Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}$ and its consequences

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Abstract

In the paper we formulate a Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}$, which holds in the iterated perfect set model, and show that it implies easily the following facts.

(a) For every $S \subset \mathbb{R}$ of cardinality continuum there exists a uniformly continuous function $g\colon \mathbb{R} \to \mathbb{R}$ with $g[S] = [0,1]$.

(b) If $S \subset \mathbb{R}$ is either perfectly meager or universally null then S has cardinality less than $\mathfrak{c}$.

(c) $\mathrm{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$, i.e., the cofinality of the measure ideal $\mathcal{N}$ is ω_1.

(d) For every uniformly bounded sequence $\langle f_n \in \mathbb{R}^{\mathbb{R}} \rangle_{n<\omega}$ of Borel functions there are the sequences: $\langle P_\xi \subset \mathbb{R} \colon \xi < \omega_1 \rangle$ of compact sets and $\langle W_\xi \in [\omega]^\omega \colon \xi < \omega_1 \rangle$ such that $\mathbb{R} = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:

  $\langle f_n \restriction P_\xi \rangle_{n \in W_\xi}$ is a monotone uniformly convergent sequence of uniformly continuous functions.

(e) Total failure of Martin's Axiom: $\mathfrak{c} > \omega_1$ and for every nontrivial ccc forcing $\mathbb{P}$ there exists ω_1-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

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1 Axiom CPA$_{\rm cube}$ and other preliminaries

Our set theoretic terminology is standard and follows that of [4]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. A Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use the term *Polish space* for a complete separable metric space `without isolated points`.

For a Polish space X, the symbol $\mathrm{Perf}(X)$ will stand for the collection of all subsets of X homeomorphic to a Cantor set $\mathfrak{C}$. We will consider $\mathrm{Perf}(X)$ as ordered by inclusion. Thus, a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *dense in* $\mathrm{Perf}(X)$ provided for every $P \in \mathrm{Perf}(X)$ there exists a $Q \in \mathcal{E}$ such that $Q \subset P$.

Axiom CPA$_{\rm cube}$ is of the form

> $\mathfrak{c} = \omega_2$ and if $\mathcal{E} \subset \mathrm{Perf}(X)$ is *appropriately* dense in $\mathrm{Perf}(X)$ then $|X \setminus \bigcup \mathcal{E}_0| < \mathfrak{c}$ for some $\mathcal{E}_0 \in [\mathcal{E}]^{\leq \omega_1}$.

If the word "appropriately" in the above is ignored, then it implies the following statement.

Naïve-CPA**:** If $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$ then $|X \setminus \bigcup \mathcal{E}| < \mathfrak{c}$.

It is a very good candidate for our axiom in the sense that it implies all the properties we are interested in. It has, however, one major flaw — *it is false!* This is the case since $S \subset X \setminus \bigcup \mathcal{E}$ for some dense set $\mathcal{E}$ in $\mathrm{Perf}(X)$ provided

> for each $P \in \mathrm{Perf}(X)$ there is a $Q \in \mathrm{Perf}(X)$ such that $Q \subset P \setminus S$.

This means that the family $\mathcal{G}$ of all sets of the form $X \setminus \bigcup \mathcal{E}$, where $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, coincides with the σ-ideal s_0 of Marczewski's sets, since $\mathcal{G}$ is clearly hereditary. Thus we have

$$s_0 = \left\{ X \setminus \bigcup \mathcal{E} \colon \ \mathcal{E} \text{ is dense in } \mathrm{Perf}(X) \right\}. \tag{1}$$

However, it is well known (see e.g. [17, thm. 5.10]) that there are s_0-sets of cardinality $\mathfrak{c}$. Thus, our Naïve-CPA "axiom" cannot be consistent with ZFC.

In order to formulate the real axiom CPA$_{\rm cube}$ we need the following terminology and notation. A subset C of a product $\mathfrak{C}^\eta$ of the Cantor set is said to be a *perfect cube* if $C = \prod_{n \in \eta} C_n$, where $C_n \in \mathrm{Perf}(\mathfrak{C})$ for each n. For a fixed Polish space X let $\mathcal{F}_{\rm cube}$ stand for the family of all continuous injections from a perfect cube $C \subset \mathfrak{C}^\omega$ onto a set P from $\mathrm{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\rm cube}$ from C onto P as a coordinate system imposed on P. We say that $P \in \mathrm{Perf}(X)$ is a *cube* if it is determined by an (implicitly given) witness function $f \in \mathcal{F}_{\rm cube}$ onto P, and Q is a *subcube of a cube* $P \in \mathrm{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\rm cube}$ is a witness function for P and $C \subset \mathrm{dom}(f) \subset \mathfrak{C}^\omega$ is a perfect cube. Here and in what follows the symbol $\mathrm{dom}(f)$ stands for the domain of f.

We say that a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *cube dense* in $\mathrm{Perf}(X)$ provided every cube $P \in \mathrm{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense provided

$$\forall f \in \mathcal{F}_{\rm cube}\ \exists g \in \mathcal{F}_{\rm cube}\ (g \subset f \ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{2}$$

It is easy to see that the notion of cube density is a generalization of the notion of density as defined in the first paragraph of this section:

$$\text{if } \mathcal{E} \text{ is cube dense in } \mathrm{Perf}(X) \text{ then } \mathcal{E} \text{ is dense in } \mathrm{Perf}(X). \tag{3}$$

On the other hand, the converse implication is not true, as shown by the following simple example.

Example 1.1 *Let $X = \mathfrak{C} \times \mathfrak{C}$ and let $\mathcal{E}$ be the family of all $P \in \mathrm{Perf}(X)$ such that either*

- *all vertical sections $P_x = \{y \in \mathfrak{C} \colon \langle x, y \rangle \in P\}$ of P are countable, or*
- *all horizontal sections $P^y = \{x \in \mathfrak{C} \colon \langle x, y \rangle \in P\}$ of P are countable.*

Then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, but it is not cube dense in $\mathrm{Perf}(X)$.

PROOF. To see that $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$ let $R \in \mathrm{Perf}(X)$. We need to find a $P \subset R$ with $P \in \mathcal{E}$. Clearly at least one of the projections $\pi_0(R)$ or $\pi_1(R)$ is uncountable. Assume that $\pi_0(R)$ is uncountable and let $p \colon \pi_0(R) \to \mathfrak{C}$ be a Borel function. (For example, if p is defined by $p(x) = \min R_x$ then $p \subset R$ is Baire class 1.) So, there is a $Q \in \mathrm{Perf}(\mathfrak{C})$ such that $p \restriction Q$ is continuous. In particular, $p \restriction Q$ (identified with its graph) is a closed subset of $X = \mathfrak{C} \times \mathfrak{C}$. So $P = p \restriction Q \in \mathcal{E}$ is a subset of R.

To see that $\mathcal{E}$ is not $\mathcal{F}_{\rm cube}$-dense in $\mathrm{Perf}(X)$ it is enough to notice that $P = X = \mathfrak{C} \times \mathfrak{C}$ considered as a cube, where the second coordinate is identified with $\mathfrak{C}^{\omega \setminus \{0\}}$, has no subcube in $\mathcal{E}$. More formally, let h be a homeomorphism from $\mathfrak{C}$ onto $\mathfrak{C}^{\omega \setminus \{0\}}$, let $g \colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}^\omega = \mathfrak{C} \times \mathfrak{C}^{\omega \setminus \{0\}}$ be given by $g(x, y) = \langle x, h(y) \rangle$, and let $f = g^{-1} \colon \mathfrak{C}^\omega \to \mathfrak{C} \times \mathfrak{C}$ be the coordinate function making $\mathfrak{C} \times \mathfrak{C} = \mathrm{range}(f)$ a cube. Then $\mathrm{range}(f)$ does not contain a subcube from $\mathcal{E}$. ■

With these notions in hand we are ready to formulate our axiom CPA$_{\rm cube}$.

CPA$_{\rm cube}$: $\mathfrak{c} = \omega_2$ and for every Polish space X and every cube dense family $\mathcal{E} \subset \mathrm{Perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \le \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$.

The proof that CPA$_{\rm cube}$ holds in the iterated perfect set model can be found in [6] and [7].

It is also worth noticing that in order to check that $\mathcal{E}$ is cube dense it is enough to consider in condition (2) only functions f defined on the entire space $\mathfrak{C}^\omega$, that is

Fact 1.2 *$\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense if and only if*

$$\forall f \in \mathcal{F}_{\rm cube},\ \mathrm{dom}(f) = \mathfrak{C}^\omega,\ \exists g \in \mathcal{F}_{\rm cube}\ (g \subset f \ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{4}$$

PROOF. To see this, let Φ be the family of all bijections $h = \langle h_n \rangle_{n<\omega}$ between perfect subcubes $\prod_{n\in\omega} D_n$ and $\prod_{n\in\omega} C_n$ of $\mathfrak{C}^\omega$ such that each h_n is a homeomorphism between D_n and C_n. Then

$$f \circ h \in \mathcal{F}_{\rm cube} \text{ for every } f \in \mathcal{F}_{\rm cube} \text{ and } h \in \Phi \text{ with } \mathrm{range}(h) \subset \mathrm{dom}(f). \tag{5}$$

Now take an arbitrary $f\colon C\to X$ from $\mathcal{F}_{\rm cube}$ and choose an $h\in\Phi$ mapping $\mathfrak{C}^\omega$ onto C. Then $\hat{f}=f\circ h\in\mathcal{F}_{\rm cube}$ maps $\mathfrak{C}^\omega$ into X and, using (4), we can find $\hat{g}\in\mathcal{F}_{\rm cube}$ such that $\hat{g}\subset\hat{f}$ and $\mathrm{range}(\hat{g})\in\mathcal{E}$. Then $g=f\restriction h[\mathrm{dom}(\hat{g})]$ satisfies condition (2). ■

Next, let us consider

$$\begin{aligned} s_0^{\rm cube} &= \left\{X\setminus\bigcup\mathcal{E}\colon\ \mathcal{E}\text{ is cube dense in }\mathrm{Perf}(X)\right\} \\ &= \{S\subset X\colon\forall\text{ cube }P\in\mathrm{Perf}(X)\ \exists\text{ subcube }Q\subset P\setminus S\}\,. \end{aligned} \tag{6}$$

Notice that

Fact 1.3 $[X]^{<\mathfrak{c}}\subset s_0^{\rm cube}\subset s_0$ *for every Polish space X.*

An easy proof of this Fact 1.3 can be found in [7]. It can be also shown in ZFC that $s_0^{\rm cube}$ forms a σ-ideal. However, neither of these facts will be used in the sequel. On the other hand we will be interested in the following proposition.

Proposition 1.4 *If* CPA$_{\rm cube}$ *holds then $s_0^{\rm cube}=[X]^{\le\omega_1}$.*

Proof. It is obvious that CPA$_{\rm cube}$ implies $s_0^{\rm cube}\subset[X]^{<\mathfrak{c}}$. We will not be interested in the other inclusion, though it follows immediately from Fact 1.3. ■

Remark 1.5 $s_0^{\rm cube}\neq[X]^{\le\omega_1}$ in a model obtained by adding Sacks numbers side-by-side. In particular CPA$_{\rm cube}$ is false in this model.

Proof. This follows from the fact that $s_0^{\rm cube}=[X]^{\le\omega_1}$ implies the property (A) (see Corollary 2.2) while it is false in the model mentioned above, as noticed by Miller in [16, p. 581]. (In this model the set X of all Sacks generic numbers cannot be mapped continuously onto $[0,1]$.) ■

2 Continuous images of sets of cardinality $\mathfrak{c}$

An important quality of the ideal $s_0^{\rm cube}$, and so the power of the assumption $s_0^{\rm cube}=[X]^{<\mathfrak{c}}$, is well depicted by the following fact.

Proposition 2.1 *If X is a Polish space and $S\subset X$ does not belong to $s_0^{\rm cube}$ then there exist a $T\in[S]^{\mathfrak{c}}$ and a uniformly continuous function h from T onto $\mathfrak{C}$.*

Proof. Take an S as above and let $f\colon\mathfrak{C}^\omega\to X$ be a continuous injection such that $f[C]\cap S\neq\emptyset$ for every perfect cube C. Let $g\colon\mathfrak{C}\to\mathfrak{C}$ be a continuous function such that $g^{-1}(y)$ is perfect for every $y\in\mathfrak{C}$. Then $h_0=g\circ\pi_0\circ f^{-1}\colon f[\mathfrak{C}^\omega]\to\mathfrak{C}$ is uniformly continuous. Moreover, if $T=S\cap f[\mathfrak{C}^\omega]$ then $h_0[T]=\mathfrak{C}$ since

$$T\cap h_0^{-1}(y)=T\cap f[\pi_0^{-1}(g^{-1}(y))]=S\cap f[g^{-1}(y)\times\mathfrak{C}\times\mathfrak{C}\times\cdots]\neq\emptyset$$

for every $y\in\mathfrak{C}$. ■

Corollary 2.2 *Assume $s_0^{\rm cube}=[X]^{<\mathfrak{c}}$ for a Polish space X. If $S\subset X$ has cardinality $\mathfrak{c}$ then there exists a uniformly continuous function $f\colon X\to[0,1]$ such that $f[S]=[0,1]$.*

In particular, CPA$_{\rm cube}$ *implies property* (a).

PROOF. If S is as above then, by CPA$_{\rm cube}$, $S\notin s_0^{\rm cube}$. Thus, by Proposition 2.1 there exists a uniformly continuous function h from a subset of S onto $\mathfrak{C}$. Consider $\mathfrak{C}$ as a subset of $[0,1]$ and let $\hat{h}\colon X\to[0,1]$ be a uniformly continuous extension of h. If $g\colon[0,1]\to[0,1]$ is continuous and such that $g[\mathfrak{C}]=[0,1]$ then $f=g\circ\hat{h}$ is as desired. ■

The fact that (a) holds in the iterated perfect set model was first proved by A. Miller in [16].

It is worth to note here that the function f in Corollary 2.2 cannot be required to be either monotone or in the class "D^1" of all functions having finite or infinite derivative at every point. This follows immediately from the following proposition, since each function which is either monotone or "D^1" belongs to the Banach class

$$(T_2)=\left\{f\in\mathcal{C}(\mathbb{R})\colon\{y\in\mathbb{R}\colon|f^{-1}(y)|>\omega\}\in\mathcal{N}\right\}.$$

(See [10] or [19, p. 278].)

Proposition 2.3 *There exists, in ZFC, an $S\in[\mathbb{R}]^{\mathfrak{c}}$ such that $[0,1]\not\subset f[S]$ for every $f\in(T_2)$.*

PROOF. Let $\{f_\xi\colon\xi<\mathfrak{c}\}$ be an enumeration of all functions from (T_2) whose range contains $[0,1]$. Construct by induction a sequence $\langle\langle s_\xi,y_\xi\rangle\colon\xi<\mathfrak{c}\rangle$ such that for every $\xi<\mathfrak{c}$

(i) $y_\xi\in[0,1]\setminus f_\xi[\{s_\zeta\colon\zeta<\xi\}]$ and $|f_\xi^{-1}(y_\xi)|\le\omega$.

(ii) $s_\xi\in\mathbb{R}\setminus\left(\{s_\zeta\colon\zeta<\xi\}\cup\bigcup_{\zeta\le\xi}f_\zeta^{-1}(y_\zeta)\right)$.

Then the set $S=\{s_\xi\colon\xi<\mathfrak{c}\}$ is as required since $y_\xi\in[0,1]\setminus f_\xi[S]$ for every $\xi<\mathfrak{c}$. ■

3 Perfectly meager and universally null sets

The fact that (b) holds in the iterated perfect set model was first proved by A. Miller in [16].

Theorem 3.1 *If $S\subset\mathbb{R}$ is either perfectly meager or universally null then $S\in s_0^{\rm cube}$.*

In particular, CPA$_{\rm cube}$ *implies property* (b).

Proof. Take an $S \subset \mathbb{R}$ which is either perfectly meager or universally null and let $f\colon \mathfrak{C}^\omega \to \mathbb{R}$ be a continuous injection. Then $S \cap f[\mathfrak{C}^\omega]$ is either meager or null in $f[\mathfrak{C}^\omega]$. Thus $G = \mathfrak{C}^\omega \setminus f^{-1}(S)$ is either comeager or of full measure in $\mathfrak{C}^\omega$. Hence the theorem follows immediately from the following claim. ∎

Claim 3.2 *Consider $\mathfrak{C}^\omega$ with standard topology and standard product measure. If G is a Borel subset of $\mathfrak{C}^\omega$ which is either of second category or of positive measure then G contains a perfect cube $\prod_{i<\omega} P_i$.*

The measure version of the claim is a variant the following theorem:

(m) for every full measure subset H of $[0,1]\times[0,1]$ there are a perfect set $P \subset [0,1]$ and a positive inner measure subset $\hat{H}$ of $[0,1]$ such that $P \times \hat{H} \subset H$

which was proved by Eggleston [9] and, independently, by Brodskiĭ [3]. The category version of the claim is a consequence of the category version of (m):

(c) for every Polish space X and every comeager subset G of $X \times X$ there are a perfect set $P \subset X$ and a comeager subset $\hat{G}$ of X such that $P \times \hat{G} \subset G$.

This well known result can be found in [12, Exercise 19.3]. (Its version for $\mathbb{R}^2$ is also proved, for example, in [8, condition ($\star$), p. 416].) For completeness, we will show here in detail how to deduce the claim from (m) and (c).

We will start the argument with a simple fact, in which we will use the following notations. If X is a Polish space endowed with a Borel measure then $\psi_0(X)$ will stand for the sentence

$\psi_0(X)$**:** For every full measure subset H of $X \times X$ there are a perfect set $P \subset X$ and a positive inner measure subset $\hat{H}$ of X such that $P \times \hat{H} \subset H$.

Thus $\psi_0([0,1])$ is is a restatement of (m). We will also use the following seemingly stronger variants of $\psi_0(X)$.

$\psi_1(X)$**:** For every full measure subset H of $X \times X$ there are a perfect set $P \subset X$ and a subset $\hat{H}$ of X of full measure such that $P \times \hat{H} \subset H$.

$\psi_2(X)$**:** For a subset H of $X \times X$ of positive inner measure there are a perfect set $P \subset X$ and a positive inner measure subset $\hat{H}$ of X such that $P \times \hat{H} \subset H$.

Fact 3.3 *Let $n = 1, 2, 3, \ldots$.*

(i) *If $E \subset \mathbb{R}^n$ has a positive Lebesgue measure then $\mathbb{Q}^n + E = \bigcup_{q \in \mathbb{Q}^n}(q+E)$ has a full measure.*

(ii) *$\psi_k(X)$ holds for all $k < 3$ and $X \in \{[0,1], (0,1), \mathbb{R}, \mathfrak{C}\}$.*

Proof. (i) Let λ be the Lebesgue measure on $\mathbb{R}^n$ and for $\varepsilon > 0$ and $x \in \mathbb{R}^n$ let $B(x,\varepsilon)$ be an open ball in $\mathbb{R}^n$ of radius ε centered at x. By way of contradiction assume that there exists a positive measure set $A \subset \mathbb{R}^n$ disjoint with $\mathbb{Q}^n + E$. Let $a \in A$ and $x \in E$ be the Lebesgue density points of A and X, respectively. Take an $\varepsilon>0$ such that $\lambda(A\cap B(a,\varepsilon))>(1-4^{-n})\lambda(B(a,\varepsilon))$ and $\lambda(E\cap B(x,\varepsilon))>(1-4^{-n})\lambda(B(x,\varepsilon))$. Now, if $q\in\mathbb{Q}^n$ is such that $q+x\in B(a,\varepsilon/2)$ then $A\cap(q+E)\cap B(a,\varepsilon/2)\neq\emptyset$ since $B(a,\varepsilon/2)\subset B(a,\varepsilon)\cap B(q+x,\varepsilon)$ and thus $\lambda(A\cap(q+E)\cap B(a,\varepsilon/2))>\lambda(B(a,\varepsilon/2))-2\cdot 4^{-n}\lambda(B(a,\varepsilon))\geq 0$. Hence $A\cap(\mathbb{Q}^n+E)\neq\emptyset$ contradicting the choice of A.

(ii) First note that $\psi_k(\mathbb{R})\Leftrightarrow\psi_k((0,1))\Leftrightarrow\psi_k([0,1])\Leftrightarrow\psi_k(\mathfrak{C})$ for every $k<3$. This is justified by the fact that for the mappings $f\colon(0,1)\to\mathbb{R}$ given by $f(x)=\cot(x\pi)$, the identity mapping $id\colon(0,1)\to[0,1]$, and $d\colon\mathfrak{C}\to[0,1]$ given by $d(x)=\sum_{i<\omega}\frac{x(i)}{2^{i+1}}$, the image and the preimage of measure zero set (full measure set) is of measure zero (full measure).

Since, by (m), $\psi_0([0,1])$ is true, we have also that $\psi_0(X)$ holds also for $X\in\{(0,1),\mathbb{R},\mathfrak{C}\}$. To finish the proof it is enough to show that $\psi_0(\mathbb{R})$ implies $\psi_1(\mathbb{R})$ and $\psi_2(\mathbb{R})$.

To prove $\psi_1(\mathbb{R})$ let H be a full measure subset of $\mathbb{R}\times\mathbb{R}$ and let us define $H_0=\bigcap_{q\in\mathbb{Q}}(\langle 0,q\rangle+H)$. Then H_0 is still of full measure so, by $\psi_0(\mathbb{R})$, there are perfect set $P\subset\mathbb{R}$ and a positive inner measure subset $\hat{H}_0$ of $\mathbb{R}$ such that $P\times\hat{H}_0\subset H_0$. Thus, for every $q\in\mathbb{Q}$ we also have $P\times(q+\hat{H}_0)\subset\langle 0,q\rangle+H_0=H_0$. Let $\hat{H}=\bigcup_{q\in\mathbb{Q}}(q+\hat{H}_0)$. Then $P\times\hat{H}\subset H_0\subset H$ and, by (i), $\hat{H}$ has full measure. So, $\psi_1(\mathbb{R})$ is proved.

To prove $\psi_2(\mathbb{R})$ let $H\subset\mathbb{R}\times\mathbb{R}$ be of positive inner measure. Decreasing H, if necessary, we can assume that H is compact. Let $H_0=\mathbb{Q}^2+H$. Then, by (i), H_0 is of full measure so, by $\psi_0(\mathbb{R})$, there are perfect set $P_0\subset\mathbb{R}$ and a positive inner measure subset $\hat{H}_0$ of $\mathbb{R}$ such that $P_0\times\hat{H}_0\subset H_0$. Once again, decreasing P_0 and $\hat{H}_0$ if necessary, we can assume that they are homeomorphic to $\mathfrak{C}$ and that no relatively open subset of $\hat{H}_0$ has measure zero. Since $P_0\times\hat{H}_0\subset\bigcup_{q\in\mathbb{Q}^2}(q+H)$ is covered by countably many compact sets $(P_0\times\hat{H}_0)\cap(q+H)$ with $q\in\mathbb{Q}^2$, there is a $q=\langle q_0,q_1\rangle\in\mathbb{Q}^2$ such that $(P_0\times\hat{H}_0)\cap(q+H)$ has a non-empty interior in $P_0\times\hat{H}_0$. Let U and V be non-empty clopen subsets of P_0 and $\hat{H}_0$, respectively, such that $U\times V\subset(P_0\times\hat{H}_0)\cap(q+H)\subset\langle q_0,q_1\rangle+H$. Then U and V are perfect and V has positive measure. Let $P=-q_0+U$ and $\hat{H}=-q_1+V$. Then $P\times\hat{H}=(-q_0+U)\times(-q_1+V)=-\langle q_0,q_1\rangle+(U\times V)\subset H$, so $\psi_2(\mathbb{R})$ holds. ∎

PROOF OF CLAIM 3.2. Since the natural homeomorphism between $\mathfrak{C}$ and $\mathfrak{C}^{\omega\setminus\{0\}}$ preserves product measure, we can identify $\mathfrak{C}^\omega=\mathfrak{C}\times\mathfrak{C}^{\omega\setminus\{0\}}$ with $\mathfrak{C}\times\mathfrak{C}$ considered with its usual topology and its usual product measure. With this identification the result follows easily, by induction on coordinates, from the following fact.

- (•) For every Borel subset H of $\mathfrak{C}\times\mathfrak{C}$ which is of second category (of positive measure) there are a perfect set $P\subset\mathfrak{C}$ and a second category (positive measure) subset $\hat{H}$ of $\mathfrak{C}$ such that $P\times\hat{H}\subset H$.

The measure version of (•) is a restatement of $\psi_2(\mathfrak{C})$, which was proved in Fact 3.3(ii). To see the category version of (•) let H be a Borel subset of $\mathfrak{C}\times\mathfrak{C}$ of second category. Then there are clopen subsets U and V of $\mathfrak{C}$ such that $H_0=H\cap(U\times V)$ is comeager in $U\times V$. Since U and V are homeomorphic to

$\mathfrak{C}$, we can apply (c) to H_0 and $U \times V$ we can find a perfect set $P \subset U$ and a comeager Borel subset $\hat{H}$ of V such that $P \times \hat{H} \subset H_0 \subset H$, finishing the proof. ■

4 $\mathrm{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$

Next we show that CPA$_{\rm cube}$ implies that $\mathrm{cof}(\mathcal{N}) = \omega_1$. So, under CPA$_{\rm cube}$, all cardinals from Cichoń's diagram (see e.g. [1]) are equal to ω_1. The fact that this holds in the iterated perfect set model has been well known.

Let $\mathcal{C}_H$ be the family of all subsets $\prod_{n<\omega} T_n$ of ω^ω such that $T_n \in [\omega]^{\leq n+1}$ for all $n < \omega$. We will use the following characterization.

Proposition 4.1 (Bartoszyński [1, thm. 2.3.9])

$$\mathrm{cof}(\mathcal{N}) = \min\left\{|\mathcal{F}|\colon \mathcal{F} \subset \mathcal{C}_H \ \& \ \bigcup \mathcal{F} = \omega^\omega\right\}.$$

Lemma 4.2 *The family $\mathcal{C}_H^* = \{X \subset \omega^\omega\colon\ X \subset T \ \text{ for some } \ T \in \mathcal{C}_H\}$ is $\mathcal{F}_{\rm cube}$-dense in* $\mathrm{Perf}(\omega^\omega)$.

Proof. Let $f\colon \mathfrak{C}^\omega \to \omega^\omega$ be a continuous function. By (4) it is enough to find a perfect cube C in $\mathfrak{C}^\omega$ such that $f[C] \in \mathcal{C}_H^*$.

Construct, by induction on $n < \omega$, the families $\{E^i_s\colon s \in 2^n \ \& \ i < \omega\}$ of non-empty clopen subsets of $\mathfrak{C}$ such that for every $n < \omega$ and $s, t \in 2^n$

(i) $E^i_s = E^i_t$ for every $n \leq i < \omega$;

(ii) $E^i_{s\hat{}0}$ and $E^i_{s\hat{}1}$ are disjoint subsets of E^i_s for every $i < n+1$;

(iii) for every $\langle s_i \in 2^n\colon i < \omega\rangle$

$$f(x_0) \restriction 2^{(n+1)^2} = f(x_1) \restriction 2^{(n+1)^2} \quad \text{for every} \quad x_0, x_1 \in \prod_{i<\omega} E_{s_i}.$$

For each $i < \omega$ the fusion of $\{E^i_s\colon s \in 2^{<\omega}\}$ will give us the i-th coordinate set of the desired perfect cube C.

Condition (iii) can be ensured by uniform continuity of f. Indeed, let $\delta > 0$ be such that $f(x_0) \restriction 2^{(n+1)^2} = f(x_1) \restriction 2^{(n+1)^2}$ for every $x_0, x_1 \in \mathfrak{C}^\omega$ of distance less than δ. Then it is enough to choose $\{E^i_s\colon s \in 2^n \ \& \ i < \omega\}$ such that (i) and (ii) are satisfied and every set $\prod_{i<\omega} E_{s_i}$ from (iii) has diameter less than δ. This finishes the construction.

Next for every $i, n < \omega$ let $E^i_n = \bigcup\{E^i_s\colon s \in 2^n\}$ and $E_n = \prod_{i<\omega} E^i_n$. Then $C = \bigcap_{n<\omega} E_n = \prod_{i<\omega}\left(\bigcap_{n<\omega} E^i_n\right)$ is a perfect cube, since $\bigcap_{n<\omega} E^i_n \in \mathrm{Perf}(\mathfrak{C})$ for every $i < \omega$. Thus, to finish the proof it is enough to show that $f[C] \in \mathcal{C}_H^*$.

So, for every $k < \omega$ let $n < \omega$ be such that $2^{n^2} \leq k+1 < 2^{(n+1)^2}$, put

$$T_k = \{f(x)(k)\colon x \in E_n\} = \left\{f(x)(k)\colon x \in \prod_{i<\omega} E_{s_i} \ \text{ for some } \langle s_i \in 2^n\colon i < \omega\rangle\right\},$$

and notice that T_k has at most $2^{n^2} \leq k+1$ elements. Indeed, by (iii), the set $\left\{f(x)(k)\colon x \in \prod_{i<\omega} E_{s_i}\right\}$ has precisely one element for every $\langle s_i \in 2^n\colon i<\omega\rangle$ while (i) implies that $\left\{\prod_{i<\omega} E_{s_i}\colon \langle s_i \in 2^n\colon i<\omega\rangle\right\}$ has 2^{n^2} elements. Therefore $\prod_{k<\omega} T_k \in \mathcal{C}_H$.

To finish the proof it is enough to notice that $f[C] \subset \prod_{k<\omega} T_k$. ■

Corollary 4.3 *If* CPA$_{\rm cube}$ *holds then* $\mathrm{cof}(\mathcal{N}) = \omega_1$.

Proof. By CPA$_{\rm cube}$ and Lemma 4.2 there exists an $\mathcal{F} \in [\mathcal{C}_H]^{\leq\omega_1}$ such that $|\omega^\omega \setminus \bigcup \mathcal{F}| \leq \omega_1$. This and Proposition 4.1 imply $\mathrm{cof}(\mathcal{N}) = \omega_1$. ■

5 Pointwise convergent of subsequences of real-valued functions

A sequence $\langle f_n\rangle_{n<\omega}$ of real-valued functions is *uniformly bounded* provided there exists an $r \in \mathbb{R}$ such that $\mathrm{range}(f_n) \subset [-r,r]$ for every n. In 1932 Mazurkiewicz [15] proved the following variant of Egorov's theorem.

> *For every uniformly bounded sequence $\langle f_n\rangle_{n<\omega}$ of real-valued continuous functions defined on a Polish space X there exists a subsequence which is uniformly convergent on some perfect set P.*

The main result of this section is the following theorem.

Theorem 5.1 *If* CPA$_{\rm cube}$ *holds then*

(∗) *for every Polish space X and uniformly bounded sequence $\langle f_n\colon X \to \mathbb{R}\rangle_{n<\omega}$ of Borel measurable functions there are the sequences: $\langle P_\xi\colon \xi<\omega_1\rangle$ of compact subsets of X and $\langle W_\xi \in [\omega]^\omega\colon \xi<\omega_1\rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi<\omega_1$:*

> *$\langle f_n \restriction P_\xi\rangle_{n\in W_\xi}$ is a monotone uniformly convergent sequence of uniformly continuous functions.*

Theorem 5.1 is a variant of [5, theorem 2] and its corollary, according to which condition (∗) for continuous functions f_n can be deduced from the assumptions that $\mathrm{cof}(\mathcal{N}) = \omega_1$ *and there exists a selective ω_1-generated ultrafilter on ω.*

Proof. We first note that the family $\mathcal{E}$ of all $P \in \mathrm{Perf}(X)$ for which there exists a $W \in [\omega]^\omega$ such that

$$\text{the sequence } \langle f_n \restriction P\rangle_{n\in W} \text{ is monotone and uniformly convergent}$$

is $\mathcal{F}_{\rm cube}$-dense in $\mathrm{Perf}(X)$.

Indeed, let $g \in \mathcal{F}_{\rm cube}$, $g\colon \mathfrak{C}^\omega \to X$, and consider the functions $h_n = f_n \circ g$. Since $h = \langle h_n\colon n<\omega\rangle\colon \mathfrak{C}^\omega \to \mathbb{R}^\omega$ is Borel measurable, there is a dense G_δ

subset G of $\mathfrak{C}^\omega$ such that $h \restriction G$ is continuous. So, we can find a perfect cube $C \subset G \subset \mathfrak{C}^\omega$, and for this C function $h \restriction C$ is continuous. Thus, identifying the coordinate spaces of C with $\mathfrak{C}$, without loss of generality we can assume that $C = \mathfrak{C}^\omega$, that is, that each function $h_n\colon \mathfrak{C}^\omega \to \mathbb{R}$ is continuous. Now, by [21, thm. 6.9], there is a perfect cube C in $\mathfrak{C}^\omega$ and a $W \in [\omega]^\omega$ such that the sequence $\langle h_n \restriction C\rangle_{n\in W}$ is monotone and uniformly convergent.[1] So $P = g[C]$ is in $\mathcal{E}$.

Now, by CPA$_{\rm cube}$, there exists an $\mathcal{E}_0 \in [\mathcal{E}]^{\le\omega_1}$ such that $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$. Then $\{P_\xi\colon \xi < \omega_1\} = \mathcal{E}_0 \cup \{\{x\}\colon x \in X \setminus \bigcup \mathcal{E}_0\}$ is as desired: if $P_\xi \in \mathcal{E}_0$ then the existence of an appropriate W_ξ follows from the definition of $\mathcal{E}$. If P_ξ is a singleton, then the existence of W_ξ follows from the fact that every sequence of reals has a monotone subsequence. ■

6 Total failure of Martin's Axiom

In this section we prove that CPA$_{\rm cube}$ implies the total failure of Martin's Axiom, that is, the property that

> for every non-trivial ccc forcing $\mathbb{P}$ there exists ω_1-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

The consistency of this fact with $\mathfrak{c} > \omega_1$ was first proved by Baumgartner [2] in a model obtained by adding Sacks reals side-by-side. The topological and boolean algebraic formulations of the theorem follow immediately from the following proposition.

Proposition 6.1 *The following conditions are equivalent.*

(a) *For every non-trivial ccc forcing $\mathbb{P}$ there exists ω_1-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.*

(b) *Every compact ccc topological space without isolated points is a union of ω_1 nowhere dense sets.*

(c) *For every atomless ccc complete Boolean algebra B there exists ω_1-many dense sets in B such that no filter intersects all of them.*

(d) *For every atomless ccc complete Boolean algebra B there exists ω_1-many maximal antichains in B such that no filter intersects all of them.*

(e) *For every countably generated atomless ccc complete Boolean algebra B there exists ω_1-many maximal antichains in B such that no filter intersects all of them.*

[1] Actually [21, thm. 6.9] is stated for functions defined on $[0,1]^\omega$. However, the proof presented there for works also for functions defined on $\mathfrak{C}^\omega$.

PROOF. The equivalence of the conditions (a), (b), (c), and (d) is well known. In particular, equivalence (a)–(c) is explicitly given in [2, thm. 0.1]. Clearly (d) implies (e). The remaining implication, (e)$\Longrightarrow$(d), is a version of the theorem from [14, p. 158]. However, it is expressed there in a bit different language, so we include here its proof.

So, let $\langle B, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ be an atomless ccc complete Boolean algebra. For every $\sigma \in 2^{<\omega_1}$ define, by induction on the length $\mathrm{dom}(\sigma)$ of a sequence σ, a $b_\sigma \in B$ such that the following conditions are satisfied.

- $b_\emptyset = \mathbf{1}$.
- b_σ is a disjoint union of $b_{\sigma\hat{\ }0}$ and $b_{\sigma\hat{\ }1}$.
- If $b_\sigma > \mathbf{0}$ then $b_{\sigma\hat{\ }0} > \mathbf{0}$ and $b_{\sigma\hat{\ }1} > \mathbf{0}$.
- If $\lambda = \mathrm{dom}(\sigma)$ is a limit ordinal then $b_\sigma = \bigwedge_{\xi<\lambda} b_{\sigma\restriction\xi}$.

Let $T = \{s \in 2^{<\omega_1} : b_s > \mathbf{0}\}$. Then T is a subtree of $2^{<\omega_1}$; its levels determine antichains in B, so they are countable.

First assume that T has a countable height. Then T itself is countable. Let B_0 be the smallest complete subalgebra of B containing $\{b_\sigma : \sigma \in T\}$ and notice that B_0 is atomless. Indeed, if there were an atom a in B_0 then $S = \{\sigma \in T : a \leq b_\sigma\}$ would be a branch in T so that $\delta = \bigcup S$ would belong to $2^{<\omega_1}$. Since $b_\delta \geq a > \mathbf{0}$, we would also have $\delta \in T$. But then $a \leq b_\delta = b_{\delta\hat{\ }0} \vee b_{\delta\hat{\ }1}$ so that either $\delta\hat{\ }0$ or $\delta\hat{\ }1$ belongs to S, which is impossible.

Thus, B_0 is a complete, countably generated, atomless subalgebra of B. So, by (e), there exists a family $\mathcal{A}$ of ω_1-many maximal antichains in B_0 with no filter in B_0 intersecting all of them. But then each $A \in \mathcal{A}$ is also a maximal antichain in B and no filter in B would intersect all of them. So, we have (d).

Next, assume that T has height ω_1 and for every $\alpha < \omega_1$ let

$$T_\alpha = \{\sigma \in T : \mathrm{dom}(\sigma) = \alpha\}$$

be the α-th level of T. Also let $b_\alpha = \bigvee_{\sigma \in T_\alpha} b_\sigma$. Notice that $b_\alpha = b_{\alpha+1}$ for every $\alpha < \omega_1$. On the other hand, it may happen that $b_\lambda > \bigwedge_{\alpha<\lambda} b_\alpha$ for some limit $\lambda < \omega_1$; however, this may happen only countably many times, since B is ccc. Thus, there is an $\alpha < \omega_1$ such that $b_\beta = b_\alpha$ for every $\alpha < \beta < \omega_1$.

Now, let B_0 be the smallest complete subalgebra of B below $\mathbf{1} \setminus b_\alpha$ containing $\{b_\sigma \setminus b_\alpha : \sigma \in T\}$. Then B_0 is countably generated and, as before, it can be shown that B_0 is atomless. Thus, there exists a family $\mathcal{A}_0$ of ω_1-many maximal antichains in B_0 with no filter in B_0 intersecting all of them. Then no filter in B containing $\mathbf{1} \setminus b_\alpha$ intersects every $A \in \mathcal{A}_0$. But for every $\alpha < \beta < \omega_1$ the set $A^\beta = \{b_\sigma : \sigma \in T_\beta\}$ is a maximal antichain in B below b_α. Therefore, $\mathcal{A}_1 = \{A^\beta : \alpha < \beta < \omega_1\}$ is an uncountable family of maximal antichains in B below b_α with no filter in B containing b_α intersecting every $A \in \mathcal{A}_1$. Then it is easy to see that the family $\mathcal{A} = \{A_0 \cup A_1 : a_0 \in \mathcal{A}_0 \ \&\ A_1 \in \mathcal{A}_1\}$ is a family of ω_1-many maximal antichains in B with no filter in B intersecting all of them. This proves (d). ∎

Theorem 6.2 CPA$_{\rm cube}$ *implies the total failure of Martin's Axiom.*

Proof. Let $\mathcal{A}$ be a countably generated atomless ccc complete Boolean algebra and let $\{A_n : n < \omega\}$ generate $\mathcal{A}$. By Proposition 6.1 it is enough to show that $\mathcal{A}$ contains ω_1-many maximal antichains such that no filter in $\mathcal{A}$ intersects all of them.

Next let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathfrak{C} = 2^\omega$. Recall that it is a free countably generated σ-algebra, with free generators $B_i = \{s \in \mathfrak{C} : s(i) = 0\}$. Define $h_0 : \{B_n : n < \omega\} \to \{A_n : n < \omega\}$ by $h_0(B_n) = A_n$ for all $n < \omega$. Then h_0 can be uniquely extended to a σ-homomorphism $h : \mathcal{B} \to \mathcal{A}$ between σ-algebras $\mathcal{B}$ and $\mathcal{A}$. (See e.g. [20, 34.1 p. 117].) Let $\mathcal{I} = \{B \in \mathcal{B} : h[B] = \mathbf{0}\}$. Then $\mathcal{I}$ is a σ-ideal in $\mathcal{B}$ and the quotient algebra $\mathcal{B}/\mathcal{I}$ is isomorphic to $\mathcal{A}$. (Compare also Loomis-Sikorski theorem in [20, p. 117] or [13].) In particular, $\mathcal{I}$ contains all singletons and is ccc, since $\mathcal{A}$ is atomless and ccc.

It follows that we need only to consider complete Boolean algebras of the form $\mathcal{B}/\mathcal{I}$, where $\mathcal{I}$ is some ccc σ-ideal of Borel sets containing all singletons. To prove that such an algebra has ω_1 maximal antichains as desired, it is enough to prove that

(∗) $\mathfrak{C}$ is a union of ω_1 perfect sets $\{N_\xi : \xi < \omega_1\}$ which belong to $\mathcal{I}$.

Indeed, assume that (∗) holds and for every $\xi < \omega_1$ let $\mathcal{D}^*_\xi$ be a family of all $B \in \mathcal{B} \setminus \mathcal{I}$ with closures $\mathrm{cl}(B)$ disjoint from N_ξ. Then $\mathcal{D}_\xi = \{B/\mathcal{I} : B \in \mathcal{D}^*_\xi\}$ is dense in $\mathcal{B}/\mathcal{I}$, since $\mathfrak{C} \setminus N_\xi$ is σ-compact and $\mathcal{B}/\mathcal{I}$ is a σ-algebra. Let $\mathcal{A}^*_\xi \subset \mathcal{D}^*_\xi$ be such that $\mathcal{A}_\xi = \{B/\mathcal{I} : B \in \mathcal{A}^*_\xi\}$ is a maximal antichain in $\mathcal{B}/\mathcal{I}$. It is enough to show that no filter intersects all $\mathcal{A}_\xi$'s. But if there were a filter $\mathcal{F}$ in $\mathcal{B}/\mathcal{I}$ intersecting all $\mathcal{A}_\xi$'s then for every $\xi < \omega_1$ there would exist a $B_\xi \in \mathcal{A}^*_\xi$ with $B_\xi/\mathcal{I} \in \mathcal{F} \cap \mathcal{A}_\xi$. Thus, the set $\bigcap_{\xi<\omega_1} \mathrm{cl}(B_\xi)$ would be non-empty, despite the fact that it is disjoint from $\bigcup_{\xi<\omega_1} N_\xi = \mathfrak{C}$.

To finish the proof it is enough to show that (∗) follows from CPA$_{\rm cube}$. But this follows immediately from the fact that any cube P in $\mathfrak{C}$ contains a subcube $Q \in \mathcal{I}$ as any cube P can be partitioned into $\mathfrak{c}$-many disjoint subcubes and, by the ccc property of $\mathcal{I}$, only countably many of them can be outside $\mathcal{I}$. ■

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Other consequences of CPA$_{\rm cube}$ can be found in [6], [11], [18], and in the monograph in preparation [7].

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