# Strong Fubini properties for measure and category

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**Abstract**

Let (FP) abbreviate the statement that

$$\int_0^1 \left( \int_0^1 f\, dy \right) dx = \int_0^1 \left( \int_0^1 f\, dx \right) dy$$

holds for every bounded function $f\colon [0,1]^2 \to \mathbb{R}$ whenever each of the integrals involved exists. We shall denote by (SFP) the statement that the equality above holds for every bounded function $f\colon [0,1]^2 \to \mathbb{R}$

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having measurable vertical and horizontal sections. It follows from well-known results that both of (FP) and (SFP) are independent from the axioms of ZFC. We investigate the logical connections of these statements with several other strong Fubini type properties of the ideal of null sets. In particular, we establish the equivalence of (SFP) to the nonexistence of certain sets with paradoxical properties, a phenomenon that was already known for (FP). We also give the category analogues of these statements and, whenever possible, we try to put the statements in a setting of general ideals as initiated by Recław and Zakrzewski.

# 1 Introduction

In this paper we investigate the equality

$$\int_0^1 \left( \int_0^1 f\,dy \right) dx = \int_0^1 \left( \int_0^1 f\,dx \right) dy \tag{1}$$

under different conditions on the function $f\colon [0,1]^2 \to \mathbb{R}$. It was already known to Cauchy that (1) holds for any continuous function $f\colon [0,1]^2 \to \mathbb{R}$. Cauchy new also (see [1]) that this is false when we allow just one point of discontinuity for $f$: if $f(x,y) = (x^2-y^2)/(x^2+y^2)^2$ then the two sides of (1) equal $\pi/4$ and $-\pi/4$, respectively. (Another simple example is $f(x,y) = (x-y)/(x+y)^3$, where the corresponding values are $-1/2$ and $1/2$.) However, it was proved in 1883 by du Bois-Reymond [4] that if $f$ is Riemann integrable on $[0,1]^2$ then both of the integrals $\int_0^1 \left( \overline{\int_0^1} f\,dy \right) dx$ and $\int_0^1 \left( \overline{\int_0^1} f\,dx \right) dy$ are equal to the double integral $\iint_{[0,1]^2} f(x,y)\,dx\,dy$, where $\overline{\int}$ denotes the Darboux upper integral. In particular, if $f$ is Riemann integrable on $[0,1]^2$ and the integrals $\int_0^1 f_x\,dy$ and $\int_0^1 f^y\,dx$ exist for every $x, y \in [0,1]$ then (1) holds. Here $f_x, f^y\colon [0,1] \to \mathbb{R}$ are the sections of $f$; that is, $f_x(y) = f(x,y) = f^y(x)$ for every $x, y \in [0,1]$.

The most important theorem on (1) is that of Fubini [8][1] stating that (1) holds provided that $f$ is summable on $[0,1] \times [0,1]$.

[1] According to [9, p. 161] Fubini's proof was defective. Later different proofs were provided by Hobson [10] and de la Vallée-Poussin [3].

The first theorem stating (1) without assuming the measurability of $f$ was proved in 1911 by L. Lichtenstein [20],[2] who showed that (1) holds for every bounded function $f$ for which $f^y$ is Riemann integrable for every $y \in [0,1]$, and $f_x$ is measurable for every $x \in [0,1]$.

Motivated by Lichtenstein's result Sierpiński [22] proved that, under the continuum hypothesis (CH), there exists a function $f\colon [0,1]^2 \to \{0,1\}$ for which $\int_0^1 f^y\,dx = 0$ and $\int_0^1 f_x\,dy = 1$ for every $x, y \in [0,1]$. In particular, the iterated integrals for this function exist but are not equal, and so (1) fails. Actually, Sierpiński proves the following statement under CH:

$S^2$ : there exists a set $H \subset [0,1]^2$ such that the horizontal sections of $H$ are of measure zero and the vertical sections of $H$ are of full measure. That is, $\lambda(H^y) = \lambda([0,1] \setminus H_x) = 0$ for every $x, y \in [0,1]$, where $\lambda$ is the Lebesgue measure, $H^y = \{x\colon \langle x,y\rangle \in H\}$ and $H_x = \{y\colon \langle x,y\rangle \in H\}$.

Then Sierpiński notes that the characteristic function of $H$ has the properties described above. This example shows that in the theorems of Fubini and Lichtenstein we cannot omit the conditions that $f$ is summable or that the sections $f^y$ are Riemann integrable.

Can Sierpiński's example be constructed in ZFC? The negative answer was given in 1980 by Friedman [7], who proved that there are models of set theory ZFC in which the following statement, which we call *Fubini property*, holds.

(FP) If $f\colon [0,1]^2 \to \mathbb{R}$ is a bounded function such that $f_x$ and $f^y$ are measurable for every $x, y \in [0,1]$ and the mappings $[0,1] \ni x \mapsto \int_0^1 f(x,y)\,dy$ and $[0,1] \ni y \mapsto \int_0^1 f(x,y)\,dx$ are measurable then (1) holds.

Later it was shown independently by Laczkovich [16] and Freiling [5] that (FP) is, in fact, *equivalent* to the negation of the statement $S^2$. Now it is easy to see that $S^2$ implies the inequality $\operatorname{cov}(\mathcal{N}) \le \operatorname{non}(\mathcal{N})$. Here $\operatorname{cov}(\mathcal{N})$ denotes the minimum cardinality of a system of null sets that can cover $\mathbb{R}$ and $\operatorname{non}(\mathcal{N})$ is the minimum cardinality of a set of positive outer measure. It was shown by Kunen [14] that the inequality $\operatorname{cov}(\mathcal{N}) > \operatorname{non}(\mathcal{N})$ is consistent with ZFC. Since this inequality implies $\neg S^2$ which, in turn, is equivalent to (FP), we find that the consistency of (FP) also follows from Kunen's theorem.

[2]Note that Sierpiński in [22] incorrectly refers this result to another paper of Lichtenstein of 1910. We like to thank Dr. J. Trzeciak for helping us to spot this error.

We may also ask whether or not the following stronger version of (FP), which we call *strong Fubini property*, is consistent with ZFC.

(SFP) If $f\colon [0,1]^2 \to \mathbb{R}$ is a bounded function such that $f_x$ and $f^y$ are measurable for every $x, y \in [0,1]$ then the mappings $[0,1] \ni x \mapsto \int_0^1 f(x,y)\,dy$ and $[0,1] \ni y \mapsto \int_0^1 f(x,y)\,dx$ are measurable and (1) holds.

Although we did not find in the literature (SFP) in its specific, it follows easily from the following *strong approximation property* (for measure).

(SAP) If $f\colon [0,1]^2 \to \mathbb{R}$ is such that $f_x$ and $f^y$ are measurable for every $x, y \in [0,1]$ then there is a measurable function $F\colon [0,1]^2 \to \mathbb{R}$ such that $f_x = F_x$ a.e. and $f^y = F^y$ a.e. for every $x, y \in [0,1]$.

In [21, p. 139], the theorem stating the consistency of (SAP) is attributed to H. Woodin. The first published proof can be found in [21, Theorem 2.17(ii)]. Therefore (SAP) and, consequently, (SFP) are both consistent with ZFC.

As we shall see later, the statements (SAP) and (SFP) are, in fact, equivalent. One of the goals of this paper will be to find several other statements equivalent to (SFP), including one that states the nonexistence of certain sets with paradox properties, analogously to the equivalence (FP) $\iff \neg S^2$. Since all these statements can be expressed in a setting of general ideals on Polish spaces, we will do this in the next section.

# 2 Kuratowski-Ulam type theorems for ideals

We use standard set theoretic terminology as in [2].

Let $X$ be a Polish space and let $\mathcal{I}$ be an ideal in $X$. We shall denote by $\mathcal{B}_\mathcal{I}$ the $\sigma$-algebra generated by $\mathcal{I}$ and by the Borel subsets $\mathcal{B}$ of $X$. The ideals that are the most interesting for us are the $\sigma$-ideals $\mathcal{N}$ of Lebesgue measure zero subsets of $\mathbb{R}$ and $\mathcal{M}$ of meager subsets of $\mathbb{R}$. Then $\mathcal{B}_\mathcal{N}$ and $\mathcal{B}_\mathcal{M}$ are the families of measurable sets, and the sets with the Baire property, respectively. The sections of a set $A \subset X^2$ are defined by

$$A_x = \{y \in X\colon \langle x,y\rangle \in A\} \quad \text{and} \quad A^y = \{x \in X\colon \langle x,y\rangle \in A\}$$

for every $x, y \in X$.

Recall that the Kuratowski-Ulam theorem says that for every $A \subset \mathbb{R}^2$ with the Baire property, if $A_x \in \mathcal{M}$ for every $x \in \mathbb{R}$ then $\{y\colon A^y \notin \mathcal{M}\} \in \mathcal{M}$.

Thus the following *Kuratowski-Ulam property* of an arbitrary ideal $\mathcal{I}$ is a stronger version of the theorem, when considered with $\mathcal{I} = \mathcal{M}$.

(KU) If $A \subset X^2$ is such that $A_x \in \mathcal{I}$ and $A^y \in \mathcal{B}_\mathcal{I}$ for every $x, y \in X$ and $\{y\colon A^y \notin \mathcal{I}\} \in \mathcal{B}_\mathcal{I}$, then $\{y\colon A^y \notin \mathcal{I}\} \in \mathcal{I}$.

We will also consider the following two *strong Kuratowski-Ulam properties*.

(SKU) If $A \subset X^2$ is such that $A_x \in \mathcal{I}$ and $A^y \in \mathcal{B}_\mathcal{I}$ for every $x, y \in X$, then $\{y\colon A^y \notin \mathcal{I}\} \in \mathcal{I}$.

(SKU$^*$) If $A \subset X^2$ is such that $A_x \in \mathcal{B}_\mathcal{I}$ and $A^y \in \mathcal{B}_\mathcal{I}$ for every $x, y \in X$, then $\{y\colon A^y \notin \mathcal{I}\} \in \mathcal{B}_\mathcal{I}$.

Note that Recław and Zakrzewski in [21] refer to (SKU) as "strong Fubini property" and denote it as (SFP). However, we prefer to reserve the adjective "Fubini" to the properties that refer explicitly to the properties of integrals.

Our first aim is to establish the logical connections between these three statements and the nonexistence of sets having some paradoxical properties with respect to the ideal $\mathcal{I}$. We shall say that a set $H \subset A \times B$ is a *0–1 set (in $A \times B$)* provided $H^y \in \mathcal{I}$ for every $y \in B$ and $B \setminus H_x \in \mathcal{I}$ for every $x \in A$. Note that

$$\text{if there is a 0–1 set in } A \times B \text{ then there is one also in } B \times A. \tag{2}$$

Indeed, if $H$ is a 0–1 set in $A \times B$ then $\{\langle x, y\rangle \in B \times A\colon \langle y, x\rangle \notin H\}$ is a 0–1 set in $B \times A$.

We shall use the notation $A^c = X \setminus A$ for every $A \subset X$. We will also use the following propositions concerning the existence of various 0–1 sets.

$S^2_\mathcal{I}$: There exists a 0–1 set in $X \times X$.

$S^{2w}_\mathcal{I}$: There exists a set $A \subset X$ not in $\mathcal{I}$ such that $A \times X$ contains a 0–1 set.

$E_\mathcal{I}$: There exist sets $A, B \subset X$ not in $\mathcal{I}$ such that both $A \times B$ and $A^c \times B^c$ contain 0–1 sets.

To state the next theorem, we need also the following classical cardinal numbers connected to $\mathcal{I}$.

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) &= \min\{|\mathcal{F}|\colon \mathcal{F} \subset \mathcal{I} \ \&\ \bigcup \mathcal{F} \notin \mathcal{I}\}\\
\operatorname{cov}(\mathcal{I}) &= \min\{|\mathcal{F}|\colon \mathcal{F} \subset \mathcal{I} \ \&\ \bigcup \mathcal{F} = X\}\\
\operatorname{non}(\mathcal{I}) &= \min\{|A|\colon A \subset X \ \&\ A \notin \mathcal{I}\}\\
\operatorname{shr}(\mathcal{I}) &= \min\{\kappa\colon \forall A \subset X (A \notin \mathcal{I} \Longrightarrow [A]^\kappa \setminus \mathcal{I} \neq \emptyset)\}.
\end{aligned}$$

In other words, $\operatorname{shr}(\mathcal{I})$ is the smallest cardinal $\kappa$ such that any subset $A$ of $X$ not belonging to $\mathcal{I}$ contains a subset $B$ such that $|B| \leq \kappa$ and $B \notin \mathcal{I}$. Clearly $\operatorname{non}(\mathcal{I}) \leq \operatorname{shr}(\mathcal{I})$.

**Theorem 1** *Let $\mathcal{I}$ be an ideal in an uncountable Polish space $X$. Then we have the following implications:*

$$\begin{array}{ccccccc}
 & & & & (\mathrm{SKU}) & \longrightarrow & (\mathrm{KU}) \\
 & & & & \downarrow & & \downarrow \\
[\operatorname{shr}(\mathcal{I}) < \operatorname{cov}(\mathcal{I})] & \longrightarrow & \neg E_{\mathcal{I}} & \longrightarrow & \neg S^{2w}_{\mathcal{I}} & \longrightarrow & \neg S^2_{\mathcal{I}} \longrightarrow [\operatorname{non}(\mathcal{I}) < 2^\omega \ \& \\
 & & & & & & \operatorname{add}(\mathcal{I}) < \operatorname{cov}(\mathcal{I})]
\end{array}$$

*Moreover, if $\mathcal{I}$ satisfies the following condition:*

$(*)$ *every set $B \in \mathcal{B}_{\mathcal{I}} \setminus \mathcal{I}$ contains a subset $S \notin \mathcal{B}_{\mathcal{I}}$*

*then the chart can be expanded as*

$$\begin{array}{ccccccccc}
 & & & & (\mathrm{SKU}^*) & \longrightarrow & (\mathrm{SKU}) & \longrightarrow & (\mathrm{KU}) \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
[\operatorname{shr}(\mathcal{I}) < \operatorname{cov}(\mathcal{I})] & \longrightarrow & \neg E_{\mathcal{I}} & \longrightarrow & \neg S^{2w}_{\mathcal{I}} & \longrightarrow & \neg S^2_{\mathcal{I}} & \longrightarrow & [\operatorname{non}(\mathcal{I}) < 2^\omega \ \& \\
 & & & & & & & & \operatorname{add}(\mathcal{I}) < \operatorname{cov}(\mathcal{I})]
\end{array}$$

Proof. (SKU$^*$) $\Longrightarrow$ (SKU): The argument for this implication is closely related to that for [21, Theorem 2.11]. Let $A \subset X^2$ and suppose that $A_x \in \mathcal{I}$ and $A^y \in \mathcal{B}_{\mathcal{I}}$ for every $x, y \in X$. By (SKU$^*$) we have $C := \{y\colon A^y \notin \mathcal{I}\} \in \mathcal{B}_{\mathcal{I}}$. If $C \notin \mathcal{I}$ then, by $(*)$, there is a subset $D \subset C$ such that $D \notin \mathcal{B}_{\mathcal{I}}$. Put

$$H = A \cap (X \times D).$$

Then $H^y \in \mathcal{B}_\mathcal{I}$ for every $y \in X$, since $y \in D$ implies $H^y = A^y$ and $y \notin D$ implies $H^y = \emptyset$. Since $H \subset A$, we also have $H_x \in \mathcal{I}$ for every $x \in X$. Applying (SKU$^*$) again we find that

$$\{y\colon H^y \notin \mathcal{I}\} = D \in \mathcal{B}_\mathcal{I},$$

which is impossible.

(SKU$^*$) $\Longrightarrow \neg E_\mathcal{I}$: Suppose that $E_\mathcal{I}$ holds. Then there are sets $A, B \subset X$, $H \subset A \times B$, and $K \subset A^c \times B^c$ such that $A \notin \mathcal{I}$, $B \notin \mathcal{I}$, $H$ is a 0–1 set in $A \times B$, and $K$ is a 0–1 set in $A^c \times B^c$.

First assume that $A, B \in \mathcal{B}_\mathcal{I}$. Then $H_x, H^y \in \mathcal{B}_\mathcal{I}$ for every $x, y \in X$. By $(*)$, there is a set $D \subset B$ such that $D \notin \mathcal{B}_\mathcal{I}$. Putting $E = H \cup (A \times D)$ we have $E_x, E^y \in \mathcal{B}_\mathcal{I}$ for every $x, y \in X$, and $\{y\colon E^y \notin \mathcal{I}\} = D \notin \mathcal{B}_\mathcal{I}$. This, however, contradicts (SKU$^*$).

Therefore we may assume that at least one of $A$ and $B$ does not belong to $\mathcal{B}_\mathcal{I}$. By symmetry we may suppose $B \notin \mathcal{B}_\mathcal{I}$. Put

$$M = H \cup ((A^c \times B^c) \setminus K) \cup (A \times B^c)$$

and notice that $M$ refutes (SKU$^*$). Indeed, to see that the sections of $M$ belong to $\mathcal{B}_\mathcal{I}$ note that if $y \in B$ then $M^y = H^y \in \mathcal{I}$ and for $y \in B^c$ we have $X \setminus M^y = K^y \in \mathcal{I}$. Similarly, if $x \in A$ then $X \setminus M_x = B \setminus H_x \in \mathcal{I}$ and for $x \in A^c$ we have $M_x = A^c \setminus K_x \in \mathcal{I}$. However, $\{y\colon M^y \notin \mathcal{I}\} = B \notin \mathcal{B}_\mathcal{I}$ contradicts (SKU$^*$).

(SKU) $\Longrightarrow$ (KU): This implication is immediate from the definitions.

All the remaining implications will be proved by contraposition.

(SKU) $\Longrightarrow \neg S^{2w}_\mathcal{I}$: Suppose that $S^{2w}_\mathcal{I}$ holds. Then, by (2), there are sets $A \subset X$ and $H \subset X \times A$ such that $A \notin \mathcal{I}$ and $H$ is a 0–1 set in $X \times A$. Let $K = (X \times A) \setminus H$. Then $K_x \in \mathcal{I}$ and $K^y \in \mathcal{B}_\mathcal{I}$ for every $x, y \in X$ while $\{y\colon K^y \notin \mathcal{I}\} = A \notin \mathcal{I}$, which contradicts (SKU).

(KU) $\Longrightarrow \neg S^2_\mathcal{I}$: If $S^2_\mathcal{I}$ holds then there is a 0–1 set in $X^2$ which clearly contradicts (KU).

$\neg S^2_\mathcal{I} \Longrightarrow [\mathrm{non}(\mathcal{I}) < 2^\omega \ \&\ \mathrm{add}(\mathcal{I}) < \mathrm{cov}(\mathcal{I})]$: A proof that there is a 0–1 set if either $\mathrm{non}(\mathcal{I}) = 2^\omega$ or $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I})$ can be found, for example, in [17, Theorem 2]. (Note that the proof given in [17] is valid for every ideal.)

$\neg S^{2w}_\mathcal{I} \Longrightarrow \neg S^w_\mathcal{I}$: The implication $S^2_\mathcal{I} \Longrightarrow S^{2w}_\mathcal{I}$ is obvious.

$\neg E_{\mathcal{I}} \Longrightarrow \neg S^{2w}_{\mathcal{I}}$: Assume that $S^{2w}_{\mathcal{I}}$ holds. Then there are sets $A \subset X$ and $H \subset A \times X$ such that $A \notin \mathcal{I}$ and $H$ is a 0–1 set in $A \times X$. If $A^c \in \mathcal{I}$ then $E_{\mathcal{I}}$ is satisfied with $B = A$. If $A^c \notin \mathcal{I}$ then $E_{\mathcal{I}}$ is satisfied with $B = A^c$, since $H \cap (A \times A^c)$ is a 0–1 set in $A \times A^c$, and the existence of a 0–1 set in $A^c \times A$ follows from (2).

$[\mathrm{shr}(\mathcal{I}) < \mathrm{cov}(\mathcal{I})] \Longrightarrow \neg E_{\mathcal{I}}$: For the proof of $E_{\mathcal{I}} \Longrightarrow [\mathrm{shr}(\mathcal{I}) \geq \mathrm{cov}(\mathcal{I})]$ we shall need the following lemma, which is implicit in [5, p. 193] and [21, Lemma 2.7]. For the sake of completeness we give the simple proof. In the lemma we use the notation $\mathcal{I}|A$ to denote the ideal $\{B \in \mathcal{I}\colon B \subset A\}$.

**Lemma 2** *If $A, B \subset X$, $A, B \notin \mathcal{I}$, and there is a 0–1 set in $A \times B$ then* $\mathrm{cov}(\mathcal{I}|A) \leq \mathrm{non}(\mathcal{I}|B)$.

Proof. Let $H$ be a 0–1 set in $A \times B$. Then $H^y \in \mathcal{I}$ for every $y \in B$ and $B \setminus H_x \in \mathcal{I}$ for every $x \in A$. If $D \subset B$, $|D| = \mathrm{non}(\mathcal{I}|B)$, and $D \notin \mathcal{I}$ then $A = \bigcup_{y \in D} H^y$, since $x \in A \setminus \bigcup_{y \in D} H^y$ would imply $H_x \cap D = \emptyset$ contradicting $B \setminus H_x \in \mathcal{I}$. Thus $\mathrm{cov}(\mathcal{I}|A) \leq |D| = \mathrm{non}(\mathcal{I}|B)$. □

Now we turn to the proof of $E_{\mathcal{I}} \Longrightarrow [\mathrm{shr}(\mathcal{I}) \geq \mathrm{cov}(\mathcal{I})]$. By $E_{\mathcal{I}}$, there are sets $A, B \notin \mathcal{I}$ such that both of $A \times B$ and $A^c \times B^c$ contain 0–1 sets. If $A^c \notin \mathcal{I}$ and $B^c \notin \mathcal{I}$ then Lemma 2 gives $\mathrm{cov}(\mathcal{I}|A) \leq \mathrm{non}(\mathcal{I}|B) \leq \mathrm{shr}(\mathcal{I})$ and $\mathrm{cov}(\mathcal{I}|A^c) \leq \mathrm{non}(\mathcal{I}|B^c) \leq \mathrm{shr}(\mathcal{I})$, and hence

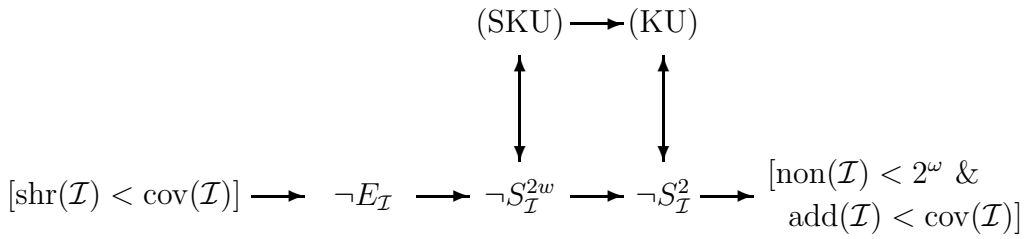
$$\mathrm{cov}(\mathcal{I}) \leq \max\{\mathrm{cov}(\mathcal{I}|A), \mathrm{cov}(\mathcal{I}|A^c)\} \leq \mathrm{shr}(\mathcal{I}).$$

On the other hand, if $A^c \in \mathcal{I}$ then there is a 0–1 set in $X \times B$ and thus $\mathrm{cov}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I}|B) \leq \mathrm{shr}(\mathcal{I})$. We have the same conclusion if $B^c \in \mathcal{I}$. □

For a semigroup $G$ of Borel functions from $X$ to $X$ and an ideal $\mathcal{I}$ on $X$ we say that $\mathcal{I}$ is *$G$-invariant* provided $g^{-1}(A) \in \mathcal{I}$ for every $g \in G$ and $A \in \mathcal{I}$; and $\mathcal{I}$ is *$G$-ergodic* when $X \setminus \bigcup_{g \in G} g^{-1}(A) \in \mathcal{I}$ for every $A \in \mathcal{B}_{\mathcal{I}} \setminus \mathcal{I}$. Clearly if $G$ is the group of rational translations in $\mathbb{R}$ then the ideals $\mathcal{N}$ and $\mathcal{M}$ are $G$-invariant and $G$-ergodic.

If we strengthen a bit the assumptions of the main part of Theorem 1 then we can get few more implications.

**Corollary 3** *Let $\mathcal{I}$ be a $\sigma$-ideal in the Polish space $X$ and suppose that there is a countable semigroup $G$ of Borel functions from $X$ to $X$ such that $\mathcal{I}$ is $G$-invariant and $G$-ergodic. Then we have the following implications:*




PROOF. The equivalence (SKU) $\iff \neg S^{2w}_{\mathcal{I}}$ is proved in [21, Lemma 2.8]. The same argument gives $\neg S^2_{\mathcal{I}} \iff$ (KU). The other implications follow from Theorem 1. □

We do not know if the hypothesis above also implies (SKU$^*$) $\iff \neg E_{\mathcal{I}}$ even in the presence of $(*)$. However, as we shall see later, this equivalence holds for $\mathcal{I} = \mathcal{N}$ and $\mathcal{I} = \mathcal{M}$.

The condition $(*)$ was only used in the proof of Theorem 1 in the implications (SKU$^*$) $\implies \neg E_{\mathcal{I}}$ and (SKU$^*$) $\implies$ (SKU). The following example shows that it is really needed in these arguments.

**Example 4** *Let $X = \mathbb{R}$ and let $\mathcal{I}$ be a prime ideal containing all sets of cardinality less than $2^\omega$. For this ideal* (SKU$^*$) *holds while all other statements of the diagram are false.*

PROOF. Since $\mathcal{I}$ is prime, we have $\mathcal{B}_{\mathcal{I}} = \mathcal{P}(\mathbb{R})$. Thus (SKU$^*$) holds. On the other hand, clearly $\operatorname{non}(\mathcal{I}) = 2^\omega$, so the statements of the diagram are false. □

We shall also investigate the following *Borel approximation property*:

(BAP) if $A \subset X^2$ is such that $A_x \in \mathcal{B}_{\mathcal{I}}$ and $A^y \in \mathcal{B}_{\mathcal{I}}$ for every $x, y \in X$, then there is a Borel set $B \subset X^2$ such that $(A \Delta B)_x \in \mathcal{I}$ for $\mathcal{I}$-a.e. $x \in X$ and $(A \Delta B)^y \in \mathcal{I}$ for $\mathcal{I}$-a.e. $y \in X$.

We will leave the following simple fact without a proof.

**Fact 5** *If $\mathcal{I}$ is a $\sigma$-ideal on $X$ then* (BAP) *is equivalent to the following statement.*

- *If $f\colon X^2 \to \mathbb{R}$ is such that $f_x$ and $f^y$ are $\mathcal{B}_{\mathcal{I}}$-measurable for every $x, y \in X$ then there is a Borel function $g\colon X^2 \to \mathbb{R}$ such that $f_x = g_x$ $\mathcal{I}$-a.e. and $f^y = g^y$ $\mathcal{I}$-a.e. for $\mathcal{I}$-almost every $x, y \in X$.*

*Also,* (BAP) *is equivalent to* (SAP) *for* $\mathcal{I} = \mathcal{N}$ *and for* $\mathcal{I} = \mathcal{M}$.

We state the equivalence to (SAP) only for the ideals $\mathcal{N}$ and $\mathcal{M}$ to avoid a problem of defining the meaning of $\mathcal{B}_\mathcal{I}$-measurability of a function from $X^2$ into $\mathbb{R}$. (A generalization of (SAP) to the category case is natural.)

Notice that it would not be reasonable to require in (BAP) the existence of a Borel set $B$ such that $(A\Delta B)_x \in \mathcal{I}$ and $(A\Delta B)^y \in \mathcal{I}$ for *every* $x$ and $y$. In fact, this stronger form of (BAP) is satisfied neither by $\mathcal{N}$ nor by $\mathcal{M}$ as the following simple example shows.

Let $H$ be a non-Borel subset of the Cantor set, and put $A = H \times [0,1]$. Then $A_x \in \mathcal{B}_\mathcal{I}$ and $A^y \in \mathcal{B}_\mathcal{I}$ hold for every $x, y \in [0,1]$ for both $\mathcal{I} = \mathcal{N}$ and $\mathcal{I} = \mathcal{M}$. Suppose there is a Borel set $B \subset [0,1]^2$ such that $(A\Delta B)_x \in \mathcal{N}$ for every $x \in [0,1]$. Then $\{x \in [0,1] \colon B_x \notin \mathcal{N}\} = H$ contradicting the fact that this set must be Borel whenever $B$ is Borel. (See e.g. [12, (16.1) Theorem, p. 94].) The same argument works for $\mathcal{I} = \mathcal{M}$.

In what follows we will prove that (SAP) is equivalent to (SKU$^*$) for $\mathcal{I} = \mathcal{N}$ and $\mathcal{I} = \mathcal{M}$. We start in this direction noticing the following simple fact, which will be left without a proof.

**Proposition 6** *Suppose that* $\{y \colon A^y \notin \mathcal{I}\} \in \mathcal{B}_\mathcal{I}$ *for every Borel set* $A \subset X^2$. *Then* (BAP) $\Longrightarrow$ (SKU$^*$).

Note that the assumption of Proposition 6 is not satisfied by every ideal. For example, if $\mathcal{I}$ equals the ideal of countable sets or that of the first category null sets than there is a Borel set $A \subset \mathbb{R}^2$ such that $\{y \colon A^y \notin \mathcal{I}\} \notin \mathcal{B}_\mathcal{I}$.

We do not know whether or not (BAP) implies (SKU$^*$) for every ideal.

# 3 Strong Fubini properties for measure

It was proved by C. Freiling in [5] (see also Laczkovich [16]) that (FP) is equivalent to $\neg S^2_\mathcal{N}$. Our aim here is to give a similar characterization of (SFP), by showing that it is equivalent to $\neg E_\mathcal{N}$ as well as to several other strong Fubini type properties, including (BAP) for $\mathcal{N}$. Recall that $\mathcal{B}_\mathcal{N}$ coincides with the $\sigma$-algebra of measurable subsets of $\mathbb{R}$.

Notice also that the chart implies that all the properties considered there are independent of the axioms of set theory ZFC. This is the case, as since the relations $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = 2^\omega$ and $\mathrm{shr}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$

are both consistent with ZFC: the first is a consequence of the Martin's Axiom, while the second holds in the random reals model. (For the proof of $\mathrm{shr}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$ see [18] or [11]. This was certainly already known to Kunen [14].)

**Theorem 7** *The following statements are equivalent to each other.*

(i) *If $f$ is a bounded function on $[0,1]^2$ such that the sections $f_x$ and $f^y$ are measurable for every $x, y \in [0,1]$ then the mapping $x \mapsto \int_0^1 f(x,y)\,dy$ is measurable on $[0,1]$.*

(ii) *The strong Fubini property* (SFP) *holds.*

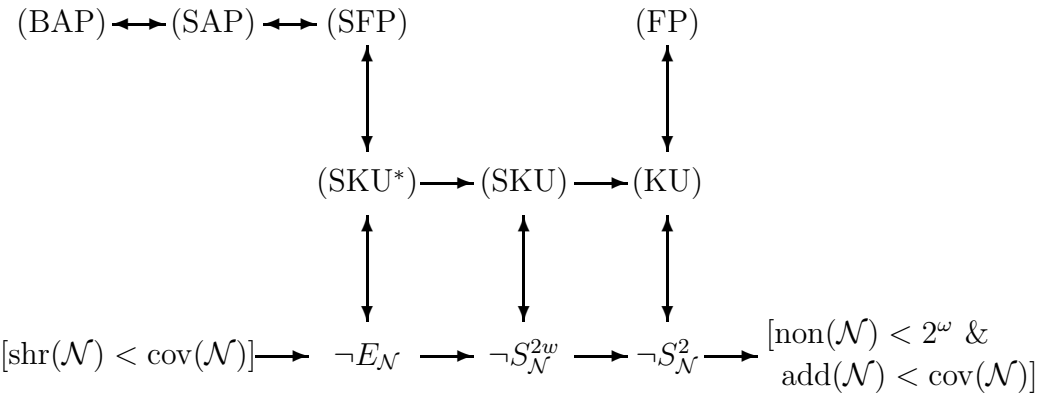
(iii) *$\mathcal{N}$ has property* (SAP).

(iv) *$\mathcal{N}$ has property* (BAP).

(v) *$\mathcal{N}$ has property* (SKU$^*$).

(vi) *$E_{\mathcal{N}}$ is false.*

*In particular, for $\mathcal{I} = \mathcal{N}$ we have the following relations.*




Proof. The implications in the chart follow from the main part of the theorem, from Freiling's result quoted above, as well as from Theorem 1 and Corollary 3.

In the course of the proof the sign $\int$ will abbreviate $\int_0^1$ unless another domain is indicated.

(i)$\Longrightarrow$(ii): Let $f$ be a bounded function on $[0,1]^2$ such that the sections $f_x$ and $f^y$ are measurable for every $x, y \in [0,1]$. By (i), for every $x, y \in [0,1]$ the mappings $x \mapsto \int f(x,y)\,dy$ and $y \mapsto \int f(x,y)\,dx$ are measurable. We need to prove that the integrals $I_1 = \int (\int f(x,y)\,dy)\,dx$ and $I_2 = \int (\int f(x,y)\,dx)\,dy$ are equal.

By way of contradiction assume that $I_1 \neq I_2$. Then, by Freiling's theorem [5], there exists a 0–1 set $A \subset [0,1]^2$. After removing from $A$ some vertical sections we obtain a set $B$ such that $B^y \in \mathcal{N}$ for every $y$, and $\{x \in [0,1]\colon [0,1] \setminus B_x \in \mathcal{N}\}$ is not measurable. But then the characteristic function of $B$ refutes (i).

(ii)$\Longrightarrow$(iii): Suppose (SFP). Let $f\colon [0,1]^2 \to \mathbb{R}$ be a function with measurable sections. We have to show that there is a measurable function $F\colon [0,1]^2 \to \mathbb{R}$ such that $F_x = f_x$ a.e. for every $x \in [0,1]$ and $F^y = f^y$ a.e. for every $y \in [0,1]$. In the proof we may assume that $f$ is bounded. For $x, y \in [0,1]$ put $G(x,y) = \int_0^y f(x,t)\,dt$. Then $G$ is well defined and measurable. Indeed, by (SFP), $G^y$ is measurable for every $y \in [0,1]$. Also, $G_x$ is continuous for every $x \in [0,1]$. Thus, by Lebesgue's theorem [19], $G$ is measurable. Let $H = \frac{\partial}{\partial y}G$ where $\frac{\partial}{\partial y}G$ exists, and $H = 0$ elsewhere. Then $H$ is measurable, and for every $x \in [0,1]$ we have $H_x(y) = f_x(y)$ for almost all $y \in [0,1]$. Then, by (SFP) and the Fubini theorem,

$$\begin{aligned}\int_0^b \left(\int_0^a f^y(x)\,dx\right) dy &= \int_0^a \left(\int_0^b f_x(y)\,dy\right) dx \\ &= \int_0^a \left(\int_0^b H_x(y)\,dy\right) dx \\ &= \int_0^b \left(\int_0^a H^y(x)\,dx\right) dy\end{aligned}$$

for every $a, b \in [0,1]$. Thus, for every $a \in [0,1]$ and almost every $y \in [0,1]$ we have $\int_0^a f^y(x)\,dx = \int_0^a H^y(x)\,dx$. So, for almost every $y \in [0,1]$ we have $\int_0^a f^y(x)\,dx = \int_0^a H^y(x)\,dx$ for every rational $a \in [0,1]$. Thus, by the continuity of the integral, for almost every $y \in [0,1]$ we have $\int_0^a f^y(x)\,dx = \int_0^a H^y(x)\,dx$ for every $a \in [0,1]$. Then, for almost every $y \in [0,1]$, that is, for $y'$s outside some null set $T \subset [0,1]$, we have $H^y(x) = f^y(x)$ for almost every $x \in [0,1]$. To get $F$ as desired define it as equal to $f$ on $[0,1] \times T$ and equal to $H$ otherwise.

(iii)$\Longrightarrow$(iv): This is obvious, and follows from Fact 5.

(iv)$\Longrightarrow$(v): If $A \subset [0,1]^2$ is Borel then the set $\{y \in [0,1]\colon A^y \notin \mathcal{N}\}$ is measurable by Fubini's theorem. Thus we may apply Proposition 6.

(v)$\Longrightarrow$(vi): This was proved in Theorem 1.

(vi)$\Longrightarrow$(i): Suppose that (i) is false. Then there exists a bounded function $f\colon [0,1]^2 \to \mathbb{R}$ such that the sections of $f$ are measurable, but the function $h(x) = \int f_x\,dy$, defined for $x \in [0,1]$, is not. Fix a real number $b$ such that the set $H = \{x \in [0,1]\colon h(x) < b\}$ is not measurable, and define $K = \{x \in [0,1]\colon h(x) \geq b\}$. Then $\lambda(H) + \lambda(K) > 1$, where $\lambda$ stands for Lebesgue outer measure. Thus, there exists a real number $a < b$ such that $\lambda(H_a) + \lambda(K) > 1$, where $H_a = \{x \in [0,1]\colon h(x) \leq a\}$.

Select measurable sets $M$ and $N$ such that $H_a \subset M$, $\lambda(H_a) = \lambda(M)$, $K \subset N$, and $\lambda(K) = \lambda(N)$. Then $\lambda(M) + \lambda(N) = \lambda(H_a) + \lambda(K) > 1$, and thus $P = M \cap N$ is of positive measure. Clearly, $\lambda(P) = \lambda(H_a \cap P) = \lambda(K \cap P)$. The measure space $\langle P, \frac{1}{\lambda(P)}(\lambda \restriction P)\rangle$ is isomorphic to $\langle [0,1], \lambda\rangle$. Let $\phi$ be an isomorphism from $[0,1]$ onto $P$, and put $f_1(x,y) = f(\phi(x), y)$ for $x, y \in [0,1]$. Then the sections of $f_1$ are measurable and both of the sets $U = \{x \in [0,1]\colon \int f_1(x,y)\,dy \leq a\}$ and $V = \{x \in [0,1]\colon \int f_1(x,y)\,dy \geq b\}$ have outer measure 1.

We shall denote by $\Omega$ the product space $[0,1]^\omega$ equipped with the product measure $\mu$ and define $\overline{\mu}(H) = \inf\{\mu(M)\colon H \subset M \text{ and } M \text{ is } \mu\text{-measurable}\}$ for every $H \subset \Omega$. For $x \in [0,1]$ and $v = \langle v_1, v_2, \ldots\rangle \in \Omega$ let

$$F(x,v) = \limsup_{n\to\infty} \frac{f_1(x,v_1) + \cdots + f_1(x,v_n)}{n}.$$

By the strong law of large numbers for every $x \in [0,1]$ the equation $F(x,v) = \int f_1(x,y)\,dy$ holds for $\mu$-a.e. $v \in \Omega$. Thus, $F_x$ is $\mu$-a.e. constant for every $x \in [0,1]$ since $F_x(v) = \int f_1(x,y)\,dy = F_x(v')$ holds for $\mu$-a.e. $v, v' \in \Omega$. Moreover, for $x \in U$ we have $F(x,v) \leq a$ for $\mu$-a.e. $v \in \Omega$, and for $x \in V$ we have $F(x,v) \geq b$ for $\mu$-a.e. $v \in \Omega$. Also, $F^v(x)$ is measurable for every $v \in \Omega$, being a lim sup of measurable functions $\frac{1}{n}[(f_1)^{v_1}(x) + \cdots + (f_1)^{v_n}(x)]$. Now put

$$G(u,v) = \limsup_{n\to\infty} \frac{F(u_1,v) + \cdots + F(u_n,v)}{n}$$

for every $u, v \in \Omega$. Then, by the strong law of large numbers, for every $v \in \Omega$ we have $G(u,v) = \int F^v dx$ for $\mu$-a.e. $u \in \Omega$. In particular, $G^v$ is $\mu$-a.e. constant for every $v \in \Omega$ since $G^v(u) = \int F^v dx = G^v(u')$ holds for $\mu$-a.e. $u, u' \in \Omega$. Also, $G_u(v)$ is $\mu$-a.e. constant for every $u \in \Omega$ since it is a lim sup of $\mu$-a.e. constant functions $\frac{1}{n}[F_{u_1}(v) + \cdots + F_{u_n}(v)]$.

Put $E = \{\langle u, v\rangle \in \Omega^2\colon G(u,v) \leq a\}$. Since every vertical and horizontal section of $G$ is $\mu$-a.e. constant, it follows that every vertical and

horizontal section of $E$ is either null or of full measure. Let us define $A_1 = \{u \in \Omega\colon \mu(E_u) = 1\}$, $A_2 = \{u \in \Omega\colon \mu(E_u) = 0\}$, $B_1 = \{v \in \Omega\colon \mu(E^v) = 1\}$, and $B_2 = \{v \in \Omega\colon \mu(E^v) = 0\}$. By [6, 254 L Theorem, p. 249], we have $\overline{\mu}(U^\omega) = \overline{\mu}(V^\omega) = 1$. If $u \in U^\omega$, then $G(u, v) \leq a$ for $\mu$-a.e. $v$; that is, $\mu(E_u) = 1$. Similarly, if $u \in V^\omega$ then $\mu(E_u) = 0$. Therefore $U^\omega \subset A_1$ and $V^\omega \subset A_2$ and thus $\overline{\mu}(A_1) = \overline{\mu}(A_2) = 1$.

Suppose $\overline{\mu}(B_2) > 0$. Then $E \cap (A_1 \times B_2)$ is a 0–1 set in $A_1 \times B_2$ and $(A_2 \times B_1) \setminus E$ is a 0–1 set in $A_2 \times B_1$. On the other hand, if $\overline{\mu}(B_1) > 0$, then $(A_2 \times B_1) \setminus E$ is a 0–1 set in $A_2 \times B_1$ and $E \cap (A_1 \times B_2)$ is a 0–1 set in $A_1 \times B_2$. We obtain that the statement of $E_{\mathcal{N}}$ is true apart from the fact that the sets $A_i$ and $B_i$ are in $\Omega$ instead of $[0, 1]$. However, the measure spaces $\Omega$ and $[0, 1]$ are isomorphic, therefore we can find sets in $[0, 1]$ with the same properties. Thus $E_{\mathcal{N}}$ holds, which completes the proof. □

# 4 The category case

In this section we will prove that the category analogue of Theorem 7 is true with one obvious modification: the integral conditions (FP) and (SFP) have no meaning in this case.

Recall that $\mathcal{B}_{\mathcal{M}}$ coincides with the $\sigma$-algebra of sets $H \subset \mathbb{R}$ having the Baire property, and that (SAP) is equivalent to the following statement: if $A \subset [0, 1]^2$ is such that $A_x$ and $A^y$ have the Baire property for every $x, y \in [0, 1]$ then there is a set $B \subset [0, 1]^2$ having the Baire property such that $(A \Delta B)_x \in \mathcal{M}$ and $(A \Delta B)^y \in \mathcal{M}$ for every $x, y \in [0, 1]$.

Notice also that the chart below implies that all the properties considered there are independent of the axioms of set theory ZFC. This is the case, as since the relations $\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = 2^\omega$ and $\mathrm{shr}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ are both consistent with ZFC: the first is a consequence of the Martin's Axiom, while the second holds in the Cohen model. (For the proof of $\mathrm{shr}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ see [13] or [11].)

**Theorem 8** *The following statements are equivalent to each other.*

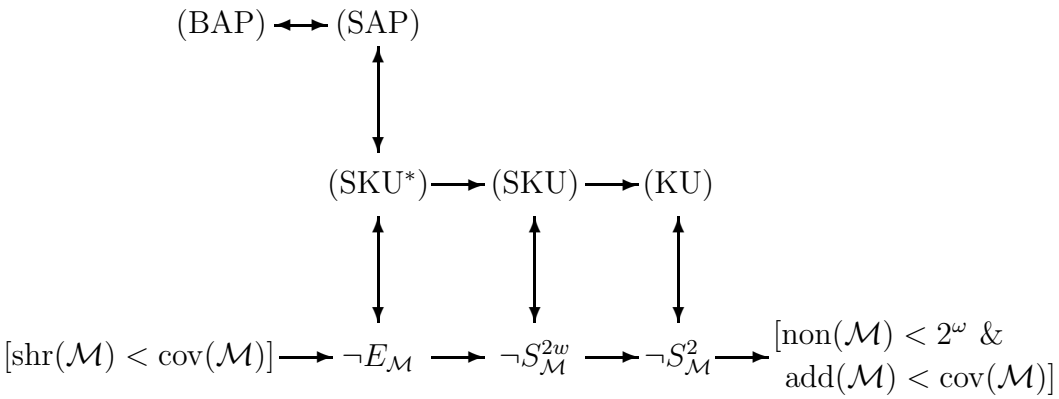
(i) $\mathcal{M}$ *has property* (SAP).

(ii) $\mathcal{M}$ *has property* (BAP).

(iii) $\mathcal{M}$ *has property* (SKU$^*$).

(iv) $E_{\mathcal{M}}$ *is false.*

*In particular, for* $\mathcal{I} = \mathcal{M}$ *we have the following relations.*




In the proof of the theorem we will need the following lemma.

**Lemma 9** *If* $A \subset [0,1]^2$ *is Borel then the set* $\{y \in [0,1]\colon A^y \notin \mathcal{M}\}$ *has the Baire property.*

Proof. There is an open set $G \subset [0,1]^2$ such that $M = G\Delta A$ is meager. Then the set $\{y \in [0,1]\colon G^y \notin \mathcal{M}\} = \{y \in [0,1]\colon G^y \neq \emptyset\}$ is open, and the set $\{y \in [0,1]\colon M^y \notin \mathcal{M}\}$ is meager by the Kuratowski-Ulam theorem. Since

$$\{y \in [0,1]\colon A^y \notin \mathcal{M}\}\Delta\{y \in [0,1]\colon G^y \notin \mathcal{M}\} \subset \{y \in [0,1]\colon M^y \notin \mathcal{M}\},$$

it follows that $\{y \in [0,1]\colon A^y \notin \mathcal{M}\}$ has the Baire property. □

Proof of Theorem 8. The implications in the chart follow from the main part of the theorem, Theorem 1, and Corollary 3.

(i)$\Longleftrightarrow$(ii): This follows from Fact 5.

(ii)$\Longrightarrow$(iii): This follows from Lemma 9 and Proposition 6.

(iii)$\Longrightarrow$(ii): Suppose that the sections of a set $A \subset [0,1]^2$ have the Baire property. It is clear from (SKU*) that for every interval $J$ the set $H_J = \{y \in [0,1]\colon J \setminus A^y \in \mathcal{M}\}$ has the Baire property. Let

$$E = \bigcup\{J \times H_J\colon J = (p,q) \subset [0,1] \text{ for some rational numbers } p < q\}.$$

Then $E$ has the Baire property, and it is easy to see that for $F = E\Delta A$ we have $F^y = E^y\Delta A^y \in \mathcal{M}$ for every $y \in [0,1]$. It is also clear that $F_x = E_x\Delta A_x$ has the Baire property for every $x \in [0,1]$. Since $\mathcal{M}$ has property (SKU$^*$), by Theorem 1 it also has property (KU) and so $U := \{x\colon F_x \notin \mathcal{M}\} \in \mathcal{M}$. Let $H = E \setminus (U \times [0,1])$. Then $H$ has the Baire property and we have $(A\Delta H)_x \in \mathcal{M}$ and $(A\Delta H)^y \in \mathcal{M}$ for every $x, y \in [0,1]$.

(iii)$\Longrightarrow$(iv): This was proved in Theorem 1.

(iv)$\Longrightarrow$(iii): We prove that if (SKU$^*$) is false then $E_{\mathcal{M}}$ is true. We will work in $\mathbb{R}$ instead of $[0,1]$.

Suppose that there exists a set $A \subset \mathbb{R}^2$ such that $A_x$ and $A^y$ have the Baire property for every $x, y \in \mathbb{R}$, and $H = \{y \in \mathbb{R}\colon A^y \notin \mathcal{M}\} \notin \mathcal{B}_{\mathcal{M}}$. By Banach's theorem, there exists a nonempty open set $G \subset \mathbb{R}$ such that $H \setminus G \in \mathcal{M}$ and $H$ is of second category in every nonempty open subset of $G$. (See [15, §10, V. pp. 83–85].) Since $H \notin \mathcal{B}_{\mathcal{M}}$, the set $G \setminus H$ is not meager, and thus we can select an open interval $I \subset G$ such that $G \setminus H$ is of second category in every nonempty open subinterval of $I$. We conclude that both $H$ and $\mathbb{R} \setminus H$ are of second category in every nonempty open subinterval of $I$. Let $\phi$ be a homeomorphism from $I$ onto $\mathbb{R}$, and put

$$C = \{\langle x, \phi(y)\rangle\colon \langle x, y\rangle \in A \cap (\mathbb{R} \times I)\}.$$

Then $C_x$ and $C^y$ have the Baire property for every $x, y \in \mathbb{R}$, and the set $K = \{y \in \mathbb{R}\colon C^y \notin \mathcal{M}\}$ has the property that both $K$ and $\mathbb{R} \setminus K$ are of second category in every subinterval of $\mathbb{R}$. Let

$$D = \{\langle x + r, y\rangle\colon \langle x, y\rangle \in C \;\&\; r \in \mathbb{Q}\}.$$

Then $D^y \in \mathcal{M}$ if $y \notin K$ and $\mathbb{R} \setminus D^y \in \mathcal{M}$ if $y \in K$. Also, the set $D_x = \bigcup_{r\in\mathbb{Q}} C_{x-r}$ has the Baire property for every $x \in \mathbb{R}$. Let us put $U = \{x \in \mathbb{R}\colon \mathbb{R} \setminus D_x \in \mathcal{M}\}$. We shall distinguish between two cases.

Case I: $\mathbb{R} \setminus U \in \mathcal{M}$. Then the set $D \cap [U \times (\mathbb{R} \setminus K)]$ is a 0–1 set in $U \times (\mathbb{R} \setminus K)$, and $(\mathbb{R} \setminus U) \times K$ is a 0–1 set in $(\mathbb{R} \setminus U) \times K$. That is, $E_{\mathcal{M}}$ holds in this case.

Case II: $\mathbb{R} \setminus U \notin \mathcal{M}$. If $x \in \mathbb{R} \setminus U$ then $\mathbb{R} \setminus D_x \notin \mathcal{M}$ and, as $D_x$ has the Baire property, there exists an open interval $J$ with rational endpoints such that $D_x \cap J \in \mathcal{M}$. Since $\mathbb{R} \setminus U \notin \mathcal{M}$, it follows that we can fix an open interval $J$ with rational endpoints such that the set $V = \{x \in \mathbb{R}\colon D_x \cap J \in \mathcal{M}\}$ is of second category. Let $\psi$ be a homeomorphism from $J$ onto $\mathbb{R}$, and put

$$E = \{\langle x, \psi(y)\rangle\colon \langle x, y\rangle \in D \cap (\mathbb{R} \times J)\}.$$

Then $E^y \in \mathcal{M}$ if $y \notin \psi[K \cap J]$ and $\mathbb{R} \setminus E^y \in \mathcal{M}$ if $y \in \psi[K \cap J]$. Also, the set $W = \{x \in \mathbb{R}\colon E_x \in \mathcal{M}\}$ is of second category. Now we define

$$F = \{\langle x, y + r\rangle\colon \langle x, y\rangle \in E \ \&\ r \in \mathbb{Q}\}.$$

Then either $F^y \in \mathcal{M}$ or $\mathbb{R} \setminus F^y \in \mathcal{M}$ for every $y$. If $Y = \{y\colon \mathbb{R} \setminus F^y \in \mathcal{M}\}$ then $Y \notin \mathcal{M}$, as $\psi[K \cap J] \subset Y$. Also, we have $F_x \in \mathcal{M}$ if $x \in W$ and $\mathbb{R} \setminus F_x \in \mathcal{M}$ if $x \notin W$. Thus $(W \times Y) \setminus F$ is a 0–1 set in $W \times Y$ and $[(\mathbb{R} \setminus W) \times (\mathbb{R} \setminus Y)] \cap F$ is a 0–1 set in $(\mathbb{R} \setminus W) \times (\mathbb{R} \setminus Y)$. Therefore, $E_{\mathcal{M}}$ holds in this case as well. □

# 5 More on condition $E_{\mathcal{I}}$

The definition of $E_{\mathcal{I}}$ does not involve the topology of the space $X$; it is meaningful for any ideal of subsets of a set $X$. We begin with an equivalent form of $E_{\mathcal{I}}$. We shall use the notation $A^c = X \setminus A$.

**Proposition 10** *For every ideal $\mathcal{I}$ the statement $E_{\mathcal{I}}$ is equivalent to the following:*

$E^*_{\mathcal{I}}$ : *Either there exists a set $A \subset X$ such that $A, A^c \notin \mathcal{I}$ and there is a 0–1 set in $A \times A^c$, or there exists a set $A \subset X$ such that there are 0–1 sets both in $A \times A$ and in $A^c \times A^c$.*

Proof. Suppose $E^*_{\mathcal{I}}$. If $H$ is a 0–1 set in $A \times A^c$ where $A, A^c \notin \mathcal{I}$, then $E_{\mathcal{I}}$ is satisfied with $B = A^c$. If there are 0–1 sets both in $A \times A$ and in $A^c \times A^c$ then we may assume $A \notin \mathcal{I}$, since otherwise we replace $A$ by $A^c$. Then $E_{\mathcal{I}}$ is satisfied with $B = A$.

Next suppose $E_{\mathcal{I}}$. Suppose that $A, B \notin \mathcal{I}$, $H$ is a 0–1 set in $A \times B$ and $K$ is a 0–1 set in $A^c \times B^c$. If $A \setminus B \in \mathcal{I}$ and $B \setminus A \in \mathcal{I}$ then there are 0–1 sets both in $A \times A$ and in $A^c \times A^c$, and then $E^*_{\mathcal{I}}$ holds.

Therefore we may assume that at least one of the sets $A \setminus B$ and $B \setminus A$ does not belong to $\mathcal{I}$. By symmetry, we may assume that $D := A \setminus B \notin \mathcal{I}$. We also have $D^c \notin \mathcal{I}$, since $D^c \supset B$. We prove that there is a 0–1 set in $D \times D^c$. Indeed, we have $D^c = A^c \cup B$, and thus

$$D \times D^c = (D \times A^c) \cup (D \times B).$$

Since $(D \times A^c) \subset (B^c \times A^c)$ and $(D \times B) \subset (A \times B)$, it follows that there are 0–1 sets both in $D \times A^c$ and in $D \times B$, and then there is also one in $D \times D^c$. □

In the sequel we shall assume that $X$ is an Abelian group and $\mathcal{I}$ is a translation invariant ideal in $X$. The paper [17] investigated the logical connections between several statements including $S^2_{\mathcal{I}}$ and $S^{2w}_{\mathcal{I}}$ and the following three conditions.

$S^1_{\mathcal{I}}$ : There exists a set $A \subset X$ such that $A \notin \mathcal{I}$, $X \setminus A \notin \mathcal{I}$, and $(A+h) \setminus A \in \mathcal{I}$ for every $h \in X$.

$S^{1s}_{\mathcal{I}}$ : There exists an $f\colon X \to X$ such that $\{x \in X\colon f(x+h) - f(x) \neq 0\} \in \mathcal{I}$ for every $h \in X$, and $f^{-1}(\{y\}) \in \mathcal{I}$ for every $y \in X$.

$D_{\mathcal{I}}$ : There exists a partition $\{A, B\}$ of $X$ such that $\mathrm{cov}(\mathcal{I}|A) \leq \mathrm{non}(\mathcal{I}|B)$ and $\mathrm{cov}(\mathcal{I}|B) \leq \mathrm{non}(\mathcal{I}|A)$.

In [17, Theorem 2] it was shown that if $\mathcal{I}$ is a $\sigma$-ideal and $|X|$ is less than the first (2-valued) measurable cardinal then the following implications hold.

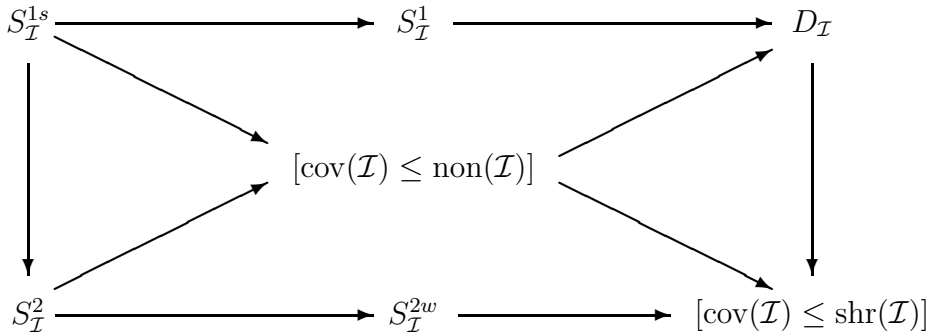



Diagram 1.

In the next theorem we try to locate the position of $E_{\mathcal{I}}$ in this diagram. As we shall see, it must be somewhere in the middle column.

**Theorem 11** *Let $\mathcal{I}$ be an invariant ideal in the Abelian group $X$. Then we have the following implications:*

$$[S^1_{\mathcal{I}} \text{ or } S^{2w}_{\mathcal{I}}] \implies E_{\mathcal{I}} \implies [S^1_{\mathcal{I}} \text{ or } (\mathrm{cov}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I})) \text{ or } S^{2w}_{\mathcal{I}}]\,.$$

In the prove of the theorem we will use the following lemma.

**Lemma 12** *Let $\mathcal{I}$ be an invariant ideal in an Abelian group $X$. If there is a partition $\{A, B\}$ of $X$ such that* $\operatorname{cov}(\mathcal{I}|A) \leq \operatorname{non}(\mathcal{I}|A)$ *and* $\operatorname{cov}(\mathcal{I}|B) \leq \operatorname{non}(\mathcal{I}|B)$ *then* $\operatorname{cov}(\mathcal{I}) \leq \operatorname{non}(\mathcal{I})$.

Proof. If $\operatorname{non}(\mathcal{I}|A) = \operatorname{non}(\mathcal{I}|B) = \kappa$ then $\operatorname{non}(\mathcal{I}) = \kappa$ and

$$\operatorname{cov}(\mathcal{I}) = \max\{\operatorname{cov}(\mathcal{I}|A), \operatorname{cov}(\mathcal{I}|B)\} \leq \kappa.$$

Therefore we may assume that $\operatorname{non}(\mathcal{I}|A) \neq \operatorname{non}(\mathcal{I}|B)$. By symmetry we may suppose $\operatorname{non}(\mathcal{I}|A) < \operatorname{non}(\mathcal{I}|B)$. Let $\operatorname{non}(\mathcal{I}|A) = \lambda$ and let $H \subset A$ be a set with $|H| = \lambda$ and $H \notin \mathcal{I}$. Then, for every $x \in X$ we have $H + x \notin \mathcal{I}$ and $|H + x| = \lambda < \operatorname{non}(\mathcal{I}|B)$. Thus $H + x$ cannot be a subset of $B$. Therefore $(H + x) \cap A \neq \emptyset$ for every $x$. This implies that for every $x$ there is an $h \in H$ such that $h + x \in A$; that is, $x \in A - h$. Thus $X = \bigcup_{h \in H}(A - h)$.

Since $A - h$ can be covered by $\operatorname{cov}(\mathcal{I}|A) \leq \lambda$ elements of $\mathcal{I}$ and $|H| = \lambda$, it follows that $X$ also can be covered by $\lambda$ elements of $\mathcal{I}$. In other words, $\operatorname{cov}(\mathcal{I}) \leq \lambda = \operatorname{non}(\mathcal{I})$. □

Proof of Theorem 11. $S^{2w}_{\mathcal{I}} \Longrightarrow E_{\mathcal{I}}$ was proved in Theorem 1.

$S^1_{\mathcal{I}} \Longrightarrow E_{\mathcal{I}}$ : Suppose $S^1_{\mathcal{I}}$, and let $A \subset X$ be a set such that $A \notin \mathcal{I}$, $X \setminus A \notin \mathcal{I}$, and $(A + h) \setminus A \in \mathcal{I}$ for every $h \in X$. We claim that there is a 0–1 set in $A \times A^c$. Indeed, let $H = \{\langle x, y\rangle \in A \times A^c \colon y \notin (A - x) \setminus A\}$. For every $x \in A$ we have $A^c \setminus H_x = (A - x) \setminus A \in \mathcal{I}$. If $y \in A^c$ then we have

$$\begin{aligned}\langle x, y\rangle \in H &\Longrightarrow y \notin A - x \Longrightarrow x \notin A - y \\ &\Longrightarrow x \in A \setminus (A - y) = [(A + y) \setminus A] - y \in \mathcal{I},\end{aligned}$$

and thus $H^y \in \mathcal{I}$ for every $y \in A^c$. Therefore, $H$ is a 0–1 set in $A \times A^c$.

Now we turn to the proof of the second implication of Theorem 11. Suppose that $E_{\mathcal{I}}$ holds. By Proposition 10, one of the following statements is true: (i) there exists a set $A \subset X$ such that $A, A^c \notin \mathcal{I}$ and there is a 0–1 set in $A \times A^c$, or (ii) there exists a set $A \subset X$ such that there are 0–1 sets both in $A \times A$ and in $A^c \times A^c$.

Suppose (ii). If $A^c \in \mathcal{I}$ then there is a 0–1 set in $X \times X$, and thus $S^{2w}_{\mathcal{I}}$ (even $S^2_{\mathcal{I}}$) holds. We have the same conclusion if $A \in \mathcal{I}$. If $A, X \setminus A \notin \mathcal{I}$ then, by Lemma 2, we obtain $\operatorname{cov}(\mathcal{I}|A) \leq \operatorname{non}(\mathcal{I}|A)$ and $\operatorname{cov}(\mathcal{I}|A^c) \leq \operatorname{non}(\mathcal{I}|A^c)$. Therefore, by Lemma 12, we get $\operatorname{cov}(\mathcal{I}) \leq \operatorname{non}(\mathcal{I})$.

Next suppose (i). Then there exists a set $A \subset X$ such that $A, A^c \notin \mathcal{I}$, and there is a 0–1 set in $A \times A^c$. If $(A+h) \setminus A \in \mathcal{I}$ for every $h \in X$ then we have $S^1_{\mathcal{I}}$. Therefore we may assume that there is an $h \in X$ such that $B := (A+h) \setminus A = (A+h) \cap A^c \notin \mathcal{I}$. Since

$$B \times X \subset [(A+h) \times A^c] \cup [A^c \times A]$$

and there are 0–1 sets both in $(A+h) \times A^c$ and $A^c \times A$, it follows that there is a 0–1 set in $B \times X$, and thus $S^{2w}_{\mathcal{I}}$ holds. □

We conclude with another simple observation.

**Proposition 13** *For every invariant ideal $\mathcal{I}$ on an Abelian group $X$ we have $S^{2w}_{\mathcal{I}} \Longrightarrow D_{\mathcal{I}}$ and $E_{\mathcal{I}} \Longrightarrow D_{\mathcal{I}}$.*

Proof. Suppose that $S^{2w}_{\mathcal{I}}$ holds. Then there exists a set $A \subset X$ such that $A \notin \mathcal{I}$, and there is a 0–1 set in $A \times X$. If $A^c \in \mathcal{I}$ then $S^2_{\mathcal{I}}$ holds, which implies $D_{\mathcal{I}}$ by Diagram 1. Therefore we may assume $A^c \notin \mathcal{I}$.

Since there is a 0–1 set in $A \times X$, there is one also in $A \times A^c$ and hence in $A^c \times A$ as well. Then, by Lemma 2, it follows that $\mathrm{cov}(\mathcal{I}|A) \leq \mathrm{non}(\mathcal{I}|A^c)$ and $\mathrm{cov}(\mathcal{I}|A^c) \leq \mathrm{non}(\mathcal{I}|A)$; that is, $D_{\mathcal{I}}$ holds.

Now each of the statements $S^1_{\mathcal{I}}$, $\mathrm{cov}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I})$, and $S^{2w}_{\mathcal{I}}$ implies $D_{\mathcal{I}}$. (For the first two of these implications we again refer to Diagram 1.) Therefore, by Theorem 11, we have $E_{\mathcal{I}} \Longrightarrow D_{\mathcal{I}}$. □

We remark that the implication $S^{2w}_{\mathcal{I}} \Longrightarrow D_{\mathcal{I}}$ was not noted in [17]. It would be interesting to decide whether or not there are other implications that can be inserted into Diagram 1.

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