\alpha}$. Also, for every function $g \in R$ it is easy to find a continuous function $h_{g}$ from range $(g)$ onto $\mathfrak{C} \times \mathfrak{C}$ such that $h_{g}^{-1}(\{c\} \times \mathfrak{C})$ and $h_{g}^{-1}(\langle c, d\rangle)$ are subprisms of range $(g)$ for every $c, d \in \mathfrak{C}$. Then the family $\mathcal{H}=\mathcal{H}_{0} \cup\left\{h_{g}: g \in R\right\}$ is as desired.

Proposition 3.3.4 implies the following stronger version of property (A). This can be considered as a version of a remark due to A. Miller [95, p. 581], who noticed that in the iterated perfect set model functions coded in the ground model can be taken as a family $\mathcal{G}$.

Corollary 3.3.5 Assume that $\mathrm{CPA}_{\text {prism }}$ holds. Then:
(A*) There exists a family $\mathcal{G}$ of uniformly continuous functions from $\mathbb{R}$ to $[0,1]$ such that $|\mathcal{G}|=\omega_{1}$ and for every $S \in[\mathbb{R}]^{\mathfrak{c}}$ there exists a $g \in \mathcal{G}$ with $g[S]=[0,1]$.

Proof. Let $\mathcal{H}$ be as in Proposition 3.3.4 for $X=\mathbb{R}, k: \mathfrak{C} \rightarrow[0,1]$ be continuous surjection, and for every $h=\left\langle h_{0}, h_{1}\right\rangle \in \mathcal{H}$ let $g_{h}: \mathbb{R} \rightarrow[0,1]$ be a continuous extension of a function $h^{*}: \operatorname{dom}(h) \rightarrow[0,1]$ defined by $h^{*}(x)=k\left(h_{1}(x)\right)$. We claim that $\mathcal{G}=\left\{g_{h}: h \in \mathcal{H}\right\}$ is as desired.

Since, by Proposition 3.3.1, $s_{0}^{\text {prism }}=[\mathbb{R}] \leq \omega_{1}$, there exists a prism $P$ in $\mathbb{R}$ such that $S$ intersects every subprism of $P$. Let $h \in \mathcal{H}$ and $c \in \mathfrak{C}$ be such that $h^{-1}(\{c\} \times \mathfrak{C})$ and $h^{-1}(\langle c, d\rangle)$ are subprisms of $P$ for every $d \in \mathfrak{C}$. Then $S$ intersects each $h^{-1}(\langle c, d\rangle)$ and so $h[S]$ contains $\{c\} \times \mathfrak{C}$. Thus $g_{h}[S]=[0,1]$.

Next we will show that $\mathrm{CPA}_{\text {prism }}$ implies ${ }^{1}$ that the additivity of the ideal $s_{0}$

$$
\operatorname{add}\left(s_{0}\right)=\min \left\{|F|: F \subset s_{0} \& \bigcup F \notin s_{0}\right\}
$$

[^0]is equal to $\omega_{1}$. This stays in contrast with Proposition 6.1.1, in which we will show that CPA implies $\operatorname{cov}\left(s_{0}\right)=\omega_{2}$.

Notice that the numbers $\operatorname{add}\left(s_{0}\right), \operatorname{cov}\left(s_{0}\right), \operatorname{non}\left(s_{0}\right)$, and $\operatorname{cof}\left(s_{0}\right)$ have been intensively studied. (See, e.g., [71].) It is known that $\operatorname{cof}\left(s_{0}\right)>\mathfrak{c}$ (see [71, thm. 1.3]) and that $\operatorname{non}\left(s_{0}\right)=\mathfrak{c}$ since there are $s_{0}$-sets of cardinality $\mathfrak{c}$. There are models of ZFC+MA with $\mathfrak{c}=\omega_{2}$ and $\operatorname{cov}\left(s_{0}\right)=\omega_{1}$, while the proper forcing axiom, PFA, implies that $\operatorname{add}\left(s_{0}\right)=\mathfrak{c}$.

In what follows we need the following useful fact, which is essentially [71, lem. 1.1]. (Compare also [111, thm 2.4(1)].)

Fact 3.3.6 For any open dense subset $\mathcal{D}$ of $\operatorname{Perf}(\mathfrak{C})$ (considered as ordered by inclusion) there exists a maximal antichain $\mathcal{A} \subset \mathcal{D}$ consisting of pairwise disjoint sets such that every $P \in \operatorname{Perf}(\bigcup \mathcal{A})$ is covered by less than continuum many sets from $\mathcal{A}$.

Proof. Let $\operatorname{Perf}(\mathfrak{C})=\left\{P_{\alpha}: \alpha<\mathfrak{c}\right\}$. We will build inductively a sequence $\left\langle\left\langle A_{\alpha}, x_{\alpha}\right\rangle \in \mathcal{D} \times \mathfrak{C}: \alpha<\mathfrak{c}\right\rangle$ aiming for $\mathcal{A}=\left\{A_{\alpha}: \alpha<\mathfrak{c}\right\}$. At step $\alpha<\mathfrak{c}$, given already $\left\langle\left\langle A_{\beta}, x_{\beta}\right\rangle: \beta<\alpha\right\rangle$, we look at $P_{\alpha}$.

Choice of $x_{\alpha}$ : If $P_{\alpha} \subset \bigcup_{\beta<\alpha} A_{\beta}$, we take $x_{\alpha}$ as an arbitrary element of $\mathfrak{C}$; otherwise we pick $x_{\alpha} \in P_{\alpha} \backslash \bigcup_{\beta<\alpha} A_{\beta}$.

Choice of $A_{\alpha}$ : If there is a $\beta<\alpha$ such that $P_{\alpha} \cap A_{\beta}$ is uncountable, we let $A_{\alpha}=A_{\beta}$; otherwise pick $A_{\alpha} \in \mathcal{D}$ below $P_{\alpha}$ and notice that we can refine it, if necessary, to be disjoint with $\bigcup_{\beta<\alpha} A_{\beta} \cup\left\{x_{\beta}: \beta \leq \alpha\right\}$.

It is easy to see that $\mathcal{A}=\left\{A_{\alpha}: \alpha<\mathfrak{c}\right\}$ is as required.

Corollary 3.3.7 $\mathrm{CPA}_{\text {prism }}$ implies that $\operatorname{add}\left(s_{0}\right)=\omega_{1}$.
Proof. Let $\mathcal{H}=\left\{h_{\xi}: \xi<\omega_{1}\right\}$ be as in Proposition 3.3 .4 with $X=\mathfrak{C}$. For every $\xi<\omega_{1}$ put $\mathcal{A}_{\xi}^{0}=\left\{h_{\xi}^{-1}(\{c\} \times \mathfrak{C}): c \in \mathfrak{C}\right\}$. Then each $\mathcal{A}_{\xi}^{0}$ is a family of pairwise disjoint sets and $\mathcal{A}^{0}=\bigcup_{\xi<\omega_{1}} \mathcal{A}_{\xi}^{0}$ is dense in $\operatorname{Perf}(\mathfrak{C})$.

For each $\xi<\omega_{1}$ let $\mathcal{A}_{\xi}^{*}$ be a maximal antichain extending $\mathcal{A}_{\xi}^{0}$, define $\mathcal{D}_{\xi}=\left\{P \in \operatorname{Perf}(\mathfrak{C}): P \subset A\right.$ for some $\left.A \in \mathcal{A}_{\xi}^{*}\right\}$, and let $\mathcal{A}_{\xi} \subset \mathcal{D}_{\xi}$ be as in Fact 3.3.6. Then $\mathcal{A}=\bigcup_{\xi<\omega_{1}} \mathcal{A}_{\xi}$ is still dense in $\operatorname{Perf}(\mathfrak{C})$.

For each $\xi<\omega_{1}$ let $\left\{P_{\xi}^{\alpha}: \alpha<\mathfrak{c}\right\}$ be an enumeration of $\mathcal{A}_{\xi}$. (Note that each $\mathcal{A}_{\xi}$ has cardinality $\mathfrak{c}$, since this was the case for the sets $\mathcal{A}_{\xi}^{0}$.) Pick $x_{\xi}^{\alpha}$ from each $P_{\xi}^{\alpha}$ and put $A_{\xi}=\left\{x_{\xi}^{\alpha}: \alpha<\mathfrak{c}\right\}$. Then $A_{\xi} \in s_{0}$ for every $\xi<\omega_{1}$. However, $A=\bigcup_{\xi<\omega_{1}} A_{\xi} \notin s_{0}$ since it intersects every element of a dense set $\mathcal{A}$.

It can be also shown that $\mathrm{CPA}_{\text {prism }}$, with the help of Proposition 3.3.4,
implies that the Sacks forcing $\mathbb{P}=\langle\operatorname{Perf}(\mathfrak{C}), \subset\rangle$ collapses $\mathfrak{c}$ to $\omega_{1}$. However, this also follows immediately from a theorem of P . Simon [122], that $\mathbb{P}$ collapses $\mathfrak{c}$ to $\mathfrak{b}$ while $\mathrm{CPA}_{\text {cube }}$ already implies that $\mathfrak{b} \leq \operatorname{cof}(\mathcal{N})=\omega_{1}$.

Note also that although the family $\mathcal{F}$ from Proposition 3.3.4 is $\mathcal{F}_{\text {prism }}{ }^{-}$ dense, we certainly cannot repeat the proof of Corollary 3.3 .7 to show that, under $\mathrm{CPA}_{\text {prism }}, \operatorname{add}\left(s_{0}^{\text {prism }}\right)=\omega_{1}$ - this clearly contradicts Proposition 3.3.1. The place where the proof breaks is Fact 3.3.6, which cannot be proved for a simple density being replaced by an $\mathcal{F}_{\text {prism }}$-density.

### 3.4 Intersections of $\omega_{1}$ many open sets

The results presented in this section come from K. Ciesielski and J. Pawlikowski [41].

For a Polish space $X$ let $G_{\omega_{1}}$ be the collection of the intersections of $\omega_{1}$ many open subsets of $X$. Next we are going to prove the following theorem.

Theorem 3.4.1 $\mathrm{CPA}_{\text {prism }}$ implies that the following property holds for every Polish space $X$.
$\left(\mathrm{N}^{*}\right)$ If $G$ is a $G_{\omega_{1}}$ subset of $X$ and $|G|=\mathfrak{c}$, then $G$ contains a perfect set.
Theorem 3.4.1 provides an affirmative answer to a question of J. Brendle, who asked us, in [11], whether $\left(\mathrm{N}^{*}\right)$ can be deduced from our axiom CPA. The fact that $\left(\mathrm{N}^{*}\right)$ holds in the iterated perfect set model is proved by J. Brendle, P. Larson, and S. Todorcevic [12]. The argument presented below is considerably simpler.

Before we prove Theorem 3.4.1, we would like to note that in property $\left(\mathrm{N}^{*}\right)$ we can replace the class of open sets with a considerably larger class $\Pi_{2}^{1}$.

Corollary 3.4.2 Assume that $\mathrm{CPA}_{\text {prism }}$ holds and $X$ is a Polish space.

- If $G$ is an intersection of $\omega_{1}$ many $\Pi_{2}^{1}$ sets from $X$ and $|G|=\mathfrak{c}$, then $G$ contains a perfect set.

Proof. Let $G=\bigcap_{\xi<\omega_{1}} T_{\xi}$, where each $T_{\xi} \subset X$ is a $\Pi_{2}^{1}$ set. Then we have $X \backslash G=\bigcup_{\xi<\omega_{1}}\left(X \backslash T_{\xi}\right)$ and each set $X \backslash T_{\xi}$ is in the class $\Sigma_{2}^{1}$; so, by Fact 1.1.7, it is a union of $\omega_{1}$ many compact sets. Thus, each $T_{\xi}$ is an intersection of $\omega_{1}$ open sets.

Theorem 3.4.1 follows easily from the following combinatorial fact concerning iterated perfect sets.

## $\mathrm{CPA}_{\text {prism }}$ and coverings with smooth functions

This chapter is based on K. Ciesielski and J. Pawlikowski [38]. Below we will use standard notation for the classes of differentiable partial functions from $\mathbb{R}$ into $\mathbb{R}$. Thus, if $X$ is an arbitrary subset of $\mathbb{R}$ without isolated points, we will write $\mathcal{C}^{0}(X)$ or $\mathcal{C}(X)$ for the class of all continuous functions $f: X \rightarrow \mathbb{R}$ and $D^{1}(X)$ for the class of all differentiable functions $f: X \rightarrow \mathbb{R}$, that is, those for which the limit

$$
f^{\prime}\left(x_{0}\right)=\lim _{x \rightarrow x_{0}, x \in X} \frac{f(x)-f\left(x_{0}\right)}{x-x_{0}}
$$

exists and is finite for all $x_{0} \in X$. Also, for $0<n<\omega$ we will write $D^{n}(X)$ to denote the class of all functions $f: X \rightarrow \mathbb{R}$ that are $n$ times differentiable with all derivatives being finite and $\mathcal{C}^{n}(X)$ for the class of all $f \in D^{n}(X)$ whose $n$-th derivative $f^{(n)}$ is continuous. The symbol $\mathcal{C}^{\infty}(X)$ will be used for all infinitely many times differentiable functions from $X$ into $\mathbb{R}$. In addition, we say that a function $f: X \rightarrow \mathbb{R}$ is in the class " $D^{n}(X)$ " if $f \in \mathcal{C}^{n-1}(X)$ and it has the $n$-th derivative, which can be infinite; $f$ is in the class " $\mathcal{C}^{n}(X)$ " when $f$ is in " $D^{n}(X)$ " and its $n$ th derivative is continuous when its range $[-\infty, \infty]$ is considered with the standard topology. " $\mathcal{C}^{\infty}(X)$ " will stand for all functions $f: X \rightarrow \mathbb{R}$ that are either in $\mathcal{C}^{\infty}(X)$ or, for some $0<n<\omega$, are in " $\mathcal{C}{ }^{n}(X)$ " and $f^{(n)}$ is constant and equal to $\infty$ or $-\infty$. (Thus, in general, " $\mathcal{C}^{\infty}(X)$ " is not a subclass of " $\mathcal{C}^{n}(X)$.") In addition, we assume that functions defined on a singleton are in the $\mathcal{C}^{\infty}$ class, that is, $\mathcal{C}^{\infty}(\{x\})=\mathbb{R}^{\{x\}}$. We will use these symbols mainly for $X$ 's that are either in the class $\operatorname{Perf}(\mathbb{R})$ or are the singletons. In particular, $\mathcal{C}_{\text {perf }}^{n}$ will stand for the union of all $\mathcal{C}^{n}(P)$ for which $P \subset \mathbb{R}$ is either in $\operatorname{Perf}(\mathbb{R})$ or is a singleton. The classes $D_{\text {perf }}^{n}$, $\mathcal{C}_{\text {perf }}^{\infty}$, and " $\mathcal{C}_{\text {perf }}^{\infty}$ " are defined in a similar way. We will drop the parameter $X$ if $X=\mathbb{R}$. In particular, $D^{n}=D^{n}(\mathbb{R})$ and $\mathcal{C}^{n}=\mathcal{C}^{n}(\mathbb{R})$. The relations
between these classes for $n<\omega$ are given in the following chart, where the arrows $\longrightarrow$ indicate the strict inclusions $\subsetneq$.


## Chart 1.

In addition, for $F \subset \mathbb{R}^{2}$ we define $F^{-1}=\{\langle y, x\rangle:\langle x, y\rangle \in F\}$ and for $\mathcal{F} \subset \mathcal{P}\left(\mathbb{R}^{2}\right)$ we put $\mathcal{F}^{-1}=\left\{F^{-1}: F \in \mathcal{F}\right\}$.

### 4.1 Chapter overview; properties ( $\mathrm{H}^{*}$ ) and (R)

The main result of this chapter is the following theorem.
Theorem 4.1.1 The following facts follow from $\mathrm{CPA}_{\text {prism }}$.
(a) For every Borel measurable function $g: \mathbb{R} \rightarrow \mathbb{R}$ there exists a family of functions $\left\{f_{\xi} \in " \mathcal{C}_{\text {perf }}^{\infty} ": \xi<\omega_{1}\right\}$ such that

$$
g=\bigcup_{\xi<\omega_{1}} f_{\xi}
$$

Moreover, for each $\xi<\omega_{1}$ there exists an extension $\bar{f}_{\xi}: \mathbb{R} \rightarrow \mathbb{R}$ of $f_{\xi}$ such that
(i) $\bar{f}_{\xi} \in{ }^{\prime} \mathcal{C}^{1}$ " and
(ii) either $\bar{f}_{\xi} \in \mathcal{C}^{1}$ or $\bar{f}_{\xi}$ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ such that $\bar{f}_{\xi}^{-1} \in \mathcal{C}^{1}$.
(b) There exists a sequence $\left\{f_{\xi} \in \mathbb{R}^{\mathbb{R}}: \xi<\omega_{1}\right\}$ of $\mathcal{C}^{1}$ functions such that

$$
\mathbb{R}^{2}=\bigcup_{\xi<\omega_{1}}\left(f_{\xi} \cup f_{\xi}^{-1}\right)
$$

Clearly parts (a) and (b) of the theorem imply properties (R) and ( $\mathrm{H}^{*}$ ), respectively. In particular, we have

Corollary 4.1.2 $\mathrm{CPA}_{\text {prism }}$ implies properties $(\mathrm{R})$ and $\left(\mathrm{H}^{*}\right)$.

Note also that, by Corollary 5.0.2, under $\mathrm{CPA}_{\text {prism }}^{\text {game }}$, the functions $f_{\xi}$ in Theorem 4.1.1(a) may be chosen to have disjoint graphs. Also, $\mathbb{R}^{2}$ can be covered by $\omega_{1}$ pairwise disjoint sets $P$ such that either $P$ or $P^{-1}$ is a function in the class $\mathcal{C}_{\text {perf }}^{1} \cap$ " $\mathcal{C}_{\text {perf }}^{\infty}$ ".

The essence of Theorem 4.1.1 lies in the following real analysis fact. Its proof is combinatorial in nature and uses no extra set-theoretical assumptions.

Proposition 4.1.3 Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be Borel and $0<\alpha<\omega_{1}$.
(a) For every continuous injection $h: \mathfrak{C}^{\alpha} \rightarrow \mathbb{R}$ there exists an $E \in \mathbb{P}_{\alpha}$ such that $g \upharpoonright h[E] \in$ "C ${ }_{\text {perf }}^{\infty}$ " and there is an extension $f: \mathbb{R} \rightarrow \mathbb{R}$ of $g \upharpoonright h[E]$ such that $f \in$ " $C^{1}$ " and either $f \in \mathcal{C}^{1}$ or $f$ is an autohomeomorphism of $\mathbb{R}$ with $f^{-1} \in \mathcal{C}^{1}$.
(b) For every continuous injection $h: \mathfrak{C}^{\alpha} \rightarrow \mathbb{R}^{2}$ there exists an $E \in \mathbb{P}_{\alpha}$ such that either $F=h[E] \subset \mathbb{R}^{2}$ or its inverse, $F^{-1}$, is a function that can be extended to a $\mathcal{C}^{1}$ function $f: \mathbb{R} \rightarrow \mathbb{R}$.

With Proposition 4.1.3 in hand the proof of Theorem 4.1.1 becomes an easy exercise.

Proof of Theorem 4.1.1. (a) Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function and let $\mathcal{E}$ be the family of all $P \in \operatorname{Perf}(\mathbb{R})$ such that
$g \upharpoonright P \in{ }^{\prime} \mathcal{C}_{\text {perf }}^{\infty}$ " and there is an extension $f: \mathbb{R} \rightarrow \mathbb{R}$ of $g \upharpoonright P$ such that $f \in{ }^{\prime} \mathcal{C}^{1}$ " and either $f \in \mathcal{C}^{1}$ or $f$ is an autohomeomorphism of $\mathbb{R}$ with $f^{-1} \in \mathcal{C}^{1}$.

By Proposition 4.1.3(a), the family $\mathcal{E}$ is $\mathcal{F}_{\text {prism }}$-dense: If $P \in \operatorname{Perf}(\mathbb{R})$ is a prism and $h: \mathfrak{C}^{\alpha} \rightarrow \mathbb{R}$ from $\mathcal{F}_{\text {prism }}$ witnesses it, then $Q=h[E]$ as in the proposition is a subprism of $P$ with $Q \in \mathcal{E}$. So, by $\mathrm{CPA}_{\text {prism }}$, there exists an $\mathcal{E}_{0} \in[\mathcal{E}] \leq \omega_{1}$ such that $\left|\mathbb{R} \backslash \bigcup \mathcal{E}_{0}\right| \leq \omega_{1}$. Let $\mathcal{E}_{1}=\mathcal{E}_{0} \cup\left\{\{r\}: r \in \mathbb{R} \backslash \bigcup \mathcal{E}_{0}\right\}$. Then the family $\left\{g \upharpoonright P: P \in \mathcal{E}_{1}\right\}$ satisfies the theorem.
(b) Let $\mathcal{E}$ be the family of all $P \in \operatorname{Perf}\left(\mathbb{R}^{2}\right)$ such that either $P$ or $P^{-1}$ is a function that can be extended to a $\mathcal{C}^{1}$ function $f: \mathbb{R} \rightarrow \mathbb{R}$. By Proposition $4.1 .3(\mathrm{~b})$ the family $\mathcal{E}$ is $\mathcal{F}_{\text {prism-dense, }}$, so there exists an $\mathcal{E}_{0} \in[\mathcal{E}] \leq \omega_{1}$ such that $\left|\mathbb{R} \backslash \bigcup \mathcal{E}_{0}\right| \leq \omega_{1}$. Let $\mathcal{E}_{1}=\mathcal{E}_{0} \cup\left\{\{x\}: x \in \mathbb{R}^{2} \backslash \bigcup \mathcal{E}_{0}\right\}$. For every $P \in \mathcal{E}_{1}$ let $f_{P}: \mathbb{R} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$ function that extends either $P$ or $P^{-1}$. Then the family $\left\{f_{P}: P \in \mathcal{E}_{1}\right\}$ is as desired.

The proof of Proposition 4.1 .3 will be left to the next sections of this chapter. Meanwhile, we want to present a discussion of Theorem 4.1.1.

First we reformulate Theorem 4.1.1 in a language of a covering number cov defined below, where $X$ is an infinite set (in our case $X \subset \mathbb{R}^{2}$ with $|X|=\mathfrak{c})$ and $\mathcal{A}, \mathcal{F} \subset \mathcal{P}(X):$

$$
\operatorname{cov}(\mathcal{A}, \mathcal{F})=\min \left(\left\{\kappa:(\forall A \in \mathcal{A})\left(\exists \mathcal{G} \in[\mathcal{F}]^{\leq \kappa}\right) A \subset \bigcup \mathcal{G}\right\} \cup\left\{|X|^{+}\right\}\right)
$$

If $A \subset X$, we will write $\operatorname{cov}(A, \mathcal{F})$ for $\operatorname{cov}(\{A\}, \mathcal{F})$. Notice the following monotonicity of the cov operator: For every $A \subset B \subset X, \mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(X)$, and $\mathcal{F} \subset \mathcal{G} \subset \mathcal{P}(X)$,
$\operatorname{cov}(\mathcal{A}, \mathcal{G}) \leq \operatorname{cov}(\mathcal{B}, \mathcal{G}) \leq \operatorname{cov}(\mathcal{B}, \mathcal{F}) \& \operatorname{cov}(A, \mathcal{G}) \leq \operatorname{cov}(B, \mathcal{G}) \leq \operatorname{cov}(B, \mathcal{F})$.
In terms of the cov operator, Theorem 4.1.1 can be expressed in the following form, where Borel stands for the class of all Borel functions $f: \mathbb{R} \rightarrow \mathbb{R}$.

Corollary 4.1.4 $\mathrm{CPA}_{\text {prism }}$ implies that
(a) $\operatorname{cov}\left(\right.$ Borel, " $\mathcal{C}_{\text {perf }}^{\infty}$ " $)=\omega_{1}<\mathfrak{c}$;
(b) $\operatorname{cov}\left(\right.$ Borel, " $\mathcal{C}^{1 "}$ ") $=\omega_{1}<\mathfrak{c}$;
(c) $\operatorname{cov}\left(\right.$ Borel, $\left.\mathcal{C}^{1} \cup\left(\mathcal{C}^{1}\right)^{-1}\right)=\omega_{1}<\mathfrak{c}$;
(d) $\operatorname{cov}\left(\mathbb{R}^{2}, \mathcal{C}^{1} \cup\left(\mathcal{C}^{1}\right)^{-1}\right)=\omega_{1}<\mathfrak{c}$.

Proof. The fact that all numbers $\operatorname{cov}(\mathcal{A}, \mathcal{G})$ listed above are $\leq \omega_{1}$ follows directly from Theorem 4.1.1. The other inequalities follow from Examples 4.5.6 and 4.5.8.

Theorem 4.1.1(b) and Corollary 4.1.4(d) can be treated as generalizations of a result of J. Steprāns [124] who proved that in the iterated perfect set model we have cov $\left(\mathbb{R}^{2},\left(" D^{1} "\right) \cup\left(" D^{1 "}\right)^{-1}\right) \leq \omega_{1}$. This clearly follows from Corollary 4.1.4(d) since $\mathcal{C}^{1} \subsetneq " D^{1 "}$. (See survey article [15]. For more information on how to "locate" Steprāns' result in [124], see also [32, cor. 9].)

The following proposition shows that Theorem 4.1.1 is, in a way, the best possible. (Parts (i), (ii), and (iii) relate, respectively, to items (b), (c) together with (d), and (a) from Corollary 4.1.4.)

Proposition 4.1.5 The following are true in ZFC.
(i) $\operatorname{cov}\left(\right.$ Borel, $\left.\mathcal{C}^{1}\right)=\operatorname{cov}\left({ }^{\prime} \mathcal{C}^{1} ", \mathcal{C}^{1}\right)=\operatorname{cov}\left(" \mathcal{C}^{1} ", D_{\text {perf }}^{1}\right)=\mathfrak{c}$. Moreover, $\operatorname{cov}\left(" \mathcal{C}^{n} ", \mathcal{C}^{n}\right)=\operatorname{cov}\left(" \mathcal{C}^{n} ", D_{\text {perf }}^{n}\right)=\mathfrak{c}$ for every $0<n<\omega$.
(ii) $\operatorname{cov}\left(\right.$ Borel, $\left.\mathcal{C}^{2} \cup\left(\mathcal{C}^{2}\right)^{-1}\right)=\operatorname{cov}\left(" \mathcal{C}^{2} ", D_{\text {perf }}^{2} \cup\left(D_{\text {perf }}^{2}\right)^{-1}\right)=\mathfrak{c}$ and $\operatorname{cov}\left(\mathbb{R}^{2}, \mathcal{C}^{2} \cup\left(\mathcal{C}^{2}\right)^{-1}\right)=\operatorname{cov}\left({ }^{\prime} \mathcal{C}^{2} ", D_{\text {perf }}^{2} \cup\left(D_{\text {perf }}^{2}\right)^{-1}\right)=\mathfrak{c}$.
(iii) $\operatorname{cov}\left(\right.$ Borel, $\left.\mathcal{C}_{\text {perf }}^{\infty}\right)=\operatorname{cov}\left(\right.$ " $\left.\mathcal{C}^{1} ", \mathcal{C}_{\text {perf }}^{\infty}\right)=\operatorname{cov}\left(" \mathcal{C}^{1} ", D_{\text {perf }}^{1}\right)=\mathfrak{c}$ and $\operatorname{cov}($ Borel, $" \mathcal{C} \infty ")=\operatorname{cov}\left(\mathcal{C}^{1}, " \mathcal{C}{ }^{\infty} "\right)=\operatorname{cov}\left(\mathcal{C}^{1}, " D^{2} "\right)=\mathfrak{c}$. Moreover, $\operatorname{cov}\left(\mathcal{C}^{n}, " D^{n+1} "\right)=\mathfrak{c}$ for every $0<n<\omega$.

Proof. Part (i) follows immediately from Examples 4.5.2 and 4.5.3.
Part (ii) follows from the monotonicity of the cov operator and Example 4.5.1.

The first part of (iii) follows from (i). The remaining two parts follow, respectively, from Examples 4.5.4 and 4.5.5.

Corollary 4.1.4 and Proposition 4.1.5 establish the values of the cov operator for all classes in Chart 1 except for $\operatorname{cov}\left(D^{n}, \mathcal{C}^{n}\right)$ and $\operatorname{cov}\left(" D^{n} ", ~ " \mathcal{C}{ }^{n} "\right)$. These are established in the following theorem, the proof of which will be left to the latter sections of this chapter.

Theorem 4.1.6 If $\mathrm{CPA}_{\text {prism }}$ holds, then, for every $0<n<\omega$,

$$
\operatorname{cov}\left(D^{n}, \mathcal{C}^{n}\right)=\operatorname{cov}\left(" D^{n ",} " \mathcal{C}^{n} "\right)=\omega_{1}<\mathfrak{c}
$$

Note also that, by Corollary 5.0.2, under $\mathrm{CPA}_{\text {prism }}^{\text {game }}$, covering functions in Theorem 4.1.6 may be chosen to have disjoint graphs.

With this theorem in hand we can summarize the values of the cov operator between the classes from Chart 1 in the following graphical form. Here the mark " $c$ " next to the arrow means that the covering of the larger class by the functions from the smaller class is equal to $\mathfrak{c}$ and that this can be proved in ZFC. The mark " $<\mathfrak{c}$ " next to the arrow means that it is consistent with ZFC (and it follows from $\mathrm{CPA}_{\text {prism }}$ ) that the appropriate cov number is $<\mathfrak{c}$. (From Examples 4.5.6, 4.5.7, and 4.5.8 it follows that all these numbers are greater than or equal to $\min \{\operatorname{cov}(\mathcal{M}), \operatorname{cov}(\mathcal{N})\}>\omega$. So, under the continuum hypothesis CH or Martin's axiom, all these numbers are equal to $c$.)


Chart 2. Values of the cov operator for $n=0$ (left) and $n>0$ (right).
The values of cov operator next to the vertical arrows are justified by $\operatorname{cov}\left(" \mathcal{C} n ", D^{n}\right)=\mathfrak{c}$ (Proposition 4.1.5(i)), while the marks " $<\mathfrak{c}$ " below the upper horizontal arrows and that directly below them follow from Theorem 4.1.6. The remaining arrow in the right part of the chart is the restatement of the last part of Proposition 4.1.5(iii), while its counterpart in the left part of the chart follows from Corollary 4.1.4(b): $\operatorname{cov}\left(\mathcal{C}, " \mathcal{C}^{1} "\right)=\operatorname{cov}($ Borel, " $\mathcal{C} ")<\mathfrak{c}$ is a consequence of $\mathrm{CPA}_{\text {prism }}$. Finally, let us mention that in Corollary 4.1.4(b) there is no chance of increasing the Borel family in any essential way and keep the result. This follows from the following fact:

$$
\begin{equation*}
\operatorname{cov}(\mathrm{Sc}, \mathcal{C})=\operatorname{cov}\left(\mathbb{R}^{\mathbb{R}}, \mathcal{C}\right) \geq \operatorname{cof}(\mathfrak{c}) \tag{4.1}
\end{equation*}
$$

where the symbol Sc stands for the family of all symmetrically continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ that are, in particular, continuous outside of some set of measure zero and the first category. (See K. Ciesielski [28, cor. 1.1] and the remarks below on the operator dec.)

The number $\operatorname{cov}(\mathcal{A}, \mathcal{F})$ is very closely related to the decomposition num$\operatorname{ber} \operatorname{dec}(\mathcal{A}, \mathcal{F})$ defined as

$$
\min \left(\left\{\kappa \geq \omega:(\forall A \in \mathcal{A})\left(\exists \mathcal{G} \in[\mathcal{F}]^{\kappa}\right) \mathcal{G} \text { is a partition of } A\right\} \cup\left\{|X|^{+}\right\}\right)
$$

which was first studied by J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki [24] for the Baire class $\alpha$ functions. (More information on $\operatorname{dec}(\mathcal{F}, \mathcal{G})$ can be found in a survey article $[26, \sec .4]$.) It is easy to see that if $\mathcal{A}$ and $\mathcal{F}$ are some classes of partial functions and $\mathcal{F}_{r}$ denotes all possible restrictions of functions from $\mathcal{F}$, then $\operatorname{cov}(\mathcal{A}, \mathcal{F})=\operatorname{dec}\left(\mathcal{A}, \mathcal{F}_{r}\right)$. In particular, for all situations relevant to our discussion above, the operators cov and dec have the same values.

Our number cov is also related to the following general class of problems. We say that the families $\mathcal{A}, \mathcal{F} \subset \mathcal{P}(X)$ satisfy the Intersection Theorem,
which we denote by

$$
\operatorname{IntTh}(\mathcal{A}, \mathcal{F})
$$

if for every $A \in \mathcal{A}$ there exists an $F \in \mathcal{G}$ such that $|A \cap F|=|X|$. If $\mathcal{A}=\{A\}$, we write $\operatorname{IntTh}(A, \mathcal{F})$ in place of $\operatorname{IntTh}(\mathcal{A}, \mathcal{F})$. This kind of theorem has been studied for a large part of the twentieth century. In particular, in the early 1940s S. Ulam asked in the Scottish Book [90, problem 17.1] if $\operatorname{IntTh}(\mathcal{C}$, Analytic) holds, that is, whether for every $f \in \mathcal{C}$ there exists a real analytic function $g: \mathbb{R} \rightarrow \mathbb{R}$ that agrees with $f$ on a perfect set. (See [127].) In 1947, Z. Zahorski [130] gave a negative answer to this question by proving that the proposition $\operatorname{IntTh}\left(\mathcal{C}^{\infty}\right.$, Analytic) is false. In the same paper he also raised a natural question, which has become known as the Ulam-Zahorski Problem: Does $\operatorname{IntTh}(\mathcal{C}, \mathcal{G})$ hold for $\mathcal{G}=\mathcal{C}^{\infty}$ (or $\mathcal{G}=\mathcal{C}^{n}$ or $\mathcal{G}=D^{n}$ )? Here is a quick summary of what is known about this problem. (See [15].)

## Proposition 4.1.7

(a) $\neg \operatorname{IntTh}\left(\mathcal{C}^{\infty}\right.$, Analytic) (Z. Zahorski [130])
(b) $\operatorname{IntTh}\left(\mathcal{C}, \mathcal{C}^{1}\right)$ (S. Agronsky, A. M. Bruckner, M. Laczkovich, and D. Preiss [1])
(c) $\operatorname{IntTh}\left(\mathcal{C}^{1}, \mathcal{C}^{2}\right)(A$. Olevskiǐ [109])
(d) $\neg \operatorname{IntTh}\left(\mathcal{C}, \mathcal{C}^{2}\right)$ and $\neg \operatorname{IntTh}\left(\mathcal{C}^{n}, \mathcal{C}^{n+1}\right)$ for $n \geq 2$. (A. Olevskiǐ [109])

We are interested in these problems because for the families $\mathcal{A}, \mathcal{F} \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ of uncountable Borel sets

$$
\begin{equation*}
\neg \operatorname{IntTh}(\mathcal{A}, \mathcal{F}) \Longrightarrow \operatorname{cov}(\mathcal{A}, \mathcal{F})=\mathfrak{c} \tag{4.2}
\end{equation*}
$$

as, in this situation, if $\neg \operatorname{Int} \operatorname{Th}(\mathcal{A}, \mathcal{F})$, then there exists an $A \in \mathcal{A},|A|=\mathfrak{c}$, such that $|A \cap F| \leq \omega$ for every $F \in \mathcal{F}$. Thus in the examples relevant to Proposition 4.1.5, instead of proving $\operatorname{cov}(\mathcal{A}, \mathcal{F})=\mathfrak{c}$ we will in fact be showing a stronger fact that $\neg \operatorname{Int} \operatorname{Th}\left(A_{0}, \mathcal{F}\right)$ for an appropriate choice of $A_{0} \subset A \in \mathcal{A}$.

### 4.2 Proof of Proposition 4.1.3

Proposition 4.1.3 will be deduced from the following fact, which is a generalization of a theorem of M. Morayne [99]. (Morayne proved his results for $E$ and $E_{1}$ being perfect sets, that is, for $\alpha=1$.) For a set $X$ we will use the symbol $\Delta_{X}$ to denote the diagonal in $X \times X$, that is, $\Delta_{X}=\{\langle x, x\rangle: x \in X\}$. We will usually write simply $\Delta$ in place of $\Delta_{X}$, since $X$ is always clear from the context.

## 5

## Applications of $\mathrm{CPA}_{\text {prism }}^{\text {game }}$

First notice that the proof that is identical to that for Theorem 2.1.1 also gives its prism version which reads as follows.

Theorem 5.0.1 Assume that $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ holds and let $X$ be a Polish space. If $\mathcal{D} \subset \operatorname{Perf}(X)$ is $\mathcal{F}_{\text {prism-dense }}$ and it is closed under perfect subsets, then there exists a partition of $X$ into $\omega_{1}$ disjoint sets from $\mathcal{D} \cup\{\{x\}: x \in X\}$.

Notice that, by using Theorem 5.0.1, we can obtain the following generalizations of Theorems 4.1.1 and 4.1.6.

Corollary 5.0.2 The graphs of covering functions in Theorems 4.1.1 and 4.1.6 can be chosen as pairwise disjoint.

### 5.1 Nice Hamel bases

The results presented in this section are based on K. Ciesielski and J. Pawlikowski [40].

In the next two sections we will consider $\mathbb{R}$ as a linear space over $\mathbb{Q}$. For $Z \subset \mathbb{R}$, its linear span with respect to this structure will be denoted by $\operatorname{LIN}(Z)$. Notice that if $L_{m}$, for $0<m<\omega$, is the collection of all functions $\ell: \mathbb{R}^{m} \rightarrow \mathbb{R}$ given by a formula

$$
\begin{equation*}
\ell\left(x_{0}, \ldots, x_{m-1}\right)=\sum_{i<m} q_{i} x_{i}, \text { where } q_{i} \in \mathbb{Q} \backslash\{0\} \text { for all } i<m \tag{5.1}
\end{equation*}
$$

then

$$
\operatorname{LIN}(Z)=\bigcup_{0<m<\omega} \bigcup_{\ell \in L_{m}} \ell\left[Z^{m}\right]
$$

Also, $Z \subset \mathbb{R}$ is linearly independent (over $\mathbb{Q}$ ) provided $\ell\left(x_{0}, \ldots, x_{m-1}\right) \neq 0$ for every $\ell \in L_{m}, 0<m<\omega$, and $\left\{x_{0}, \ldots, x_{m-1}\right\} \in[Z]^{m}$. It should be
clear that the linear independence over $\mathbb{Q}$ is an $\mathcal{F}$-independence for the family $\mathcal{F}$ of closed symmetric relations $F_{\ell}$, with $\ell \in L_{n}$, defined by
$F_{\ell}\left(x_{0}, \ldots, x_{n-1}\right) \Leftrightarrow \ell\left(x_{\pi(0)}, \ldots, x_{\pi(n-1)}\right)=0$ for some permutation $\pi$ of $n$.
We also have $\operatorname{LIN}(Z)=\operatorname{cl}_{\mathcal{F}}(Z)$ for every $Z \subset \mathbb{R}$. Recall also that the subset $H$ of $\mathbb{R}$ is a Hamel basis provided it is a linear basis of $\mathbb{R}$ over $\mathbb{Q}$, that is, it is linearly independent and $\operatorname{LIN}(H)=\mathbb{R}$.

The first result we prove in this section is that CPA $\mathrm{prism}_{\text {pame }}^{\text {ame }}$ implies the existence of a Hamel basis that is a union of $\omega_{1}$ pairwise disjoint perfect sets. This can be viewed as a generalization of Theorem 5.0.1. To prove this we will need, as usual, some prism density results.

For a Polish space $X$ and a family $\mathcal{F}$ of finitary relations on $X$, we say that $\mathcal{F}$ has countable character provided for every $n$-ary relation $F \in \mathcal{F}$ and every $F$-independent set $\left\{x_{1}, \ldots, x_{n}\right\} \in[X]^{n}$ the set

$$
\left\{x \in X: F\left(x, x_{1}, \ldots, x_{n-1}\right)\right\}
$$

is countable. It should be clear that the linear independence family $\mathcal{F}$ defined above is of countable character. Similarly, algebraic independence can be expressed in this language. (See page 65.)

The next fact can be considered a first approximation of what we will need for finding our Hamel basis. However, it is not strong enough for what we need: Both its assumptions and its conclusion are too strong.

Proposition 5.1.1 Let $X$ be a Polish space and $\mathcal{F}$ be a countable family of closed finitary relations on $X$ such that $\mathcal{F}$ has countable character. Then for every prism $P$ in $X$ there is a subprism $Q$ of $P$ such that $Q$ is $\mathcal{F}$-independent.

Proof. This follows easily from Proposition 3.2.3. Indeed, pick an $h \in \Phi_{\text {prism }}(\alpha)$ such that $P=h\left[\mathfrak{C}^{\alpha}\right]$, for every $n$-ary relation $F$ on $X$ let

$$
F^{*}=\left\{\left\langle p_{1}, \ldots, p_{n}\right\rangle \in\left(\mathfrak{C}^{\alpha}\right)^{n}: F\left(h\left(p_{1}\right), \ldots, h\left(p_{n}\right)\right) \text { holds }\right\}
$$

and put $\mathcal{F}^{*}=\left\{F^{*}: F \in \mathcal{F}\right\}$. The countable character of $\mathcal{F}$ implies that (ex) holds with $G=\mathfrak{C}^{\alpha}$ for every $F^{*} \in \mathcal{F}^{*}$. So, by Proposition 3.2.3, there exists an $\mathcal{F}^{*}$-independent $E \in \mathbb{P}_{\alpha}$. But this means that $Q=h[E]$ is an $\mathcal{F}$-independent subprism of $P$.

From Proposition 5.1.1 and the remark preceding its statement we immediately conclude that

Corollary 5.1.2 For every prism $P$ in $\mathbb{R}$ there is a subprism $Q$ of $P$ such that $Q$ is algebraically (so linearly) independent.

From Theorem 5.0.1 and Corollary 5.1.2 we can also easily deduce the following fact.

Corollary 5.1.3 $\mathrm{CPA}_{\mathrm{prism}}^{\text {game }}$ implies that $\mathbb{R}$ is a union of $\omega_{1}$ disjoint, closed, algebraically independent sets.

Remark 5.1.4 Note that Corollary 5.1.2 is false if we replace prisms with cubes. In particular, there is a cube $P$ in $\mathbb{R}$ without a linearly independent subcube.

Proof. Indeed, let $P_{1}$ and $P_{2}$ be disjoint perfect subsets of $\mathbb{R}$ such that $P_{1} \cup P_{1}$ is linearly independent over $\mathbb{Q}$. Let $f: P_{1} \times P_{2} \rightarrow \mathbb{R}$ be defined by a formula $f\left(x_{1}, x_{2}\right)=x_{1}+x_{2}$. Then $P=f\left[P_{1} \times P_{2}\right]$ is a cube in $\mathbb{R}$. To see that $P$ has no linearly independent subcube, let $Q=Q_{1} \times Q_{2}$ be a subcube of $P$ and choose different $a_{1}, b_{1} \in Q_{1}$ and $a_{2}, b_{2} \in Q_{2}$. Then the set $\left\{a_{1}+a_{2}, a_{1}+b_{2}, b_{1}+a_{2}, b_{1}+a_{2}\right\} \subset Q$ is clearly linearly dependent.

It seems that the conclusion from Corollary 5.1.3 is already close to the existence of a Hamel basis that is union of $\omega_{1}$ disjoint closed sets. However, the sets from Corollary 5.1 .3 can be pairwise highly linearly dependent. Thus, in order to prove the Hamel basis result, we need a density result that is considerably stronger than that from Corollary 5.1.2.

Lemma 5.1.5 Let $M \subset \mathbb{R}$ be a $\sigma$-compact and linearly independent. Then for every prism $P$ in $\mathbb{R}$ there exist a subprism $Q$ of $P$ and a compact subset $R$ of $P \backslash M$ such that $M \cup R$ is a maximal linearly independent subset of $M \cup Q$.

Proof. Let $\mathcal{F}$ be the linear independent family defined at the beginning of this section and let $\bar{M}=\left\langle M_{n}: n<\omega\right\rangle$ be an increasing family of compact sets such that $M=\bigcup_{n<\omega} M_{n}$. Let $\mathcal{F}_{\bar{M}}=\bigcup_{n<\omega} \mathcal{F}_{M_{n}}$, where each $\mathcal{F}_{M_{n}}$ is defined as on page 65 , that is, $\mathcal{F}_{M_{n}}$ is the the collection of all possible projections of the relations from $\mathcal{F}$ along $M_{n}$.

If $M \cap P$ is of second category in $P$, then we can choose a subprism $Q$ of $P$ with $Q \subset M$. Then $Q$ and $R=\emptyset$ have the desired properties. On the other hand, if $M \cap P$ is of the first category in $P$, then, by Claim 1.1.5, we can find a subprism $P_{1}$ of $P$ disjoint with $M$.

Now, applying Lemma 3.2 .7 we can find a $\operatorname{subprism} Q$ of $P_{1}$ and a
compact $\mathcal{F}_{\bar{M}}$-independent set $R \subset P_{1} \subset P \backslash M$ such that $Q \subset \operatorname{cl}_{\mathcal{F}_{\bar{M}}}(R)$. But then $M \cup R$ is $\mathcal{F}$-independent; see (3.18). Moreover,

$$
Q \subset \operatorname{cl}_{\mathcal{F}_{\bar{M}}}(R)=\operatorname{cl}_{\mathcal{F}}(M \cup R)=\operatorname{LIN}(M \cup R)
$$

So, $M \cup Q \subset \operatorname{LIN}(M \cup R)$, proving that $Q$ and $R$ are as desired.

Remark 5.1.6 In Lemma 5.1.5 we cannot require $R=Q$.
Proof. Let $P_{1}, P_{2}$, and $f$ be as in Remark 5.1.4. If $M=P_{2}$, then $P$ has no subprism $Q$ such that $M \cup Q$ is linearly independent, since any vertical section of $Q$ is a translation of a portion of $M$.

The next theorem represents a generalization of Proposition 5.1.1 and Theorem 5.0.1.

Theorem 5.1.7 $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ implies that there exists a family $\mathcal{H}$ of $\omega_{1}$ pairwise disjoint perfect subsets of $\mathbb{R}$ such that $H=\bigcup \mathcal{H}$ is a Hamel basis.

Proof. For a linearly independent $\sigma$-compact set $M \subset \mathbb{R}$ and a prism $P$ in $\mathbb{R}$, let $Q(M, P)=Q$ and $R(M, P)=R \subset P \backslash M$ be as in Lemma 5.1.5. Consider Player II strategy $S$ given by

$$
S\left(\left\langle\left\langle P_{\eta}, Q_{\eta}\right\rangle: \eta<\xi\right\rangle, P_{\xi}\right)=Q\left(\bigcup\left\{R_{\eta}: \eta<\xi\right\}, P_{\xi}\right)
$$

where the $R_{\eta}$ 's are defined inductively by $R_{\eta}=R\left(\bigcup\left\{R_{\zeta}: \zeta<\eta\right\}, P_{\eta}\right)$.
By $\mathrm{CPA}_{\text {prism }}^{\text {game }}$, strategy $S$ is not a winning strategy for Player II. So there exists a game $\left\langle\left\langle P_{\xi}, Q_{\xi}\right\rangle: \xi<\omega_{1}\right\rangle$ played according to $S$ in which Player II loses, that is, $\mathbb{R}=\bigcup_{\xi<\omega_{1}} Q_{\xi}$.

Let $\mathcal{H}=\left\{R_{\xi}: \xi<\omega_{1}\right\}$ and notice that $\bigcup \mathcal{H}$ is a Hamel basis. Indeed, clearly $\cup \mathcal{H}$ is linearly independent. To see that it spans $\mathbb{R}$ it is enough to notice that $\operatorname{LIN}\left(\bigcup_{\eta<\xi} Q_{\eta}\right) \subset \operatorname{LIN}\left(\bigcup_{\eta<\xi} R_{\eta}\right)$ for every $\xi<\omega_{1}$.

Although sets in $\mathcal{H}$ need not be perfect, they are clearly pairwise disjoint and compact. Thus, the theorem follows immediately from the following remark.

Remark 5.1.8 If there exists a family $\mathcal{H}$ of $\omega_{1}$ pairwise disjoint compact subsets of $\mathbb{R}$ such that $\bigcup \mathcal{H}$ is a Hamel basis, then there exists such an $\mathcal{H}$ with $\mathcal{H} \subset \operatorname{Perf}(\mathbb{R})$.

Proof. Let $\mathcal{H}_{0}$ be a family of $\omega_{1}$ pairwise disjoint compact subsets of $\mathbb{R}$ such that $\bigcup \mathcal{H}_{0}$ is a Hamel basis. Partitioning each $H \in \mathcal{H}_{0}$ into its perfect part and singletons from its scattered part we can assume that
$\mathcal{H}_{0}$ contains only perfect sets and singletons. To get $\mathcal{H}$ as required, fix a perfect set $P_{0} \in \mathcal{H}_{0}$ and an $x \in P_{0}$ and notice that if we replace each $P \in \mathcal{H}_{0} \backslash\left\{P_{0}\right\}$ with $p x+q P$ for some $p, q \in \mathbb{Q} \backslash\{0\}$, then the resulting family will still be pairwise disjoint with union being a Hamel basis. Thus, without loss of generality, we can assume that every open interval in $\mathbb{R}$ contains $\omega_{1}$ perfect sets from $\mathcal{H}_{0}$. Now, for every singleton $\{x\}$ in $\mathcal{H}_{0}$, we can choose a sequence $P_{1}^{x}>P_{2}^{x}>P_{3}^{x}>\cdots$ from $\mathcal{H}_{0}$ converging to $x$ and replace a family $\{x\} \cup\left\{P_{n}^{x}: n<\omega\right\}$ with its union. (We assume that we choose different sets $P_{n}^{x}$ for different singletons.) If $\mathcal{H}$ is such a modification of $\mathcal{H}_{0}$, then $\mathcal{H}$ is as desired.

Recall that a subset $T$ of $\mathbb{R}$ is a transcendental basis of $\mathbb{R}$ over $\mathbb{Q}$ provided $T$ is a maximal algebraically independent subset of $\mathbb{R}$. The proof that is identical to that for Theorem 5.1.7 also gives the following result.

Theorem 5.1.9 $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ implies that there exists a family $\mathcal{T}$ of $\omega_{1}$ pairwise disjoint perfect subsets of $\mathbb{R}$ such that $T=\bigcup \mathcal{T}$ is a transcendental basis of $\mathbb{R}$ over $\mathbb{Q}$.

Next, we will present two interesting consequences of the existence of a Hamel basis described in Theorem 5.1.7. Let $\mathcal{I}$ be a translation invariant ideal on $\mathbb{R}$. We say that a subset $X$ of $\mathbb{R}$ is $\mathcal{I}$-rigid provided $X, \mathbb{R} \backslash X \notin \mathcal{I}$ but $X \triangle(r+X) \in \mathcal{I}$ for every $r \in \mathbb{R}$. An easy inductive construction gives a nonmeasurable subset $X$ of $\mathbb{R}$ without the Baire property, which is $[\mathbb{R}]^{<\boldsymbol{c}}$-rigid. (The first such construction, under CH , can be found in a paper [119] by W. Sierpiński. Compare also [68].) Thus, under CH or MA there are $\mathcal{N} \cap \mathcal{M}$-rigid sets. Recently these sets have been studied by M. Laczkovich [83] and J. Cichoń, A. Jasiński, A. Kamburelis, and P. Szczepaniak [23]. In particular, M. Laczkovich's result from [83, thm. 2] implies that there is no $\mathcal{N} \cap \mathcal{M}$-rigid set in the random and Cohen models. The next corollary shows that the existence of such sets follows from CPA ${ }_{\text {prism }}^{\text {game }}$.

Corollary 5.1.10 $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ implies there exists an $\mathcal{N} \cap \mathcal{M}$-rigid set $X$ that is neither measurable nor has the Baire property.

Proof. Let $\mathcal{H}=\left\{Q_{\xi}: \xi<\omega_{1}\right\}$ be from Theorem 5.1.7 and for every $\xi<\omega_{1}$ let $L_{\xi}=\operatorname{LIN}\left(\bigcup_{\eta<\xi} Q_{\eta}\right)$. Then $\mathbb{R}$ is an increasing union of $L_{\xi}$ 's and each $L_{\xi}$ belongs to $\mathcal{N} \cap \mathcal{M}$, since it is a proper Borel subgroup of $\mathbb{R}$.

Since $\operatorname{CPA}_{\text {prism }}^{\text {game }}$ implies that $\operatorname{cof}(\mathcal{N})=\operatorname{cof}(\mathcal{M})=\omega_{1}$, there exists a family $\left\{C_{\xi}: \xi<\omega_{1}\right\} \subset \mathcal{M} \cup \mathcal{N}$ such that every $S \in \mathcal{M} \cup \mathcal{N}$ is a subset of
some $C_{\xi}$. By induction choose $X_{0}=\left\{x_{\xi}: \xi<\omega_{1}\right\} \subset \mathbb{R}$ such that

$$
x_{\xi} \notin C_{\xi} \cup \operatorname{LIN}\left(L_{\xi} \cup\left\{x_{\zeta}: \zeta<\xi\right\}\right)
$$

Then $X_{0}$ intersects the complement of every set from $\mathcal{M} \cup \mathcal{N}$. Define

$$
X=\bigcup_{\xi<\omega_{1}}\left(x_{\xi}+L_{\xi}\right)
$$

and notice that $X_{0} \subset X$ and $2 X_{0} \subset \mathbb{R} \backslash X$. Therefore, both $X$ and $\mathbb{R} \backslash X$ intersect the complement of every set from $\mathcal{M} \cup \mathcal{N}$. In particular, $X, \mathbb{R} \backslash X \notin \mathcal{M} \cup \mathcal{N}$.

Next notice that for every $r \in L_{\zeta}$

$$
X \triangle(r+X) \subset \bigcup_{\xi<\zeta}\left[\left(x_{\xi}+L_{\xi}\right) \cup\left(r+x_{\xi}+L_{\xi}\right)\right] \in \mathcal{N} \cap \mathcal{M}
$$

Thus, $X$ is $\mathcal{N} \cap \mathcal{M}$-rigid, but also $\mathcal{N}$-rigid and $\mathcal{M}$-rigid. These last two facts imply that $X$ is neither measurable nor does it have the Baire property.

Our next application of Theorem 5.1.7 is the following.
Corollary 5.1.11 $\mathrm{CPA}_{\text {prism }}^{\text {game }}$ implies there exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that for every $h \in \mathbb{R}$ the difference function $\Delta_{h}(x)=f(x+h)-f(x)$ is Borel; however, for every $\alpha<\omega_{1}$ there is an $h \in \mathbb{R}$ such that $\Delta_{h}$ is not of the Borel class $\alpha$.

Note that, answering a question of M. Laczkovich from [82], R. Filipów and I. Recław [57] gave an example of such an $f$ under CH. I. Recław also asked (private communication) whether such a function can be constructed in the absence of CH. Corollary 5.1.11 gives an affirmative answer to this question. It is an open question whether such a function exists in ZFC.

Proof. The proof is quite similar to that for Corollary 5.1.10.
Let $\mathcal{H}=\left\{Q_{\xi}: \xi<\omega_{1}\right\}$ be from Theorem 5.1.7. For every $\xi<\omega_{1}$ define $L_{\xi}=\operatorname{LIN}\left(\bigcup_{\eta<\xi} Q_{\eta}\right)$ and choose a Borel subset $B_{\xi}$ of $Q_{\xi}$ of Borel class greater than $\xi$. Define

$$
X=\bigcup_{\xi<\omega_{1}}\left(B_{\xi}+L_{\xi}\right)
$$

and let $f$ be the characteristic function $\chi_{X}$ of $X$.
To see that $f$ is as required note that

$$
\Delta_{-h}(x)=\left[\chi_{(h+X) \backslash X}-\chi_{X \backslash(h+X)}\right](x)
$$

So, it is enough to show that each of the sets $(h+X) \backslash X$ and $X \backslash(h+X)$
is Borel, though they can be of arbitrary high class. For this, notice that for every $h \in L_{\alpha+1} \backslash L_{\alpha}$ we have

$$
h+X=h+\bigcup_{\xi<\omega_{1}}\left(B_{\xi}+L_{\xi}\right)=\bigcup_{\xi \leq \alpha}\left(h+B_{\xi}+L_{\xi}\right) \cup \bigcup_{\alpha<\xi<\omega_{1}}\left(B_{\xi}+L_{\xi}\right)
$$

and that the sets $\bigcup_{\xi \leq \alpha}\left(h+B_{\xi}+L_{\xi}\right) \subset L_{\alpha+1}$ and $\bigcup_{\alpha<\xi<\omega_{1}}\left(B_{\xi}+L_{\xi}\right)$ are disjoint. So

$$
(h+X) \backslash X=\bigcup_{\xi \leq \alpha}\left(h+B_{\xi}+L_{\xi}\right) \backslash X=\bigcup_{\xi \leq \alpha}\left(h+B_{\xi}+L_{\xi}\right) \backslash \bigcup_{\xi \leq \alpha}\left(B_{\xi}+L_{\xi}\right)
$$

is Borel, since each set $B_{\xi}+L_{\xi}$ is Borel. (It is a subset of $Q_{\xi}+L_{\xi}$, which is homeomorphic to $Q_{\xi} \times L_{\xi}$ via the addition function.) Similarly, set $X \backslash(h+X)$ is Borel.

Finally notice that for $h \in Q_{\alpha} \backslash B_{\alpha}$ the set

$$
(h+X) \backslash X=\bigcup_{\xi \leq \alpha}\left(h+B_{\xi}+L_{\xi}\right)
$$

is of the Borel class greater than $\alpha$, since $\left(h+Q_{\alpha}\right) \cap[(h+X) \backslash X]=h+B_{\alpha}$ has the same property. Thus, $\Delta_{h}(x)$ can be of an arbitrarily high Borel class.

### 5.2 Some additive functions and more on Hamel bases

The results presented in this section come from K. Ciesielski and J. Pawlikowski [42].

The proof of the next application is essentially more involved than those presented so far and requires considerably more preparation. However, it can be viewed as a "model example" of how some CH proofs can be modified to the proofs from $\mathrm{CPA}_{\mathrm{prism}}^{\text {game }}$. Recall that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is almost continuous provided any open subset $U$ of $\mathbb{R}^{2}$ that contains the graph of $f$ also contains a graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. It is known that if $f$ is almost continuous, then its graph is connected in $\mathbb{R}^{2}$ (i.e., $f$ is a connectivity function) and that $f$ has the intermediate value property (i.e., $f$ is Darboux). (See, e.g., [105] or [29].) Recall also that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is additive provided $f(x+y)=f(x)+f(y)$ for every $x, y \in \mathbb{R}$. It is well known that every function defined on a Hamel basis can be uniquely extended to an additive function. (See, e.g., [25, thm. 7.3.2].)

Our next goal will be to construct an additive discontinuous, almost continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ whose graph is of measure zero. In fact, we will show that, under $\mathrm{CPA}_{\text {prism }}^{\text {game }}$, such an $f$ can be found inside a set


[^0]:    ${ }^{1}$ In fact, this also follows from $\mathrm{CPA}_{\text {cube }}$.

