UNCOUNTABLE INTERSECTIONS OF OPEN SETS UNDER $\mathrm{CPA}_{\mathrm{prism}}$

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Abstract. We prove that the Covering Property Axiom $\mathrm{CPA}_{\mathrm{prism}}$, which holds in the iterated perfect set model, implies the following facts.

- If G is an intersection of ω_1-many open sets of a Polish space and G has cardinality continuum, then G contains a perfect set.
- There exists a subset G of the Cantor set which is an intersection of ω_1-many open sets but is not a union of ω_1-many closed sets.

The example from the second fact refutes a conjecture of Brendle, Larson, and Todorcevic.

1. Preliminaries and axiom $\mathrm{CPA}_{\mathrm{prism}}$

Our set-theoretic terminology is standard and follows that of [3]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use term *Polish space* for a complete separable metric space `without isolated points`. For a Polish space X the symbol $\mathrm{Perf}(X)$ will stand for a collection of all subsets of X homeomorphic to the Cantor set $\mathfrak{C}$. For a fixed $0 < \alpha < \omega_1$ and $0 < \beta \leq \alpha$ a symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$.

Axiom $\mathrm{CPA}_{\mathrm{prism}}$ was introduced by the authors in [5], where it is shown that it holds in the iterated perfect set model. Also, $\mathrm{CPA}_{\mathrm{prism}}$ is a simpler version of the axiom CPA, which is described in monograph [6]. (See also [4].) For the reader's convenience, we will restate the axiom in the next few paragraphs.

The main notions needed for the axiom are that of *prism* and *prism-density*. Let $0 < \alpha < \omega_1$ and let $\Phi_{\mathrm{prism}}(\alpha)$ be the family of all continuous injections $f\colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ with the property that

$$f(x) \restriction \beta = f(y) \restriction \beta \iff x \restriction \beta = y \restriction \beta \qquad \text{for all } \beta \in \alpha \text{ and } x, y \in \mathfrak{C}^\alpha$$

or, equivalently, such that for every $\beta < \alpha$

$$f \restriction\restriction \beta \stackrel{\text{def}}{=} \{\langle x \restriction \beta, y \restriction \beta \rangle \colon \langle x, y \rangle \in f\}$$

is a one-to-one function from $\mathfrak{C}^\beta$ into $\mathfrak{C}^\beta$. Functions f from $\Phi_{\mathrm{prism}}(\alpha)$ were first introduced, in a more general setting, in [8], where they are called *projection-keeping*

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homeomorphisms. Note that $\Phi_{\rm prism}(\alpha)$ is closed under compositions and that, for every $0<\beta<\alpha$, if $f\in\Phi_{\rm prism}(\alpha)$, then $f\restriction\restriction\beta\in\Phi_{\rm prism}(\beta)$. Let

$$\mathbb{P}_\alpha=\{\mathrm{range}(f)\colon f\in\Phi_{\rm prism}(\alpha)\},$$

and note that if $f\in\Phi_{\rm prism}(\alpha)$ and $E\in\mathbb{P}_\alpha$, then $f[E]\in\mathbb{P}_\alpha$. In [11] the elements of $\mathbb{P}_\alpha$ are called I-perfect, where I is the ideal of countable sets.

The simplest elements of $\mathbb{P}_\alpha$ are *perfect cubes*, that is, the sets of the form $C=\prod_{\beta<\alpha}C_\beta$, where $C_\beta\in\mathrm{Perf}(\mathfrak{C})$ for each $\beta<\alpha$. (This is justified by a function $f=\langle f_\beta\rangle_{\beta<\alpha}\in\Phi_{\rm prism}(\alpha)$, where each f_β is a homeomorphism from $\mathfrak{C}$ onto C_β.)

To state $\mathrm{CPA}_{\rm prism}$ we need a few more definitions. For a fixed Polish space X we say that $P\in\mathrm{Perf}(X)$ is a *prism* if we consider it with an (implicitly given) continuous injection f from $\mathfrak{C}^\alpha$, $0<\alpha<\omega_1$, onto P. We say that Q is a *subprism of a prism* $P\in\mathrm{Perf}(X)$ provided $Q=f[C]$, where f is as above and $C\in\mathbb{P}_\alpha$. A family $\mathcal{E}\subset\mathrm{Perf}(X)$ is *prism-dense in* X provided every prism in X contains a subprism $Q\in\mathcal{E}$. Now we are ready to state the axiom.

$\mathrm{CPA}_{\rm prism}$: $\mathfrak{c}=\omega_2$ and for every Polish space X and every prism-dense family $\mathcal{E}\subset\mathrm{Perf}(X)$ there is an $\mathcal{E}_0\subset\mathcal{E}$ such that $|\mathcal{E}_0|\le\omega_1$ and $|X\setminus\bigcup\mathcal{E}_0|\le\omega_1$.

If in the definition above we restrict our attention only to prisms which are perfect cubes in $\mathfrak{C}^\omega$, we get a notion of cube-density which is stronger than that of prism-density. This naturally leads to a weaker version of $\mathrm{CPA}_{\rm prism}$, known as $\mathrm{CPA}_{\rm cube}$, which is obtained from $\mathrm{CPA}_{\rm prism}$ by replacing the word "prism" with "cube." Thus, any consequence of axiom $\mathrm{CPA}_{\rm cube}$, which has been studied in [7, 4, 10, 6], follows also from $\mathrm{CPA}_{\rm prism}$.

2. Intersections of ω_1-many open sets

For a Polish space X let G_{ω_1} be the collection of the intersections of ω_1-many open subsets of X. We are going to prove the following theorem.

Theorem 2.1. $\mathrm{CPA}_{\rm prism}$ *implies that the following property holds for every Polish space X.*

($*$) *If G is a G_{ω_1} subset of X and $|G|=\mathfrak{c}$, then G contains a perfect set.*

Theorem 2.1 provides an affirmative answer to a question of Jörg Brendle, who asked us in [1] whether ($*$) can be deduced from our axiom CPA. The fact that ($*$) holds in the iterated perfect set model is proved in [2]. The argument presented below is considerably simpler.

Before we prove Theorem 2.1 we would like to notice, in Corollary 2.4, that in property ($*$) we can replace the class of open sets by the considerably larger class Π^1_2. Here Σ^1_1 stands for the class of analytic sets, that is, continuous images of Borel sets; Π^1_1 for the class of co-analytic sets, the complements of analytic sets; Σ^1_2 for continuous images of co-analytic sets, and Π^1_2 for the class of all complements of Σ^1_2 sets. For the argument that follows we need also to recall a theorem of Sierpiński that every Σ^1_2 set is the union of ω_1 Borel sets. (See, e.g., [9, p. 324].)

To argue for Corollary 2.4 we need the following two results.

Claim 2.2. *If $G\subset\mathfrak{C}^\omega$ is comeager in $\mathfrak{C}^\omega$, then it contains a perfect cube $\prod_{i<\omega}P_i$.*

An argument for the claim can be found in [7], [4], or [6].

Fact 2.3. *Under* $\mathrm{CPA}_{\mathrm{cube}}$ *(so, also under* $\mathrm{CPA}_{\mathrm{prism}}$*) the following holds:*

For every Σ^1_2 subset B of a Polish space X there exists a family $\mathcal{P}$ of ω_1-many compact sets such that $B = \bigcup \mathcal{P}$.

Proof. Since every Σ^1_2 set is a union of ω_1 Borel sets, we can assume that B is Borel. Let $\mathcal{E}$ be the family of all $P \in \mathrm{Perf}(X)$ such that either $P \subset B$ or $P \cap B = \emptyset$. We claim that $\mathcal{E}$ is $\mathcal{F}_{\mathrm{cube}}$-dense. Indeed, if $f\colon \mathfrak{C}^\omega \to X$ is a continuous injection, then $f^{-1}(B)$ is Borel in $\mathfrak{C}^\omega$. Thus, there exists a basic open set U in $\mathfrak{C}^\omega$, which is homeomorphic to $\mathfrak{C}^\omega$, such that either $U \cap f^{-1}(B)$ or $U \setminus f^{-1}(B)$ is comeager in U. Apply Claim 2.2 to this comeager set to find a perfect cube P contained in it. Then $f[P] \in \mathcal{E}$ is a subcube of range(f). So, $\mathcal{E}$ is $\mathcal{F}_{\mathrm{cube}}$-dense.

By $\mathrm{CPA}_{\mathrm{cube}}$ there exists an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \le \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$. Let $\mathcal{P}_0 = \{P \in \mathcal{E}_0 \colon P \subset B\}$ and $\mathcal{P} = \mathcal{P}_0 \cup \{\{x\} \colon x \in B \setminus \bigcup \mathcal{E}_0\}$. Then $\mathcal{P}$ is as desired. ■

Corollary 2.4. *Assume that* $\mathrm{CPA}_{\mathrm{prism}}$ *holds and X is a Polish space.*

- *If G is an intersection of ω_1-many Π^1_2 sets from X and $|G| = \mathfrak{c}$, then G contains a perfect set.*

Proof. Let $G = \bigcap_{\xi<\omega_1} T_\xi$, where each $T_\xi \subset X$ is a Π^1_2 set. Then

$$X \setminus G = \bigcup_{\xi<\omega_1} (X \setminus T_\xi)$$

and each set $X \setminus T_\xi$ is in the class Σ^1_2, so, by Fact 2.3, it is a union of ω_1-many compact sets. Thus, each T_ξ is an intersection of ω_1 open sets. ■

Theorem 2.1 follows easily from the following combinatorial fact concerning iterated perfect sets.

Proposition 2.5. *Let $0 < \alpha < \omega_1 < \mathfrak{c}$ and $H = \mathfrak{C}^\alpha \setminus \bigcup_{\xi<\omega_1} F_\xi$, where each set F_ξ is compact. Then either H contains a perfect set P, or else there exists an $E \in \mathbb{P}_\alpha$ disjoint with H.*

Proof of Theorem 2.1. Let $G = X \setminus \bigcup_{\xi<\omega_1} T_\xi$, where each set T_ξ is closed in X, and assume that G does not contain a perfect set.

Let $\mathcal{E} = \{P \in \mathrm{Perf}(X) \colon P \cap G = \emptyset\}$. We will show that $\mathcal{E}$ is prism-dense. This will finish the proof since then, by $\mathrm{CPA}_{\mathrm{prism}}$, $X \setminus \bigcup \mathcal{E} \supset G$ has cardinality $\le \omega_1 < \mathfrak{c}$.

So, let $f\colon \mathfrak{C}^\alpha \to X$ be a continuous injection. We need to find an $E \in \mathbb{P}_\alpha$ for which $f[E] \in \mathcal{E}$, that is, $f[E] \cap G = \emptyset$. Let $F_\xi = f^{-1}(T_\xi)$ for $\xi < \omega_1$. Then $H = \mathfrak{C}^\alpha \setminus \bigcup_{\xi<\omega_1} F_\xi$ is equal to $f^{-1}(G)$. If H contains a perfect set P, then so does $G \supset f[P]$, contradicting our assumption. Thus, by Proposition 2.5, there exists an $E \in \mathbb{P}_\alpha$ disjoint with $H = f^{-1}(G)$. So, $f[E]$ is disjoint with G. ■

To prove Proposition 2.5 we need some auxiliary terminology. For a proper ideal I of subsets of $\mathfrak{C}$ and an ordinal $0 < \alpha < \omega_1$ we say that a tree $T \subset \mathfrak{C}^{\le\alpha}$ (ordered by inclusion) is a *co-I tree*, $T \in \mathcal{T}^\alpha_I$, provided

- for every $\beta < \alpha$ and $t \in T \cap \mathfrak{C}^\beta$ we have $\mathfrak{C} \setminus \mathrm{succ}_T(t) \in I$, where $\mathrm{succ}_T(t) = \{s(\beta) \colon t \subset s \in T \cap \mathfrak{C}^{\beta+1}\}$ is the set of all immediate successors of t in T, and
- for every limit ordinal $\lambda \le \alpha$ the λ-th level $T \cap \mathfrak{C}^\lambda$ of T consists all the branches of $T \cap \mathfrak{C}^{<\lambda}$, that is, $t \in T \cap \mathfrak{C}^\lambda$ if and only if $t \restriction \gamma \in T$ for every $\gamma < \lambda$.

Also let $\mathcal{K}_I^\alpha = \{T \cap \mathfrak{C}^\alpha \colon T \in \mathcal{T}_I^\alpha\}$ and let $I^\alpha = \bigcup_{K \in \mathcal{K}_I^\alpha} \mathcal{P}(\mathfrak{C}^\alpha \setminus K)$. It is easy to see that I^α is an ideal on $\mathfrak{C}^\alpha$. We will call I^α the *α-th Fubini power* of the ideal I. This notion, with slightly different emphasis, appears also in Zapletal [11].

We will be interested in these notions only for I of the form $I_\kappa = [\mathfrak{C}]^{\leq\kappa}$, where $\kappa < \mathfrak{c}$ is equal to either ω or ω_1. It is easy to see that the families $\mathcal{T}_{I_\kappa}^\alpha$ and $\mathcal{K}_{I_\kappa}^\alpha$ are closed under the intersection of κ-many of their elements. So $(I_\kappa)^\alpha$ is a κ^+-additive ideal on $\mathfrak{C}^\alpha$.

A big part of the difficulty behind our proof of Theorem 2.1 lies in the following lemma generalizing the Cantor-Bendixon theorem that an uncountable closed set contains a perfect set. In fact, this result is true not only for closed sets, but also for analytic sets. This version of the lemma, which can be viewed as a generalization of the Suslin theorem that an uncountable analytic set contains a perfect set, can be found in [11, Lemma 4.1.29(2)]. (See also [11, Lemma 3.3.1].)

Lemma 2.6. *Let $0 < \alpha < \omega_1$. If F is a closed subset of $\mathfrak{C}^\alpha$ and $F \notin (I_\omega)^\alpha$, then F contains an iterated perfect set, that is, there exists an $E \in \mathbb{P}_\alpha$ such that $E \subset F$.*

Proof of Proposition 2.5. If there is a $\xi < \omega_1$ such that $F_\xi \notin (I_\omega)^\alpha$, then, by Lemma 2.6, there is an $E \in \mathbb{P}_\alpha$ such that $E \subset F_\xi$. Clearly this E is disjoint with $H = \mathfrak{C}^\alpha \setminus \bigcup_{\xi<\omega_1} F_\xi$. So, assume that $F_\xi \in (I_\omega)^\alpha$ for every $\xi < \omega$. We will show that H contains a perfect set.

For every $\xi < \omega_1$, since $F_\xi \in (I_\omega)^\alpha$, there is a $T_\xi \in \mathcal{T}_{I_\omega}^\alpha$ disjoint with F_ξ. But then $T = \bigcap_{\xi<\omega_1} T_\xi \in \mathcal{T}_{I_{\omega_1}}^\alpha$ is disjoint with $\bigcup_{\xi<\omega_1} F_\xi$, so T is a subset of H. Thus, it is enough to show that T contains a perfect set. But this follows immediately from the assumption that $\omega_1 < \mathfrak{c}$.

If α is a successor ordinal, say $\alpha = \beta + 1$, then take $t \in \pi_\beta[T]$ and a perfect set $C \subset \mathrm{succ}_T(t)$. Then $P = \{t\} \times C$ is a perfect subset of T.

If α is a limit ordinal, take an increasing sequence $\langle \alpha_n \colon n < \omega \rangle$ cofinal with α. By induction on $n < \omega$ choose a sequence $\langle t_\sigma \in \pi_{\alpha_n}[T] \colon n < \omega \ \&\ \sigma \in 2^n \rangle$ such that $t_{\sigma\hat{}0}$ and $t_{\sigma\hat{}1}$ are distinct extensions of t_σ. Then

$$P = \{t \in \mathfrak{C}^\alpha \colon t \restriction \alpha_n \in \{t_\sigma \colon \sigma \in 2^n\} \text{ for every } n < \omega\}$$

is a perfect subset of T. ■

Note that Lemma 2.6 (see [11, Lemma 3.3.1]) and appropriate CPA versions hold also for many other "nicely definable" forcings. (See [11, chapter 5].) The form of these CPA's remains the most similar to our axiom for the forcings for which Lemma 3.4 remains valid. This, for example, includes Miller forcing. For these forcings the argument remains intact, so appropriate CPA's also imply $(*)$.

3. The complements of G_{ω_1} sets

Brendle, Larson, and Todorcevic conjectured in [2, conj. 5.11] that, in the iterated perfect set model, the complement of a G_{ω_1} set is also G_{ω_1}, that is, that any G_{ω_1} set is a union of ω_1-many closed sets. In this section we will prove that the conjecture is false, by showing that the existence of a counterexample follows from $\mathrm{CPA}_{\mathrm{prism}}$. More precisely, this follows from the following theorem, which is interesting in its own right.

Let $\mathcal{X}$ be the family of all continuous functions f from an uncountable G_δ subset of $\mathfrak{C}$ into $\mathfrak{C}$.

Theorem 3.1. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that there exists a sequence* $\langle g_\alpha \in \mathcal{X} \colon \alpha < \omega_1 \rangle$ *such that for every* $g \in \mathcal{X}$ *there is an* $\alpha < \omega_1$ *such that* $|g \cap g_\alpha| = \mathfrak{c}$.

Before we prove the theorem we notice that it easily implies the following corollary, which refutes [2, conj. 5.11]. It also gives a negative answer to [2, Question 5.13]. Note that in the proof of the corollary we use from Theorem 3.1 only the fact that $g \cap g_\alpha \neq \emptyset$.

Corollary 3.2. *If* $\omega_1 < \mathfrak{c}$ *and* $G = \mathfrak{C}^2 \setminus \bigcup_{\alpha<\omega_1} g_\alpha$, *where the* g_α*'s are from Theorem 3.1, then* G *is not a union of* ω_1*-many closed sets.*

In particular, $\mathrm{CPA}_{\mathrm{prism}}$ *implies that* G *is* G_{ω_1} *but* $\mathfrak{C}^2 \setminus G$ *is not.*

Proof. To see that G is not a union of ω_1-many closed sets, notice that

$$\text{if } P \subset G \text{ is compact then } \pi[P] \text{ is countable.} \tag{1}$$

To see this, take a compact subset P of $\mathfrak{C}^2$ for which $\pi[P]$ is uncountable. We need to show that $P \cap \bigcup_{\alpha<\omega_1} g_\alpha \neq \emptyset$. So, take a Borel selection $g \colon \pi[P] \to \mathfrak{C}$ for P. (For example, put $g(x) = \min\{y \in \mathfrak{C} \colon \langle x, y\rangle \in P\}$.) Then, there exists a dense G_δ subset D of $\pi[P]$ such that $g \restriction D$ is continuous. Thus, $g \restriction D \in \mathcal{X}$ and $(g \restriction D) \cap g_\alpha \neq \emptyset$ for some $\alpha < \omega_1$. In particular, $P \cap \bigcup_{\alpha<\omega_1} g_\alpha \supset (g \restriction D) \cap g_\alpha \neq \emptyset$.

Next, let $\mathcal{P} = \{P_\xi \colon \xi < \omega_1\}$ be a family of compact subsets of G. We need to show that $G \neq \bigcup \mathcal{P}$. But, by (1), there exists an $x \in \mathfrak{C} \setminus \bigcup_{\xi<\omega_1} \pi[P_\xi]$. Let $p \in (\{x\} \times \mathfrak{C}) \setminus \bigcup_{\alpha<\omega_1} g_\alpha$. Then $p \in G \setminus \bigcup \mathcal{P}$.

To see the additional part, first note that $\mathfrak{C}^2 \setminus G$ is not G_{ω_1}, by the above argument. To see that G is G_{ω_1}, it is enough to show that its complement $\bigcup_{\alpha<\omega_1} g_\alpha$ is a union of ω_1-many compact sets. But every g_α is a G_δ subset of $\mathfrak{C}^2$, so it is a Polish space. Thus g_α is a union of ω_1 compact sets — this follows directly from the formulation of $\mathrm{CPA}_{\mathrm{prism}}$ (as well as from $\mathrm{CPA}_{\mathrm{cube}}$). ■

The proof[1] of Theorem 3.1 is based on the following simple observation. Note also that if we are willing to prove the theorem only with the conclusion that $g \cap g_\alpha \neq \emptyset$ (i.e., a version needed to prove Corollary 3.2), then in Lemma 3.3 one need only require that $F \cap f \neq \emptyset$ for every continuous $f \colon \mathfrak{C} \to \mathfrak{C}$. The argument for this version is a bit simpler, and in this form Lemma 3.3 is more likely to be known.

Lemma 3.3. *There exists a continuous function* F *from a* G_δ *subset* T *of* $\mathfrak{C}$ *into* $\mathfrak{C}$ *such that* $|F \cap f| = \mathfrak{c}$ *for every continuous* $f \colon \mathfrak{C} \to \mathfrak{C}$.

Proof. Let $\mathcal{C}$ be the family of all continuous functions $f \colon \mathfrak{C} \to \mathfrak{C}$, considered with the sup norm. Then $\mathcal{C}$ is homeomorphic to the Baire space ω^ω, and so is $\mathcal{C} \times \mathfrak{C}$. (To see this use, for example, [9, thm. 7.7].) Let T be a G_δ subset of $\mathfrak{C}$ homeomorphic to ω^ω and let $h = \langle h_1, h_2\rangle \colon T \to \mathcal{C} \times \mathfrak{C}$ be a homeomorphism. Define $F \colon T \to \mathfrak{C}$ by

$$F(t) = [h_1(t)](t).$$

Clearly F is continuous. To see the other part, take an $f \in \mathcal{C}$ and notice that $P = (h_1)^{-1}(f)$ is uncountable. It is enough to show that $F \restriction P = f \restriction P$.

Indeed, if $t \in P$, then $h_1(t) = f$ and $F(t) = [h_1(t)](t) = f(t)$. ■

[1] It was pointed to us by the referee that "Theorem 3.1 is of the syntactical form isolated in [11] for the consequences of CPA, while Corollary 3.2 is not of this form." This certainly sheds more light on our work. However, in order to use the results from [11] to deduce Theorem 3.1, in addition to checking "the syntactical form," one needs also to know how to force the conclusion of Theorem 3.1. So, one cannot use [11] without actually using forcing technic.

One of the most important properties of prisms, distinguishing them from cubes, is the following fact, which is a particular case of [8, thm. 20]. In its current form it has been used in [5]. Its proof can be also found in [6, Lemma 3.2.2].

Lemma 3.4. *For every $0 < \alpha < \omega_1$, a Polish space X, and a continuous function $f\colon \mathfrak{C}^\alpha \to X$ there exist $0 < \beta \le \alpha$ and $E \in \mathbb{P}_\alpha$ such that $f \circ \pi_\beta^{-1}$ is a function on $\pi_\beta[E] \in \mathbb{P}_\beta$ which is either one-to-one or constant.*

We will also need the following result.

Proposition 3.5. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that there exists a family $\mathcal{F} = \{f_\alpha \colon \alpha < \omega_1\}$ of continuous injections from $\mathfrak{C}^2$ into $\mathfrak{C}$ such that $\{f_\alpha[\{x\} \times \mathfrak{C}] \colon x \in \mathfrak{C} \;\&\; \alpha < \omega_1\}$ is dense in* $\mathrm{Perf}(\mathfrak{C})$.

Proof. Let $\mathcal{C}$ be the family of all continuous functions from $\mathfrak{C}$ into $\mathfrak{C}$ endowed with the uniform convergence topology, and let X be a subspace of $\mathcal{C}$ consisting of all continuous injections from $\mathfrak{C}$ into $\mathfrak{C}$. Note that X is a G_δ subset of X and that X is a Polish space. Thus, there exists a metric on X which makes it a Polish space.

For every prism P in X witnessed by a continuous injection f from $\mathfrak{C}^\alpha$ onto P there exists a subprism P^* defined as follows. Let $f^* \colon \mathfrak{C}^{\alpha+1} \to \mathfrak{C}$ be defined by $f^*(x) = f(x \restriction \alpha)(x(\alpha))$. Clearly f^* is continuous. Apply Lemma 3.4 to the function f^* to find a $\beta \le \alpha+1$ and an $E \in \mathbb{P}_{\alpha+1}$ such that $f^* \circ \pi_\beta^{-1}$ is a function on $\pi_\beta[P] \in \mathbb{P}_\beta$ which is either one-to-one or constant. Note that

$$\beta = \alpha + 1 \text{ and } f^* \text{ is one-to-one on } E,$$

since there are distinct $x, y \in E$ with $x \restriction \alpha = y \restriction \alpha$ for which we have $f^*(x) \ne f^*(y)$. Let $P^* = f[\pi_\alpha[E]]$ and $f_{P^*} = f^* \restriction E$.

Since $\mathcal{E} = \{P^* \colon P \text{ is a prism in } X\}$ is prism-dense, by $\mathrm{CPA}_{\mathrm{prism}}$ we can find an $\mathcal{E}_0 \in [\mathcal{E}]^{\le\omega_1}$ such that $Y = X \setminus \bigcup \mathcal{E}$ has cardinality at most ω_1.

For $Q \in \mathcal{E}_0$, if f_Q is defined on $E \in \mathbb{P}_{\alpha+1}$ let $g \in \Phi_{\mathrm{prism}}(\alpha+1)$ be a projection-keeping homeomorphism from $\mathfrak{C}^{\alpha+1}$ into E. If in $\mathfrak{C}^{\alpha+1} = \mathfrak{C}^\alpha \times \mathfrak{C}$ we identify $\mathfrak{C}^\alpha$ with $\mathfrak{C}$, then $g_Q = f_Q \circ g$ is an injection from $\mathfrak{C}^2$ into $\mathfrak{C}$, and for every $h \in Q$ there is an $x \in \mathfrak{C}$ such that $g_Q[\{x\} \times \mathfrak{C}] \subset h[\mathfrak{C}]$. Now, if the domains of the functions from Y are identified with $\mathfrak{C}^2$, then $\mathcal{F} = Y \cup \{g_Q \colon Q \in \mathcal{E}_0\}$ is as desired. ■

Proof of Theorem 3.1. Let $\{f_\alpha \colon \alpha < \omega_1\}$ be as in Proposition 3.5 and let $F \colon T \to \mathfrak{C}$ be as in Lemma 3.3. For $\alpha < \omega_1$ let $K_\alpha = f_\alpha[\mathfrak{C} \times T]$. Define $g_\alpha \colon K_\alpha \to \mathfrak{C}$ by

$$g_\alpha(f_\alpha(x,t)) = F(t) \quad \text{for every } \langle x, t\rangle \in \mathfrak{C} \times T.$$

Clearly the functions g_α are continuous and defined on G_δ sets.

Fix a $g \in \mathcal{X}$. We need to find an $\alpha < \omega_1$ such that $|g \cap g_\alpha| = \mathfrak{c}$. Since the domain of g is uncountable, it contains a perfect set P. So, there are $\alpha < \omega_1$ and $x_0 \in \mathfrak{C}$ such that $f_\alpha[\{x_0\} \times \mathfrak{C}] \subset P$. Let $f \in \mathcal{C}$ be defined by $f(y) = g(f_\alpha(x_0, y))$. Then, by Lemma 3.3, there is a $Q \in \mathrm{Perf}(T)$ such that $F \restriction Q = f \restriction Q$. Thus, for every $t \in Q \subset T$ we have

$$g_\alpha(f_\alpha(x_0,t)) = F(t) = f(t) = g(f_\alpha(x_0,t)).$$

Thus, $g_\alpha \restriction f_\alpha[\{x_0\} \times Q] = g \restriction f_\alpha[\{x_0\} \times Q]$. ■

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[2]Preprints marked by * are available in electronic form from *Set Theoretic Analysis Web Page:* http://www.math.wvu.edu/homepages/kcies/STA/STA.html