Polish spaces, computable approximations, and bitopological spaces

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Abstract

Answering a question of J. Lawson (formulated also earlier, in 1984, by Kamimura and Tang [16]) we show that every Polish space admits a bounded complete computational model, as defined below. This results from our construction, in each Polish space $\langle X, \tau\rangle$, of a countable family $\mathcal{C}$ of non-empty closed subsets of X such that:

(cp) each subset of $\mathcal{C}$ with the finite intersection property has non-empty intersection;

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(br) if $x \in T$ and $T \in \tau$ then there exists $C \in \mathcal{C}$ such that $x \in \text{int}(C)$ and $C \subset T$, and

(r*) for every $C \in \mathcal{C}$ and $x \in X \setminus C$ there is a $D \in \mathcal{C}$ such that $C \subset \text{int}(D)$ and $x \notin D$.

These conditions assure us that there is another compact topology $\tau^* \subset \tau$ on X such that the bitopological space $\langle X, \tau, \tau^* \rangle$, is pairwise regular. The existence of such a topology is also shown equivalent to admitting a bounded complete computational model.

1 Background: what is a bounded complete computational model?

In this section we will introduce the notions coming from theoretical computer science and necessary for understanding the main problem. These notions are standard in domain theory, but are unknown to many topologists. Thus, we take extra time and space to explain the motivation behind these notions.

In the past few decades theoretical computer science has considered the basic problem: What is the best way to approximate mathematical objects? One of the most fundamental of such questions is about the representation of a real number. A common theoretical approach to this problem is to identify each real number r with a collection of intervals whose intersection is $\{r\}$. In such a representation a smaller interval gives more information about a number than a bigger interval. So an interval I carries more information than an interval J, which we represent by writing $J \leq I$, provided that $J \supset I$.

An approximation of a number (some knowledge accumulated about it) is stored in the partially ordered set $\langle P_{\mathbb{R}}, \leq \rangle$ whose elements are $\mathbb{R}$, and all its closed bounded intervals including singletons, and whose partial order, $\leq$, is reverse set inclusion, $\supset$. The numbers themselves are represented by singletons, denoted here by $\text{Max}(P_{\mathbb{R}})$, since they are the maximal elements of $P_{\mathbb{R}}$. Each element of $P_{\mathbb{R}}$ is below a maximal element.

More generally, certain partially ordered sets $\langle P, \leq \rangle$ each of whose elements is below a maximal element, can be considered as models for approximating their maximal elements. This idea has been explored by many authors (see e.g. Scott [19], Edalat [6], Edalat and Heckmann [7], or Lawson [18]) and led to a field known as domain theory. An authoritative reference in this area is [2], which has set much of the standard notation in the subject.

To make approximation in a model computationally feasible a poset P must have several nice properties. The most fundamental is that after we go through all the work of approximation, we have actually approximated an object. We see that this is embodied by the following:

Definition 1 A poset $\langle P, \leq \rangle$ is *directed complete* (abbreviated as *dcpo*) provided each directed subset D of P has a supremum $\bigvee D$. It is *bounded complete* (abbreviated as *bcpo*) if it is a dcpo and each subset which is bounded above has a supremum.

The importance of the notion of dcpo is that when increasingly fine approximations are obtained, they indeed approximate some object; for example, this would be false if we used $P_{\mathbb{Q}}$ to try to compute rational numbers.

For a dcpo $\langle P, \leq \rangle$, each $x \in P$ is below the join (i.e., supremum) of a maximal chain of elements $\geq x$, which is certainly an element of $\mathrm{Max}(P)$.

The definition of dcpo also requires the existence of a bottom element, $\bigvee \emptyset$, which in the case of $P_{\mathbb{R}}$ is equal to $\mathbb{R}$. Certainly $P_{\mathbb{R}}$ is bounded complete with $\bigvee D = \bigcap D$ for any directed or bounded subset of $P_{\mathbb{R}}$.

As we shall see, bounded completeness has important consequences, although its theoretical value is less clear. Note that a dcpo in which pairs that are bounded above have suprema, is bounded complete (since for any bounded set, the set of suprema of its finite subsets then is directed, thus must have a supremum and that is the supremum of the original set).

The next issue is that of "observability;" the idea that we should be able to see whether r is in one of its supposed approximations. For example, if r is an endpoint of the interval I, no magnification of the real line would make it possible to see whether r is actually in I or not. Similarly, for another interval $J \in P_{\mathbb{R}}$, if either the left endpoints of I and J are identical, or their right endpoints are, it will not be possible under any magnification of the real line to see whether one of the intervals contains the other. This problem has an obvious answer involving topology: given two intervals I, J in the poset $\langle P_{\mathbb{R}}, \supset \rangle$, J is observably below I if I is a subset of the interior $\mathrm{int}(J)$ of J. But this can be expressed just in terms of posets:

Definition 2 For a dcpo and $x, y \in P$ we say that x *is way-below* y (written $x \ll y$) if whenever $y \leq \bigvee D$ and D is directed, then there is some $z \in D$ such that $x \leq z$.

The compactness of the elements of $P_{\mathbb{R}}$ immediately implies that $K \in P_{\mathbb{R}}$ is way-below $M \in P_{\mathbb{R}}$ if and only if $M \subset \operatorname{int}(K)$. The reader should check that if P is the collection of all compact subsets of a locally compact topological space X, then $\langle P \cup \{X\}, \supset \rangle$ is a dcpo, and $M \subset \operatorname{int}(K)$ if and only if $K \ll M$.

In any dcpo, the bottom element, $\bigvee \emptyset$, is way-below itself. This is the only element of $P_{\mathbb{R}}$ that is way-below itself, and in what follows we will be mainly concerned with posets for which $x \ll x$ only for the bottom element. However, there is much interest, both in domain theory and in algebra, in continuous posets in which each element x is the supremum of the set $\{y \leq x : y \ll y\}$, and this set is directed. These are called *algebraic posets,* and include the algebraic lattices (such as the collection of all ideals of a ring, ordered by $\subset$, and many other examples). See [13] or [2] for further discussion, and [11] for discussion of a topological example of interest in domain theory (ultrametric separable spaces).

The interpretation of the definition of continuous dcpo which follows, is that sufficient information needed to compute any object is available in the objects way-below it.

Definition 3 For $A \subset P$ we define $\Uparrow A = \{x \in P : a \ll x \text{ for some } a \in A\}$ and $\Downarrow A = \{x \in P : x \ll a \text{ for some } a \in A\}$. For $a \in P$ the symbols $\Uparrow a$ and $\Downarrow a$ stand for $\Uparrow \{a\}$ and $\Downarrow \{a\}$, respectively.

A *continuous dcpo* is a dcpo P such that for every $x \in P$, $\Downarrow x$ is directed and $x = \bigvee(\Downarrow x)$.

Clearly for $[p, q] \in P_{\mathbb{R}}$ we have $\bigvee(\Downarrow [p, q]) = \bigcap\{[r, s] : r < p \leq q < s\} = [p, q]$, so $P_{\mathbb{R}}$ is a continuous dcpo.

Let us note that $\ll$ satisfies a transitivity condition and it is stronger than $\leq$:

(str) if $x \ll y$ then $x \leq y$;

(trans) if $w \leq x \ll y \leq z$ then $w \ll z$.

(To see the (str) condition take $D = \{y\}$ in the definition of $\ll$.)

The properties (str) and (trans) immediately imply that $\Downarrow (\Downarrow x) \subset \Downarrow x$. The reverse inclusion is not automatic, however it holds for continuous dcpo's.

Fact 4 *If P is a continuous dcpo then $\Downarrow (\Downarrow x) = \Downarrow x$ for every $x \in P$.*

Proof. We need only show that $\Downarrow x \subset \Downarrow (\Downarrow x)$. So, first note that

$$\Downarrow (\Downarrow x) \text{ is directed.}$$

Indeed, if $y \ll y' \ll x$ and $z \ll z' \ll x$ then, since $\Downarrow x$ is directed, we can find a $w \in \Downarrow x$ such that $y', z' \leq w$. Thus, by (trans), $y, z \ll w$. Since $\Downarrow w$ is directed, we can find a $v \ll w$ for which $y, z \leq v$. But $v \in \Downarrow (\Downarrow x)$. So $\Downarrow (\Downarrow x)$ is directed.

Next note that if $y \in \Downarrow x$ then $\Downarrow y \subset \Downarrow (\Downarrow x)$, so $y = \bigvee \Downarrow y \leq \bigvee \Downarrow (\Downarrow x)$. Thus $x = \bigvee \Downarrow x \leq \bigvee \Downarrow (\Downarrow x)$, so by definition of $\ll$, if $y \in \Downarrow x$ (i.e., $y \ll x$), then $y \leq w$ for some $w \in \Downarrow (\Downarrow x)$, and so $y \in \Downarrow (\Downarrow x)$. ■

We restate the conclusion of Fact 4 in a form in which we will use it:

(interpolation) if $x \ll y$ then there exists a $z \in P$ such that $x \ll z \ll y$.

In the case of $P_{\mathbb{R}}$ the interpolation property is obvious. Later we will consider similarly-defined posets for more general topological spaces, and the interpolation property for these will follow from the normality of the topology.

Finally, computation requires the existence in P of a nice countable subset B (called a basis) whose elements may be used to recursively approximate maximal elements of P. The full information on a maximal element x of P can be represented as the filter $\mathcal{F}_x$ of all elements in B which are above x. However, we should imagine that at any particular moment of approximating x we have access only to the elements of $\mathcal{F}_x$ but not to the entire $\mathcal{F}_x$. (The situation is quite similar to that in forcing – a generic number is represented by a generic filter $\mathcal{F}$, but in the ground model we have access only to elements of $\mathcal{F}$, but not the entire $\mathcal{F}$.)

Definition 5 Following [13, page 168] we say that a subset D of a dcpo P is a *basis* for P provided for every $x \ll y$ from P there exists a d in D such that $x \leq d \ll y$. A poset P is *ω-continuous* provided it is a continuous dcpo and has a countable basis.

Notice that if D is a basis for a dcpo P then

$$x = \bigvee (D \cap \Downarrow x) \text{ for every } x \in P.$$

It is also easy to see that if P has the interpolation property then D is a basis for P if and only if D is $\ll$-dense in P in the sense that if $x \ll y$ then there exists a $d \in D$ such that $x \ll d \ll y$.

Clearly the family of all intervals with rational endpoints form a countable $\ll$-dense subset of $P_{\mathbb{R}}$.

Note, that the property $x = \bigvee(D \cap \Downarrow x)$ means that x is uniquely determined by $F(x) = D \cap \Downarrow x$ which is a filter in D. This means that the "learning process" about the object $x \in \mathrm{Max}(P)$ can be done by coding the incoming information using the elements from the countable set D. In fact, we do not need to know the entire structure of P to recover the elements of $\mathrm{Max}(P)$; we just need to know the full order structure of the set D. Moreover, notice that our knowledge about x is "continuously approaching" full information, since x is a limit of $D \cap \Downarrow x$. Thus, the bounded ω-continuous dcpo's (or, more precisely, their $\ll$-dense subsets) are a tool to recover the information on the structure of $\mathrm{Max}(P)$. That is, the knowledge gathered in an ω-continuous poset allows us to reconstruct the set $\mathrm{Max}(P)$.

Confronted with the situation described above to compute real numbers, it is natural to ask when we can find a similar model for a topological space X: an ω-continuous poset $\langle P, \leq \rangle$ which approximates the elements of X. Can the structure on P also encode the topological structure on X?

The topological spaces for which such a bounded ω-continuous dcpo can be found were studied by Lawson in [18], where he calls such spaces maximal point spaces. To define the notion of a maximal point space precisely we need to recall that each poset P can be equipped with the information-motivated *Scott topology* σ; certainly, it is natural to think of a set, C, as "knowledge-closed" (= Scott-closed) if, whenever $x \leq y \in C$, then $x \in C$, and whenever $D \subset C$ is directed, then its supremum $\bigvee D \in C$. Of course, then a set T is Scott-open if, as the complement of a Scott-closed set, whenever $y \geq x \in T$, then $y \in T$, and whenever D is directed and $\bigvee D \in T$, then D meets T. For a poset with the interpolation property, it is easy to check that the collection of sets $\Uparrow x$ with $x \in P$, is a base for the topology σ.

Definition 6 A topological space $\langle X, \tau \rangle$ is a *maximal point space* provided there exists an ω-continuous dcpo P and a bijection $i \colon X \to \mathrm{Max}(P)$ such that

(i) i is a homeomorphism between $\langle X, \tau \rangle$ and $\mathrm{Max}(P)$ considered with a subspace topology of $\langle P, \sigma \rangle$;

(ii) for every $x \in P$ the set $i^{-1}(\{y \in \mathrm{Max}(P) \colon x \leq y\})$ is τ-closed.

Such a poset P is a *computational model* for X, and if the poset P is bounded complete, then P is a *bounded complete computational model* for X.

It is easy to see that for each locally compact space X the poset P formed with X and all compact subsets of X, and ordered by the reverse inclusion, is a bcpo. If further, X is a separable locally compact metrizable space, then P is a bounded complete computational model for X.

Lawson [18] shows that a topological space is a maximal point space if and only if it is a Polish space. Also, using "formal balls," Edalat and Heckmann [7] provide a simple explicit construction of a maximal point space P_X for every Polish space X. Lawson's characterization and the Edalat-Heckmann construction are remarkable achievements, but they lack some desirable properties. In particular, posets P_X constructed by them are not bounded complete. Thus, at the North Bay Summer Conference, Jimmie Lawson asked whether every Polish space is the maximal point space of a *bounded complete* ω-continuous poset. (The same question was also posed earlier, in 1984, by Kamimura and Tang [16].) The goal of this paper is to give an affirmative answer for this question. It should be pointed out that the property of bounded completeness of the representation P_X of X gives advantages that are not present if P_X is just directed complete. For example, given a Scott continuous function from a maximal point space P_X into another, P_Y, its restriction to $\mathrm{Max}(X)$ (identified with X) is a continuous function from X into Y. It is desirable (cf, Escardó [9]) that every continuous map $X \to Y$ also extends to a Scott continuous function from P_X into P_Y. This is the case if P_X and P_Y are bounded complete computational models for X and Y, respectively.[1]

2 Topological reduction of the problem

The motivation for the definitions stated above came from a situation, which we now describe in the language of general Hausdorff topological spaces $\langle X, \tau\rangle$. We considered a family P_X of non-empty closed subsets of X whose interiors formed a base for X. We ordered P_X by reverse inclusion, introduced in P_X a way-below relation $\ll$, and noted that in our particular case

[1]To see this it is enough to notice that $\mathrm{Max}(P_X)$ is dense in the Scott topology and every continuous function defined on a dense subset of a bounded complete ω-continuous poset P (considered with the Scott topology σ) can be extended continuously to P [13, Exercise II, 3.19].

$K \ll M$ was equivalent to $M \subset \mathrm{int}(K)$. Then we found a $\ll$-dense subfamily D of P_X and identified each $x \in X$ with the filter $F(x) = D \cap \Downarrow x$. In the case we considered, the interiors of sets from D also formed a base for sets from D, so for each $K \in D$ we could also define the following filter in $\langle D, \supset \rangle$

$$j(M) = D \cap \Downarrow M = \{K \in D \colon K \ll M\}$$

and note that $j(M)$ still uniquely determines M, since $M = \bigcap j(M)$. Now, let $P_X^*(D)$ (we will write only P_X^* where D is clear from the context) be the family of all filters $\mathcal{F}$ in $\langle D, \supset \rangle$ with the property that

$$\text{for every } F \in \mathcal{F} \text{ there exists a } K \in \mathcal{F} \text{ such that } K \subset \mathrm{int}(F).^2 \tag{1}$$

P_X^* is ordered by the inclusion $\subset$.

It is not difficult to see that if X is locally compact and P_X is the family of all compact sets, then P_X^* is a bounded complete computational model for X with j (restricted to singletons) being a homeomorphism witnessing it. The main reason for this is that in this particular situation the mapping $k \colon P_X^* \to P_X$, given by $k(F) = \bigcap F$, is an order isomorphism between P_X^* and P_X. If X is a Polish space which is not locally compact the mapping k will need not even be one-to-one. The next theorem gives (implicitly) the properties of the families P_X and D (denoted there by Γ) which imply that P_X^* is a bounded complete computational model for X.

Of course, each family D generates a smallest topology τ^* on X such that all sets in D are closed. Since sets in D are closed in τ, we have $\tau^* \subset \tau$. Note that even in the case of $P_{\mathbb{R}}$, τ^* was strictly smaller than τ. So our notion of bounded complete computational model carries the bitopological structure $\langle X, \tau, \tau^* \rangle$. In our next theorem we will show that such a structure is not just a convenience — a bitopological structure is always associated with a computational model.

In what follows we will need the following definition (see [17]):

Definition 7 Given a property Q, a bitopological space $\langle X, \tau, \tau^* \rangle$ is *pairwise* Q if both it and its *bitopological dual*, $\langle X, \tau^*, \tau \rangle$ are Q.

Let $\langle X, \tau, \tau^* \rangle$ be a bitopological space. We say it is *regular* provided that for each $x \in U \in \tau$ there is a $V \in \tau$ such that $x \in V$ and $\mathrm{cl}_{\tau^*}(V) \subset U$.

[2] For $D = \mathcal{P}(X)$ filters satisfying (1) are sometimes called *round filters* (in a topological space X).

It is *normal* provided for every pair of disjoint sets: τ-closed E and τ^*-closed F^*, there exist disjoint sets $U^* \in \tau^*$ and $V \in \tau$ such that $E \subset U^*$ and $F^* \subset V$.

In fact, if $\langle X, \tau, \tau^* \rangle$ is normal, then notice that it is pairwise normal. Below, we use the terminology "pairwise normal" for this situation, and "normal" only for topological spaces.

In what follows we will need the following fact.

Fact 8 *If a bitopological space $\langle X, \tau, \tau^* \rangle$ is pairwise regular and X considered with the join topology $\tau \vee \tau^*$ is Lindelöf then $\langle X, \tau, \tau^* \rangle$ is pairwise normal.*

PROOF. This can be shown by a small adjustment of the usual proof that a regular Lindelöf space is normal:

Take disjoint sets E and F^* such that E is τ-closed and F^* is τ^*-closed. By pairwise regularity, and since E and F^* are $\tau \vee \tau^*$-closed, we can find a family $\mathcal{C}_{F^*} = \{C_i : i < \omega\}$ of τ^*-closed sets such that

$$E \subset \bigcup_{i<\omega} (X \setminus C_i) \;\;\&\;\; F^* \subset \bigcap_{i<\omega} \operatorname{int}_\tau(C_i)$$

and a family $\mathcal{B}_E = \{B_i : i < \omega\}$ of τ-closed sets such that

$$F^* \subset \bigcup_{i<\omega} (X \setminus B_i) \;\;\&\;\; E \subset \bigcap_{i<\omega} \operatorname{int}_{\tau^*}(B_i).$$

Now, define the sets $U^* \in \tau^*$ and $V \in \tau$ as in the standard proof that every Lindelöf space is normal:

$$U^* = \bigcup_{n<\omega} \left(\operatorname{int}_{\tau^*}(B_n) \setminus \bigcap_{i \le n} C_i \right) \in \tau^* \quad \& \quad V = \bigcup_{n<\omega} \left(\operatorname{int}_\tau(C_n) \setminus \bigcap_{i \le n} B_i \right) \in \tau.$$

But then $U^* \supset E$ and $V \supset F^*$ are disjoint. So, $\langle X, \tau, \tau^* \rangle$ is pairwise normal. ■

Theorem 9 *The following are equivalent for a topological space $\langle X, \tau \rangle$.*

(1) X has a bounded complete computational model.

(2) There is a countable family $\mathcal{C}$ of nonempty τ-closed subsets of X such that:

(cp) each subset of $\mathcal{C}$ with the finite intersection property has nonempty intersection,

(br) if $x \in T$ and $T \in \tau$ then there exists $C \in \mathcal{C}$ such that $x \in \mathrm{int}(C)$ and $C \subset T$, and

(r) if $x \in X \setminus C$ for some $C \in \mathcal{C}$ then there exists a $D \in \mathcal{C}$ such that $x \notin D$ and $C \subset \mathrm{int}(D)$.*

(3) $\langle X, \tau \rangle$ is second countable and T_1, and there is a compact topology $\tau^* \subset \tau$ on X such that $\langle X, \tau, \tau^* \rangle$ is pairwise regular.

PROOF. (3)⇒(2): By the regularity of $\langle X, \tau^*, \tau \rangle$, for every τ^*-closed set F and $x \in X \backslash F$ there exists a τ^*-open set T_x such that $x \in T_x$ and $\mathrm{cl}_\tau T_x \subset X \backslash F$. Since $\tau = \tau \vee \tau^*$ is second countable, so is its restriction to the subspace $X \backslash F$; thus this restriction is Lindelöf. In particular, there exists a countable subfamily of $\{T_x : x \in X \setminus F\}$ which covers $X \setminus F$. Let $\mathcal{C}_F$ be the set of complements of elements of this countable family. Then $\mathcal{C}_F$ is countable and

$$F \subset \mathrm{int}(C) \ \text{ for every } C \in \mathcal{C}_F \ \ \& \ \ F = \bigcap \mathcal{C}_F. \tag{2}$$

Let $\mathcal{B}$ be a countable base for $\langle X, \tau \rangle$ and $\mathcal{C}_0 = \{\mathrm{cl}_{\tau^*}(B) : B \in \mathcal{B}\}$. Define a sequence $\langle \mathcal{C}_n : n < \omega \rangle$ by putting

$$\mathcal{C}_{n+1} = \mathcal{C}_n \cup \bigcup_{F \in \mathcal{C}_n} \mathcal{C}_F$$

for every $n < \omega$. Then each $\mathcal{C}_n$ is a countable family of τ^*-closed sets. Thus $\mathcal{C} = \bigcup_{n<\omega} \mathcal{C}_n$ is also a countable family of τ^*-closed sets and it is easy to see that $\mathcal{C}$ is as required.

To show (2) ⇒ (1) first note that, by (br), τ-interiors of the sets from $\mathcal{C}$ form a base for τ. Thus $\langle X, \tau \rangle$ is second countable. Next, let τ^* be the topology generated by the complements of sets from $\mathcal{C}$. Then condition (cp) implies that $\langle X, \tau^* \rangle$ is compact.

Note also that (br) implies also that $\langle X, \tau, \tau^* \rangle$ is regular, while the regularity of $\langle X, \tau^*, \tau \rangle$ follows from (r*). Thus $\langle X, \tau, \tau^* \rangle$ is pairwise regular. Moreover, $\tau \vee \tau^* = \tau$ is Lindelöf (as second countable) so, by Fact 8, $\langle X, \tau, \tau^* \rangle$

is pairwise normal. Thus, for every pair $\langle A, B\rangle$ of subsets of X where A is τ^*-closed and $A \subset \mathrm{int}_\tau(B)$ there exists a τ^*-closed set $c(A,B)$ such that $A \subset \mathrm{int}(c(A,B))$ and $c(A,B) \subset \mathrm{int}(B)$.

Let Γ be the closure of $\mathcal{C}$ under the binary operations of union $\cup$, intersection $\cap$, and c defined above. More directly, we put $\Gamma_0 = \mathcal{C} \cup \{X\}$, for each $k \in \omega$ let

$$\Gamma_{k+1} = \bigcup \{\{c(A,B), B \cup C, B \cap C\} \colon A, B, C \in \Gamma_k \;\&\; B \cap C \neq \emptyset \;\&\; A \subset \mathrm{int}(B)\}$$

and define Γ as $\bigcup_{k\in\omega} \Gamma_k$. Then Γ is a countable family of τ^*-closed sets which satisfies conditions (br), (r*), and (cp), while it is closed under finite intersections, finite unions, and the operation c.

Let $P_X^* = P_X^*(\Gamma)$ be defined as in (1) near the beginning of this section. We will show that P_X^* is a bounded complete computational model for X.[3]

First note that for every $A \in \Gamma$ the filter $j(A) = \{B \in \Gamma \colon A \subset \mathrm{int}_\tau(B)\}$ belongs to P_X^*, since Γ is closed under the operation c.

It should also be clear that if $\mathbf{S} \subset P_X^*$ is directed then $\bigcup \mathbf{S}$ is a filter, in which case $\bigcup \mathbf{S} = \bigvee \mathbf{S} \in P_X^*$. In particular, P_X^* is a dcpo. It is also bounded complete: if $\mathbf{S} \subset P_X^*$ is bounded by an $\mathcal{F} \in P_X^*$, then $u(\mathbf{S}) = \{\bigcup F \colon F \text{ a finite subset of } \bigcup \mathbf{S}\}$ is a directed subset of $\mathcal{F}$, so $\bigvee \mathbf{S} = \bigcup u(\mathbf{S}) \in P_X^*$.

Next note that for every $\mathcal{E}, \mathcal{F} \in P_X^*$

$$\mathcal{E} \ll \mathcal{F} \iff (\exists F \in \mathcal{F})\ \mathcal{E} \subset j(F). \tag{3}$$

To see this first assume that there exists an $F \in \mathcal{F}$ such that $\mathcal{E} \subset j(F)$ and let $\mathbf{S} \subset P_X^*$ be a directed set with $\mathcal{F} \subset \bigvee \mathbf{S} = \bigcup \mathbf{S}$. Then there exists an $\mathcal{F}_0 \in \mathbf{S}$ with $F \in \mathcal{F}_0$. So, $\mathcal{E} \subset j(F) \subset \mathcal{F}_0$.

To see the other implication assume that $\mathcal{E} \ll \mathcal{F}$ and consider the family $\mathbf{S} = \{j(F) \colon F \in \mathcal{F}\}$. Clearly $\mathbf{S}$ is directed and, by (1), $\mathcal{F} = \bigcup \mathbf{S} = \bigvee \mathbf{S}$.

With (3) in hand it is clear that P_X^* is a continuous dcpo: if $\mathcal{F} \in P_X^*$ then $\Downarrow \mathcal{F} = \{\mathcal{E} \in P_X^* \colon (\exists F \in \mathcal{F})\ \mathcal{E} \subset j(F)\}$ and so, by (1), $\mathcal{F} = \bigcup \Downarrow \mathcal{F}$.

The above shows also immediately that the family $\mathcal{D} = \{j(A) \colon A \in \Gamma\}$ forms a basis for P_X^*. Thus, P_X^* is a bounded complete ω-continuous dcpo. To finish the proof it is enough to show that P_X^* is a complete computational model for X.

[3]This construction is closely related to that of rounded ideal completion, which is discussed in some detail in [2].

We do this by showing that a homeomorphism $i\colon X \to \mathrm{Max}(P_X^*)$ can be defined by $i(x) = j(\{x\})$.

To see the maximality of each $i(x)$, let $i(x) \subset \mathcal{F} \in P_X^*$ and, by way of contradiction, assume that there is an $A \in \mathcal{F} \setminus i(x)$. Then there is a $D \in \mathcal{F} \cap \Gamma$ such that $D \subset \mathrm{int}(A)$; if $D \in i(x)$ then $A \in i(x)$, contradicting our assumption. Thus $x \notin D$, and so by (br), there is a $C \in \Gamma$ so that $x \in \mathrm{int}(C) \subset C \subset X \setminus D$, so $X \setminus D \in i(x) \subset \mathcal{F}$, a contradiction to $D \in \mathcal{F}$. Thus $i(x)$ is maximal.

Since $\langle X, \tau\rangle$ is T_1, $\{x\} = \bigcap i(x)$, so i is one-to-one. To see that so i is onto $\mathrm{Max}(P_X^*)$ take an $\mathcal{F} \in \mathrm{Max}(P_X^*)$. The compactness of τ^* guarantees that $\bigcap \mathcal{F} \neq \emptyset$. If $x \in \bigcap \mathcal{F}$ and $\mathcal{F} \neq i(x)$, then $\mathcal{F}$ is a proper subset of $i(x)$, contradicting the maximality of $\mathcal{F}$. Thus $\mathcal{F} = i(x)$, so i is onto.

To see that i is a homeomorphism we need to show that the sets

$$U(\mathcal{F}) = \{x \in X\colon j(\{x\}) \in \Uparrow \mathcal{F}\} = \{x \in X\colon \mathcal{F} \ll j(\{x\})\}$$

with $\mathcal{F} \in P_X^*$ form a base for τ. But, by (3),

$$U(\mathcal{F}) = \{x \in X\colon \exists D_x \in j(\{x\}),\ \mathcal{F} \subset j(D_x)\}.$$

Thus the $U(\mathcal{F})$ are open: for note that if $x \in U(\mathcal{F})$ then $x \in \mathrm{int}_\tau(D_x) \subset U(\mathcal{F})$. On the other hand, by (br), for every $W \in \tau$ and $x \in W$ there exists a $D \in \Gamma$ with $x \in \mathrm{int}_\tau(D) \subset W$ and it is easy to see that $x \in U(j(D)) \subset \mathrm{int}_\tau(D) \subset W$. Thus, i is a homeomorphism.

Finally we need to show that for every $\mathcal{F} \in P_X^*$ the set

$$K(\mathcal{F}) = i^{-1}(\{\mathcal{E} \in \mathrm{Max}(P_X^*)\colon \mathcal{F} \subset \mathcal{E}\}) = \{x \in X\colon \mathcal{F} \subset j(\{x\})\}$$

is τ-closed. For this it is enough to prove that

$$K(\mathcal{F}) = \bigcap \mathcal{F}.$$

But if $x \in K(\mathcal{F})$ and $F \in \mathcal{F}$ then $F \in j(\{x\})$ implying that $x \in \mathrm{int}_\tau(F) \subset F$. So, $K(\mathcal{F}) \subset \bigcap \mathcal{F}$.

Conversely, assume that $x \in \bigcap \mathcal{F}$ and let $F \in \mathcal{F}$. Then, by (1), there exists an $E \in \mathcal{F}$ with $E \subset \mathrm{int}_\tau(F)$. Since $x \in \bigcap \mathcal{F} \subset E$ we conclude that $F \in j(\{x\})$.

(1)$\Rightarrow$(3): Assume $\langle P, \leq\ \rangle$ is a bounded complete computational model for $\langle X, \tau\rangle$ as in Definition 6. We will identify $\langle \mathrm{Max}(P), \sigma\rangle$, with $\langle X, \tau\rangle$, since they are homeomorphic. Let D be a countable $\ll$-dense subset of P. Then, for every $p \in P$, by interpolation:

$$(\Uparrow p) \cap \mathrm{Max}(P) = \bigcup \{(\Uparrow q) \cap \mathrm{Max}(P) \colon p \ll q \ \& \ q \in D\}.$$

The sets $(\Uparrow q) \cap \mathrm{Max}(P)$, $q \in D$, form a countable base for $\langle \mathrm{Max}(P), \sigma \rangle$. So, $\langle X, \tau \rangle$ is second countable.

To see that $\langle X, \tau \rangle$ is T_1 take an $x \in \mathrm{Max}(P)$ and recall that by the continuity of P we have $x = \bigvee (\Downarrow x)$, so that

$$\{x\} = \bigcap_{z \ll x} \{y \in \mathrm{Max}(P) \colon z \leq y\}.$$

Since the sets $\{y \in \mathrm{Max}(P) \colon z \leq y\}$ are τ-closed, $\langle X, \tau \rangle$ is T_1.

Now, let $\mathcal{C}$ be the family of all sets $C_d = \{y \in \mathrm{Max}(P) \colon d \leq y\}$ with $d \in D$ and let τ^* be the smallest topology for which all sets from $\mathcal{C}$ are closed. Thus, $\langle X, \tau^* \rangle$ is second countable, since it is generated by the countable subbase $\mathcal{B} = \{X \setminus C \colon C \in \mathcal{C}\}$. Since $\mathcal{B} \subset \tau$, we also have $\tau^* \subset \tau$.

Next we will show that $\langle X, \tau^* \rangle$ is compact. For this first note that

the family $\mathcal{C}$ satisfies the condition (cp).

Indeed, if $D_0 \subset D$ is such that $\mathcal{C}_0 = \{C_d \colon d \in D_0\}$ has the finite intersection property then the set D_0 is directed: for if D_1 is a finite subset of D_0 and $x \in \bigcap_{d \in D_1} C_d$ then $\{x\}$ is an upper bound of D_1. Since P is a dcpo, the supremum $\bigvee D_0$ is well defined. Now, let $\{x\} \in \mathrm{Max}(P)$ be such that $\bigvee D_0 \leq \{x\}$. Then $x \in \bigcap \mathcal{C}_0$. Now, the Alexander subbasis theorem implies that $\langle X, \tau^* \rangle$ is compact.

To see that $\langle X, \tau, \tau^* \rangle$ is regular, take $x \in U \in \tau$. Clearly, we can assume that U^* is a basic open set, say $U = (\Uparrow p) \cap \mathrm{Max}(P)$. Therefore $p \ll x$ and we can find a $d \in D$ with $p \ll d \ll x$. Then $V = (\Uparrow d) \cap \mathrm{Max}(P)$ is as desired, since $x \in V$ and $\mathrm{cl}_{\tau^*}(V) \subset C_d \subset U$.

For the regularity of $\langle X, \tau^*, \tau \rangle$, take $x \in U^* \in \tau^*$. We need to find a $V^* \in \tau^*$ for which $x \in V^*$ and $\mathrm{cl}_\tau(V^*) \subset U^*$. Clearly it will do to prove this for every U^* from the subbase $\mathcal{B}$. So, assume that $U^* = X \setminus C_d$ for some $d \in D$. Thus $x \notin C_d$. Since $d = \bigvee (\Downarrow d)$ we have that

$$C_d = \bigcap_{z \ll d} \{y \in \mathrm{Max}(P) \colon z \leq y\}.$$

Thus, there is $z \ll d$ such that $x \notin \{y \in \mathrm{Max}(P): z \leq y\}$. Take $d_0, d_1 \in D$ such that $z \ll d_0 \ll d_1 \ll d$. Then we have $C_d \subset (\Uparrow d_1) \cap \mathrm{Max}(P) \in \tau$ and $x \in X \setminus C_{d_0} \in \tau^*$. So, $V^* = X \setminus C_{d_0}$ is as desired. ∎

3 Construction of the other topology

By Theorem 9, in order to learn whether each Polish space has a bounded complete computational model we must determine whether or not it has a countable family $\mathcal{C}$ of τ-closed subsets satisfying (cp), (br) and (r*). Indeed, it does:

Theorem 10 *Every Polish space $\langle X, \tau \rangle$ has a bounded complete computational model.*

Proof. It is enough to show that for every Polish space X there exists a countable collection $\mathcal{C}$ of closed sets satisfying conditions (cp), (br), and (r*) from Theorem 9(2).

The set theoretic and topological terminology and notation used are standard and follow [3] and [8], respectively. For a subset K of a metric space $\langle M, d \rangle$ and a number $r > 0$, the symbol $B_r(K)$ will denote the open ball centered in K with radius r, that is, $B_r(K) = \{x \in M: d(x, K) < r\}$. For $x \in M$ we will write $B_r(x)$ for $B_r(\{x\})$.

Since X is Polish, there exists a compact metrizable space $\langle M, \tau_d \rangle$ with metric d such that X is a dense G_δ-subspace of M. Thus there are dense open subsets $W_0 \supset W_1 \supset W_2 \supset \cdots$ of M such that $X = \bigcap_{n<\omega} W_n$. For every $i < \omega$ let $\mathcal{B}_i$ be a finite cover of M by open balls of diameter $\leq 2^{-i}$ and let $\{B_n: n < \omega\}$ be an enumeration of $\mathcal{B} = \bigcup_{i<\omega} \mathcal{B}_i$. Note that $\mathcal{B}$ is a base for M and that the sequence $\langle \mathrm{diam}(B_n): n < \omega \rangle$ of diameters of B_n's converges to 0. In addition for every $n, i < \omega$ define the sets

$$K_n^i = \{x \in M: B_{2^{-i}}(x) \subset B_n \cap W_n\} = \{x \in M: d(x, M \setminus (B_n \cap W_n)) \geq 2^{-i}\}.$$

Then

$$\text{each } \; K_n^i \text{ is closed}, \quad K_n^i \subset \mathrm{int}(K_n^{i+1}), \quad \text{and} \quad \bigcup_{i<\omega} K_n^i = B_n \cap W_n. \tag{4}$$

To begin constructing our family $\mathcal{C}$ we need the following notions. Let

$$S = \left\{ s \in \bigcup_{n=1}^{\infty} \mathbb{Z}^n : s(0) \geq 0 > s(i) \text{ for every } i > 0 \right\}.$$

Thus, S is the set of finite nonempty sequences of integers, whose first entry is nonnegative and others are negative. Then S is totally ordered by the lexicographic order $\preceq$. For future use note that for any $s,t \in S$ if $s \subset t$ (i.e., t is an extension of s) then $s \preceq t$; also let $\prec$ denote the strict order defined by: $s \prec t$ when $s \preceq t$ and $s \neq t$. We sometimes denote such sequences as $\langle i_0, \dots, i_{n-1} \rangle$ (simply $\langle i \rangle$ if in ω^1); if $s = \langle i_0, \dots, i_{n-1} \rangle \in S$ and $0 > i \in \mathbb{Z}$ then $s\hat{\ }i$ denotes $\langle i_0, \dots, i_{n-1}, i \rangle$.

Of course, if for $0 < n < \omega$ we set

$$S_n = \bigcup_{k=0}^{n-1} \left(\{0, \dots, n-1\} \times \{-(n-1), \dots, -1\}^k \right) = S \cap (-n,n)^{\leq n},$$

then $S = \bigcup_{n=1}^{\infty} S_n$. Below, we inductively define finite collections $\mathcal{F}_n$, indexed by $\{0, \dots, n-1\} \times S_n$: $\mathcal{F}_n = \{C_k^s : s \in S_n, k < n\}$, and consisting of closed sets. The sequence $\langle \mathcal{F}_n : n < \omega \rangle$ is to satisfy six properties. Here are the first three, which are used to show (br) and (r*).

(i) $K_n^i \subset C_n^{\langle i \rangle} \subset \operatorname{int}(K_n^{i+1})$ for $s = \langle i \rangle \in S$.

(ii) If $s \in S$ and $0 > i \in \mathbb{Z}$, then $C_n^{s\hat{\ }i} \subset B_{2^i}(C_n^s)$.

(iii) For $s,t \in S$ if $s \prec t$ then $C_n^s \subset \operatorname{int}(C_n^t)$.

With all the $\mathcal{F}_n$'s (so $\bigcup_{n<\omega} \mathcal{F}_n = \{C_n^s : s \in S \ \& \ n < \omega\}$) constructed, we define $\mathcal{C}_n = \{C_n^s : s \in S\}$, $\hat{\mathcal{C}} = \bigcup_{n<\omega} \mathcal{C}_n$, and $\mathcal{C} = \{C \cap X : C \in \hat{\mathcal{C}}\}$. Then we have the following:

Lemma 11 *If $\mathcal{C}$ is defined as above and the conditions (i)–(iii) hold then $\mathcal{C}$ satisfies (br) and (r*).*

Proof. For (br) first notice that, by (i) and (iii), $K_n^i \subset C_n^{\langle i \rangle}$ and $C_n^s \subset C_n^{\langle s(0)+1 \rangle} \subset \operatorname{int}\left(K_n^{s(0)+1}\right)$ for every $s \in S$ and $n, i < \omega$. So, by (4),

$$\bigcup \{\operatorname{int}(C) : C \in \mathcal{C}_n\} = \bigcup \mathcal{C}_n = B_n \cap W_n \tag{5}$$

for each $n < \omega$. If $x \in T$ and T is an open subset of X, then let U be an open subset of M for which $T = U \cap X$. Since the B_i's form a base for M, there exists an $n < \omega$ such that $x \in B_n \subset U$. So $x \in B_n \cap W_n \subset U \cap W_n$. Thus, by (5), there is a $C \in \mathcal{C}_n \subset \hat{\mathcal{C}}$ for which $x \in \mathrm{int}(C) \subset C \subset B_n \cap W_n \subset U \cap W_n$. In particular, $x \in \mathrm{int}_X(C \cap X)$ and $C \cap X \subset U \cap X = T$, i.e., $C \cap X \in \mathcal{C}$ satisfies (br).

To see (r*), if $x \in X \setminus C$ for some $C = C_n^s \in \hat{\mathcal{C}}$, there is some negative integer i such that $B_{2^i}(x) \subset X \setminus C$, so $x \notin B_{2^i}(C)$. By (ii) and (iii), $D = C_n^{s\hat{\ }i}$ satisfies (r*). ∎

To state properties (iv)–(vi), which are used to show (cp), we need a definition. Recall that a closed set C is regular closed if $C = \mathrm{cl}(\mathrm{int}(C))$. We will say that the family $\mathcal{F}$ of subsets of M is *meet-regular* provided $\bigcap \mathcal{G}$ is regular closed for every finite subfamily $\mathcal{G}$ of $\mathcal{F}$. Moreover for each $n < \omega$ we will choose $\varepsilon_n > 0$ and make sure that in addition to (i)-(iii), the following conditions are satisfied.

(iv) $\mathcal{F}_n$ is meet-regular.

(v) For every $\mathcal{G} \subset \mathcal{F}_n$ if $\bigcap \mathcal{G} = \emptyset$ then $\bigcap_{C \in \mathcal{G}} B_{\varepsilon_n}(C) = \emptyset$.

(vi) If $k < n$, $t \in S_{n+1}$, and s is the largest element of S_n with $s \prec t$ then $B_{\varepsilon_{n+1}}(C_k^t) \subset B_{\varepsilon_n}(C_k^s)$.

Before we describe the details of the construction we show (cp):

Lemma 12 *If $\mathcal{C}$ is defined as above and the conditions (i)–(vi) hold, then $\mathcal{C}$ satisfies (cp).*

PROOF. Let $\hat{\mathcal{D}} \subset \hat{\mathcal{C}}$ be such that $\mathcal{D} = \{C \cap X : C \in \hat{\mathcal{D}}\}$ has the finite intersection property. We have to show that $\bigcap \mathcal{D} \neq \emptyset$. Consider the set $\Gamma = \{n < \omega : \mathcal{D} \cap \mathcal{C}_n \neq \emptyset\}$. We will consider two cases:

Case 1: Γ is infinite. Clearly $\bigcap \hat{\mathcal{D}} \neq \emptyset$, since M is compact. Since (5) holds, $\bigcap \hat{\mathcal{D}} \subset B_n$ for every $n \in \Gamma$ and the diameters of B_n's tend to 0, so we conclude that $\bigcap \hat{\mathcal{D}}$ is a singleton, say $\bigcap \hat{\mathcal{D}} = \{x\}$. If $x \in X$ then $\bigcap \mathcal{D} = \{x\} \neq \emptyset$. So, by way of contradiction assume that $x \in M \setminus X$. Choose an $n \in \Gamma$ such that $x \in M \setminus W_n$ and let $C \in \mathcal{D} \cap \mathcal{C}_n$. Then $\bigcap \hat{\mathcal{D}} \subset C \subset W_n$ and so, $\bigcap \hat{\mathcal{D}} = \bigcap \hat{\mathcal{D}} \cap W_n = \{x\} \cap W_n = \emptyset$, a contradiction.

Case 2: Γ is finite. Let $n < \omega$ be such that for every $k \in \Gamma$, $k < n$ and there exists a $t \in S_n$ so that $C_k^t \in \hat{\mathcal{D}}$. For $k \in \Gamma$ let $s_k \in S_n$ be a $\prec$-maximal element of S_n such that $s_k \preceq t$ for each $t \in S$ with $C_k^t \in \hat{\mathcal{D}}$ (there is such an element since $0 < n$, so S_n is a nonempty, finite set ordered by $\preceq$ and $\langle 0 \rangle \in S_n$ is the $\prec$-least element of S). Thus, if $\tilde{s}_k$ is the immediate $\prec$-successor of s_k in S_n then there exists a $t_k \in S$ such that $s_k \preceq t_k \prec \tilde{s}_k$ and $C_k^{t_k} \in \hat{\mathcal{D}}$. Moreover, if $t_k \in S_m$ then applying (vi) at most $m - n$ many times we note that $C_k^{t_k} \subset B_{\varepsilon_n}(C_k^{s_k})$. By (iii), if $C_k^t \in \hat{\mathcal{D}}$, then $k \in \Gamma$, $C_k^{s_k} \subset C_k^t$, so:

$$\bigcap_{k\in\Gamma} C_k^{s_k} \subset \bigcap \hat{\mathcal{D}} \subset \bigcap_{k\in\Gamma} C_k^{t_k} \subset \bigcap_{k\in\Gamma} B_{\varepsilon_n}(C_k^{s_k}).$$

In particular $\bigcap_{k\in\Gamma} B_{\varepsilon_n}(C_k^{s_k})$ is non-empty since, $\bigcap \hat{\mathcal{D}} \neq \emptyset$. Hence, applying (v) to $\mathcal{G} = \{C_k^{s_k} : k \in \Gamma\} \subset \mathcal{C}_n$ we conclude that $\bigcap_{k\in\Gamma} C_k^{s_k} \neq \emptyset$. So, by (iv), $\operatorname{int}\left(\bigcap_{k\in\Gamma} C_k^{s_k}\right) \neq \emptyset$ and, by the density of X in M we conclude that $\emptyset \neq \operatorname{int}\left(\bigcap_{k\in\Gamma} C_k^{s_k}\right) \cap X \subset \bigcap \hat{\mathcal{D}} \cap X = \bigcap \mathcal{D}$. ■

For the inductive construction we will need two facts. The first is a special case of [15, lemma 4.3] (this lemma is actually stated for finite families of open sets, arbitrary unions of which are regular open; we use it on the set of complements of our closed sets):

Lemma 13 *Let $\mathcal{F}$ be a meet-regular finite family of closed subsets of a metric space. For every open set U and closed set $D \subset U$ there is a closed regular set C such that $D \subset \operatorname{int}(C) \subset C \subset U$ and $\mathcal{F} \cup \{C\}$ is meet-regular.*

We now show the second:

Lemma 14 *For every finite family $\mathcal{F}$ of closed subsets of a compact metric space there exists an $\varepsilon > 0$ such that for every $\mathcal{G} \subset \mathcal{F}$ if $\bigcap \mathcal{G} = \emptyset$ then $\bigcap_{C\in\mathcal{G}} B_\varepsilon(C) = \emptyset$.*

Proof. Given a compact metric space $\langle M, d\rangle$, and a finite family $\mathcal{H}$ of subsets of M let $d_{\mathcal{H}}: M \to \mathbb{R}$ be defined by $d_{\mathcal{H}}(x) = \sum_{H\in\mathcal{H}} d(x, H)$. Certainly,

$$\text{if } d_{\mathcal{H}}(x) > 0 \text{ then } x \notin \bigcap \mathcal{H}. \tag{6}$$

Moreover, if $\mathcal{H}$ is a family of closed sets then

$$\bigcap \mathcal{H} = \emptyset \text{ if and only if } 0 \notin d_{\mathcal{H}}[M]. \tag{7}$$

Let $\mathcal{F}$ be as in the lemma and fix $\mathcal{G} \subset \mathcal{F}$ such that $\bigcap \mathcal{G} = \emptyset$. Then, by (7) and the compactness of M, there is an $\varepsilon_{\mathcal{G}} > 0$ such that $[0, \varepsilon_{\mathcal{G}}) \cap d_{\mathcal{G}}[M] = \emptyset$. It is also easy to see that if n is the cardinality of $\mathcal{F}$ then for every $x \in M$ and $\varepsilon > 0$

$$d_{\{B_\varepsilon(G): G \in \mathcal{G}\}}(x) \geq d_{\mathcal{G}}(x) - n\varepsilon.$$

In particular, if $\delta_{\mathcal{G}} \in (0, \varepsilon_{\mathcal{G}}/(n+1))$ then

$$d_{\{B_{\delta_{\mathcal{G}}}(G): G \in \mathcal{G}\}}(x) \geq d_{\mathcal{G}}(x) - n\delta_{\mathcal{G}} \geq \delta_{\mathcal{G}}.$$

So, by (6), $\bigcap_{G \in \mathcal{G}} B_{\delta_{\mathcal{G}}}(G) = \emptyset$. Let $\varepsilon = \min\{\delta_{\mathcal{G}}: \mathcal{G} \subset \mathcal{F} \ \& \ \bigcap \mathcal{G} = \emptyset\} > 0$. Then $\bigcap_{G \in \mathcal{G}} B_\varepsilon(G) = \emptyset$ for each $\mathcal{G} \subset \mathcal{F}$ such that $\bigcap \mathcal{G} = \emptyset$, showing the lemma. ∎

We start our inductive construction with $\mathcal{F}_0 = \emptyset$. Assume now that we have $\mathcal{F}_n = \{C^s_k: s \in S_n, k < n\}$ and $\varepsilon_0, \ldots, \varepsilon_n$ satisfying (i)–(vi). We will first construct $\mathcal{F}_{n+1} = \{C^s_k: s \in S_{n+1}, k < n+1\}$ satisfying (i)–(iv), and then find an $\varepsilon_{n+1} > 0$ which will guarantee (v) and (vi).

We find it useful to let $\{\langle m_0, v_0\rangle, \ldots, \langle m_{p-1}, v_{p-1}\rangle\}$ be the enumeration of the set $\{0, \ldots, n\} \times S_{n+1} \setminus \{0, \ldots, n-1\} \times S_n$ such that if $0 \leq i < j < p$ then:

$$\text{either } m_j < m_i \text{ or } m_j = m_i \text{ and } v_j \prec v_i. \tag{8}$$

Then for each $i = 0, \ldots, p$, let $R_i = (\{0, \ldots, n-1\} \times S_n) \cup \{v_j: j < i\}$. Thus $R_0 = \{0, \ldots, n-1\} \times S_n$ and $R_p = \{0, \ldots, n\} \times S_{n+1}$. We will next show inductively that for each $i \leq p$ there is a family $\mathcal{E} = \mathcal{E}(R_i) = \{C^s_k: \langle k, s\rangle \in R_i\}$ containing $\mathcal{F}_n$ and satisfying (i)–(iv).

First we notice that for each such R_i, the following fact holds: whenever $\langle m, v\rangle, \langle m, s\hat{\ }j\rangle \in \{0, \ldots, n\} \times S_{n+1}$, and $s \neq \emptyset$,

$$\text{if } v \prec s\hat{\ }j,\ \langle m, v\rangle \in R_i, \text{ and } \langle m, s\hat{\ }j\rangle \notin R_i \text{ then } v \preceq s. \tag{9}$$

To see (9) we use the traditional identification $n = \{0, \ldots, n-1\}$, and notice that $v \prec s\hat{\ }j$ if and only if there exists a $k < \mathrm{dom}(s\hat{\ }j)$ such that

$$v \restriction k = s\hat{\ }j \restriction k \quad \text{and} \quad \text{either } \ \mathrm{dom}(v) = k \ \text{ or } \ v(k) < s\hat{\ }j(k). \tag{10}$$

If either $k < \mathrm{dom}(s)$ or $k = \mathrm{dom}(s) = \mathrm{dom}(v)$ then $v \preceq s$. The remaining case is when $k = \mathrm{dom}(s) < \mathrm{dom}(v)$ in which case

$$v \restriction k = s \ \text{ and } \ v(k) < j. \tag{11}$$

Now, by way of contradiction, suppose that $v \prec s\hat{\ }j$, $\langle m, v\rangle \in R_i$, and $\langle m, s\hat{\ }j\rangle \notin R_i$, while $v \not\preceq s$. First note that $\langle m, v\rangle \in R_0$ is impossible, since then, by (11) we would have $\langle m, s\rangle \in R_0$, and so $v(k) < s\hat{\ }j(k) = j = -n$. Thus, $\langle m, v\rangle, \langle m, s\hat{\ }j\rangle \notin R_0$ and, by (8), $s\hat{\ }j \prec v$, another contradiction. This shows (9).

We now show that the assignment $\mathcal{E}$ on R_{i-1} can be extended to one on R_i, that is, setting $t = v_{i-1}$, that there exists a C^t_m such that $\mathcal{E} \cup \{C^t_m\}$ is meet-regular and satisfies (i)–(iii).

To do this, we first choose finite families $\mathbb{D}^t_m$ and $\mathbb{U}^t_m$ of closed sets and of open sets, respectively, such that $D = \bigcup \mathbb{D}^t_m \subset U = \bigcap \mathbb{U}^t_m$ and then apply Lemma 13 to $\mathcal{E}$, D, and U letting $C^t_m = C$. This will guarantee meet-regularity. To ensure (i)–(iii) we will choose $\mathbb{D}^t_m$ and $\mathbb{U}^t_m$ as follows. (We write (i-iii·u) for the upper estimates and (i-iii·d) for the lower estimates; (iid) is taken care of by (iiid).)

(id) If $t = \langle j\rangle \in S^1$ then $K^j_m \in \mathbb{D}^t_m$.

(iu) If $t = \langle j\rangle \in S^1$ then $\mathrm{int}(K^{j+1}_m) \in \mathbb{U}^t_m$.

(iiu) If $C^s_m \in \mathcal{E}$ and $t = s\hat{\ }j$ then $B_{2^j}(C^s_m) \in \mathbb{U}^t_m$.

(iiid) If $C^v_m \in \mathcal{E}$ and $v \prec t$ then $C^v_m \in \mathbb{D}^t_m$.

(iiiu) If $C^u_m \in \mathcal{E}$ and $t \prec u$ then $\mathrm{int}(C^u_m) \in \mathbb{U}^t_m$.

We now show that $D \subset U$, so this construction is possible, and the family $\mathcal{E} \cup \{C^t_m\}$ is meet-regular. We prove that $D \subset U$ by showing that each element of $\mathbb{D}^t_m$ is a subset of each element of $\mathbb{U}^t_m$. There are six cases, three involving (id) and three involving (iiid):

(id)-(iu): This holds since we already know that $K^j_m \subset \mathrm{int}(K^{j+1}_m)$.

(id)-(iiu): This holds trivially, since it never can occur that $\langle j\rangle = s\hat{\ }k$.

(id)-(iiiu): If $\langle j\rangle = t \prec u$, then $j < u(0)$ or $j = u(0)$ and $u \neq \langle j\rangle$; we then have inductively in the first case that $K^j_m \subset \mathrm{int}\left(K^{u(0)}_m\right) \subset \mathrm{int}(C^u_m)$ and in the second that $K^j_m \subset \mathrm{int}\left(C^{\langle u(0)\rangle}_m\right) \subset \mathrm{int}(C^u_m)$.

(iiid)-(iu): If $v \prec t = \langle j\rangle$ then $v(0) < j$ so by (i), $C^v_m \subset K^{v(0)+1}_m \subset \mathrm{int}(K^{j+1}_m)$.

(iiid)-(iiu): If $v \prec t = s\hat{\ }j$ then, by (9), $v \preceq s$, so by inductive assumption, $C^v_m \subset C^s_m \subset B_{2^j}(C^s_m)$.

(iiid)-(iiiu): If $v \prec t$ and $t \prec u$, then $v \prec u$ so inductively $C^v_m \subset \mathrm{int}(C^u_m)$.

Next, notice that by inductive hypothesis on $\mathcal{E}$, (id) and (iu), $\mathcal{E} \cup \{C^t_m\}$ satisfies (i). Similarly, using (iiu), $\mathcal{E} \cup \{C^t_m\}$ satisfies (ii); using (iiid) and (iiiu), we conclude $\mathcal{E} \cup \{C^t_m\}$ satisfies (iii). This contradicts the maximality of $\mathcal{E}$, showing that $\mathcal{F}_n$ can be extended to $\mathcal{F}_{n+1}$ satisfying (i)–(iv).

We now choose ε_{n+1} so as to ensure (v) and (vi). First apply Lemma 14 to the family $\mathcal{F}_{n+1}$ to obtain an $\varepsilon > 0$ so that for every $\mathcal{G} \subset \mathcal{F}_{n+1}$ if $\bigcap \mathcal{G} = \emptyset$ then $\bigcap_{C \in \mathcal{G}} B_\varepsilon(C) = \emptyset$. For such an ε any $\varepsilon_{n+1} \leq \varepsilon$ guarantees (v). Now there are only finitely many triples $\langle k, s, t\rangle$ relevant for (vi) and for each of them we have $C^t_k \subset B_{\varepsilon_n}(C^s_k)$, so there is an $\varepsilon_{k,s,t} > 0$ for which $B_{\varepsilon_{k,s,t}}(C^t_k) \subset B_{\varepsilon_n}(C^s_k)$. Then choose an $\varepsilon_{n+1} > 0$ less than ε and all relevant $\varepsilon_{k,s,t}$. Now (i)–(vi) hold for $\mathcal{F}_{n+1}$ and $\varepsilon_0, \ldots, \varepsilon_{n+1}$ satisfy (i)–(vi), completing the proof. ∎

4 Final remarks

Note that by Lawson's ([18]) result that for a topological space X

> X has a computational model if and only if X is Polish,

each space with a bounded complete computational model is Polish. Thus by Theorem 9 we have that

> X has a bounded complete computational model if and only if X is Polish,

and we immediately obtain the following corollary:

Corollary 15 *A topological space $\langle X, \tau\rangle$ is Polish if and only if $\langle X, \tau\rangle$ is second countable and T_1, and there is a compact topology $\tau^* \subset \tau$ on X such that $\langle X, \tau, \tau^*\rangle$ is pairwise regular.*

There is a second, somewhat older road to this converse. In [1], (1970), it was shown (in somewhat different terminology) that any metrizable space $\langle X, \tau\rangle$ is topologically complete if and only if there is a second, compact T_1 topology on X, $\tau^* \subset \tau$, such that $\langle X, \tau, \tau^*\rangle$ is regular. But by 9 each space with a bounded complete computational model is second countable and has such a topology (with the additional property that $\langle X, \tau^*, \tau \rangle$, is regular). Thus the space is Polish.

This leads to a question: if a metrizable space $\langle X, \tau \rangle$ is complete must there be a second, compact T_1 topology τ^* on X such that $\langle X, \tau, \tau^* \rangle$ is *pairwise* regular (as we have shown in the separable case)?

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