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Continuous images of big sets and additivity of s_0 under $\text{CPA}_{\text{prism}}$

Abstract

We prove that the Covering Property Axiom $\text{CPA}_{\text{prism}}$, which holds in the iterated perfect set model, implies the following facts.

- There exists a family \mathcal{G} of uniformly continuous functions from \mathbb{R} to $[0, 1]$ such that $|\mathcal{G}| = \omega_1$ and for every $S \in [\mathbb{R}]^{\mathfrak{c}}$ there exists a $g \in \mathcal{G}$ with $g[S] = [0, 1]$.
- The additivity of the Marczewski's ideal s_0 is equal to $\omega_1 < \mathfrak{c}$.

1 Preliminaries and axiom $\text{CPA}_{\text{prism}}$

Our set theoretic terminology is standard and follows that of [1]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set 2^ω will be denoted by a symbol \mathfrak{C} . We use the term *Polish space* for a complete separable metric space **without isolated points**. For a Polish space X the symbol $\text{Perf}(X)$ will stand for the collection of all subsets of X homeomorphic to the Cantor set \mathfrak{C} . For a fixed $0 < \alpha < \omega_1$ and $0 < \beta \leq \alpha$ the symbol π_β will stand for the projection from \mathfrak{C}^α onto \mathfrak{C}^β .

Axiom $\text{CPA}_{\text{prism}}$ was introduced by the authors in [3], where it is shown that it holds in the iterated perfect set model. Also, $\text{CPA}_{\text{prism}}$ is a simpler

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version of the axiom CPA which is described in a monograph [4]. (See also [2].) For the reader's convenience, we will restate the axiom in the next few paragraphs.

The main notions needed for the axiom are that of *prism* and *prism-density*. Let $0 < \alpha < \omega_1$ and let $\Phi_{\text{prism}}(\alpha)$ be the family of all continuous injections $f: \mathfrak{C}^\alpha \rightarrow \mathfrak{C}^\alpha$ with the property that

$$f(x) \upharpoonright \beta = f(y) \upharpoonright \beta \Leftrightarrow x \upharpoonright \beta = y \upharpoonright \beta \quad \text{for all } \beta \in \alpha \text{ and } x, y \in \mathfrak{C}^\alpha$$

or, equivalently, such that for every $\beta < \alpha$

$$f \upharpoonright \beta \stackrel{\text{def}}{=} \{\langle x \upharpoonright \beta, y \upharpoonright \beta \rangle : \langle x, y \rangle \in f\}$$

is a one-to-one function from \mathfrak{C}^β into \mathfrak{C}^β . Functions f from $\Phi_{\text{prism}}(\alpha)$ were first introduced, in more general setting, in [7] where they are called *projection-keeping homeomorphisms*. Note that $\Phi_{\text{prism}}(\alpha)$ is closed under compositions and that for every $0 < \beta < \alpha$ if $f \in \Phi_{\text{prism}}(\alpha)$ then $f \upharpoonright \beta \in \Phi_{\text{prism}}(\beta)$. Let

$$\mathbb{P}_\alpha = \{\text{range}(f) : f \in \Phi_{\text{prism}}(\alpha)\}$$

and note that if $f \in \Phi_{\text{prism}}(\alpha)$ and $E \in \mathbb{P}_\alpha$ then $f[E] \in \mathbb{P}_\alpha$. We will also define $\mathbb{P}_{\omega_1} = \bigcup_{0 < \alpha < \omega_1} \mathbb{P}_\alpha$. We will refer to elements of \mathbb{P}_{ω_1} as *iterated perfect sets*. (In [12] the elements of \mathbb{P}_{ω_1} are called *I-perfect*, where *I* is the ideal of countable sets.)

The simplest elements of \mathbb{P}_α are *perfect cubes*, that is, the sets of the form $C = \prod_{\beta < \alpha} C_\beta$, where $C_\beta \in \text{Perf}(\mathfrak{C})$ for each $\beta < \alpha$. (This is justified by a function $f = \langle f_\beta \rangle_{\beta < \alpha} \in \Phi_{\text{prism}}(\alpha)$, where each f_β is a homeomorphism from \mathfrak{C} onto C_β .)

One of the most important properties of iterated perfect sets, distinguishing them from perfect cubes, is the following fact, which is a particular case of [7, thm. 20]. In its current form it has been used in [3]. Its proof can be also found in [4, Lemma 3.2.2].

Lemma 1.1 *For every $0 < \alpha < \omega_1$, $E \in \mathbb{P}_\alpha$, a Polish space X , and a continuous function $f: E \rightarrow X$ there exist $0 < \beta \leq \alpha$ and $P \in \mathbb{P}_\alpha$, $P \subset E$, such that $f \circ \pi_\beta^{-1}$ is a function on $\pi_\beta[P] \in \mathbb{P}_\beta$ which is either one-to-one or constant.*

To state $\text{CPA}_{\text{prism}}$ we need a few more definitions. For a fixed Polish space X let $\mathcal{F}_{\text{prism}}(X)$ stand for the family of all continuous injections from an $E \in \mathbb{P}_{\omega_1}$ onto perfect subsets of X . Each such injection f is called a *prism* and is considered as a coordinate system imposed on $P = \text{range}(f)$.¹ We will

¹In a language of forcing a coordinate function f is simply a nice name for an element from X .

usually abuse this terminology and refer to P itself as a *prism* (in X) and to f as a *witness function* for P . A function $g \in \mathcal{F}_{\text{prism}}(X)$ is *subprism* of f provided $g \subset f$. In the above spirit we call $Q = \text{range}(g)$ a *subprism of a prism* P . Thus, when we say that Q a *subprism of a prism* $P \in \text{Perf}(X)$ we mean that $Q = f[E]$, where f is a witness function for P and $E \subset \text{dom}(f)$ is an iterated perfect set. A family $\mathcal{E} \subset \text{Perf}(X)$ is *prism-dense in X* provided every prism in X contains a subprism $Q \in \mathcal{E}$. It is easy to see (using the fact that $\Phi_{\text{prism}}(\alpha)$ is closed under the composition) that we can assume that a witness function of a prism is always defined on the entire space \mathfrak{C}^α for an appropriate α .

Now we are ready to state the axiom.

$\text{CPA}_{\text{prism}}$: $\mathfrak{c} = \omega_2$ and for every Polish space X and every prism-dense family $\mathcal{E} \subset \text{Perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \leq \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \leq \omega_1$.

If in the definition above we restrict our attention only to prisms which domains are perfect cubes in \mathfrak{C}^ω , we get a notion of cube-density which is stronger than that of prism-density. This naturally leads to a weaker version of $\text{CPA}_{\text{prism}}$, known as CPA_{cube} , obtained from $\text{CPA}_{\text{prism}}$ by replacing the word “prism” with “cube.” Thus, any consequence of axiom CPA_{cube} , which has been studied in [5, 2, 10, 4], follows also from $\text{CPA}_{\text{prism}}$.

Next, let us consider the following ideals on \mathfrak{C} :

$$s_0^{\text{prism}} = \left\{ \mathfrak{C} \setminus \bigcup \mathcal{E} : \mathcal{E} \text{ is prism-dense in } \text{Perf}(\mathfrak{C}) \right\}$$

and

$$s_0^{\text{cube}} = \left\{ \mathfrak{C} \setminus \bigcup \mathcal{E} : \mathcal{E} \text{ is cube-dense in } \text{Perf}(\mathfrak{C}) \right\}.$$

Clearly they are the variants of the Marczewski’s ideal s_0 of subsets of \mathfrak{C} , that is, the family of all sets $S \subset \mathfrak{C}$ such that for every $P \in \text{Perf}(\mathfrak{C})$ there exists a $Q \in \text{Perf}(P)$ disjoint with S . It is not difficult to see that

$$[X]^{<\mathfrak{c}} \subset s_0^{\text{cube}} \subset s_0^{\text{prism}} \subset s_0.$$

(The proof that $[X]^{<\mathfrak{c}} \subset s_0^{\text{cube}} \subset s_0$ can be found in [5, Fact 1.3] or [4]. Inclusion $s_0^{\text{cube}} \subset s_0^{\text{prism}}$ follows immediately from the fact that any cube-dense family is also prism-dense. The proof that $s_0^{\text{prism}} \subset s_0$ is identical to that of $s_0^{\text{cube}} \subset s_0$.)

Obviously $\text{CPA}_{\text{prism}}$, used with $X = \mathfrak{C}$, implies that $s_0^{\text{prism}} \subset [\mathfrak{C}]^{\leq \omega_1}$. So, we get

Proposition 1.2 *If $\text{CPA}_{\text{prism}}$ holds then $s_0^{\text{prism}} = [\mathfrak{C}]^{\leq \omega_1}$.*

This distinguishes ideal s_0^{prism} from s_0 , since there exist ZFC examples of s_0 -sets of cardinality \mathfrak{c} . (See e.g. [9, thm. 5.10].) Cube analog of Proposition 1.2 has been proved in [5].

2 Continuous images of sets of cardinality continuum

In [8] A. Miller proved that the following holds in the iterated perfect set model

- (A) for every subset S of \mathbb{R} of cardinality \mathfrak{c} there exists a (uniformly) continuous function $f: \mathbb{R} \rightarrow [0, 1]$ such that $f[S] = [0, 1]$.

This result was refined by the authors in [5] by showing that (A) follows already from CPA_{cube} . The main goal of this section is to show that $\text{CPA}_{\text{prism}}$ implies the following stronger version of (A).

Theorem 2.1 $\text{CPA}_{\text{prism}}$ implies that

- (A*) there exists a family \mathcal{G} of uniformly continuous functions from \mathbb{R} to $[0, 1]$ such that $|\mathcal{G}| = \omega_1$ and for every $S \in [\mathbb{R}]^{\mathfrak{c}}$ there exists a $g \in \mathcal{G}$ with $g[S] = [0, 1]$.

This also constitutes a version of a remark due to Miller [8, p. 581], who noticed that in the Sacks model functions coded in the ground model can be taken as a family \mathcal{G} .

To prove the theorem we need some auxiliary results. For a fixed Polish space X and $0 < \alpha < \omega_1$ let \mathcal{F}^α denote the family of all continuous injections from \mathfrak{C}^α into X . Note that if we consider \mathcal{F}^α with the topology of uniform convergence then

$$\mathcal{F}^\alpha \text{ is a Polish space.} \tag{1}$$

To prove (1) it is enough to show that \mathcal{F}^α is a G_δ subset of the space $\mathcal{C} = \mathcal{C}(\mathfrak{C}^\alpha, X)$ of all continuous functions from \mathfrak{C}^α into X . But \mathcal{F}^α is the intersection of the open sets G_n , $n < \omega$, where sets G_n are constructed as follows. Fix a finite partition \mathcal{P}_n of \mathfrak{C}^α into clopen sets each of the diameter less than 2^{-n} , and let \mathcal{H}_n be the family of all mappings h from \mathcal{P}_n into the topology of X such that $h(P) \cap h(P') = \emptyset$ for distinct $P, P' \in \mathcal{P}_n$. We put

$$G_n = \bigcup_{h \in \mathcal{H}_n} \{f \in \mathcal{C}: (\forall P \in \mathcal{P}_n)(\forall x \in P) f(x) \in h(P)\}.$$

This completes the argument for (1).

Lemma 2.2 *Let X be a Polish space and $0 < \alpha < \omega_1$. Then every function $f: \mathfrak{C}^\beta \rightarrow \mathcal{F}^\alpha$ from $\mathcal{F}_{\text{prism}}(\mathcal{F}^\alpha)$ has a restriction $f^* \in \mathcal{F}_{\text{prism}}(\mathcal{F}^\alpha)$ with the property that there exists an $\hat{f} \in \mathcal{F}_{\text{prism}}(X)$ defined on a subset of $\mathfrak{C}^{\beta+\alpha}$ such that*

- (a) $\hat{f}(s, t) = f^*(s)(t)$ for all $\langle s, t \rangle \in (\mathfrak{C}^\beta \times \mathfrak{C}^\alpha) \cap \text{dom}(\hat{f})$, and
- (b) for each $s \in \text{dom}(f^*)$ the function $\hat{f}(s, \cdot): \{t \in \mathfrak{C}^\alpha: \langle s, t \rangle \in \text{dom}(\hat{f})\} \rightarrow X$ is a restriction of $f^*(s)$ and belongs to $\mathcal{F}_{\text{prism}}(X)$.

PROOF. Let $f: \mathfrak{C}^\beta \rightarrow \mathcal{F}^\alpha$, $f \in \mathcal{F}_{\text{prism}}(\mathcal{F}^\alpha)$, and define a function g from a set $\mathfrak{C}^\beta \times \mathfrak{C}^\alpha = \mathfrak{C}^{\beta+\alpha}$ into X by $g(s, t) = f(s)(t)$ for $\langle s, t \rangle \in \mathfrak{C}^\beta \times \mathfrak{C}^\alpha$. It is easy to see that g is continuous.

Apply Lemma 1.1 to $E = \mathfrak{C}^{\beta+\alpha} \in \mathbb{P}_{\beta+\alpha}$ and to the function g to find a $\gamma \leq \beta + \alpha$ and a subset $P \in \mathbb{P}_{\beta+\alpha}$ of E such that $g \circ \pi_\gamma^{-1}$ is a function on $\pi_\gamma[P] \in \mathbb{P}_\gamma$ which is either one-to-one or constant. Let $f^* = f \upharpoonright \pi_\beta[P]$. We will show that it is as desired.

First note that

$$\gamma = \beta + \alpha \text{ and } g \text{ is one-to-one on } P.$$

Indeed, if $z \in \text{range}(f^*) \cap \mathcal{F}_{\text{prism}}(X)$ and $z = f^*(s)$ then for every different $t_0, t_1 \in \mathfrak{C}^\alpha$ with $\langle s, t_0 \rangle, \langle s, t_1 \rangle \in P$ we have $g(s, t_0) = f(s)(t_0) = z(t_0) \neq z(t_1) = g(s, t_1)$. So, g cannot be constant and if $\gamma < \beta + \alpha$ then we can find t_0 and t_1 such that $\pi_\gamma(\langle s, t_0 \rangle) = \pi_\gamma(\langle s, t_1 \rangle)$ contradicting the above calculation.

It is easy to see that $\hat{f} = g \upharpoonright P$ is as desired. \blacksquare

Lemma 2.2 implies the following useful fact.

Proposition 2.3 $\text{CPA}_{\text{prism}}$ *implies that for every Polish space X there exists a family \mathcal{H} of continuous functions from compact subsets of X onto $\mathfrak{C} \times \mathfrak{C}$ such that $|\mathcal{H}| \leq \omega_1$ and*

- for every prism P in X there are $h \in \mathcal{H}$ and $c \in \mathfrak{C}$ such that $h^{-1}(\{c\} \times \mathfrak{C})$ and $h^{-1}(\{c, d\})$ are subprisms of P for every $d \in \mathfrak{C}$.

In particular, $\mathcal{F} = \{h^{-1}(\{c\} \times \mathfrak{C}): h \in \mathcal{H} \ \& \ c \in \mathfrak{C}\}$ is prism-dense in X .

PROOF. Let $0 < \alpha < \omega_1$. We use the notation as in Lemma 2.2.

Since the family of all sets $\text{range}(f^*)$ is prism-dense in \mathcal{F}^α by $\text{CPA}_{\text{prism}}$ we can find a family $\mathcal{G}_\alpha = \{f_\xi^*: \xi < \omega_1\}$ such that $R_\alpha = \mathcal{F}^\alpha \setminus \bigcup_{\xi < \omega_1} \text{range}(f_\xi^*)$ has cardinality less than or equal to ω_1 .

If $f^* \in \mathcal{G}_\alpha$ then \hat{f} maps injectively a $P = P_f \in \mathbb{P}_{\beta+\alpha}$ onto $Q = Q_f \subset X$. Moreover, for every $z \in \mathcal{F}^\alpha \setminus R_\alpha$ there are $f^* \in \mathcal{G}_\alpha$ and $s \in \text{dom}(f^*)$ such that $z = f^*(s)$ and $\hat{f}(s, \cdot) \in \mathcal{F}_{\text{prism}}(X)$ is a restriction of z .

Now, let $H_f \in \Phi_{\text{prism}}(\beta + \alpha)$ be from $\mathfrak{C}^{\beta+\alpha}$ onto P and consider the composition $\hat{f} \circ H_f: \mathfrak{C}^{\beta+\alpha} \rightarrow Q$. Then functions $(\hat{f} \circ H_f)^{-1}: Q_f \rightarrow \mathfrak{C}^{\beta+\alpha}$ are our desired functions modulo some projections. More precisely, let $k_0: \mathfrak{C}^\beta \rightarrow \mathfrak{C}$ be a homeomorphism and let $k_1: \mathfrak{C} \rightarrow \mathfrak{C}$ be such that $k_1^{-1}(c) \in \text{Perf}(\mathfrak{C})$ for every $c \in \mathfrak{C}$. Define $h_f^\alpha: Q_f \rightarrow \mathfrak{C} \times \mathfrak{C}$ by

$$h_f^\alpha(x) = \langle (k_0 \circ \pi_\beta)((\hat{f} \circ H_f)^{-1}(x)), k_1((\hat{f} \circ H_f)^{-1}(x))(\beta) \rangle.$$

Then family $\mathcal{H}_0 = \{h_f^\alpha: \alpha < \omega_1 \ \& \ f^* \in \mathcal{G}_\alpha\}$ works for all functions not in $R = \bigcup_{0 < \alpha < \omega_1} R_\alpha$. Also, for every function $g \in R$ it is easy to find a continuous function h_g from $\text{range}(g)$ onto $\mathfrak{C} \times \mathfrak{C}$ such that $h_g^{-1}(\{c\} \times \mathfrak{C})$ and $h_g^{-1}(\langle c, d \rangle)$ are subprisms of $\text{range}(g)$ for every $c, d \in \mathfrak{C}$. Then $\mathcal{H} = \mathcal{H}_0 \cup \{h_g: g \in R\}$ is as desired. \blacksquare

PROOF OF THEOREM 2.1. Let \mathcal{H} be as in Proposition 2.3 used with $X = \mathbb{R}$, let $k: \mathfrak{C} \rightarrow [0, 1]$ be continuous surjection, and for every $h = \langle h_0, h_1 \rangle \in \mathcal{H}$ let $g_h: \mathbb{R} \rightarrow [0, 1]$ be a continuous extension of a function $h^*: \text{dom}(h) \rightarrow [0, 1]$ defined by $h^*(x) = k(h_1(x))$. We claim that $\mathcal{G} = \{g_h: h \in \mathcal{H}\}$ is as desired.

To see it, let $S \in [\mathbb{R}]^\mathfrak{c}$ and let $\mathcal{E} = \{P \in \text{Perf}(\mathbb{R}): P \cap S = \emptyset\}$. Since $\mathbb{R} \setminus \bigcup \mathcal{E}$ contains S , it has cardinality \mathfrak{c} . So, from $\text{CPA}_{\text{prism}}$ we conclude that \mathcal{E} is not prism-dense. (Compare with Proposition 1.2.) Thus, there exists a prism P in \mathbb{R} such that S intersects every subprism of P . Let $h \in \mathcal{H}$ and $c \in \mathfrak{C}$ be such that $h^{-1}(\{c\} \times \mathfrak{C})$ and $h^{-1}(\langle c, d \rangle)$ are subprisms of P for every $d \in \mathfrak{C}$. Then S intersects each $h^{-1}(\langle c, d \rangle)$ so $h[S]$ contains $\{c\} \times \mathfrak{C}$. Thus $g_h[S] = [0, 1]$. \blacksquare

3 $\text{CPA}_{\text{prism}}$ implies that $\text{add}(s_0) = \omega_1$

Recall that the additivity number is defined as

$$\text{add}(s_0) = \min \left\{ |F|: F \subset s_0 \ \& \ \bigcup F \notin s_0 \right\}.$$

Numbers $\text{add}(s_0)$, $\text{cov}(s_0)$, $\text{non}(s_0)$, and $\text{cof}(s_0)$ has been intensively studied. (See e.g. [6].) It is known that $\text{cof}(s_0) > \mathfrak{c}$ (see e.g. [6, thm. 1.3]) and that $\text{non}(s_0) = \mathfrak{c}$ since there are s_0 -sets of cardinality \mathfrak{c} . There are models of ZFC+MA with $\mathfrak{c} = \omega_2$ and $\text{cov}(s_0) = \omega_1$, while Proper Forcing Axiom implies that $\text{add}(s_0) = \mathfrak{c}$. Here we prove that $\text{CPA}_{\text{prism}}$ implies $\text{add}(s_0) = \omega_1$. Note also that a stronger form of CPA implies that $\text{cov}(s_0) = \omega_2$. (See [4, prop. 6.1.1].)

In the proof we will use the following fact in which the assumption that \mathcal{D} is an open subset of $\text{Perf}(\mathfrak{C})$ means that $\text{Perf}(P) \subset \mathcal{D}$ for every $P \in \mathcal{D}$.

Fact 3.1 For any open dense subset \mathcal{D} of $\text{Perf}(\mathfrak{C})$ (considered as ordered by inclusion) there exists a maximal antichain $\mathcal{A} \subset \mathcal{D}$ consisting of pairwise disjoint sets such that every $P \in \text{Perf}(\bigcup \mathcal{A})$ is covered by less than continuum many sets from \mathcal{A} .

PROOF. Let $\text{Perf}(\mathfrak{C}) = \{P_\alpha: \alpha < \mathfrak{c}\}$. We will build inductively a sequence $\langle \langle A_\alpha, x_\alpha \rangle \in \mathcal{D} \times \mathfrak{C}: \alpha < \mathfrak{c} \rangle$ aiming for $\mathcal{A} = \{A_\alpha: \alpha < \mathfrak{c}\}$. At step $\alpha < \mathfrak{c}$, given already $\langle \langle A_\beta, x_\beta \rangle: \beta < \alpha \rangle$ we look at P_α .

Choice of x_α : If $P_\alpha \subset \bigcup_{\beta < \alpha} A_\beta$ we take x_α as an arbitrary element of \mathfrak{C} ; otherwise we pick $x_\alpha \in P_\alpha \setminus \bigcup_{\beta < \alpha} A_\beta$.

Choice of A_α : If there is a $\beta < \alpha$ such that $P_\alpha \cap A_\beta$ is uncountable, we let $A_\alpha = A_\beta$; otherwise pick $A_\alpha \in \mathcal{D}$ below P_α and notice that we can refine it, if necessary, to be disjoint with $\bigcup_{\beta < \alpha} A_\beta \cup \{x_\beta: \beta \leq \alpha\}$.

It is easy to see that $\mathcal{A} = \{A_\alpha: \alpha < \mathfrak{c}\}$ is as required. ■

Theorem 3.2 $\text{CPA}_{\text{prism}}$ implies that $\text{add}(s_0) = \omega_1$.

PROOF. Let $\mathcal{H} = \{h_\xi: \xi < \omega_1\}$ be as in Proposition 2.3 with $X = \mathfrak{C}$. For every $\xi < \omega_1$ put $\mathcal{A}_\xi^0 = \{h_\xi^{-1}(\{c\} \times \mathfrak{C}): c \in \mathfrak{C}\}$. Then each \mathcal{A}_ξ^0 is a family of pairwise disjoint sets and $\mathcal{A}^0 = \bigcup_{\xi < \omega_1} \mathcal{A}_\xi^0$ is dense in $\text{Perf}(\mathfrak{C})$.

For each $\xi < \omega_1$ let \mathcal{A}_ξ^* be a maximal antichain extending \mathcal{A}_ξ^0 , define $\mathcal{D}_\xi = \{P \in \text{Perf}(\mathfrak{C}): P \subset A \text{ for some } A \in \mathcal{A}_\xi^*\}$, and let $\mathcal{A}_\xi \subset \mathcal{D}_\xi$ be as in Fact 3.1. Then $\mathcal{A} = \bigcup_{\xi < \omega_1} \mathcal{A}_\xi$ is still dense in $\text{Perf}(\mathfrak{C})$.

For each $\xi < \omega_1$ let $\{P_\xi^\alpha: \alpha < \mathfrak{c}\}$ be an enumeration of \mathcal{A}_ξ . (Note that each \mathcal{A}_ξ has cardinality \mathfrak{c} , since this was the case for sets \mathcal{A}_ξ^0 .) Pick x_ξ^α from each P_ξ^α and put $A_\xi = \{x_\xi^\alpha: \alpha < \mathfrak{c}\}$. Then $A_\xi \in s_0$ for every $\xi < \omega_1$. However, $A = \bigcup_{\xi < \omega_1} A_\xi \notin s_0$ since it intersects every element of a dense set \mathcal{A} . ■

It can be also shown that $\text{CPA}_{\text{prism}}$, with a help of Proposition 2.3, implies that the Sacks forcing $\mathbb{P} = \langle \text{Perf}(\mathfrak{C}), \subset \rangle$ collapses \mathfrak{c} to ω_1 . However, this also follows immediately from a theorem of P. Simon [11] that \mathbb{P} collapses \mathfrak{c} to \mathfrak{b} while already CPA_{cube} implies that $\mathfrak{b} \leq \text{cof}(\mathcal{N}) = \omega_1$.

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