MAXIMAL LINEABILITY OF SEVERAL SUBCLASSES OF DARBOUX-LIKE MAPS ON $\mathbb{R}$

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Abstract. The class $\mathbb{D}$ of generalized continuous functions on $\mathbb{R}$ known under the common name of *Darboux-like functions* is usually described as consisting of eight families of maps: Darboux, connectivity, almost continuous, extendable, peripherally continuous, those having perfect road, and having either the Cantor Intermediate Value Property (CIVP) or the Strong Cantor Intermediate Value Property (SCIVP). The goal of this paper is to show that all Darboux-like subclasses of $(\mathrm{PC} \setminus \mathrm{D}) \cup (\mathrm{AC} \setminus \mathrm{Ext})$ in the algebra generated by $\mathbb{D}$ are $2^{\mathfrak{c}}$-lineable, that is, have maximal lineability.

Contents

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1. Background and the summary of presented results

Over the last two decades, a lot of mathematicians have been interested in finding the largest possible vector spaces that are contained in various families of real functions, see e.g. survey [8], monograph [6], and the literature cited there. (More recent work in this direction include [5, 11, 21].) Specifically, given a (finite or infinite) cardinal number κ, a subset M of a vector space X is said to be *κ-lineable (in X)* provided there exists a linear space $Y \subset M \cup \{0\}$ of dimension κ. This notion was first studied by Vladimir Gurariĭ [29], even though he did not use the term lineability. He showed that the set of continuous nowhere differentiable functions on [0,1], together with the constant 0 function, contains an infinite dimensional vector space, that is, it is ω-lineable.

In what follows we consider only real-valued functions and no distinction is made between a function and its graph. Standard set-theoretic notation and terminology is used throughout the paper. The reader can check[12] for basic definitions. Also, we will denote the collection of Darboux-like classes of maps by the symbol $\mathbb{D}$, that is, we put $\mathbb{D} := \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}, \mathrm{PC}\}$. For the sake of completeness, we provide below the full definitions of these classes.

- D of all *Darboux functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that $f[C]$ is connected (i.e., an interval) for every connected $C \subset \mathbb{R}$. (Equivalently, $f \in \mathrm{D}$ provided it has the intermediate value property.) This class was first systematically investigated by Jean-Gaston Darboux (1842–1917) in his 1875 paper [23].
- PC of all *peripherally continuous functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that for every number $x \in \mathbb{R}$ there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ with $\lim_{n\to\infty} f(s_n) = f(x) = \lim_{n\to\infty} f(t_n)$. This class was introduced in a 1907 paper [40] of John Wesley Young (1879–1932). The name comes from the papers [30, 31, 39]. Note that any function with a graph dense in $\mathbb{R}^2$ is PC.
- PR of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with perfect road*, that is, such that for every $x \in \mathbb{R}$ there exists a perfect $P \subset \mathbb{R}$ having x as a bilateral limit point (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) such that $f \restriction P$ is continuous at x. This class was introduced in a 1936 paper [33] of Isaie Maximoff, where he proved that $\mathrm{D} \cap \mathcal{B}_1 = \mathrm{PR} \cap \mathcal{B}_1$, where $\mathcal{B}_1$ is the class of Baire class 1 functions.
- Conn of all *connectivity functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that the graph of f restricted to any connected $C \subset \mathbb{R}$ is a connected subset of $\mathbb{R}^2$. This notion can be traced to a 1956 problem [34] stated by John Forbes Nash (1928–2015). We also refer to [31, 38]. Connectivity maps on $\mathbb{R}^2$ are defined in a similar fashion.
- AC of all *almost continuous functions* $f \in \mathbb{R}^{\mathbb{R}}$ (in the sense of Stallings), that is, such that every open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. This class was first seriously studied in a 1959 paper [38] of John Robert Stallings (1935–2008); however, it appeared already in a 1957 paper [31] by Olan H. Hamilton (1899–1976).
- Ext of all *extendable functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that there exists a connectivity function $g: \mathbb{R} \times [0, 1] \to \mathbb{R}$ with $f(x) = g(x, 0)$ for all $x \in \mathbb{R}$. The notion of extendable functions (without the name) first appeared in a 1959 paper [38] of J. Stallings, where he asks a question whether every connectivity function defined on [0, 1] is extendable.

CIVP of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$. This class was first introduced in a 1982 paper [27] of Richard G. Gibson and Fred William Roush.

SCIVP of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with Strong Cantor Intermediate Value Property*, that is, such that for all $p, q \in \mathbb{R}$ with $p \neq q$ and $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$ and $f \restriction P$ is continuous. This notion was introduced in a 1992 paper [37] of Harvey Rosen, R. Gibson, and F. Roush to help distinguish extendable and connectivity functions on $\mathbb{R}$.

The diagram in Fig. 1 shows the relations between the classes in $\mathbb{D}$. The arrows denote strict inclusions.

$$\begin{tikzcd}
 & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] &
\end{tikzcd}$$

FIGURE 1. All inclusions, indicated by arrows, among the Darboux-like classes $\mathbb{D}$. The only inclusions among the intersections of these classes are those that follow trivially from this schema. (See [17, 26].)

The inclusions $\mathrm{Conn} \subset \mathrm{D} \subset \mathrm{PC}$, $\mathrm{PR} \subset \mathrm{PC}$, and $\mathrm{SCIVP} \subset \mathrm{CIVP}$ are obvious from the previous definitions. On the other hand, the remaining inclusions are less obvious. Among them the inclusions $\mathrm{Ext} \subset \mathrm{AC} \subset \mathrm{Conn}$ were proved by Stallings [38], while $\mathrm{CIVP} \subset \mathrm{PR}$ was stated without proof in [28] (although its proof can be found in [26, theorem 3.8]). The inclusion $\mathrm{Ext} \subset \mathrm{SCIVP}$ comes from [37].

The inclusions indicated in Fig. 1 are the only inclusions among these classes even when we add to the considerations the intersections of the classes from the top and bottom rows of Fig. 1. This is well described in the expository papers [13, 17, 26]. Specifically, $\mathrm{AC} \setminus \mathrm{CIVP} \neq \emptyset$ and $\mathrm{CIVP} \setminus \mathrm{AC} \neq \emptyset$ was shown in a 1982 paper [27]. The fact that $\mathrm{Conn} \setminus \mathrm{AC} \neq \emptyset$ is the trickiest to prove and is related to late 1960's papers: [36] of John Henderson Roberts, [22] of James L. Cornette, [32] of F. Burton Jones and Edward S. Thomas Jr., and [9] of J. Brown. The result $\mathrm{D} \setminus \mathrm{Conn} \neq \emptyset$ can be traced to 1965 paper [10] of Andrew M. Bruckner and Jack Gary Ceder (see also [9]), while examples for $\mathrm{PC} \setminus \mathrm{D} \neq \emptyset$, $\mathrm{PR} \setminus \mathrm{CIVP} \neq \emptyset$, and $\mathrm{PC} \setminus \mathrm{PR} \neq \emptyset$ to a 2000 paper [17] of K. C. Ciesielski and Jan Jastrzębski.

The inclusions indicated in Fig. 1 suggest a natural split of $\mathbb{D}$ into two subclasses: $\mathbb{U} := \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}\}$ and $\mathbb{L} := \{\mathrm{Ext}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}, \mathrm{PC}\}$, each consisting of the families that are mutually comparable by inclusion. In particular, the algebra $\mathcal{A}(\mathbb{U})$ of subsets of PC generated by the classes in $\mathbb{U}$ has 5 atoms:

$$\{\mathrm{PC} \setminus \mathrm{D}, \mathrm{D} \setminus \mathrm{Conn}, \mathrm{Conn} \setminus \mathrm{AC}, \mathrm{AC} \setminus \mathrm{Ext}, \mathrm{Ext}\}.$$

Similarly, $\mathcal{A}(\mathbb{L})$ generated by the classes in $\mathbb{L}$ has also 5 atoms:

$$\{\mathrm{PC}\setminus\mathrm{PR}, \mathrm{PR}\setminus\mathrm{CIVP}, \mathrm{CIVP}\setminus\mathrm{SCIVP}, \mathrm{SCIVP}\setminus\mathrm{Ext}, \mathrm{Ext}\}.$$

This means that the algebra $\mathcal{A}(\mathbb{D})$ has theoretically 25 atoms, the intersections $L\cap U$, where $L\in\mathcal{A}(\mathbb{L})$ and $U\in\mathcal{A}(\mathbb{U})$. However, if $\mathrm{Ext}\in\{U,L\}$, then $L\cap U=\varnothing$ unless $L=U=\mathrm{Ext}$. Thus, $\mathcal{A}(\mathbb{D})=\mathcal{A}(\mathbb{U}\cup\mathbb{L})$ has actually 17 atoms: Ext and the 16 atoms presented in Table 1, where for $\mathcal{F}\subset\mathbb{R}^{\mathbb{R}}$ we use the symbol $\neg\mathcal{F}$ to denote the complement of $\mathcal{F}$ with respect to $\mathbb{R}^{\mathbb{R}}$, that is, $\neg\mathcal{F}:=\mathbb{R}^{\mathbb{R}}\setminus\mathcal{F}$.

$\bigcap$	$\mathrm{PC}\setminus\mathrm{PR}$	$\mathrm{PR}\setminus\mathrm{CIVP}$	$\mathrm{CIVP}\setminus\mathrm{SCIVP}$	$\mathrm{SCIVP}\setminus\mathrm{Ext}$
$\mathrm{PC}\setminus\mathrm{D}$	$\mathrm{PC}\cap\neg(\mathrm{PR}\cup\mathrm{D})$	$\mathrm{PR}\cap\neg(\mathrm{CIVP}\cup\mathrm{D})$	$\mathrm{CIVP}\cap\neg(\mathrm{SCIVP}\cup\mathrm{D})$	$\mathrm{SCIVP}\setminus\mathrm{D}$
$\mathrm{D}\cap\neg\mathrm{Conn}$	$\mathrm{D}\cap\neg(\mathrm{PR}\cup\mathrm{Conn})$	$\mathrm{D}\cap\mathrm{PR}\cap\neg(\mathrm{CIVP}\cup\mathrm{Conn})$	$\mathrm{D}\cap\mathrm{CIVP}\cap\neg(\mathrm{SCIVP}\cup\mathrm{Conn})$	$\mathrm{D}\cap\mathrm{SCIVP}\cap\neg\mathrm{Conn}$
$\mathrm{Conn}\cap\neg\mathrm{AC}$	$\mathrm{Conn}\cap\neg(\mathrm{PR}\cup\mathrm{AC})$	$\mathrm{Conn}\cap\mathrm{PR}\cap\neg(\mathrm{CIVP}\cup\mathrm{AC})$	$\mathrm{Conn}\cap\mathrm{CIVP}\cap\neg(\mathrm{SCIVP}\cup\mathrm{AC})$	$\mathrm{Conn}\cap\mathrm{SCIVP}\cap\neg\mathrm{AC}$
$\mathrm{AC}\cap\neg\mathrm{Ext}$	$\mathrm{AC}\setminus\mathrm{PR}$	$\mathrm{AC}\cap\mathrm{PR}\cap\neg\mathrm{CIVP}$	$\mathrm{AC}\cap\mathrm{CIVP}\cap\neg\mathrm{SCIVP}$	$\mathrm{AC}\cap\mathrm{SCIVP}\cap\neg\mathrm{Ext}$

TABLE 1. All atoms of $\mathcal{A}(\mathbb{D})$ with exception of Ext.

The goal of this paper is to show that all classes presented in rows 1 and 4 of Table 1 are $2^{\mathfrak{c}}$-lineable, see Table 2 below. This work is a part of the general study of lineability of all classes presented in Table 1, which is spread over the papers [1–4, 14] and is expected to lead to a Ph.D. dissertation of the first author, written under the supervision of the second author.

The lineabilities of the classes in $\mathbb{D}$ (including some of their differences) have been previously studied in 2005 paper [7] and 2010 papers [24, 25]. (See also 2014 survey [8].) The fact that all classes in $\mathbb{D}$ are $2^{\mathfrak{c}}$-lineable was established in a 2014 article [16]. The systematic study of the atoms of $\mathcal{A}(\mathbb{D})$ has been initiated in a 2021 paper [18], but did not directly concern their lineabilities.

Table 2 summarizes the results presented in this paper and in [1, 2, 4, 14].

$\bigcap$	$\mathrm{PC}\setminus\mathrm{PR}$ $2^{\mathfrak{c}}$	$\mathrm{PR}\setminus\mathrm{CIVP}$ $2^{\mathfrak{c}}$	$\mathrm{CIVP}\setminus\mathrm{SCIVP}$ $2^{\mathfrak{c}}$	$\mathrm{SCIVP}\setminus\mathrm{Ext}$ $2^{\mathfrak{c}}$
$\mathrm{PC}\setminus\mathrm{D}$ $2^{\mathfrak{c}}$	$2^{\mathfrak{c}}$ Thm 2.5	$2^{\mathfrak{c}}$ Thm 2.10	$2^{\mathfrak{c}}$ Thm 2.15	$2^{\mathfrak{c}}$ Thm 2.18
$\mathrm{D}\setminus\mathrm{Conn}$ $2^{\mathfrak{c}}$, [14, Thm 2.1]	$2^{\mathfrak{c}}$ [2]	$2^{\mathfrak{c}}$ [2]	$2^{\mathfrak{c}}$ [2]	$\mathfrak{c}^+$ when $\mathfrak{c}$ is regular, [4] $\mathfrak{c}$ in ZFC, [1][a]
$\mathrm{Conn}\setminus\mathrm{AC}$ $\mathfrak{c}$, [1]	$\mathfrak{c}$ [1]	$\mathfrak{c}$ [1]	$\mathfrak{c}$ [1]	$\mathfrak{c}$ [1]
$\mathrm{AC}\setminus\mathrm{Ext}$ $2^{\mathfrak{c}}$	$2^{\mathfrak{c}}$ Thm 2.8	$2^{\mathfrak{c}}$ Thm 2.11	$2^{\mathfrak{c}}$ Thm 2.16	$2^{\mathfrak{c}}$ Thm 2.20

TABLE 2. The values of lineability for all the classes in Table 1 and references to these results.

[a]The authors just proved, in ZFC, that this number is $2^{\mathfrak{c}}$, to appear in [2].

2. The results

For an $f \in \mathbb{R}^{\mathbb{R}}$ its *support* is defined as $\operatorname{supp}(f) := \{x \in \mathbb{R} : f(x) \neq 0\}$. The following definition will constitute the main tool used in this paper.

Definition 2.1. For a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of functions with pairwise disjoint supports we define the *canonical linear space* $W_{\mathcal{F}}$ over $\mathbb{R}$, a subspace of $\mathbb{R}^{\mathbb{R}}$, as

$$W_{\mathcal{F}} := \bigcup_{n \in \mathbb{N}} \left\{ \sum_{i<n} a_i \varphi_i : a_i \in \mathbb{R} \ \& \ \varphi_i \in V_{\mathcal{F}} \text{ for every } i < n \right\},$$

where $V_{\mathcal{F}} = \left\{ \sum_{f \in \mathcal{F}} h(f) f : h \in \{0,1\}^{\mathcal{F}} \right\}$. That is, $W_{\mathcal{F}}$ is spanned by $V_{\mathcal{F}}$.

Notice that each element in $V_{\mathcal{F}}$ is well defined, since maps in $\mathcal{F}$ have pairwise disjoint supports. Also, if $|\mathcal{F}| = \mathfrak{c}$, then $|V_{\mathcal{F}}| = 2^{\mathfrak{c}}$. So, the following remark is obvious.

Remark 2.2. *If $|\mathcal{F}| = \mathfrak{c}$, then $W_{\mathcal{F}}$ has dimension $2^{\mathfrak{c}}$.*

Notice $W_{\mathcal{F}}$ is strictly contained in the vector space $\mathcal{L}_{\mathcal{F}} = \left\{ \sum_{f \in \mathcal{F}} s(f) \cdot f : s \in \mathbb{R}^{\mathcal{F}} \right\}$ naturally associated with the family $\mathcal{F}$ and earlier considered in the literature. (See [1] and the literature cited therein.)

In what follows we will repeatedly use the following simple fact that we will leave without a proof.

Remark 2.3. *If $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is a family of functions with pairwise disjoint supports and $g \in \mathcal{L}_{\mathcal{F}}$ is non-zero, then there is an $f \in \mathcal{F}$ and a non-zero $c \in \mathbb{R}$ such that $g \restriction \operatorname{supp}(f) = c\, f \restriction \operatorname{supp}(f)$.*

2.1. Lineability of $\mathrm{PC} \setminus (\mathrm{D} \cup \mathrm{PR})$.

Here and in what follows $\{B^r_\xi : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ is a fixed partition of $\mathbb{R}$ into Bernstein sets. For a dense subset D of $\mathbb{R}$ and $\xi < \mathfrak{c}$ define

(α) $\alpha^D_\xi := \sum_{d \in D} d\, \chi_{B^d_\xi}$,

where χ_B denotes the characteristic function of $B \subset \mathbb{R}$. Clearly the supports of maps in the family $\mathcal{F}(\alpha^D) := \{\alpha^D_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact that we will leave without a proof.

Fact 2.4. *If $D \subset \mathbb{R}$ is dense, $f \in \mathcal{F}(\alpha^D)$, $c \in \mathbb{R} \setminus \{0\}$, and $g \in \mathbb{R}^{\mathbb{R}}$ is such that $g \restriction \operatorname{supp}(f) = c\, f \restriction \operatorname{supp}(f)$, then $g \restriction P$ is dense in $P \times \mathbb{R}$ for every perfect $P \subset \mathbb{R}$. In particular, g has a dense graph and belongs to $\mathrm{PC} \setminus \mathrm{PR}$.*

Theorem 2.5. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g \in \mathrm{PC} \setminus (\mathrm{D} \cup \mathrm{PR})$ for every non-zero $g \in W_{\mathcal{F}}$. In particular, $\mathrm{PC} \setminus (\mathrm{D} \cup \mathrm{PR})$ is $2^{\mathfrak{c}}$-lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\alpha^{\mathbb{Q}})$ is as needed. Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 2.3, there is an $f \in \mathcal{F}(\alpha^{\mathbb{Q}})$ and $c \in \mathbb{R} \setminus \{0\}$ with $g \restriction \operatorname{supp}(f) = c\, f \restriction \operatorname{supp}(f)$. Thus, by Fact 2.4, g has a dense graph and belongs to $\mathrm{PC} \setminus \mathrm{PR}$.

Also, if $g = \sum_{i<n} a_i \varphi_i$, with $a_i \in \mathbb{R}$ and $\varphi_i \in V_{\mathcal{F}}$, then $g[\mathbb{R}]$ is contained in $a_1\varphi_1[\mathbb{R}] + a_2\varphi_2[\mathbb{R}] + \cdots + a_n\varphi_n[\mathbb{R}] \subset a_1\mathbb{Q} + a_2\mathbb{Q} + \cdots + a_n\mathbb{Q}$, a countable set. So $g[\mathbb{R}] \subsetneq \mathbb{R}$ which, together with the density of the graph of g, implies that $g \in \neg\mathrm{D}$.

Of course, by Remark 2.2, $W_{\mathcal{F}}$ has dimension $2^{\mathfrak{c}}$. □

2.2. **Lineability of** $\mathrm{AC} \setminus \mathrm{PR}$**.** A set $B \subset \mathbb{R}^2$ is a *blocking set* provided it is closed, meets the graph of every continuous function, and is disjoint with some (arbitrary) function $h \in \mathbb{R}^{\mathbb{R}}$. In what follows, the family of all blocking sets will be denoted by $\mathbb{B}$. It is well known and easy to see that an $f \in \mathbb{R}^{\mathbb{R}}$ is in AC if, and only if, $f \cap K \neq \emptyset$ for every $K \in \mathbb{B}$. Recall that the x-axis projection of every blocking set contains a non-trivial interval, see e.g. [35]. (Compare also [18, lemma 5.1] and related history.)

As above, let $\{B^r_\xi : r \in \mathbb{R} \ \&\ \xi < \mathfrak{c}\}$ be a partition of $\mathbb{R}$ into Bernstein sets.

Fact 2.6. *For any meager set $M \subset \mathbb{R}$ and $\xi < \mathfrak{c}$ there exists a map $\beta^M_\xi \in \mathbb{R}^{\mathbb{R}}$ with $\mathrm{supp}(\beta^M_\xi) \subset \bigcup_{r \in \mathbb{R}} B^r_\xi \setminus M$ such that*

(β_1) $\left(\beta^M_\xi \restriction (B^0_\xi \setminus M)\right) \cap K \neq \emptyset$ *for every $K \in \mathbb{B}$; and*

(β_2) $\beta^M_\xi[\mathrm{supp}(\beta^M_\xi) \cap P]$ *is unbounded for every perfect P contained in $\mathbb{R} \setminus M$.*

Proof. Take a function ϕ from an $E \subset B^0_\xi \setminus M$ into $\mathbb{R}$ such that $\phi \cap K \neq \emptyset$ for every $K \in \mathbb{B}$. Such a map can be constructed by an easy transfinite induction, see e.g. [35]. Let $\Phi_\xi \in \mathbb{R}^{\mathbb{R}}$ be an extension of ϕ such that $\Phi_\xi(x) = 0$ for every $x \in \mathbb{R} \setminus E$. Then $\beta^M_\xi := \Phi_\xi + \alpha^{\mathbb{Q}}_\xi \cdot \chi_{\mathbb{R} \setminus M}$ is as needed since Φ_ξ and $\alpha^{\mathbb{Q}}_\xi$ have disjoint supports, Φ_ξ ensures (β_1), and, by Fact 2.4, $\alpha^{\mathbb{Q}}_\xi \cdot \chi_{\mathbb{R} \setminus M}$ ensures (β_2). □

Clearly the supports of maps in the family $\mathcal{F}(\beta^M) := \{\beta^M_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact that we will leave without a proof.

Fact 2.7. *If $M \subset \mathbb{R}$ is meager, $f \in \mathcal{F}(\beta^M)$, $c \in \mathbb{R} \setminus \{0\}$, and $g \in \mathbb{R}^{\mathbb{R}}$ is such that $g \restriction \mathrm{supp}(f) = c\, f \restriction \mathrm{supp}(f)$, then g has a dense graph, belongs to* AC*, and $g[P]$ is unbounded for every perfect P contained in $\mathbb{R} \setminus M$.*

Theorem 2.8. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g \in \mathrm{AC} \setminus \mathrm{PR}$ for every non-zero $g \in W_{\mathcal{F}}$. In particular,* $\mathrm{AC} \setminus \mathrm{PR}$ *is $2^{\mathfrak{c}}$-lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\beta^{\emptyset})$ is as needed. This follows from Fact 2.7 and Remarks 2.2 and 2.3. Specifically, a non-zero $g \in W_{\mathcal{F}}$ is not in PR since, by Fact 2.7, $g[P]$ is unbounded for every perfect $P \subset \mathbb{R}$ and so, $g \restriction P$ is discontinuous at every $x \in P$. □

2.3. **Lineability of** $\mathrm{PR} \setminus (\mathrm{D} \cup \mathrm{CIVP})$**.** Here and in what follows the symbol $\mathcal{B}$ denotes the standard countable basis $\{(p,q) : p < q \ \&\ p, q \in \mathbb{Q}\}$ of $\mathbb{R}$ and $\{P^I \subset I : I \in \mathcal{B}\}$ is a family of pairwise disjoint nowhere dense perfect sets. For every $I \in \mathcal{B}$ let $\{P^I_\xi : \xi < \mathfrak{c}\}$ be an enumeration of some partition of P^I into perfect sets. For the use in the later part of this paper it is convenient to put $\mathcal{P}^I := \{h_I[\{x\} \times 2^\omega] : x \in 2^\omega\}$, where h_I is a homeomorphism from $2^\omega \times 2^\omega$ onto P^I. Notice that the sets

$$M_\xi := \bigcup_{I \in \mathcal{B}} P^I_\xi,$$

are pairwise disjoint and that

$$M := \bigcup_{I \in \mathcal{B}} P^I = \bigcup_{\xi < \mathfrak{c}} M_\xi \tag{1}$$

is meager.

For every $\xi < \mathfrak{c}$ let

(γ) $\gamma_\xi := \sum_{I\in\mathcal{B}} \gamma_\xi^I$, where $\gamma_\xi^I:\mathbb{R}\to\mathbb{Q}$ has support contained in P_ξ^I and $(\gamma_\xi^I)^{-1}(q)$ contains non-empty perfect set for every $q\in\mathbb{Q}$.

Clearly the supports of maps in the family $\mathcal{F}(\gamma) := \{\gamma_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact.

Fact 2.9. *If $f\in\mathcal{F}(\gamma)$ and $g\in\mathbb{R}^\mathbb{R}$ is such that $g\restriction \operatorname{supp}(f) = c\, f\restriction\operatorname{supp}(f)$ for some $c\in\mathbb{R}\setminus\{0\}$, then g has a dense graph and belongs to* PR.

Proof. Clearly (γ) implies that $f\restriction\operatorname{supp}(f)$ is dense in $\mathbb{R}^2$, so g has a dense graph. To see that $g\in\mathrm{PR}$ choose an $x\in\mathbb{R}$ and a sequence $\langle q_n : n<\omega\rangle$ of non-zero rational numbers such that $c\cdot q_n \to_{n\to\infty} g(x)$.

Choose a sequence $\langle (a_n,b_n)\in\mathcal{B} : n<\omega\rangle$ such that $\lim_{n\to\infty} a_n = x$ and $a_0 < b_0 < a_2 < b_2 < \cdots < x < \cdots < a_3 < b_3 < a_1 < b_1$. By ($\gamma$), for every $n<\omega$ there exists a perfect set $P_n \subset (a_n,b_n)$ such that $f[P_{2n}\cup P_{2n+1}] = \{q_n\}$. Then $P := \{x\}\cup\bigcup_{n<\omega} P_n$ is a perfect set having x as a bilateral limit point and $g\restriction P$ is continuous at x. □

Theorem 2.10. *There exists a family $\mathcal{F}\subset\mathbb{R}^\mathbb{R}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g\in\mathrm{PR}\setminus(\mathrm{D}\cup\mathrm{CIVP})$ for every non-zero $g\in W_\mathcal{F}$. In particular,* $\mathrm{PR}\setminus(\mathrm{D}\cup\mathrm{CIVP})$ *is $2^\mathfrak{c}$-lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\gamma)$ is as needed. Indeed, if $g\in W_\mathcal{F}$ is non-zero, then, by Remark 2.3 and Fact 2.9, g has a dense graph and belongs to PR. Also, similarly as in the proof of Theorem 2.5 we see that $g[\mathbb{R}]$ is countable. This and the density of its graph imply that $g\in\neg\mathrm{D}$.

Finally, to see that $g\in\neg\mathrm{CIVP}$, using density of the graph of g, choose $p<q$ so that $g(p)<g(q)$. Since $g[\mathbb{R}]$ is countable, there is perfect $K\subset(g(p),g(q))\setminus g[\mathbb{R}]$. Then there is no nonempty $P\subset(p,q)$ with $g[P]\subset K$, that is, indeed $g\in\neg\mathrm{CIVP}$. □

2.4. **Lineability of** $\mathrm{AC}\cap\mathrm{PR}\setminus\mathrm{CIVP}$**.** Using the notation as above, for every $\xi<\mathfrak{c}$ define

(δ) $\delta_\xi := \gamma_\xi + \beta_\xi^M$.

Notice that the supports of γ_ξ and β_ξ^M are disjoint, the first contained in M, the second in $\mathbb{R}\setminus M$. It is also easy to see that the supports of maps in the family $\mathcal{F}(\delta) := \{\delta_\xi : \xi<\mathfrak{c}\}$ are pairwise disjoint.

Theorem 2.11. *There exists a family $\mathcal{F}\subset\mathbb{R}^\mathbb{R}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g\in\mathrm{AC}\cap\mathrm{PR}\setminus\mathrm{CIVP}$ for every non-zero $g\in W_\mathcal{F}$. In particular,* $\mathrm{AC}\cap\mathrm{PR}\setminus\mathrm{CIVP}$ *is $2^\mathfrak{c}$-lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\delta)$ is as needed. Indeed, if $g\in W_\mathcal{F}$ is non-zero, then, by Remark 2.3, there exist an $f=\delta_\xi\in\mathcal{F}(\delta)$ and a number $c\in\mathbb{R}\setminus\{0\}$ with $g\restriction\operatorname{supp}(\delta_\xi) = c\,\delta_\xi\restriction\operatorname{supp}(\delta_\xi)$. Since $\operatorname{supp}(\gamma_\xi)\subset\operatorname{supp}(\delta_\xi)$, this implies that $g\restriction\operatorname{supp}(\gamma_\xi) = c\,\delta_\xi\restriction\operatorname{supp}(\gamma_\xi) = c\,\gamma_\xi\restriction\operatorname{supp}(\gamma_\xi)$ so by Fact 2.9, g has a dense graph and belongs to PR. Similarly $\operatorname{supp}(\beta_\xi^M)\subset\operatorname{supp}(\delta_\xi)$, which implies that $g\restriction\operatorname{supp}(\beta_\xi^M) = c\,\delta_\xi\restriction\operatorname{supp}(\beta_\xi^M) = c\,\beta_\xi^M\restriction\operatorname{supp}(\beta_\xi^M)$ so, by Fact 2.7, $g\in\mathrm{AC}$.

Finally, to see that $g\in\neg\mathrm{CIVP}$, notice that $g[M]=\hat{g}[\mathbb{R}]$ for some $\hat{g}\in W_{\mathcal{F}(\gamma)}$ so, as in the proof of Theorem 2.10, $g[M]=\hat{g}[\mathbb{R}]$ is countable. Since g has a dense graph, we can choose $p<q$ so that $g(p)<g(q)$. Also, choose a nonempty perfect $K\subset\mathbb{R}\setminus g[M]$. Take a nonempty perfect $P\subset(p,q)$. It is enough to prove that $g[P]\not\subset K$. So, by way of contradiction, assume that there is a prefect $P\subset\mathbb{R}$ with

$g[P] \subset K$. Then, reducing P if necessary, we can assume that P is either contained in or disjoint with M.

But $P \subset \mathbb{R} \setminus M$ is impossible, since in such case Fact 2.6 implies that the set $g[P] \supset g[\operatorname{supp}(\beta_\xi^M) \cap P] = c\ \beta_\xi^M[\operatorname{supp}(\beta_\xi^M) \cap P]$ is unbounded, so it cannot be contained in bounded K.

Similarly, $P \subset M$ implies that $g[P] \subset g[M]$, which is disjoint with K, contradicting $g[P] \subset K$. Thus $g \in \neg\,\mathrm{CIVP}$ as needed. □

2.5. **Lineability of** $\mathrm{CIVP} \setminus (\mathrm{D} \cup \mathrm{SCIVP})$**.** Here the families $\mathcal{P}^I$, used earlier to construct functions γ_ξ, will need to be chosen more carefully with the help of the following lemma.

Lemma 2.12. *For every $I \in \mathcal{B}$ there is a subfamily $\mathcal{P}_0^I$ of $\mathcal{P}^I$ with $|\mathcal{P}_0^I| = \mathfrak{c}$ such that if $\mathcal{P}_0 := \bigcup_{I \in \mathcal{B}} \mathcal{P}_0^I$, then for every perfect $P \subset \mathbb{R}$,*

- *if $|P \cap Q| \le \omega$ for every $Q \in \mathcal{P}_0$, then $|P \setminus \bigcup \mathcal{P}_0| = \mathfrak{c}$.*

Proof. Let B be a Bernstein subset of 2^ω, that is, such that $B \cap Q \neq \varnothing \neq Q \setminus B$ for every perfect $Q \subset 2^\omega$. Clearly $|B| = \mathfrak{c}$. For every $I \in \mathcal{B}$ let h_I be a homeomorphism from $2^\omega \times 2^\omega$ onto P^I and let $\mathcal{P}_0^I := \{h_I[\{b\} \times 2^\omega] : b \in B\}$.

To see that this choice ensures •, choose a perfect $P \subset \mathbb{R}$ so that $|P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$. We need to find a $b \in B$ and an $I \in \mathcal{B}$ so that $|P \cap h_I[\{b\} \times 2^\omega]| > \omega$.

Since $\left|P \setminus \bigcup_{I \in \mathcal{B}} P^I\right| \le |P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$ there is an $I \in \mathcal{B}$ and a perfect $Q \subset P \cap P^I$. Let $\pi_1 : 2^\omega \times 2^\omega \to 2^\omega$ be the projection onto the first coordinate. If the compact set $Q_0 := \pi_1[h_I^{-1}(Q)]$ is uncountable, then the set $Q_0 \setminus B$ has cardinality $\mathfrak{c}$ and so has the set $h_I[\pi_1^{-1}(Q_0 \setminus B)] \subset P \setminus \bigcup \mathcal{P}_0$, contradicting our assumption that $|P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$. So, we can assume that Q_0 is countable. Then, for some $b \in Q_0$, the set $h_I[\pi_1^{-1}(b)] = h_I[\{b\} \times 2^\omega] \subset Q \subset P$ has cardinality $\mathfrak{c}$. By the assumption $|P \setminus \bigcup \mathcal{P}_0| < \mathfrak{c}$ we must have $b \in B$, since otherwise $h_I[\pi_1^{-1}(b)] \subset P \setminus \bigcup \mathcal{P}_0$. So, $h_I[\pi_1^{-1}(b)] \subset P \cap h_I[\{b\} \times 2^\omega]$ is uncountable, as needed. □

To ensure CIVP the range of our modified functions γ_ξ needs to intersect every perfect set while no generated function can be surjective. This will be achieved with the following lemma.

Lemma 2.13. *There exists a linear space $V \subset \mathbb{R}$ over $\mathbb{Q}$ which intersects every non-empty perfect set $P \subset \mathbb{R}$ and such that*

- *$a_1 V + \cdots + a_n V \neq \mathbb{R}$ for every $a_1, \ldots, a_n \in \mathbb{R}$.*

Proof. Let T be a transcendental basis that is also a Bernstein set—it can be constructed by an easy transfinite induction. (See e.g. [14]. Compare also [12, theorem 7.3.4], where an analogous construction of a Hamel basis that is a Bernstein set is described.) Choose countable infinite subset T_0 of T and let V be a vector space over $\mathbb{Q}$ generated by $T \setminus T_0$. Notice that it is as needed.

To see •, let $F = \mathbb{Q}(T \setminus T_0)$ be a subfield of $\mathbb{R}$ generated by $T \setminus T_0$. In particular T_0 is linearly independent over F, implying that the dimension of $\mathbb{R}$ over F is infinite. Therefore, if $a_1, a_2, \cdots, a_n \in \mathbb{R}$, then $a_1 V + \cdots + a_n V \subset a_1 F + \cdots + a_n F \subsetneq \mathbb{R}$. Clearly $V \supset T \setminus T_0$ intersects every non-empty perfect set $P \subset \mathbb{R}$. □

Let $\mathcal{P}$ be the family of all perfect subsets of $\mathbb{R}$ and $\left\{P_\xi^{I,C} \subset P^I : \xi < \mathfrak{c}\ \&\ C \in \mathcal{P}\right\}$ be an enumeration of $\mathcal{P}_0^I$. For every $\xi < \mathfrak{c}$ let

(κ) $\kappa_\xi := \sum_{\langle I,C\rangle\in\mathcal{B}\times\mathcal{P}} \kappa_\xi^{I,C}$, where $\kappa_\xi^{I,C}:\mathbb{R}\to V$ has support contained in $P_\xi^{I,C}$, $\kappa_\xi^{I,C}[P_\xi^{I,C}]\subset C\cap V$, and $\kappa_\xi^{I,C}$ is discontinuous on any perfect subset of $P_\xi^{I,C}$.[1]

Clearly the supports of the maps in the family $\mathcal{F}(\kappa) := \{\kappa_\xi : \xi<\mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact.

Fact 2.14. *If $f\in\mathcal{F}(\kappa)$ and $g\in\mathbb{R}^\mathbb{R}$ is such that $g\restriction \operatorname{supp}(f) = c\, f\restriction\operatorname{supp}(f)$ for some $c\in\mathbb{R}\setminus\{0\}$, then g has a dense graph and belongs to* CIVP.

Proof. This easily follows from our definition (κ). □

Theorem 2.15. *There exists a family $\mathcal{F}\subset\mathbb{R}^\mathbb{R}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g\in \mathrm{CIVP}\setminus(\mathrm{D}\cup\mathrm{SCIVP})$ for every non-zero $g\in W_\mathcal{F}$. In particular,* $\mathrm{CIVP}\setminus(\mathrm{D}\cup\mathrm{SCIVP})$ *is $2^\mathfrak{c}$-lineable.*

Proof. The family $\mathcal{F}:=\mathcal{F}(\kappa)$ is as needed. Indeed, if $g\in W_\mathcal{F}$ is non-zero, then, by Remark 2.3 and Fact 2.14, g has a dense graph and belongs to CIVP.

Also, if $g=\sum_{i<n} a_i\varphi_i$, with $a_i\in\mathbb{R}$ and $\varphi_i\in V_\mathcal{F}$, then $g[\mathbb{R}]$ is contained in $a_1\varphi_1[\mathbb{R}]+a_2\varphi_2[\mathbb{R}]+\dots+a_n\varphi_n[\mathbb{R}]\subset a_1V+a_2V+\dots+a_nV$ which, by Lemma 2.13, is strictly contained in $\mathbb{R}$. So $g[\mathbb{R}]\subsetneq\mathbb{R}$ which, together with the density of the graph of g, implies that $g\in\neg\mathrm{D}$.

Finally, to see that $g\in\neg\mathrm{SCIVP}$, using density of the graph of g, choose $p<q$ with $g(p)<g(q)$ and a nonempty perfect $K\subset(g(p),g(q))\setminus\{0\}$. It is enough to show that for every perfect set $P\subset(p,q)$ with $g[P]\subset K$ the restriction $g\restriction P$ is discontinuous. Indeed, $g[P]\subset K\not\ni 0$ implies that $P\subset\bigcup_{\xi<\mathfrak{c}}\operatorname{supp}(\kappa_\xi)\subset\bigcup\mathcal{P}_0$. So, by Lemma 2.12, there is a $P_\xi^{I,C}\in\bigcup\mathcal{P}_0$ with $|P\cap P_\xi^{I,C}|>\omega$. In particular, there exists a perfect set $Q\subset P\cap P_\xi^{I,C}$. Notice that $g\restriction\operatorname{supp}(\kappa_\xi)=c\,\kappa_\xi\restriction\operatorname{supp}(\kappa_\xi)$ and $c\neq 0$, since otherwise $g[Q]=\{0\}\notin K$. Since $\kappa_\xi\restriction Q=\kappa_\xi^{I,C}\restriction Q$ is discontinuous, as ensured in (κ), $g\restriction Q$ is discontinuous. □

2.6. **Lineability of** $\mathrm{AC}\cap\mathrm{CIVP}\setminus\mathrm{SCIVP}$**.** Using the notation as above, for every $\xi<\mathfrak{c}$ define

(λ) $\lambda_\xi := \kappa_\xi+\beta_\xi^M$.

Notice that the supports of κ_ξ and β_ξ^M are disjoint, the first contained in M, the second in $\mathbb{R}\setminus M$. It is also easy to see that the supports of the maps in the family $\mathcal{F}(\lambda):=\{\lambda_\xi:\xi<\mathfrak{c}\}$ are pairwise disjoint.

Theorem 2.16. *There exists a family $\mathcal{F}\subset\mathbb{R}^\mathbb{R}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g\in\mathrm{AC}\cap\mathrm{CIVP}\setminus\mathrm{SCIVP}$ for every non-zero $g\in W_\mathcal{F}$. In particular,* $\mathrm{AC}\cap\mathrm{CIVP}\setminus\mathrm{SCIVP}$ *is $2^\mathfrak{c}$-lineable.*

Proof. The family $\mathcal{F}:=\mathcal{F}(\lambda)$ is as needed.

Indeed, if $g\in W_\mathcal{F}$ is non-zero, then, by Remark 2.3, there is an $f=\lambda_\xi\in\mathcal{F}(\lambda)$ and $c\in\mathbb{R}\setminus\{0\}$ with $g\restriction\operatorname{supp}(\lambda_\xi)=c\,\lambda_\xi\restriction\operatorname{supp}(\lambda_\xi)$. Since $\operatorname{supp}(\kappa_\xi)\subset\operatorname{supp}(\lambda_\xi)$, this implies that $g\restriction\operatorname{supp}(\kappa_\xi)=c\,\lambda_\xi\restriction\operatorname{supp}(\kappa_\xi)=c\,\kappa_\xi\restriction\operatorname{supp}(\kappa_\xi)$ so by Fact 2.14, g has a dense graph and belongs to CIVP. Similarly $\operatorname{supp}(\beta_\xi^M)\subset\operatorname{supp}(\lambda_\xi)$, which

[1] $\kappa_\xi^{I,C}\restriction P_\xi^{I,C}$ is just a Sierpiński-Zygmund function from $P_\xi^{I,C}$ into $C\cap V$, which can be easily constructed by a transfinite induction, see e.g. [20].

implies that $g \restriction \mathrm{supp}(\beta_\xi^M) = c\ \lambda_\xi \restriction \mathrm{supp}(\beta_\xi^M) = c\ \beta_\xi^M \restriction \mathrm{supp}(\beta_\xi^M)$ so, by Fact 2.7, $g \in \mathrm{AC}$.

Lastly, to see $g \in \neg\mathrm{SCIVP}$, using density of the graph of g, choose $p < q$ with $g(p) < g(q)$ and a nonempty perfect $K \subset (g(p), g(q)) \setminus \{0\}$. It is enough to show that for every perfect set $P \subset (p,q)$ with $g[P] \subset K$ the restriction $g \restriction P$ is discontinuous. As in the proof of Theorem 2.11, we can assume $P \subset M$. So, $P \subset \bigcup_{\xi<\mathfrak{c}} \mathrm{supp}(\kappa_\xi) \subset \bigcup \mathcal{P}_0$ as $g[P] \subset K \not\ni 0$. A similar argument as in Theorem 2.15 shows $g \restriction P$ is discontinuous, as needed. □

2.7. **Lineability of** $\mathrm{SCIVP} \setminus \mathrm{D}$**.** To ensure SCIVP the definition of $\kappa_\xi^{I,C}$ needs to be slightly changed whereas no generated function can be surjective. For $\xi < \mathfrak{c}$ let $\xi < \mathfrak{c}$

(μ) $\mu_\xi := \sum_{\langle I,C\rangle \in \mathcal{B}\times\mathcal{P}} \mu_\xi^{I,C}$, where $\mu_\xi^{I,C} : \mathbb{R} \to C \cap V$ is defined as $\mu_\xi^{I,C} = a\chi_{P_\xi^{I,C}}$ for some $a \in C \cap V$.

Notice that the support of $\mu_\xi^{I,C}$ is contained in $P_\xi^{I,C}$. So, the supports of maps in the family $\mathcal{F}(\mu) := \{\mu_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint. Moreover, we have the following simple fact.

Fact 2.17. *If $f \in \mathcal{F}(\mu)$ and $g \in \mathbb{R}^\mathbb{R}$ is such that $g \restriction \mathrm{supp}(f) = c\ f \restriction \mathrm{supp}(f)$ for some $c \in \mathbb{R} \setminus \{0\}$, then g has a dense graph and belongs to* SCIVP.

Proof. It is straightforward from our definition (μ). □

Theorem 2.18. *There exists a family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g \in \mathrm{SCIVP} \setminus \mathrm{D}$ for every non-zero $g \in W_\mathcal{F}$. In particular,* $\mathrm{SCIVP} \setminus \mathrm{D}$ *is $2^\mathfrak{c}$-lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\mu)$ is as needed.

Indeed, if $g \in W_\mathcal{F}$ is non-zero, then, by Remark 2.3 and Fact 2.17, g has a dense graph and belongs to SCIVP.

For $g \in \neg\mathrm{D}$, the proof is an identical to that presented in Theorem 2.15. □

2.8. **Lineability of** $\mathrm{AC} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}$**.** The hardest aspect of this argument will be ensuring that the functions in $W_\mathcal{F}$ are not extendable. For this, we recall the following useful result that was proved in [19].

Theorem 2.19. *If $f : \mathbb{R} \to \mathbb{R}$ is an extendable function with a dense graph, then for every $a, b \in \mathbb{R}$, $a < b$, and for each perfect set K between $f(a)$ and $f(b)$ there is a perfect set C between a and b such that $f[C] \subset K$ and the restriction $f \restriction C$ is continuous strictly increasing.*

By using the notation as above, for every $\xi < \mathfrak{c}$

(ν) $\nu_\xi := \mu_\xi + \beta_\xi^M$.

Notice that the supports of μ_ξ and β_ξ^M are disjoint, the first contained in M, the second in $\mathbb{R} \setminus M$. It is also easy to see that the supports of maps in the family $\mathcal{F}(\nu) := \{\nu_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint.

Theorem 2.20. *There exists a family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ of $\mathfrak{c}$-many functions with nonempty pairwise disjoint supports such that $g \in \mathrm{AC} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}$ for every non-zero $g \in W_\mathcal{F}$. In particular,* $\mathrm{AC} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}$ *is $2^\mathfrak{c}$-lineable.*

Proof. The family $\mathcal{F} := \mathcal{F}(\nu)$ is as needed.

Indeed, if $g \in W_{\mathcal{F}}$ is non-zero, then, by Remark 2.3, there is an $f = \nu_\xi \in \mathcal{F}(\nu)$ and $c \in \mathbb{R} \setminus \{0\}$ with $g \restriction \mathrm{supp}(\nu_\xi) = c\, \nu_\xi \restriction \mathrm{supp}(\nu_\xi)$. Since $\mathrm{supp}(\mu_\xi) \subset \mathrm{supp}(\nu_\xi)$, this implies that $g \restriction \mathrm{supp}(\mu_\xi) = c\, \nu_\xi \restriction \mathrm{supp}(\mu_\xi) = c\, \mu_\xi \restriction \mathrm{supp}(\mu_\xi)$ so by Fact 2.17, g has a dense graph and belongs to SCIVP. Similarly $\mathrm{supp}(\beta_\xi^M) \subset \mathrm{supp}(\nu_\xi)$, which implies that $g \restriction \mathrm{supp}(\beta_\xi^M) = c\, \nu_\xi \restriction \mathrm{supp}(\beta_\xi^M) = c\, \beta_\xi^M \restriction \mathrm{supp}(\beta_\xi^M)$ so, by Fact 2.7, $g \in \mathrm{AC}$.

Finally, to see $g \in \neg\,\mathrm{Ext}$, using density of the graph of g, choose $p < q$ with $g(p) < g(q)$ and a nonempty perfect $K \subset (g(p), g(q)) \setminus \{0\}$. By Theorem 2.19, it is enough to show that for no perfect set $P \subset (p,q)$ with $g[P] \subset K$ the restriction $g \restriction P$ is strictly increasing. As in the proof of Theorem 2.11, we can assume $P \subset M$. Since $g[P] \subset K \not\ni 0$, which implies $P \subset \bigcup_{\xi<\mathfrak{c}} \mathrm{supp}(\mu_\xi) \subset \bigcup \mathcal{P}_0$. So, by Lemma 2.12, there is a $P_\xi^{I,C} \in \bigcup \mathcal{P}_0$ with $|P \cap P_\xi^{I,C}| > \omega$. Notice that $P \cap P_\xi^{I,C} \subset P_\xi^{I,C} \subset \mathrm{supp}(\mu_\xi) \subset \mathrm{supp}(\nu_\xi)$. So, $\mu_\xi \restriction P$ is not strictly increasing and the same is true for $\nu_\xi \restriction P$ and $g \restriction P$. Thus, $g \in \neg\,\mathrm{Ext}$, as needed. □

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