A CENTURY OF SIERPIŃSKI-ZYGMUND FUNCTIONS

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Abstract. Sierpiński–Zygmund (SZ) functions are the maps from $\mathbb{R}$ to $\mathbb{R}$ that have "as little continuity" as possible. In this work we discuss the history behind their discovery, their constructions through the years, and their generalizations. The presentation emphasizes the algebraic properties of SZ maps and their relation to different classes of generalized continuous-like functions. From the seminal work of H. Blumberg and the appearance of Sierpiński–Zygmund's result, we describe the current state of the art of this century-old class of functions and discuss the impact that it has had on several different directions of research. Many typical proofs used in the theory, often in a simplified format never published before, are included in the presented material. Moreover, open problems and new directions of research are indicated.

Contents

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1. Introduction: How did Sierpiński-Zygmund maps come about?

How much continuity must an arbitrary function from the real line $\mathbb{R}$ *into* $\mathbb{R}$ *have?* At a first glance, an answer to this question should be *none*, since the characteristic function $\chi_{\mathbb{Q}} \colon \mathbb{R} \to \{0, 1\}$ of the set $\mathbb{Q}$ of all rational numbers (i.e., given as $\chi_{\mathbb{Q}}(x) = 1$ for $x \in \mathbb{Q}$ and $\chi_{\mathbb{Q}}(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$), known as the *Dirichlet function*, is clearly continuous at no point. This was first observed by Peter Gustav Lejeune Dirichlet (1805–1859)[1] in 1829, [52].

Nevertheless, if we consider the restrictions $f \restriction D$ of f to a $D \subset \mathbb{R}$, then such restriction can still be continuous. In fact, independently of the choice of f, the restriction $f \restriction D$ is continuous at any isolated point of D; in particular, $f \restriction D$ is continuous when D has no limit points. However, this could be seen as "cheating," since lack of limit points in D makes the continuity of $f \restriction D$ trivial. A more sensible question is to concentrate on the restrictions $f \restriction D$, when D has no isolated points:

Q1: Is it true that for every function $f \colon \mathbb{R} \to \mathbb{R}$ there exists a dense in itself $D \subset \mathbb{R}$ such that $f \restriction D$ is continuous?

In the early 20th century Henry Blumberg (see Fig. 1) must have come across such a question, since in his 1922 work [18] he provided an affirmative answer to the question Q1 by proving the following result.

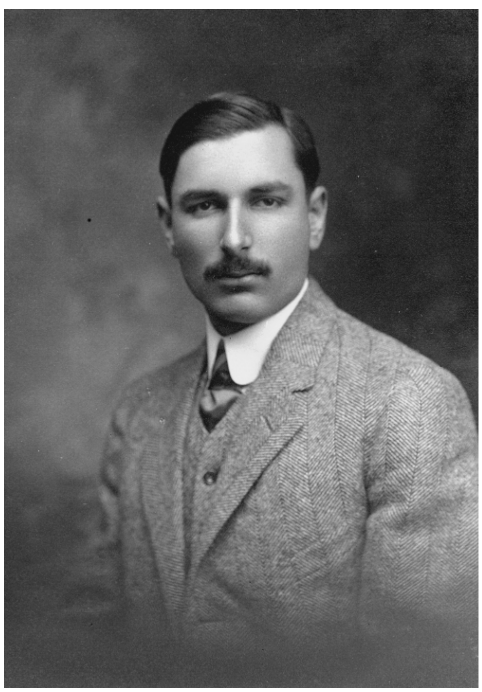

Figure 1. Henry Blumberg (1886–1950) in 1914. Born in Russia, immigrated to the USA with his parents in 1891. He received his Ph.D. in 1912 from University of Göttingen under the direction of Edmund Landau (1877–1938). He directed eight Ph.D. students between 1925 and 1950, while working at Ohio State University. Among his students was the prominent real analyst Casper Goffman (1913–2006). Interestingly, Baruch Blumberg, co-recipient of the 1976 Nobel Prize in Physiology or Medicine, was a nephew of Henry Blumberg. Photograph courtesy of the Blumberg family and Dr. George Blumberg (his great nephew).

Theorem 1.1. *For an arbitrary function* $f \colon \mathbb{R} \to \mathbb{R}$ *there exists a dense subset* D *of* $\mathbb{R}$ *such that* $f \restriction D$ *is continuous.*

The set D constructed in the original proof of Theorem 1.1 (as well as its simpler form presented in the next section) is "just" countable. In light of this fact, the following question seems natural to examine.

[1] All birth and death dates we include in this work are publicly available.

Q2: Is it true that, for every function $f\colon \mathbb{R} \to \mathbb{R}$, there exists a set $D \subset \mathbb{R}$ dense in $\mathbb{R}$ (or just in itself) which is uncountable and such that its restriction $f \restriction D$ is continuous?

Of course, one may also ask whether we can ensure that a set D in Q2 can be "big" in a sense other than cardinality, e.g., in sense of Lebesgue measure or Baire category. We will discuss the current state of knowledge on these generalized versions of Q2 in Section 2.

Question Q2 was investigated right after the publication of Theorem 1.1 by two prominent Polish mathematicians, Wacław Sierpiński (see Fig. 2) and Antoni Zygmund (see Fig. 3). They proved, in their 1923 work [104], the following result (here $\mathfrak{c}$ stands for the *continuum*, that is, the cardinality of $\mathbb{R}$).

FIGURE 2. Wacław Franciszek Sierpiński (1882–1969) was a Polish mathematician famous for contributions to topology, set theory (proving that ZF set theory together with the GCH imply the Axiom of Choice), and number theory (in 1916 he provided the first example of an absolutely normal number). He published over 700 papers and 50 books. He co-founded the famous mathematical journal Fundamenta Mathematicae. He had 9 Ph.D. students and, currently, he counts with more than 5000 mathematical descendants, one of which is the first named author of this paper, K. C. Ciesielski.

Theorem 1.2. *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is discontinuous for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.*

Nowadays, any function as in Theorem 1.2 is called a *Sierpiński-Zygmund* (or just SZ) *function.* We will also use symbol SZ to denote the class of all Sierpiński-Zygmund functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 1.2 provides a negative answer to Q2 under set theoretical assumption of the *Continuum Hypothesis, CH*, that is, the statement that any uncountable subset of $\mathbb{R}$ must have cardinality $\mathfrak{c}$. It is nowadays known that CH is consistent with, and also independent from, the standard axioms ZFC of set theory. Therefore, Theorem 1.2 shows that within the standard axioms of set theory Theorem 1.1 of Blumberg cannot be further improved.

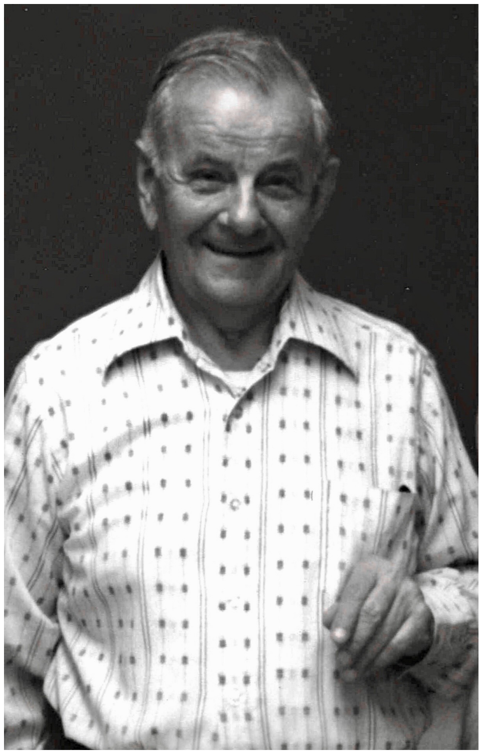

FIGURE 3. Antoni Zygmund (1900–1992) at the 1980 Summer Symposium in Real Analysis. He was a Polish mathematician and is considered as one of the greatest analysts of the 20th century. He obtained his Ph.D. in 1923 from Warsaw University. In 1940, during the World War II, he emigrated to the USA and became a professor at Mount Holyoke College in South Hadley. From 1947 until his passing he was a professor at the University of Chicago. In 1986 he received the National Medal of Science. He directed over 40 Ph.D. theses, one of which was that of Paul Cohen (1937–2007), Fields medallist in 1966. Photograph courtesy of the *Real Analysis Exchange*.

The aim of this work is to organize and fully describe the current state of the art on the century old class of Sierpiński–Zygmund functions. In order to achieve this goal, we shall present the material not only by stating and discussing existing results, but also by presenting many typical proofs used in the theory (often in a simplified never published before format). In addition, we state several open problems and point to possible new directions of research.

The paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2 and discuss their possible generalizations. Section 3 deals with the algebraic genericity of the class SZ, which has generated quite the amount of research papers during the last decade. Specifically, we discuss *lineability* problems related to SZ, by which we mean finding the largest (in the sense of dimension or systems of generators) possible algebraic structures contained in $\mathrm{SZ}\cup\{0\}$. The notion of the cardinal coefficient known as *additivity* is also introduced and related to algebraic genericity (Theorem 3.4). Section 4 focuses on the class of Sierpiński–Zygmund maps that belong also to different classes of generalized continuous functions, mainly those known as Darboux-like classes. As we will see, the existence of such functions is consistent with, but also independent from, the usual axioms of set theory. While the results presented in Sections 3 and 4 mainly depend on the behavior of a SZ-map when added to another function. In the final Section 5 we shall cover similar behaviors under the operations of product, composition, and inverse.

In the remainder of this paper we will use symbols $\mathscr{C}(X)$ and $\mathscr{B}(X)$ to denote the classes of continuous and Borel functions from a topological space X into to $\mathbb{R}$, respectively. We will also write $\mathscr{C}$ for $\mathscr{C}(\mathbb{R})$ and $\mathscr{B}$ for $\mathscr{B}(\mathbb{R})$. In addition, the symbol $|X|$ will denote the cardinality of the set X and, for a cardinality λ, we use the notation $[X]^\lambda := \{S \subset X : |X| = \lambda\}$.

2. Blumberg theorem, SZ functions, and their generalizations

In this section, partially based on the recently published paper [36] by Krzysztof Chris Ciesielski,[2] María Elena Martínez-Gómez,[3] and Juan Benigno Seoane-Sepúlveda,[4] we prove the theorems presented in the previous section and discuss their different generalizations, that can be proved under different additional set theoretical assumptions.

2.1. **Proof of Blumberg's Theorem and its ZFC generalizations.** Given an open set $U \subset \mathbb{R}$ we say that a set $Z \subset \mathbb{R}$ is *nowhere first category in U* provided $Z \cap V$ is of second category in $\mathbb{R}$ for every nonempty open $V \subset U$. We will prove the following slight generalization of Theorem 1.1 which can be found, for example, in a 1990 paper [7] of Stewart Baldwin. Used with $Z = \mathbb{R}$ it implies Blumberg's Theorem.

Theorem 2.1. *For every $Z \subset \mathbb{R}$ which is nowhere first category in $\mathbb{R}$ and an arbitrary $f : Z \to \mathbb{R}$ there exists $D \subset Z$ dense in $\mathbb{R}$ such that $f \restriction D$ is continuous.*

Recall that [36] contains a slightly shorter proof of Blumberg's Theorem than the one presented below. However, unlike the construction presented below, the one from [36] cannot be naturally generalized to the construction under Martin's axiom, which we present in Theorem 2.8.

In the remainder of this section we will assume that $Z \subset \mathbb{R}$ is nowhere first category in $\mathbb{R}$. We start with the following lemma which for $Z = \mathbb{R}$ was proved in the original Blumberg's paper [18]. (See also [36, 77].)

For an $f : Z \to \mathbb{R}$, a point $x \in Z$ is said to be *f-pleasant* provided for every open $B \ni f(x)$ there is an open $U_x^B \ni x$ such that $f^{-1}(B)$ is nowhere first category in U_x^B. Recall also that a set $G \subset Z$ is *residual in Z* provided $G = Z \setminus M$ for some first category subset M of $\mathbb{R}$.

Lemma 2.2. *For every $f : Z \to \mathbb{R}$ the set P_f of all f-pleasant points is residual in Z.*

Proof. Let $\mathcal{B}$ be a countable basis for $\mathbb{R}$. For every $B \in \mathcal{B}$ let

$$M_B := \left\{x \in f^{-1}(B) : f^{-1}(B) \text{ is not nowhere first category in any open } U \ni x\right\}$$

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[2] Krzysztof Chris Ciesielski (1957–), the first author, is a Polish American mathematician. He received his Ph.D. in 1985 from Warsaw University and the same year moved to the USA. Since 1989 he works at West Virginia University (USA) where he directed, so far, five Ph.D. students, two of which, F. Jordan and K. Płotka, contributed to this story. His research is in foundations of mathematics and, since 2004, in image processing. Around 2006 he began adding his middle name, Chris, in his publications.

[3] Current Ph.D. student of J. B. Seoane-Sepúlveda.

[4] Juan Benigno Seoane-Sepúlveda (1978–), the second author, is a Spanish mathematician. He received his first Ph.D. at the Universidad de Cádiz (Spain) jointly with Universität Karlsruhe (Germany) in 2005. His second Ph.D. was earned at Kent State University (Kent, Ohio, USA) in 2006 under the supervision of Profs. Richard M. Aron and Vladimir I. Gurariy (whose work inspired parts of this story). Since 2010 he's a professor at Universidad Complutense de Madrid (Spain) and has directed five Ph.D. theses.

and notice that M_B is of first category, in Z and in $\mathbb{R}$. Indeed, M_B is a union of two first category sets: $W \cap M_B$ and $\mathrm{bd}(W) \cap M_B$, where

$$W = \bigcup \{V \in \mathcal{B} \colon V \cap M_B \text{ is of first category}\},$$

and $\mathrm{bd}(W)$ is the boundary of W.

As $M := \bigcup_{B \in \mathcal{B}} M_B$ is of first category, it is enough to show that $Z \setminus M \subset P_f$. To see this, fix an $x \in Z \setminus M$ and an open $W \ni f(x)$. Choose $B \in \mathcal{B}$ with $f(x) \in B \subset W$. Since $x \notin M_B$, there is an open $U_x^B \ni x$ such that $f^{-1}(B)$ is nowhere first category in U_x^B. Then $f^{-1}(W) \supset f^{-1}(B)$ is also nowhere first category in U_x^B, that is, $U_x^W := U_x^B$ is as needed. $\square$

Our proof of Theorem 2.1 will be expressed in terms of partial ordered set $\langle \mathbb{P}, \leq \rangle$ and its dense subsets. Recall that D is a dense subset of $\mathbb{P}$ provided for every $p \in \mathbb{P}$ there exists a $q \in D$ such that $q \leq p$.

For an $f \colon Z \to \mathbb{R}$ let $P := P_f \setminus \mathbb{Q} \subset Z$ and notice that, by Lemma 2.2, it is residual in Z. Let $\mathcal{B}$ be a countable basis of $\mathbb{R}$ of nonempty intervals with rational endpoints. Notice that for every $B \in \mathcal{B}$ the set $B \cap P$ is clopen in P.

Let $\mathcal{B}_2 = \{U \times V \colon U, V \in \mathcal{B}\}$ and let $\mathbb{P}$ be the set of all pairs $\langle X, \mathcal{U} \rangle$ such that X is a finite subset of P, $\mathcal{U}$ is a finite subset of $\mathcal{B}_2$, and for every $U \times V, U' \times V' \in \mathcal{U}$

(a) $f[X \cap U] \subset V$ and $f^{-1}(V)$ is nowhere first category in U;
(b) either U is disjoint with U', or one of them is contained in the other;
(c) for every $x \in X \cap f^{-1}(V)$ there is $U_x \ni x$ with $U_x \times V \in \mathcal{U}$.

We order $\mathbb{P}$ by putting $\langle X, \mathcal{U} \rangle \leq \langle Y, \mathcal{V} \rangle$ if, and only if, $Y \subseteq X$ and $\mathcal{V} \subseteq \mathcal{U}$.

We will also use the following lemma.

Lemma 2.3. *For every $B, W \in \mathcal{B}$ such that $f^{-1}(W)$ is not first category in Z the following sets are dense in $\mathbb{P}$:*

$$D_B = \{\langle X, \mathcal{U} \rangle \in \mathbb{P} \colon X \cap B \neq \emptyset\},$$

$$E_W = \{\langle X, \mathcal{U} \rangle \in \mathbb{P} \colon U \times W \in \mathcal{U} \textit{ for some } U\}.$$

Proof. To see the density of D_B, fix $\langle X, \mathcal{U} \rangle \in \mathbb{P}$. It is enough to find an $x \in \mathbb{R}$ and a finite $\mathcal{V} \supset \mathcal{U}$ such that $\langle X \cup \{x\}, \mathcal{V} \rangle \in D_B$. In order to do this, choose $\hat{B} \subset B$ from $\mathcal{B}$ such that for every $U \times V \in \mathcal{U}$ either $\hat{B} \subset U$ or $\hat{B} \subset \mathbb{R} \setminus U$. Let $\mathcal{F} := \{U \times V \in \mathcal{U} \colon \hat{B} \subset U\}$.

If $\mathcal{F} \neq \emptyset$, then, by (b), there exists the smallest $\hat{U}$ with $\hat{U} \times \hat{V} \in \mathcal{F}$. Then, by (a), $f^{-1}(\hat{V}) \subset Z$ is nowhere first category in $\hat{B} \subset \hat{U}$. Since P is residual in Z, we can choose

$$x \in P \cap \hat{B} \cap f^{-1}(\hat{V}).$$

If $\mathcal{F} = \emptyset$, we simply choose $x \in P \cap \hat{B}$. Also let

$$\mathcal{W} := \{V \colon f(x) \in V \text{ for some } U \times V \in \mathcal{U}\}$$

and choose $U_0 \subset \hat{B}$ from $\mathcal{B}$ containing x such that $f^{-1}(\bigcap \mathcal{W})$ is nowhere first category in U_0. (For $\mathcal{W} = \emptyset$, we let $\bigcap \mathcal{W} = P$.) Let $\mathcal{V} := \mathcal{U} \cup \{U_0 \times V \colon V \in \mathcal{W}\}$. It is easy to see that for this choice we indeed have $\langle X \cup \{x\}, \mathcal{V} \rangle \in D_B$.

To see density of E_W, fix $\langle X, \mathcal{U} \rangle \in \mathbb{P}$, and let $X_0 = \{x \in X \colon f(x) \in W\}$. If $X_0 = \emptyset$, replace it with a singleton $\{x\} \subset P \cap f^{-1}(W)$. For every $x \in X_0$ choose $U_x \in \mathcal{B}$ containing x such that $f^{-1}(W)$ is nowhere first category in U_x and so that for every $U \times V \in \mathcal{U}$ either $U_x \subset U$ or $U_x \subset \mathbb{R} \setminus U$. Moreover, decreasing the sets U_x, if necessary, we can also assume that they are pairwise disjoint and $U_x \cap X = \{x\}$ for every $x \in X_0$. Put $\mathcal{V} = \mathcal{U} \cup \{U_x \times W : x \in X_0\}$. It is easy to see that $\langle X \cup X_0, \mathcal{V}\rangle \in E_W$ and $\langle X \cup X_0, \mathcal{V}\rangle \leq \langle X, \mathcal{U}\rangle$, proving density of E_W. □

Proof of Blumberg's Theorem 2.1. Let $\langle D^k : k \in \mathbb{N}\rangle$ be an enumeration of the family

$$\mathcal{D} = \{D_B : B \in \mathcal{B}\} \cup \{E_W : W \in \mathcal{B} \text{ and } f^{-1}(W) \text{ is not first category in } Z\}.$$

By induction, using Lemma 2.3, choose a sequence

$$\langle X_1, \mathcal{U}_1\rangle \geq \langle X_2, \mathcal{U}_2\rangle \geq \cdots \geq \langle X_k, \mathcal{U}_k\rangle \geq \cdots$$

in $\mathbb{P}$ such that each $\langle X_k, \mathcal{U}_k\rangle$ belongs to D^k. This construction constitutes a proof of the Rasiowa–Sikorski lemma (see e.g. [29]) that the filter generated by this sequence is $\mathcal{D}$-generic. (Compare also Theorem 2.8.)

Notice that the set $D := \bigcup_{k\in\mathbb{N}} X_k$ satisfies Theorem 2.1. Indeed, it is dense in $\mathbb{R}$, since for every $B \in \mathcal{B}$ there exists a $k \in \mathbb{N}$ such that $D^k = D_B$ and so, there is an $x \in X_k \subset D$ belonging to B. To see that $f \restriction D$ is continuous, choose $W \in \mathcal{B}$ and $x \in D$ such that $f(x) \in W$. It is enough to find $\hat{U} \in \mathcal{B}$ containing x such that $f[D \cap \hat{U}] \subset W$.

First notice that $f^{-1}(W)$ is not of first category in Z, as $x \in f^{-1}(W) \cap D \subset P_f$. Thus, there exists $k \in \mathbb{N}$ with $E_W = D^k$. In particular, $U \times W \in \mathcal{U}_k$ for some U. Let $\ell \geq k$ be such that $x \in X_\ell$. Then, by (c), there is an $U_x \ni x$ with $U_x \times W \in \mathcal{U}_\ell$. So, by (a), $f[X_n \cap U_x] \subset W$ for every $n \geq \ell$. Hence, $f[D \cap U_x] = f[\bigcup_{n\geq\ell} X_n \cap U_x] \subset W$, that is, $\hat{U} := U_x$ is as needed. □

Although in Blumberg's Theorem 1.1 we cannot, in ZFC, increase the size of the set D, it is still possible to improve, in a sense, its density properties. The following theorem constitutes the strongest known result in this direction and it comes from a 1996 work [72] of Aleksandra Katafiasz and Tomasz Natkaniec. Recall that for an infinite cardinal $\kappa \leq \mathfrak{c}$ we say that a set $X \subset \mathbb{R}$ is *κ-dense* provided $X \cap (a, b)$ has cardinality $\geq \kappa$ for every $a < b$.

Theorem 2.4. *Given any arbitrary function $f : \mathbb{R} \to \mathbb{R}$ there exists a $\mathfrak{c}$-dense subset W of $\mathbb{R}$ such that if C is the set of points of continuity of $f \restriction W$, then (the graph of) $f \restriction C$ is dense in (the graph of) $f \restriction W$.*

Theorem 2.4 generalizes a 1971 result of Jack B. Brown [21, Proposition C], where the author proves the existence of a $\mathfrak{c}$-dense subset W of $\mathbb{R}$ for which the set C of points of continuity of $f \restriction W$ is dense in W (and in $\mathbb{R}$). Note that this does not imply that $f \restriction C$ is dense in $f \restriction W$. Both Theorem 2.4 and its version from [21] are proved in more general setting of the real valued functions defined on the separable, complete, and dense in itself metric spaces.

2.2. **Proof of Sierpiński–Zygmund's Theorem.** The key fact needed in the construction is the following result of Kazimierz Kuratowski (1896–1980) which can be found, for instance, in [73, p. 16].

Lemma 2.5. *For every continuous function g from an $S \subset \mathbb{R}$ into $\mathbb{R}$, there exists a G_δ-set $G \supset S$ and a continuous extension $\bar{g} : G \to \mathbb{R}$ of g. In particular, g admits Borel extension $\hat{g} \in \mathscr{B}$.*

Proof. Indeed, for every $x \in \mathrm{cl}(S)$ define

$$\mathrm{osc}_g(x) := \inf\{\mathrm{diam}(g[U \cap S])\colon U \ni x \text{ is open}\}$$

and notice that the set $G := \{x \in \mathrm{cl}(S)\colon \mathrm{osc}_g(x) = 0\}$ contains S and is G_δ, since $G := \bigcap_{n\in\mathbb{N}} W_n$, where each $W_n := \{x \in \mathrm{cl}(S)\colon \mathrm{osc}_g(x) < 1/n\}$ is open in $\mathrm{cl}(S)$, so a G_δ in $\mathbb{R}$. Now, if $\mathrm{cl}(g)$ is the closure in $\mathbb{R}^2$ of the graph of g, then $\bar{g} = \mathrm{cl}(g) \cap (G \times \mathbb{R})$ is the graph of our desired function $\bar{g}$. A Borel extension $\hat{g}$ of $\bar{g}$ can be defined as 0 on $\mathbb{R} \setminus G$. □

Using Lemma 2.5 and the fact that the family $\mathscr{B}$ of all Borel functions from $\mathbb{R}$ to $\mathbb{R}$ has cardinality continuum, Theorem 1.2 follows immediately from its "folklore" generalization (see e.g. [55]) that follows.

Theorem 2.6. *For every family $\mathcal{G} \subset \mathbb{R}^\mathbb{R}$ of at most $\mathfrak{c}$ many arbitrary maps from $\mathbb{R}$ to $\mathbb{R}$ there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $g \in \mathcal{G}$ the set $[f = g] := \{x \in \mathbb{R}\colon f(x) = g(x)\}$ has cardinality $< \mathfrak{c}$.*

Proof. Let $\{x_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration, with no repetition, of $\mathbb{R}$ and let $\{g_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{G}$. For every $\xi < \mathfrak{c}$ define $f(x_\xi)$ so that

$$f(x_\xi) \in \mathbb{R} \setminus \{g_\zeta(x_\xi)\colon \zeta < \xi\}.$$

This defines our function. Indeed, for every $g \in \mathcal{G}$ there is a $\zeta < \mathfrak{c}$ such that $g_\zeta = g$. Then $[f = g] \subset \{x_\xi\colon \xi \le \zeta\}$, since $f(x_\xi) \neq g_\zeta(x_\xi) = g(x_\xi)$ for every $\xi > \zeta$. Thus, S has cardinality $< \mathfrak{c}$, as needed, and we are done. □

We should notice here that Theorem 2.6, used with the family $\mathcal{G} = \mathscr{B}$, gives a function in the class

$$\mathrm{SZ}(\mathscr{B}) := \{f \in \mathbb{R}^\mathbb{R}\colon |f \cap g| < \mathfrak{c} \text{ for every } g \in \mathscr{B}\}.$$

At the same time, Theorem 1.2 asks only for a function from the class

$$\mathrm{SZ}(\mathscr{C}) := \{f \in \mathbb{R}^\mathbb{R}\colon f \restriction X \text{ is not continuous for every } X \in [\mathbb{R}]^\mathfrak{c}\}.$$

Of course, $\mathrm{SZ}(\mathscr{B}) \subset \mathrm{SZ}(\mathscr{C})$.[5] These two notions of Sierpiński-Zygmund classes of functions were introduced in a 2016 paper [11] of Artur Bartoszewicz, Marek Bienias, Szymon Głąb, and T. Natkaniec, where they proved Theorem 2.15, discussed below, that the properness of the inclusion in $\mathrm{SZ}(\mathscr{B}) \subset \mathrm{SZ}(\mathscr{C})$ is independent of the ZFC axioms.

In what follows we will not distinguish between these two notions of Sierpiński-Zygmund functions and will use the symbol SZ to denote the class $\mathrm{SZ}(\mathscr{C})$. Nevertheless, the majority of the examples of SZ maps that we will discuss in what follows actually belong to the class $\mathrm{SZ}(\mathscr{B})$.

2.3. **Generalizations: consistent and impossible.** In some restriction theorems, the continuous restrictions $f \restriction D$ of f to "big" sets D can be further extended to the continuous maps. For an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$ this can be seen in Theorem 2.12 stated below. For measurable functions f, this follows immediately from Luzin's theorem, that there exists a compact set $P \subset [0,1]$ of arbitrary large measure less than 1 such that $f \restriction P$ is continuous. Of course, $f \restriction P$ can be extended to a continuous $g\colon \mathbb{R} \to \mathbb{R}$. Actually, if we ask P to be just uncountable, then we

[5] Indeed, by Lemma 2.5, a function $f \in \mathbb{R}^\mathbb{R}$ belongs to $\mathrm{SZ}(\mathscr{C})$ if, and only if, $|f \cap g| < \mathfrak{c}$ for every continuous g from a G_δ-set $G \subset \mathbb{R}$ into $\mathbb{R}$. Since any such g has an extension $\hat{g} \in \mathscr{B}$, we have $\mathrm{SZ}(\mathscr{C}) \supset \mathrm{SZ}(\mathscr{B})$.

can additionally require for g to be continuously differentiable, see e.g., [48, section 3.4]. (However, for an uncountable P the extension cannot be expected to be twice differentiable, even when f is continuous, see e.g. [48, section 4.2].) Thus, it seems to be natural to ask, whether the restriction $f \restriction D$ from Theorem 1.1, of Blumberg, can be ensured to have a continuous extension. However, this certainly cannot be achieved, as justified by any function with jump discontinuity, e.g., $f = \chi_{(0,\infty)}$.

As mentioned above, under the Continuum Hypothesis, Theorems 1.1 and 1.2 give a complete answer to question Q2. In particular, under CH, the following question has a negative answer in all its instances.

Q3: Can it be true that for every function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a set $D \subset \mathbb{R}$ (dense in itself, or in $\mathbb{R}$) such that $f \restriction D$ is continuous and D is uncountable? of a positive outer Lebesgue measure? of second category?

But what happens when CH is false, that is, under $\neg$CH? (Recall, that $\neg$CH is consistent with set theory ZFC.) Once again, this is not fully decided within the theory ZFC+$\neg$CH. Specifically, this follows from the following series of results.

No uncountable restrictions under $\neg$CH. The next theorem has been proved independently in 1990's by Gary Gruenhage (see the work of Ireneusz Recław (1960–2012) [97, theorem 4]) and Saharon Shelah (1945–) [102, §2].

Theorem 2.7. *In a model of ZFC obtained by adding at least ω_2 Cohen reals, we have $\neg$CH, while*

- *There exists an $f\colon \mathbb{R} \to \mathbb{R}$ for which $f \restriction X$ is discontinuous for every uncountable $X \subset \mathbb{R}$.*

Uncountable but null and meager restrictions. Martin's Axiom (MA) is a statement that is known to be consistent with ZFC+$\neg$CH. It also follows from CH. It is well known that under the Martin's Axiom every subset of $\mathbb{R}$ of cardinality $< \mathfrak{c}$ is null (i.e., of Lebesgue measure zero) and meager (i.e., of first category). The part $(*)$ of the following theorem has been proved in 1990 by S. Baldwin [7]. See also 1973 paper [103] of Juichi Shinoda, where it is shown that, under MA, there exists a set X as in $(*)$ of cardinality κ, but not necessarily dense.

Theorem 2.8. *Under MA, which is consistent with ZFC+$\neg$CH, every subset of $\mathbb{R}$ of cardinality $< \mathfrak{c}$ is null and meager, while*

$(*)$ *For every function $f\colon \mathbb{R} \to \mathbb{R}$ and every infinite cardinal $\kappa < \mathfrak{c}$ there exists a κ-dense set $X \subset \mathbb{R}$ for which $f \restriction X$ is continuous.*

In particular, under MA+$\neg$CH the set D from Blumberg's theorem can be ensured to be ω_1-dense.

Before we sketch the proof of Theorem 2.8 we need to recall the following definitions. Two elements p and q in a partially ordered set $\langle \mathbb{P}, \leq \rangle$ are *compatible* (in $\mathbb{P}$) provided there exists an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. An *antichain* in $\mathbb{P}$ is any subset A of $\mathbb{P}$ such that no two distinct elements in $\mathbb{P}$ are compatible. A partially ordered set $\langle \mathbb{P}, \leq \rangle$ is said to be *ccc* provided $\mathbb{P}$ contains no uncountable antichain. A subset $\mathcal{F}$ of $\mathbb{P}$ is said to be a *filter* provided $q \in \mathcal{F}$ whenever $q \geq p \in \mathcal{F}$ and for every $p, q \in \mathcal{F}$ there is an $r \in \mathcal{F}$ such that $r \leq p$ and $r \leq q$. For a family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ a filter $\mathcal{F}$ in $\mathbb{P}$ is *$\mathcal{D}$-generic* provided $\mathcal{F} \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

> **Martin's axiom.** *For every ccc partially ordered set $\langle \mathbb{P}, \le \rangle$ and every family $\mathcal{D}$ of cardinality less than $\mathfrak{c}$ and consisting of dense subsets of $\mathbb{P}$ there exists a $\mathcal{D}$-generic filter $\mathcal{F}$ in $\mathbb{P}$.*

Sketch of proof of Theorem 2.8. Fix an $f\colon \mathbb{R} \to \mathbb{R}$ and let and P_f be the set of all f-pleasant points. By Lemma 2.2, it is residual in $\mathbb{R}$. Let $P := P_f \setminus \mathbb{Q}$ and $\langle \mathbb{P}, \le \rangle$ be as in the proof of Theorem 2.1. A standard argument shows that it is ccc.

Let $\{Z_\xi \colon \xi < \mathfrak{c}\}$ be a partition of P into sets that are nowhere first category in $\mathbb{R}$ and fix $\kappa < \mathfrak{c}$. Similarly as in the proof of Lemma 2.3 it can be shown that for every $B \in \mathcal{B}$ and $\xi < \mathfrak{c}$ the set

$$D^\xi_B = \{\langle X, \mathcal{U} \rangle \in \mathbb{P} \colon X \cap B \cap Z_\xi \neq \emptyset\}$$

is dense in $\mathbb{P}$. Thus, the family the family

$$\mathcal{D} = \{D^\xi_B \colon B \in \mathcal{B} \ \&\ \xi < \kappa\} \cup \{E_W \colon W \in \mathcal{B} \ \&\ \ f^{-1}(W) \text{ is not first category}\}$$

has cardinality $\kappa < \mathfrak{c}$ and consists of dense subsets of $\mathbb{P}$. Thus, by MA, there exists a $\mathcal{D}$-generic filter $\mathcal{F}$ in $\mathbb{P}$. The set $D = \bigcup\{X \colon \langle X, \mathcal{U} \rangle \in \mathcal{F}\}$ is the desired set from Theorem 2.8. □

Restrictions to sets of second category. The following theorem comes from a 1995 paper [102] of S. Shelah.

Theorem 2.9. *There exists a model of ZFC+¬CH in which*

- *For every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a second category set D with $f \restriction D$ continuous.*

The issue whether the set D in the above theorem can be also dense in $\mathbb{R}$ was not addressed in the paper [102]. However, this can indeed be the case, as we indicate below.

Proposition 2.10. *The property • from Theorem 2.9 implies that*

- *For every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a nowhere first category subset D of $\mathbb{R}$ for which $f \restriction D$ continuous.*

Proof. Fix $f\colon \mathbb{R} \to \mathbb{R}$ and recall that every second category set $D \subset \mathbb{R}$ is nowhere first category on some non-empty interval. Since every non-empty open interval is homeomorphic with $\mathbb{R}$, the union of the following family of non-empty open intervals

$$\mathcal{J} := \{J \colon \text{there is nowhere first category subset } D_J \text{ of } J \text{ with } f \restriction D_J \text{ continuous}\}$$

is dense in $\mathbb{R}$. Also, any non-empty open subinterval of $J \in \mathcal{J}$ is also in $\mathcal{J}$. Thus, the maximal family $\mathcal{J}_0 \subset \mathcal{J}$ consisting of pairwise disjoint intervals has also union dense in $\mathbb{R}$. It is easy to see that the set $D = \bigcup_{J \in \mathcal{J}_0} D_J$ is as needed. □

Combining Theorem 2.9 with Proposition 2.10 gives immediately the following corollary. It can be found, without a proof, in [30, theorem 2.10].

Corollary 2.11. *There exists a model of ZFC+¬CH in which*

- *For every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a nowhere first category set D in $\mathbb{R}$ for which $f \restriction D$ continuous.*

Restrictions to sets with positive outer measure. The following theorem comes from a 2006 paper [100] of Andrzej Rosłanowski and S. Shelah.

Theorem 2.12. *There exists a model of ZFC+¬CH in which*

- *For every map $f\colon \mathbb{R} \to \mathbb{R}$ there exists a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ that agrees with f on a set D of positive Lebesgue outer measure.*

In particular, $f \restriction D$ is continuous.

The set D in the theorem can easily be assumed to be dense in itself. However, it cannot be ensured to be *measure dense*, that is, to have a positive outer measure in any non-trivial interval I. This is justified by the following 1977 example of J. Brown [22]. The presented construction comes from [30, theorem 2.11].

Theorem 2.13. *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction D$ is discontinuous for every set $D \subset \mathbb{R}$ which is nowhere measure zero, that is, such that $D \cap I$ has positive outer measure for every non-trivial interval I.*

Proof. Let $\{F_n \colon n < \omega\}$ be a partition of $\mathbb{R}$ such that F_0 is a dense G_δ-set of measure zero and F_n is nowhere dense for each $n > 0$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by putting $f(x) = n$ for $x \in F_n$. Now, $f \restriction X$ is discontinuous for any dense $X \subset \mathbb{R}$ which is nowhere measure zero.

Indeed, if $X \subset \mathbb{R}$ is dense and not of measure zero, then there is an $x \in X \setminus F_0$. Hence $f(x) = n$ for some $n > 0$. Since $\{n\}$ is open in $f[\mathbb{R}] \subset \omega$ and $f \restriction X$ is continuous, there is $U \ni x$ open in $\mathbb{R}$ such that $f[X \cap U] \subset \{n\}$. Thus, $X \cap U \subset F_n$ and $U \subset \mathrm{cl}(X \cap U) \subset \mathrm{cl}(F_n) = F_n$ in spite that F_n is nowhere dense. □

Finally, for $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, let $\mathrm{dec}(\mathcal{F}, \mathscr{C})$ denote the smallest cardinal κ such that the graph of every function in $\mathcal{F}$ can be covered by the graphs of κ-many continuous partial functions. The cardinal $\mathrm{dec}(\mathscr{B}, \mathscr{C})$, used in Theorem 2.15, was thoroughly studied in the 1991 paper [28] of Jacek Cichoń, Michał Morayne, Janusz Pawlikowski, and Sławomir Solecki. (See also also [30].) The next theorem, concerning decomposition number $\mathrm{dec}(\mathrm{SZ}, \mathscr{C})$, comes from the 1999 paper [31] of K.C. Ciesielski.

Theorem 2.14. $\mathrm{cof}(\mathfrak{c}) \leq \mathrm{dec}(\mathrm{SZ}, \mathscr{C}) = \mathrm{dec}(\mathbb{R}^{\mathbb{R}}, \mathscr{C}) \leq \mathfrak{c}$. *Moreover, for every cardinal number λ with* $\mathrm{cof}(\lambda) > \omega$*:*

(a) It is consistent with ZFC that $\mathrm{dec}(\mathrm{SZ}, \mathscr{C}) = \lambda = \mathfrak{c}$.
(b) It is consistent with ZFC that $\mathrm{dec}(\mathrm{SZ}, \mathscr{C}) = \mathrm{cof}(\mathfrak{c})$ *and* $\lambda = \mathfrak{c}$.

In particular, parts (a) and (b) of Theorem 2.14 imply, respectively, that

- It is consistent with ZFC that $\mathrm{cof}(\mathfrak{c}) < \mathrm{dec}(\mathrm{SZ}, \mathscr{C}) = \mathfrak{c}$.
- It is consistent with ZFC that $\mathrm{cof}(\mathfrak{c}) = \mathrm{dec}(\mathrm{SZ}, \mathscr{C}) < \mathfrak{c}$.

Part (a) of Theorem 2.14 holds in a Cohen model obtained by adding λ Cohen reals to a model with CH. (Compare Theorem 2.7.) Part (b) in the case $\mathrm{cof}(\lambda) = \lambda$ holds in any model of ZFC+$\mathfrak{c} = \lambda$, as implied by the main part of Theorem 2.14. In the case when $\mathrm{cof}(\lambda) < \lambda$, start with a model of ZFC+CH, choose an increasing sequence $\langle \lambda_\xi < \lambda \colon \xi < \mathrm{cof}(\lambda) \rangle$ of regular cardinals cofinal with λ, and find a generic extension of our model of ZFC+CH obtained by consecutive extensions ensuring that MA+$\mathfrak{c} = \lambda_\xi$ holds. Theorem 2.8 ensures that in a final model obtained that way the property (b) holds.

The last result here concerns the relation between the two notions of SZ maps introduced at the end of Section 2.2. It comes from [11, Theorem 4.4].

Theorem 2.15. *The properness of the inclusion* $\mathrm{SZ}(\mathscr{B}) \subset \mathrm{SZ}(\mathscr{C})$ *is independent of ZFC. More specifically,*

(1) *if* $\mathfrak{c}$ *is a successor cardinal and* $dec(\mathscr{B},\mathscr{C}) = \mathfrak{c}$*, then* $\mathrm{SZ}(\mathscr{B}) \subsetneq \mathrm{SZ}(\mathscr{C})$*;*
(2) *if* $\mathfrak{c}$ *is a regular cardinal and* $dec(\mathscr{B},\mathscr{C}) < \mathfrak{c}$*, then* $\mathrm{SZ}(\mathscr{B}) = \mathrm{SZ}(\mathscr{C})$*.*

There is also a multitude of other generalizations of Blumberg's theorem, often concerning functions between topological spaces X and Y; see, for example, [19, 21, 24, 61, 67, 71, 72, 84, 93, 108, 109]. For a placement of these results in a more general real analysis perspective see [30] or [77].

FIGURE 4. Vladimir Ilyich Gurariy (1935–2005) was born in Kharkov (Ukraine). In 1991 he moved to the USA and worked in Kent State University (Ohio) until his passing. He was a visiting Professor at universities in Germany, UK, Italy, and Venezuela. He also worked at the Institute for Advanced Study in Princeton. Author of over 120 articles in pure and applied mathematics. In addition, he was a very gifted chess player, publishing in the Russian magazine *64* and in the Latvian *Šahs* (edited by Mikhail Tal, the eighth World Chess Champion). He co-directed one of the Ph.D. theses of the second named author of this paper, J.B. Seoane-Sepúlveda. Photograph courtesy of Larisa Lev Altshuler.

3. Large algebraic structures within SZ

It is easy to see that the class SZ is closed under the multiplication by the nonzero numbers: if $c \in \mathbb{R}$ is nonzero and $f \in \mathrm{SZ}$, then $cf \in \mathrm{SZ}$.[6] Also, if $f \in \mathrm{SZ}$ and $g \in \mathbb{R}^{\mathbb{R}}$ is continuous, then clearly $f + g \in \mathrm{SZ}$. In particular, SZ is not closed under addition, as our continuous $g \notin \mathrm{SZ}$ is a sum of two SZ maps: $-f$ and $f+g$. In fact, $\mathrm{SZ} + \mathrm{SZ} = \mathbb{R}^{\mathbb{R}}$ as follows from Proposition 3.5(5) and Theorem 3.6. Similarly, SZ is not closed under multiplication, as, for the functions f and g as above, the continuous map e^g is a product of two SZ functions: e^{-f} and e^{f+g}. These simple facts, that will be put in a more general context in Subsection 3.2, come from 1997 paper [38] of K.C. Ciesielski and T. Natkaniec.

3.1. Lineability and algebrability of SZ.

The goal of this section is to investigate the largest[7] possible subfamilies of $\mathrm{SZ} \cup \{0\}$ that form either a linear subspace (over the field $\mathbb{R}$) or a sub-algebra in $\mathbb{R}^{\mathbb{R}}$. The above remarks show that neither of these structures can be realized trivially by $\mathrm{SZ} \cup \{0\}$. This direction of research—the search for large algebraic structures within nonlinear subsets of nice (in our case $\mathbb{R}^{\mathbb{R}}$) structures—is commonly referred to as *lineability research*, the term coined by

[6] This property is sometimes referred to as *being star-like*, see e.g. [56].
[7] The largest in a sense of a size of the minimal cardinality of generating set.

Vladimir I. Gurariy (see Fig. 4) in the early 2000's, see [3, 101]. It caught the interest of the mathematical community and sparked a lot of activity about a decade ago [2, 17]. However, it can be traced back to 1966 paper [62] (see, also, [63]) of Gurariy where it is proved that there is an infinite dimensional vector subspace V of the class of all continuous functions from $[0,1]$ to $\mathbb{R}$ such that every nonzero $f \in V$ is nowhere differentiable. This space can even be chosen to be closed within the space of continuous functions, as shown in [54]. Following [2, 3, 9, 12, 16, 17, 45, 53], for a cardinal number κ we say that an $F \subset \mathbb{R}^{\mathbb{R}}$ is:

- κ*-lineable* if $F \cup \{0\}$ contains a vector subspace of $\mathbb{R}^{\mathbb{R}}$, over the field $\mathbb{R}$, of dimension κ.
- κ*-algebrable* if there is an algebra $A \subset F \cup \{0\}$ for which κ is the smallest cardinality of any $B \subset A$ generating A.
- *strongly* κ*-algebrable* provided there exists a κ-generated *free* algebra $A \subset F \cup \{0\}$.

Of course, strong κ-algebrability implies κ-algebrability which, in turn, implies κ-lineability. Also, none of these implications can be reversed. A nice example of an algebrable set that is not strongly algebrable is given in [12] and consists of the family c_{00} of all eventually constant 0 sequences of real numbers. Its 1-algebrability is obvious, since it is an algebra. (In fact, it is ω-algebrable, with a canonical set $\{e_1, e_2, \ldots\}$ of generators.) However, as shown in [12], it is not strongly 1-algebrable. Also, there are sets that are lineable and not algebrable, for instance the class ES of everywhere surjective functions in $\mathbb{R}$ (i.e., $f \in \mathbb{R}^{\mathbb{R}}$ such that $f[(a,b)] = \mathbb{R}$ for all $a < b$), see e.g. [3–5].

In the above terms, the goal of this section is to establish the upper bound of the cardinal numbers κ for which SZ is κ-lineable as well as (strongly) κ-algebrable. The first results in this direction were established in a 2010 paper [55] of Jose Luis Gámez-Merino, Gustavo A. Muñoz-Fernández, Víctor M. Sánchez, and J. B. Seoane-Sepúlveda where the authors proved that SZ is $\mathfrak{c}$-algebrable and, also, the following.

Theorem 3.1. SZ *is* $\mathfrak{c}^+$*-lineable. In particular, it is consistent with ZFC, follows from the Generalized Continuum Hypothesis GCH, that* SZ *is* $2^{\mathfrak{c}}$*-lineable, which constitutes the maximal possible lineability of* SZ.

This follows immediately from Proposition 3.5(2) and Theorem 3.4. The main result of this section is presented in the following two theorems, that come from 2013 paper [57] of J.L. Gámez-Merino and J. B. Seoane-Sepúlveda. To state them we need to explain that for an infinite cardinal λ a family $\mathcal{S}$ of sets is λ*-almost disjoint* provided for any two distinct $S, T \in \mathcal{S}$ their intersection $S \cap T$ has cardinality $< \lambda$.[8] Notice that the notion of almost disjoint families has proven to be a very useful tool when it comes to lineability (see, e.g., [1]).

[8] The definition of such a family in [57] additionally assumes that each $S \in \mathcal{S}$ has cardinality λ. We do not impose it here, but apply this definition only to such families. But the distinction is important, since in the model from Theorem 3.3 there are $2^{\mathfrak{c}}$ many subsets of $\omega_1 \subset \omega_2$ which, according to our definition, are $\mathfrak{c}$-almost disjoint. Nevertheless, (4) from Theorem 3.2 fails in this model.

Theorem 3.2. *For any cardinal number κ the following are equivalent:*

1. SZ *is κ-strongly algebrable.*
2. SZ *is κ-algebrable.*
3. SZ *is κ-lineable.*
4. *There exists a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality κ.*

It should be mentioned here that the main implication, (4)$\Longrightarrow$(1), of Theorem 3.2 was not proved in [57]. The authors of [57] simply pointed out that this was proved in the 2013 paper [13, Theorem 2.6] by A. Bartoszewicz, S. Głąb, Daniel M. Pellegrino, and J. B. Seoane-Sepúlveda. The proof of this implication presented in [13] is quite intricate and, actually, uses some claims that do not hold when $\mathfrak{c}$ is a singular cardinal. (This has been corrected in the 2013 paper [9] by A. Bartoszewicz, M. Bienias, and S. Głąb.) Below we include a simplified (and correct) proof of this result.

Proof of Theorem 3.2. Clearly (1) implies (2) and (2) implies (3). Moreover, (3) implies (4) since for any family $\mathcal{F}$ justifying κ-lineability of SZ, the graphs of functions in $\mathcal{F}$ are $\mathfrak{c}$-almost disjoint subsets of $\mathbb{R}^2$, each of cardinality $\mathfrak{c}$, and they can be naturally treated as subsets of $\mathfrak{c}$.

To prove that (4) implies (1), let $\{r_\xi \colon \xi < \mathfrak{c}\}$ and $\{g_\xi \colon \xi < \mathfrak{c}\}$ be the enumerations of $\mathbb{R}$ and $\mathscr{B}$, respectively. By induction on $\xi < \mathfrak{c}$ define the set $S = \{b_\xi \colon \xi < \mathfrak{c}\}$ so that

$$b_\xi \in \mathbb{R} \setminus \mathbb{Q}(A_\xi),$$

where $A_\xi := \{b_\eta \colon \eta < \xi\} \cup \{g_\alpha(r_\beta) \colon \alpha, \beta \leq \xi\}$ and $\mathbb{Q}(A_\xi)$ is the smallest subfield of $\mathbb{R}$ containing A_ξ. This ensures that, for every $\xi < \mathfrak{c}$,

$$\{b_\eta \colon \xi \leq \eta < \mathfrak{c}\} \text{ is algebraically independent over the field } \mathbb{Q}(A_\xi). \tag{3.1}$$

(See, e.g. [81].) Now, let $\{S_\zeta \in [\mathfrak{c}]^{\mathfrak{c}} \colon \zeta < \kappa\}$ be a one-to-one enumeration of a $\mathfrak{c}$-almost disjoint family $\mathcal{F}$ from (4) and for every $\zeta < \kappa$ let f_ζ be a bijection from $\mathbb{R}$ onto $\{b_\eta \colon \eta \in S_\zeta\}$ defined inductively via formula

$$f_\zeta(r_\xi) = b_\gamma \quad \text{where } \gamma := \min\{\eta < \mathfrak{c} \colon r_\eta \in S_\zeta \setminus f_\zeta[\{r_\delta \colon \delta < \xi\}]\}. \tag{3.2}$$

Note that, in (3.2), $\gamma \geq \xi$.

The family $\mathcal{G} := \{f_\zeta \colon \zeta < \kappa\}$ has cardinality κ, since the elements of the family $\{f_\zeta[\mathbb{R}] \colon \zeta < \kappa\}$ are $\mathfrak{c}$-almost disjoint. Thus, to finish the proof, it is enough to show that the free algebra $A(\mathcal{G})$ generated by $\mathcal{G}$ is contained in SZ. So, take an arbitrary element of $A(\mathcal{G})$. It is of the form $f := p(f_{\zeta_1}, \ldots, f_{\zeta_n})$ for some $n \in \mathbb{N}$, polynomial p of n-variables, and $\zeta_1 < \cdots < \zeta_n < \kappa$. Let $g_\alpha \in \mathscr{B}$. We need to show that $|f \cap g_\alpha| < \mathfrak{c}$. To see this, choose a $\beta < \mathfrak{c}$ such that $\beta \geq \alpha$ and all coefficients of p are in $\mathbb{Q}(A_\beta)$. Also, let $X := \{r \in \mathbb{R} \colon f_{\zeta_i}(r) = f_{\zeta_j}(r) \text{ for some } 0 < i < j \leq n\}$ and define $Y := \{r \in \mathbb{R} \colon f_{\zeta_i}(r) \in \mathbb{Q}(A_\beta) \text{ for some } 0 < i \leq n\}$. Then $|X \cup Y| < \mathfrak{c}$. Thus, it is enough to show that for every $r_\xi \in \mathbb{R} \setminus X \cup Y$ with $\xi \geq \beta$ we have $f(r_\xi) \neq g_\alpha(r_\xi)$.

Indeed, we have $f(r_\xi) - g_\alpha(r_\xi) = q(f_{\zeta_1}(r_\xi), \ldots, f_{\zeta_n}(r_\xi))$, where the polynomial $q(x_1, \ldots, x_n) := p(x_1, \ldots, x_n) - g_\alpha(r_\xi)$ has all coefficients in $\mathbb{Q}(A_\xi)$. At the same time, the distinct numbers $f_{\zeta_1}(r_\xi), \ldots, f_{\zeta_n}(r_\xi)$ are, by the comment below (3.2), in $\{b_\eta \colon \xi \leq \eta < \mathfrak{c}\}$ and so, by (3.1), algebraically independent over $\mathbb{Q}(A_\xi)$. Hence, $f(r_\xi) - g_\alpha(r_\xi) = q(f_{\zeta_1}(r_\xi), \ldots, f_{\zeta_n}(r_\xi)) \neq 0$ and $f(r_\xi) \neq g_\alpha(r_\xi)$, as needed. $\square$

One of the consequences of Theorem 3.2 is that, in the case of family SZ, the three notions we consider in (1)-(3) are equivalent, which, as noted above, in general is not true. In particular, under the GCH, the situation becomes crystal clear: SZ is $2^{\mathfrak{c}}$-strongly algebrable. Actually, the same is true under a considerably weaker set theoretical assumption that $2^{<\mathfrak{c}} = \mathfrak{c}$, since under this assumption there exists a family $\mathcal{F}$ as in (4) of cardinality $2^{\mathfrak{c}}$, see e.g. [80, p. 48, theorem 1.3]. However, the condition (4) allows also to show that there are models of ZFC in which SZ is not $2^{\mathfrak{c}}$-lineable. This follows from the next theorem from [57], which is proved by forcing technique. Notice, that this result is the first result in the lineability theory that is undecidable in ZFC.

Theorem 3.3. *It is consistent with ZFC that there is no $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality $2^{\mathfrak{c}}$. In particular, the $2^{\mathfrak{c}}$-lineability of* SZ *is undecidable in ZFC.*

The proof presented below constitutes a new, hopefully less technical, presentation of the argument from [57].

Proof. The theorem follows from a general remark:

(κ) If M is a model of ZFC+GCH in which $\kappa \geq \omega_2$ is a regular cardinal number, then no ω_2-cc generic extension $M[G]$ of M can contain κ-almost disjoint family $\mathcal{F} \subset [\kappa]^{\kappa}$ with $|\mathcal{F}| = \kappa^{++}$.

By way of contradiction, assume that $M[G]$ contains a κ-almost disjoint family $\mathcal{F} \subset [\kappa]^{\kappa}$ with one-to-one enumeration $\mathcal{F} = \{E_\alpha \in [\kappa]^{\kappa} \colon \alpha < \kappa^{++}\}$. Let f be a map, in $M[G]$, from $[\kappa^{++}]^2$ into κ such that $E_\alpha \cap E_\beta \subset f(\{\alpha, \beta\})$ for every $\{\alpha, \beta\} \in [\kappa^{++}]^2$. Using [80, lemma VIII 5.6] and the ω_2-cc property of our forcing, we can find in the ground model M a map $\bar{f} \colon [\kappa^{++}]^2 \to \kappa$ so that $f(\{\alpha, \beta\}) \leq \bar{f}(\{\alpha, \beta\})$ for every $\{\alpha, \beta\} \in [\kappa^{++}]^2$.

Next, we work in M. Using in it Erdős-Rado Partition Theorem usually represented as $(2^{\kappa})^+ \to (\kappa^+)^2_{\kappa}$ and the fact that, by GCH, $(2^{\kappa})^+ = \kappa^{++}$, we find an $H \subset \kappa^{++}$ with $|H| = \kappa^+$ and a $\gamma < \kappa$ such that $\bar{f}$ on $[H]^2$ is constant with value γ.

Finally, working back in $M[G]$, we see that $\{E_\alpha \setminus \gamma \colon \alpha \in H\}$ is a family of cardinality κ^+ consisting of nonempty pairwise disjoint subsets of κ, giving a desired contradiction and finishing an argument for (κ).

Turning back to the proof of Theorem 3.3, let M be a model of ZFC+GCH and $M[G]$ its generic extension obtained by applying an Easton forcing $\mathbb{P}$ which first adds ω_4 subsets of ω_1 (by using countable supported functions $\mathrm{Fn}(\omega_4, 2, \omega_1)$) and then ω_2 Cohen reals (with $\mathrm{Fn}(\omega_2, 2, \omega)$), see [80, Ch. VIII, §4]. This forcing is ω_2-cc, see [80, lemma VIII 4.4], and in the generic extension $M[G]$ obtained by $\mathbb{P}$ we have $\mathfrak{c} = 2^{\omega} = \omega_2$, $2^{\omega_1} = \omega_4$, and also $2^{\mathfrak{c}} = 2^{\omega_2} = \omega_4$, see [80, theorem VII 4.7]. So, by (κ) used with $\kappa = \omega_2$, we see that, in $M[G]$, there is no $\mathfrak{c}$-almost disjoint $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality $2^{\mathfrak{c}}$. $\square$

Coming back to the classes SZ($\mathscr{C}$) and SZ($\mathscr{B}$) introduced in Section 2.2, we like to point out [11, corollary 4.8] that, under CH, ES $\cap$(SZ($\mathscr{C}$) $\setminus$ SZ($\mathscr{B}$)) is $\mathfrak{c}$-lineable. (See also [11, corollaries 4.6 and 4.7] about the results related to SZ functions in complex variable.) The family SZ($\mathscr{C}$) $\setminus$ SZ($\mathscr{B}$) is either empty or it has cardinality $2^{\mathfrak{c}}$, [11, corollary 4.6]. Notice also, that the techniques employed in [55, Theorem 5.6 and 5.7] can be adapted to show that, under CH, the family SZ($\mathscr{C}$) $\setminus$ SZ($\mathscr{B}$) is $\mathfrak{c}^+$-lineable.

3.2. **Additivity coefficient.** Another notion which, perhaps unexpectedly, is related to lineability is that of an *additivity cardinal coefficient* $\mathcal{A}(\mathcal{F})$ associated with any $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ and defined as the minimal cardinality $|F|$ of a family $F \subset \mathbb{R}^{\mathbb{R}}$ that cannot be shifted into $\mathcal{F}$ by any single $\varphi \in \mathbb{R}^{\mathbb{R}}$:

$$\mathcal{A}(\mathcal{F}) = \min\big(\{|F|\colon F \subset \mathbb{R}^{\mathbb{R}} \text{ and } \varphi + F \not\subset \mathcal{F} \text{ for every } \varphi \in \mathbb{R}^{\mathbb{R}}\} \cup \{(2^{\mathfrak{c}})^{+}\}\big).$$

This notion was introduced in the early 1990's by T. Natkaniec [89, 90] and thoroughly studied in a 1996 paper [69] of Francis Edmund Jordan. (See also his Ph.D. Dissertation [70], written under the supervision of K. C. Ciesielski.) It is of interest to us here, since it is related to the concept of lineability by the following theorem of Gámez, Muñoz, and Seoane-Sepúlveda [56]:

Theorem 3.4. *If $\mathcal{F} \cup \{0\} \subsetneq \mathbb{R}^{\mathbb{R}}$ is closed under the scalar multiplication and $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable.*

Proof. Choose an $h \in \mathbb{R}^{\mathbb{R}} \setminus (\mathcal{F} \cup \{0\})$. By transfinite induction on $\xi < \mathcal{A}(\mathcal{F})$ we can construct a strictly increasing sequence $\langle V_\xi \subset \mathcal{F} \cup \{0\} \colon \xi < \mathcal{A}(\mathcal{F})\rangle$ of linear spaces, each of cardinality $< \mathcal{A}(\mathcal{F})$. Then $V := \bigcup_{\xi<\mathcal{A}(\mathcal{F})} V_\xi$ witnesses $\mathcal{A}(\mathcal{F})$-lineability of $\mathcal{F}$.

For a limit ordinal ξ we define $V_\xi = \bigcup_{\zeta<\xi} V_\zeta$. For a successor ordinal $\xi + 1$ we have $|\{h\} \cup V_\xi| < \mathcal{A}(\mathcal{F})$, so there exists a $g_\xi \in \mathbb{R}^{\mathbb{R}}$ such that

$$g_\xi + (\{h\} \cup V_\xi) \subset \mathcal{F} \cup \{0\}.$$

Note that $g_\xi \notin V_\xi$ since, otherwise, we would have

$$h \in -g_\xi + V_\xi = V_\xi \subset \mathcal{F} \cup \{0\},$$

contradicting the choice of h. Then $V_{\xi+1} := \mathbb{R} \cdot (g_\xi + V_\xi) \subset \mathcal{F} \cup \{0\}$ is our desired linear space. $\square$

Before we discuss the number $\mathcal{A}(\mathrm{SZ})$, we like to list some basic properties of the operator $\mathcal{A}$. Properties (1)–(4) can be found in [44, proposition 1.1], a 1995 paper of K. C. Ciesielski and Ireneusz Recław (1960-2012). The property (5) can be found in a 1996 article [69] of F. Jordan. (Compare also [44, proposition 1.3].)

Proposition 3.5. *For every $\mathcal{F}, \mathcal{G} \subset \mathbb{R}^{\mathbb{R}}$ the following holds.*

(1) $1 \le \mathcal{A}(\mathcal{F}) \le (2^{\mathfrak{c}})^{+}$.
(2) If $\mathcal{F} \subset \mathcal{G}$, *then* $\mathcal{A}(\mathcal{F}) \le \mathcal{A}(\mathcal{G})$.
(3) $\mathcal{A}(\mathcal{F}) = 1$ *if, and only if,* $\mathcal{F} = \emptyset$.
(4) $\mathcal{A}(\mathcal{F}) = (2^{\mathfrak{c}})^{+}$ *if, and only if,* $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$.
(5) If $\mathcal{F} \neq \emptyset$, *then* $\mathcal{A}(\mathcal{F}) = 2$ *if, and only if,* $\mathcal{F} - \mathcal{F} \neq \mathbb{R}^{\mathbb{R}}$.

Proof. The properties (1), (2), (3), and (4) are straightforward from the definition of the operator $\mathcal{A}$.

To see (5), first notice that $\mathcal{F} - \mathcal{F} = \mathbb{R}^{\mathbb{R}}$ implies that $\mathcal{A}(F) > 2$. For this, fix an $F = \{f_1, f_2\}$. We need to find a $g \in \mathbb{R}^{\mathbb{R}}$ so that $g + F \subset \mathcal{F}$. Since $\mathcal{F} - \mathcal{F} = \mathbb{R}^{\mathbb{R}}$, there are $h_1, h_2 \in \mathcal{F}$ such that $f_1 - f_2 = h_1 - h_2$. Then $g = h_1 - f_1 = h_2 - f_2$ is as needed.

To see the other implication suppose that $\mathcal{F} - \mathcal{F} \neq \mathbb{R}^{\mathbb{R}}$. Since $\mathcal{F} \neq \emptyset$, it is enough to show that $\mathcal{A}(\mathcal{F}) \le 2$. Pick $h \in \mathbb{R}^{\mathbb{R}} \setminus (\mathcal{F} - \mathcal{F})$ and put $F = \{0, h\}$. Let $g \in \mathbb{R}^{\mathbb{R}}$ be arbitrary. It is enough to show that $g + F \not\subset \mathcal{F}$. Indeed, this is clear, when $g = 0 + g \notin \mathcal{F}$. At the same time, if $g = 0 + g \in \mathcal{F}$, then $h + g \notin \mathcal{F}$, since otherwise $h \in \mathcal{F} - g \subseteq \mathcal{F} - \mathcal{F}$, contradicting our choice of h. Thus, $\mathcal{A}(\mathcal{F}) = 2$. $\square$

We need to define the following cardinal numbers:

$$d_{\mathfrak{c}} := \min\{|F|\colon F \subseteq \mathfrak{c}^{\mathfrak{c}} \ \&\ (\forall g \in \mathfrak{c}^{\mathfrak{c}})(\exists f \in F)(|[f = g]| = \mathfrak{c})\},$$
$$e_{\mathfrak{c}} := \min\{|F|\colon F \subseteq \mathfrak{c}^{\mathfrak{c}} \ \&\ (\forall g \in \mathfrak{c}^{\mathfrak{c}})(\exists f \in F)(|[f = g]| < \mathfrak{c})\}.$$

The cardinal $e_{\mathfrak{c}}$ and the results of the next theorem that concern it come from 1994 paper [37] of K. C. Ciesielski and Arnold W. Miller [37]. It came in a context that we will discuss in more detail in Section 4.4. (See also Theorem 3.7.) The cardinal $d_{\mathfrak{c}}$ and Theorem 3.6 come from 1997 paper [38] of K. C. Ciesielski and T. Natkaniec.

Theorem 3.6. $\mathfrak{c}^+ \leq \mathcal{A}(\mathrm{SZ}) = d_{\mathfrak{c}} \leq 2^{\mathfrak{c}}$ *and this is all that can be proved in ZFC. Specifically,*

1. *GCH implies that* $\mathcal{A}(\mathrm{SZ}) = 2^{\mathfrak{c}}$.
2. *For any cardinals* $\lambda \geq \kappa \geq \omega_2$ *such that* $\mathrm{cof}(\lambda) > \omega_1$ *and* κ *is regular it is relatively consistent with ZFC+CH that* $\mathcal{A}(\mathrm{SZ}) = d_{\mathfrak{c}} = e_{\mathfrak{c}} = \kappa$ *and* $2^{\mathfrak{c}} = \lambda$.
3. *For any cardinal* $\lambda > \omega_2$ *such that* $\mathrm{cof}(\lambda) > \omega_1$ *it is relatively consistent with ZFC+CH that* $\mathcal{A}(\mathrm{SZ}) = \mathfrak{c}^+ < 2^{\mathfrak{c}} = \lambda = e_{\mathfrak{c}}$.

The proofs of parts (2) and (3) of the theorem require forcing arguments. However (1) follows immediately from the inequalities $\mathfrak{c}^+ \leq \mathcal{A}(\mathrm{SZ}) \leq 2^{\mathfrak{c}}$. Of these, the upper bound for $\mathcal{A}(\mathrm{SZ})$ is an immediate consequence of Proposition 3.5. The proof of the lower bound, $\mathcal{A}(\mathrm{SZ}) \geq \mathfrak{c}^+$, follows from Theorem 2.6. Indeed, if $F \subset \mathbb{R}^{\mathbb{R}}$ has cardinality $\leq \mathfrak{c}$, then $|\mathscr{B} - F| \leq \mathfrak{c}$. So, by Theorem 2.6, for every $g \in \mathcal{B} - F$ there exists $h \in \mathbb{R}^{\mathbb{R}}$ such that $|[h \cap g]| < \mathfrak{c}$. Then $h + F \subset \mathrm{SZ}$, since $|[(h + f) \cap \varphi]| = |[h \cap (\varphi - f)]| < \mathfrak{c}$ for every $f \in F$ and $\varphi \in \mathscr{B}$.

In what follows, for $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ we will use the symbol $\neg\mathcal{F}$ to denote the complement of $\mathcal{F}$ with respect to $\mathbb{R}^{\mathbb{R}}$, that is, $\neg\mathcal{F} := \mathbb{R}^{\mathbb{R}} \setminus \mathcal{F}$. The cardinal $e_{\mathfrak{c}}$ is directly related to the class SZ through the following theorem of F. Jordan, see [69, theorems 9-11].

Theorem 3.7. $\mathfrak{c}^+ \leq e_{\mathfrak{c}} \leq \mathcal{A}(\neg\,\mathrm{SZ}) \leq 2^{\mathfrak{c}}$. *If* $2^{<\mathfrak{c}} = \mathfrak{c}$, *then* $\mathcal{A}(\neg\,\mathrm{SZ}) = e_{\mathfrak{c}}$. *Moreover, if there exists a cardinal* λ *such that* $\mathfrak{c} = \lambda^+ = 2^{\lambda}$, *then* $\mathcal{A}(\mathrm{SZ}) = d_{\mathfrak{c}} \leq e_{\mathfrak{c}} = \mathcal{A}(\neg\,\mathrm{SZ})$.

Of course, from Theorems 3.6 and 3.7 it is easy to conclude that

- It is consistent with ZFC that $\mathcal{A}(\mathrm{SZ}) = \mathfrak{c}^+ < 2^{\mathfrak{c}} = \mathcal{A}(\neg\,\mathrm{SZ})$.
- It is consistent with ZFC that $\mathfrak{c}^+ \leq \mathcal{A}(\mathrm{SZ}) = \mathcal{A}(\neg\,\mathrm{SZ}) \leq 2^{\mathfrak{c}}$ and each of these inequalities can be, independently, either strict or not.

3.3. **The lineability coefficient.** While examining κ-lineability of some class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, we are naturally interested in the largest κ for which $\mathcal{F}$ is κ-lineable. The problem is, that such largest κ might not exist, as was noticed in a 2005 paper [3] of R.M. Aron, V. I. Gurariy, and J. B. Seoane–Sepúlveda. (Their example consists of a family $\mathcal{F}$ of polynomials of one variable of the form $x^{n!}p(x)$, where $n \in \mathbb{N}$ and $p(x)$ is a polynomial of the degree $\leq n$. Such $\mathcal{F}$ is n-lineable for every $n < \omega$, but is not ω-lineable.) Thus, instead of looking for the largest κ for which $\mathcal{F}$ is κ-lineable, it is only natural to look for the smallest κ for which $\mathcal{F}$ is *not* κ-lineable, as such a number is always well defined. This leads to the following definition of a *lineability coefficient* that comes from a 2013 paper [10] of A. Bartoszewicz and S. Głąb (see also [33]):

$$\mathcal{L}(\mathcal{F}) := \min\{\kappa\colon \mathcal{F} \text{ is not } \kappa\text{-lineable}\}.$$

Notice a similarity to the definition of the coefficient $\mathcal{A}(\mathcal{F})$, which is defined as the smallest cardinality κ for which there exist a subsets $F \subset \mathbb{R}^{\mathbb{R}}$ admitting *no shift* into $\mathcal{F}$.

It is easy to see that $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ is κ-lineable if, and only if, $\kappa < \mathcal{L}(\mathcal{F})$. Also, the family $\mathcal{F}$ admits the largest cardinal κ for which $\mathcal{F}$ is κ-lineable if, and only if, $\mathcal{L}(\mathcal{F})$ is a successor cardinal and equals to κ^+.

In this notation Theorem 3.4 can be expressed as follows:

Corollary 3.8. *If $\mathcal{F} \cup \{0\} \subsetneq \mathbb{R}^\mathbb{R}$ is closed under the scalar multiplication and $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$, then $\mathcal{L}(\mathcal{F}) > \mathcal{A}(\mathcal{F})$.*

Similarly, the equation $\mathcal{A}(\mathrm{SZ}) = d_\mathfrak{c}$ from Theorem 3.6 and the equivalence of parts (3) and (4) from Theorem 3.2 reduce to the statement

$$\mathcal{L}(\mathrm{SZ}) = \min\{\kappa\colon \text{ there is no } \mathfrak{c}\text{-almost disjoint } \mathcal{F} \subset \mathbb{R}^2 \text{ with } |\mathcal{F}| = \kappa\}.$$

Finally, Theorem 3.1 reduces to the inequality $\mathcal{L}(\mathrm{SZ}) > \mathfrak{c}^+$, while Theorem 3.3 to the statement that the equality $\mathcal{L}(\mathrm{SZ}) = (2^\mathfrak{c})^+$ is independent of ZFC.

It is also worth to mention here that in a 2015 paper [96] Krzysztof Płotka generalized the definition of a lineability coefficient to an arbitrary subfield E of $\mathbb{R}$:

$$\mathcal{L}_E(\mathcal{F}) := \min\{\kappa\colon \mathcal{F} \text{ is not } \kappa\text{-lineable over } E\}$$

(so that $\mathcal{L}(\mathcal{F}) = \mathcal{L}_\mathbb{R}(\mathcal{F})$) and proved that $\mathcal{L}_\mathbb{Q}(\mathrm{SZ}) = \mathcal{L}(\mathrm{SZ})$.

4. SZ maps which are Darboux-like

Clearly SZ functions are very far from being continuous. But can they be measurable in some sense? Not in the standard sense: no $f \in \mathrm{SZ}$ can be either Baire or Lebesgue measurable, since such functions have continuous restrictions to perfect subsets of $\mathbb{R}$. Nevertheless, in a 2006 paper [76] Alexander B. Kharazishvili proved that there exist SZ-functions measurable with respect to some translation invariant extensions of the Lebesgue measure. (For more on this subject, see also [75, 77, 78].)

Here, and in the remainder of this paper, we will concentrate on a question whether SZ-functions can be continuous in some generalized sense. Once again, the answer is negative for many classes of generalized continuous functions (like that of approximately or $\mathcal{I}$-approximately continuous functions, see e.g. [35]), since such functions usually have continuous restrictions to perfect (even residual) subsets of $\mathbb{R}$. However, there is a large class of generalized continuous functions, known under the common name of *Darboux-like functions*, for which (mainly) this is not the case. Thus, the subject of this section is to examine what is known about Darboux-like SZ-functions.

Recall, that a function $f \in \mathbb{R}^\mathbb{R}$ is called *Darboux* provided $f[C]$ is connected (i.e., an interval) for every connected $C \subset \mathbb{R}$. In other words, $f \in \mathbb{R}^\mathbb{R}$ is Darboux if, and only if, f has the intermediate value property. The class of all Darboux functions $f \in \mathbb{R}^\mathbb{R}$ will be denoted as $\mathscr{D}$. The name is used in honor of Jean Gaston Darboux (1842–1917)[9] who in his 1875 paper [50] showed that all derivatives, including those that are discontinuous, are in the class $\mathscr{D}$.

The study of the class $\mathrm{SZ} \cap \mathscr{D}$ was initiated in the 1997 paper [6] of Marek Balcerzak, K. C. Ciesielski, and T. Natkaniec where it was proved that existence of

[9]Darboux made many important contributions to geometry and mathematical analysis. His *alma mater* was the Ecole Normale Supérieure (in Paris). He was a biographer of Henri Poincaré (1854–1912). In 1908, he was a plenary speaker at the International Congress of Mathematicians in Rome.

Darboux SZ-functions is independent of ZFC, as expressed in Theorems 4.1 and 4.2. To state the first of these theorems more precisely, define

$$\mathrm{cov}_{\mathcal{M}} := \{\kappa\colon \text{ union of less than } \kappa\text{-many meager sets has empty interior}\}.$$

Recall that $\mathrm{cov}_{\mathcal{M}}$ is equal to the smallest cardinality of a family of meager sets whose union covers $\mathbb{R}$ and that the property

$$\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$$

is independent of ZFC: it follows from CH and, more generally, from the Martin's Axiom; at the same time, it is known that there are models of ZFC where it is false.

Theorem 4.1. *It is consistent with ZFC, follows from* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$*, that there exists a Darboux* SZ*-function, that is, that* $\mathrm{SZ} \cap \mathscr{D} \neq \emptyset$*.*

Notice, that Theorem 4.1 follows easily from the forthcoming Theorem 4.5. See also Theorem 4.4.

Theorem 4.2. *It is consistent with ZFC, holds in the iterated perfect set (Sacks) model, that there does not exist Darboux* SZ*-function, that is, that* $\mathrm{SZ} \cap \mathscr{D} = \emptyset$*.*

Proof. We will show that, in the iterated perfect set (Sacks) model, there are not Darboux SZ-functions. We are going to describe here only the properties of this model that are necessary to us.

Thus, let V be a model of ZFC+CH and let $V[G_{\omega_2}]$ be a model of ZFC+$\mathfrak{c} = \omega_2$ obtained as a generic extension of V over the forcing $\mathbb{P}$, which is a countable support iteration of the perfect set (Sacks) forcing. Then, V and $V[G_{\omega_2}]$ have the same cardinals. Moreover, there exists an increasing sequence $\{V[G_\alpha] : \alpha \leq \omega_2\}$ of models of ZFC such that:

(1) CH holds in $V[G_\alpha]$ for every $\alpha < \omega_2$.
(2) For every $\alpha < \omega_2$ of uncountable cofinality and every $p \in (\mathbb{R}^{\mathbb{Q}} \cup \mathbb{R}) \cap V[G_\alpha]$ there exists a $\beta < \alpha$ such that $p \in V[G_\beta]$.
(3) For every $a, b \in \mathbb{R}$ with $a < b$ and ordinal number $\alpha < \omega_2$ and there exists an $s \in (a,b) \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$ (a Sacks number over $V[G_\alpha]$) such that for every $x \in \mathbb{R} \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$ there exists a continuous function $g \in \mathbb{R}^{\mathbb{R}}$ coded in $V[G_\alpha]$ (i.e., such that $g|_{\mathbb{Q}} \in V[G_\alpha]$) with the property that $g(x) = s$.

(1) follows immediately from the fact that CH holds in V and we iterate forcings of cardinality $\mathfrak{c}$. The properties (2) and (3) can be found in [15] and in [86], respectively.

Next, let $h \in \mathbb{R}^{\mathbb{R}}$ be a SZ-function in $V[G_{\omega_2}]$ and let $a = \inf h[\mathbb{R}]$, $b = \sup h[\mathbb{R}]$. Then $-\infty \leq a < b \leq \infty$. We will show that $(a,b) \not\subset h[\mathbb{R}]$. To see this, define for every $\beta < \omega_2$ the set S_β as:

$$h\big[\mathbb{R} \cap V[G_\beta]\big] \cup \bigcup \big\{\{x,y\}\colon (\exists g \in \mathscr{C})\big(g|_{\mathbb{Q}} \in V[G_\beta] \ \&\ \langle x,y\rangle \in g \cap h \cap V[G_{\omega_2}]\big)\big\}.$$

Notice that, by (1), $\big|(\mathbb{R} \cup \mathbb{R}^{\mathbb{Q}}) \cap V[G_\beta]\big| \leq \omega_1$. Also $|g \cap h \cap V[G_{\omega_2}]| \leq \omega_1$ for every $g \in \mathscr{C}$ with $g|_{\mathbb{Q}} \in V[G_\beta]$. Thus, $|S_\beta| \leq \omega_1$ for every $\beta < \omega_2$. Define $\Gamma : \omega_2 \to \omega_2$ by putting $\Gamma(\beta) := \sup\{\gamma(x) : x \in S_\beta\}$, where $\gamma(x) = \min\{\xi : x \in V[G_\xi]\}$, and let $\alpha < \omega_2$ be of uncountable cofinality such that $\Gamma(\beta) < \alpha$ for every $\beta < \alpha$. Then, by (2),

(i) $h(x) \in V[G_\alpha]$ for every $x \in \mathbb{R} \cap V[G_\alpha]$.
(ii) $h \cap g \subset V[G_\alpha]$ for every $g \in \mathscr{C}$ with $g|_{\mathbb{Q}} \in V[G_\alpha]$.

Now, let $s \in (a,b) \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$ be a number from (3). It is enough to prove that $s \notin h[\mathbb{R}]$.

But $s \notin h[\mathbb{R} \cap V[G_\alpha]]$ by (i). So, let $x \in \mathbb{R} \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$. It is enough to show that $h(x) \neq s$. But, by (3), there exists a continuous function $g \in \mathbb{R}^{\mathbb{R}}$ coded in $V[G_\alpha]$ such that $g(x) = s$. So, $h(x) \neq s$, since otherwise $\langle x, s\rangle \in h \cap g$ and, by (ii), $s \in V[G_\alpha]$. This contradiction finishes the proof. $\square$

It is also worth mentioning that Theorem 4.2 follows also immediately from the next theorem that is presented in a 2004 monograph [42, section 6.2] of K. C. Ciesielski and J. Pawlikowski. (See also [41].)

Theorem 4.3. *The Covering Property Axiom* CPA*, which holds in the iterated perfect set (Sacks) model, implies that for every* SZ*-function f its range $f[\mathbb{R}]$ contains no perfect set.*

Of course, Theorems 4.1 and 4.2 imply that the statement $\mathrm{SZ} \cap \mathscr{D} \neq \emptyset$ is independent of the ZFC axioms. Recently, the question of whether this statement could be equivalent to the property $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$ was raised.

A simple construction presented in the next theorem shows that this is not the case, since is consistent (holds in the model obtained by adding ω_2 random reals to the model for CH, see e.g. [14, pages 403-404]) that $\mathrm{cov}_{\mathcal{N}} = \mathfrak{c} > \mathrm{cov}_{\mathcal{M}}$. Recall that, by $\mathcal{N}$ we mean the ideal of null (Lebesgue measure zero) subsets of $\mathbb{R}$ and $\mathrm{cov}_{\mathcal{N}}$ is the smallest cardinal κ such that $\mathbb{R}$ (equivalently, any its subset of positive measure) cannot be covered by $< \kappa$ sets in $\mathcal{N}$.

Theorem 4.4. $\mathrm{cov}_{\mathcal{N}} = \mathfrak{c}$ *implies that there exists $\bar f\colon \mathbb{R} \to \mathbb{R}$ which is Darboux and* SZ*. In fact, $\bar f \in \mathrm{SZ}(\mathscr{B})$ and maps any non-empty open set onto $\mathbb{R}$.*

Proof. Let $\mathcal{J} = \{(p,q) \times \{r\} \colon r \in \mathbb{R} \ \&\ p,q \in \mathbb{Q} \ \&\ p < q\}$. Also, let $\{g_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of the family $\mathscr{B}$ of all Borel maps in $\mathbb{R}^{\mathbb{R}}$. For every $\xi < \mathfrak{c}$ let $\mathcal{J}_\xi$ be the family of all $J \in \mathcal{J}$ such that $\mathrm{dom}(J \cap g_\xi) \notin \mathcal{N}$. Notice that each $\mathcal{J}_\xi$ is at most countable, since there exists at most countable many $r \in \mathbb{R}$ such that $g_\xi^{-1}(r) \notin \mathcal{N}$.

By transfinite induction, we construct a sequence $\langle D_\xi \in [\mathbb{R}]^{\le\omega} \colon \xi < \mathfrak{c}\rangle$ and the functions $f_\xi \colon D_\xi \to \mathbb{R}$ as follows. For every $\xi < \mathfrak{c}$ let $\mathbb{I}_\xi = \mathcal{J}_\xi \setminus \bigcup_{\zeta<\xi} \mathcal{J}_\zeta$. If $\mathbb{I}_\xi = \emptyset$ we put $D_\xi = \emptyset$. Otherwise, we let $\{J_n \colon n < \omega\}$ be an enumeration of $\mathbb{I}_\xi$ and define, by a simple induction, a set $D_\xi = \{x_n \colon n < \omega\}$ so that

$$x_n \in \mathrm{dom}(J_n \cap g_\xi) \setminus \left(\{x_i \colon i < n\} \cup \bigcup_{\zeta<\xi} (D_\zeta \cup \mathrm{dom}(J_n \cap g_\zeta))\right).$$

The choice can be made, since $\mathrm{cov}_{\mathcal{N}} = \mathfrak{c}$, $\mathrm{dom}(J_n \cap g_\xi) \notin \mathcal{N}$, and the set of non-allowed points is a union of $< \mathfrak{c}$ null sets: singletons and $\mathrm{dom}(J_n \cap g_\zeta) \in \mathcal{N}$.

For every $n < \omega$ define $f_\xi(x_n)$ as the unique number with $\langle x_n, f_\xi(x_n)\rangle \in J_n$. This finishes the inductive construction.

Next let $f := \bigcup_{\xi<\mathfrak{c}} f_\xi$ and notice that this is a partial function defined on $D := \bigcup_{\xi<\mathfrak{c}} D_\xi$. Let $\bar f \in \mathbb{R}^{\mathbb{R}}$ be an arbitrary extension of f so that $\bar f \restriction (\mathbb{R} \setminus D)$ is $\mathrm{SZ}(\mathscr{B})$. Then $\bar f$ is as needed. $\square$

4.1. **All Darboux-like classes within** SZ**.** The term *Darboux-like* classes of functions (within $\mathbb{R}^{\mathbb{R}}$) usually refers to the eight classes shown in Fig. 5. All these classes coincide (i.e., are equal) when restricted to the class of Baire class 1 functions. (See

[23] or [34, theorem 1.1] and the references therein.) However, the graph presented in Fig. 5 remains almost[10] unchanged, when we restrict Darboux-like classes of functions to either Baire class 2 or Borel functions. (See [34, theorem 1.2]).

$$\begin{tikzcd}
 & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathscr{D} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] &
\end{tikzcd}$$

FIGURE 5. All inclusions, indicated by arrows, among the Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$. The only inclusions among the intersections of these classes are those that follows trivially from this schema. (See [34, 60].)

The seven so far undefined classes of Darboux-like functions, presented in order of their chronological appearance in the literature, are as follows.

- PC of all *peripherally continuous functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that for every number $x \in \mathbb{R}$ there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ with $\lim_{n\to\infty} f(s_n) = f(x) = \lim_{n\to\infty} f(t_n)$. This class was introduced in a 1907 paper [111] of John Wesley Young (1879–1932).[11] The name comes from the papers [64, 66, 110].
- PR of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with perfect road*, that is, such that for every $x \in \mathbb{R}$ there exists a perfect $P \subset \mathbb{R}$ having x as a bilateral limit point (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) such that $f \restriction P$ is continuous at x. This class was introduced in a 1936 paper [85] of Isaie Maximoff, where he proved that $\mathscr{D} \cap \mathcal{B}_1 = \mathrm{PR} \cap \mathcal{B}_1$, where $\mathcal{B}_1$ is the class of Baire class 1 functions.
- Conn of all *connectivity functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that the graph of f restricted to any connected $C \subset \mathbb{R}$ is a connected subset of $\mathbb{R}^2$. This notion can be traced to a 1956 problem [88] stated by John Forbes Nash (1928–2015).[12] See also [66, 105]. Connectivity maps on $\mathbb{R}^2$ are defined similarly.
- AC of all *almost continuous functions* $f \in \mathbb{R}^{\mathbb{R}}$ (in the sense of Stallings), that is, such that every open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. This class was first seriously studied in a 1959 paper [105] of John Robert Stallings (1935–2008);[13] however, it appeared already in a 1957 paper [66] of Olan H. Hamilton (1899–1976).

[10] Except that we, additionally, get $\mathscr{D} \subset \mathrm{SCIVP} = \mathrm{CIVP}$.

[11] Young was co–founder and a president of the MAA. He was also editor of the Bulletin of the American Mathematical Society.

[12] Nash shared the 1994 Nobel Memorial Prize in Economic Sciences with game theorists Reinhard Selten and John Harsanyi. In 2015, he also shared the Abel Prize with Louis Nirenberg for his work on nonlinear PDEs.

[13] Stallings' contributions include a proof, in a 1960 paper, of the Poincaré Conjecture in dimensions greater than six.

Ext of all *extendable functions* $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that there exists a connectivity function $g\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ with $f(x) = g(x,0)$ for all $x \in \mathbb{R}$. The notion of extendable functions (without the name) first appeared in 1959 paper [26] of J. Stallings, where he asks a question whether every connectivity function defined on [0, 1] is extendable.

CIVP of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set C between p and q such that $f[C] \subset K$. This class was first introduced in a 1982 paper [58] of Richard G. Gibson and Fred William Roush.

SCIVP of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with Strong Cantor Intermediate Value Property*, that is, such that for all $p, q \in \mathbb{R}$ with $p \neq q$ and $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f[C] \subset K$ and $f \restriction C$ is continuous. This notion was introduced in a 1992 paper [99] of Harvey Rosen, R. Gibson, and F. Roush to help distinguish extendable and connectivity functions on $\mathbb{R}$.

The inclusions

$$\mathrm{Conn} \subset \mathscr{D} \subset \mathrm{PC}, \quad \mathrm{PR} \subset \mathrm{PC}, \quad \text{and } \mathrm{SCIVP} \subset \mathrm{CIVP}$$

are obvious from the definitions. The inclusions Ext $\subset$ AC $\subset$ Conn were proved by Stallings [105], while CIVP $\subset$ PR was stated without proof in [59]; its proof can be found in [60, theorem 3.8]. The inclusion Ext $\subset$ SCIVP comes from [99].

All inclusions indicated in Fig. 5 by the arrows are strict. In fact, this remains true even when we add to the considerations the intersections of the classes from the top and bottom rows of Fig. 6. This is well described in survey papers [30, 34, 60]. Specifically, $\mathrm{AC} \setminus \mathrm{CIVP} \neq \emptyset$ and $\mathrm{CIVP} \setminus \mathrm{AC} \neq \emptyset$ was shown in a 1982 work [58]. The fact that $\mathrm{Conn} \setminus \mathrm{AC} \neq \emptyset$ is the trickiest to prove and is related to late 1960's papers: [98] of John Henderson Roberts, [49] of James L. Cornette, [68] of F. Burton Jones and Edward S. Thomas Jr., and [20] of J. Brown. The result $\mathscr{D} \setminus \mathrm{Conn} \neq \emptyset$ can be traced to 1965 paper [25] of Andrew M. Bruckner and Jack Gary Ceder (see also [20]), while examples for $\mathrm{PC} \setminus \mathscr{D} \neq \emptyset$, $\mathrm{PR} \setminus \mathrm{CIVP} \neq \emptyset$, and $\mathrm{PC} \setminus \mathrm{PR} \neq \emptyset$ to 2000 paper [34] of K. C. Ciesielski and Jan Jastrzębski. All of these examples will be discussed also below in relation to SZ-functions.

Clearly $\mathrm{SZ} \cap \mathrm{SCIVP} = \emptyset$. This also implies that $\mathrm{SZ} \cap \mathrm{Ext} = \emptyset$. In particular, the classes Ext and SCIVP are not of interest in our context and will be removed from our further considerations below. This leaves us with the six classes from Fig. 5 which, as we will see below, can contain an SZ-maps. This reduces Fig. 5 to the following Figure 6, to which we added class SZ, indicating the results from Theorems 4.1 and 4.2.

The six Darboux-like classes of functions presented in Fig. 6 are naturally split into two subclasses:

$$\mathbb{U} := \{\mathrm{AC}, \mathrm{Conn}, \mathscr{D}\},$$

whose non-empty intersection with SZ can be proven only consistently, and

$$\mathbb{L} := \{\mathrm{CIVP}, \mathrm{PR}, \mathrm{PC}\}$$

that admit SZ functions in ZFC. As such, we often treat these two groups separately.

Our first goal in what follows is to show two most fundamental results related to our discussion: that $\mathrm{SZ} \cap \mathrm{CIVP} \cap \neg\, \mathscr{D} \neq \emptyset$ (Theorem 4.6(i)) and that it is consistent

$$
\begin{tikzcd}
\mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathscr{D} \arrow[dd] \arrow[dr, dotted] & \\
 & & & \neg\,\mathrm{SZ} \\
\mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[r] & \mathrm{PC} &
\end{tikzcd}
$$

FIGURE 6. Six Darboux-like classes of functions that contain SZ-maps. Inclusions are indicated by solid arrows. The dotted arrow indicates inclusion that is independent of the ZFC, Theorems 4.1 and 4.2.

with ZFC that $\mathrm{SZ}\cap\mathrm{AC}\cap\mathrm{CIVP}\neq\emptyset$ (Theorem 4.5). Both of these results are proved in a stronger form, which involve the following class:

Add of all *additive functions* $f\in\mathbb{R}^{\mathbb{R}}$, that is, such that $f(x+y)=f(x)+f(y)$ for every $x,y\in\mathbb{R}$. For every $f\in\mathrm{Add}$ we also have $f(qx)=qf(x)$ for every $x\in\mathbb{R}$ and $q\in\mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers. The study of this class dates back to the work of A. M. Legendre whose aim was to determine the solution of the Cauchy functional equation $f(x+y)=f(x)+f(y)$ for $x,y\in\mathbb{R}$. However, the systematic study of the additive Cauchy functional equation was initiated by Augustin-Louis Cauchy (1789–1857) in his famous 1821 seminal work [26], where he proved that any continuous additive f is of the form $f(x)=cx$ for some $c\in\mathbb{R}$. The fist construction of discontinuous $f\in\mathrm{Add}$ was given in the 1905 paper [65] by Georg Karl Wilhelm Hamel (1877–1954).

The following theorem comes from a 2005 paper [92, example 10] of T. Natkaniec and H. Rosen. The example of an $f\in\mathrm{SZ}\cap\mathrm{AC}\cap\mathrm{CIVP}$, under the same assumption, was also constructed in a 1999 paper [8] of Krzysztof Banaszewski and T. Natkaniec.

Theorem 4.5. *If* $\mathrm{cov}_{\mathcal{M}}=\mathfrak{c}$*, then* $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{AC}\cap\mathrm{CIVP}\neq\emptyset$*.*

The results related to this theorem have a long history. In 1981 paper [27] J. Ceder showed that CH implies that $\mathrm{SZ}\cap\mathrm{Conn}\neq\emptyset$ and Kenneth R. Kellum, in 1982 paper [74], noticed that Ceder's function is in fact almost continuous. Theorem 4.5 generalizes also a result from a 1997 paper [6] of M. Balcerzak, K. C. Ciesielski, and T. Natkaniec where it is proved, under $\mathrm{cov}_{\mathcal{M}}=\mathfrak{c}$ assumption, that $\mathrm{SZ}\cap\mathscr{D}\cap\mathrm{PR}\neq\emptyset$, compare Theorem 4.1. This last paper was was written to answer a problem from a 1993 paper of Udayan B. Darji [51], which we discuss below.

The next theorem concerns classes from the family $\mathbb{L}$.

Theorem 4.6. *The following can be proved in ZFC.*

(i) $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{CIVP}\setminus\mathscr{D}\neq\emptyset$.
(ii) $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{PR}\setminus(\mathscr{D}\cup\mathrm{CIVP})\neq\emptyset$.
(iii) $\emptyset\neq\mathrm{Add}\cap\mathrm{SZ}\setminus(\mathscr{D}\cup\mathrm{PR})\subset\mathrm{PC}$.

The inclusion in (iii) follows from the fact that

$$\mathrm{Add}\cap\mathrm{SZ}\subset\mathrm{PC}, \tag{4.1}$$

see e.g. [92, remark 3]. This holds, since every function with a graph dense in $\mathbb{R}^2$ is clearly PC and is well known that every discontinuous additive function (so, one from $\mathrm{Add}\cap\mathrm{SZ}$) has a dense graph, see e.g. [79].

The constructions of functions justifying (i), (ii), and (iii) can be found, respectively, in [92, examples 6, 7, and 4]. (The explicit statements describing [92, examples 7 and 4], which justify (ii) and (iii), do not mention that the maps are not in $\mathscr{D}$. However, in both cases they are, as stated, injections. As such, being additive and discontinuous, they must be in $\neg\mathscr{D}$.)

Theorem 4.6 generalizes a 1993 result of U. B. Darji [51], who gave a ZFC example of an SZ-function with perfect road. A ZFC example of an additive SZ-function with perfect road can be found in a 1997 paper [6] of M. Balcerzak, K. C. Ciesielski, and T. Natkaniec.

4.2. **Proofs of Theorems 4.6(i) and 4.5.** To illustrate a typical methodology used in the construction of different SZ Darboux-like functions we present below two of such constructions: the first one within ZFC and a second one under the assumption that $\mathrm{cov}_{\mathcal{M}}=\mathfrak{c}$.

For $A\subset\mathbb{R}$ let $\mathrm{lin}_{\mathbb{Q}}(A)$ denote the $\mathbb{Q}$–linear subspace of $\mathbb{R}$ spanned by A. It is well known (see, e.g., [29]) that if $A\subset\mathbb{R}$ is $\mathbb{Q}$–linearly independent, then any map $f\colon A\to\mathbb{R}$ has a unique additive extension $\hat{f}\colon \mathrm{lin}_{\mathbb{Q}}(A)\to\mathbb{R}$. A standard method of constructing discontinuous additive functions (and one that we shall use in the proofs below) is to, first, define an f on some Hamel basis H (i.e., a basis of $\mathbb{R}$ as a $\mathbb{Q}$–linear space) and, then, extend it to its unique additive extension $\hat{f}\in\mathrm{Add}$. Let us, then, proceed.

Proof of Theorem 4.6(i). Let H be a Hamel basis for which there exists a family $\{H_n\subset H\colon n<\omega\}$ of pairwise disjoint perfect sets such that each nonempty open interval contains one H_n. Such a basis exists, since there is a perfect set which is linearly (even algebraically) independent (see the original work by John Von Neumann [107], Jan Mycielski's paper [87] and, also, the monographs [79] and [42, theorem 5.1.9].) For each $n<\omega$ let $\left\{H_n^K\subset H\colon K\in\mathrm{Perf}\right\}$ be a partition of H_n into perfect sets and, for every $x\in\mathbb{R}$, let $P_x=K$ when $x\in H_n^K$ for some n and perfect K, and $P_x=\mathbb{R}$ otherwise.

Let $\{x_\xi\colon\xi<\mathfrak{c}\}$ and $\{g_\xi\colon\xi<\mathfrak{c}\}$ be enumerations of H and $\mathscr{B}$, respectively. By induction on $\xi<\mathfrak{c}$, choose

$$f(x_\xi)\in P_{x_\xi}\setminus\mathrm{lin}_{\mathbb{Q}}\left(\{f(x_\zeta)\colon\zeta<\xi\}\cup\bigcup_{\zeta\le\xi}g_\zeta[\mathrm{lin}_{\mathbb{Q}}(\{x_\zeta\colon\zeta\le\xi\})]\}\right). \tag{4.2}$$

This defines $f\colon H\to\mathbb{R}$. Let $\hat{f}$ be the unique extension of f in Add. We claim, that it is as needed, that is, that $\hat{f}\in\mathrm{SZ}\cap\mathrm{CIVP}\cap\neg\mathscr{D}$. This would complete the proof.

To see that $\hat{f}\in\mathrm{CIVP}$ notice that (4.2) ensures that, for every $K\in\mathrm{Perf}$, our function $\hat{f}$ maps $\bigcup_{n<\omega}H_n^K$ into K. This also shows that $\hat{f}$ is discontinuous.

Next, notice that $\hat{f}\in\neg\mathscr{D}$. Indeed, by (4.2), f is injective and $f[H]$ is linearly independent over $\mathbb{Q}$. Thus, $\hat{f}$ is also injective. Being discontinuous and additive, $\hat{f}$ is not Darboux.

Finally, we will show that $\hat{f}\in\mathrm{SZ}$. For this, choose $g\in\mathscr{B}$ and let $\zeta<\omega$ be such that $g=g_\zeta$. It is enough to show that $[\hat{f}=g_\zeta]\subset\mathrm{lin}_{\mathbb{Q}}(\{x_\eta\colon\eta<\zeta\})$.

In order to see this, let us proceed by contradiction. Then, there would exist $\xi \geq \zeta$ and $w \in \mathrm{lin}_{\mathbb{Q}}(\{x_\eta : \eta \leq \xi\}) \setminus \mathrm{lin}_{\mathbb{Q}}(\{x_\eta : \eta < \xi\})$ such that $\hat{f}(w) = g_\zeta(w)$. We claim that this contradicts the choice of $f(x_\xi)$ as in (4.2). Indeed, let $q \in \mathbb{Q} \setminus \{0\}$ and $v \in \mathrm{lin}_{\mathbb{Q}}(\{x_\eta : \eta < \xi\})$ be such that $w = qx_\xi + v$. Then

$$qf(x_\xi) + \hat{f}(v) = \hat{f}(w) = g_\zeta(w)$$

and

$$f(x_\xi) = q^{-1}(g_\zeta(w) - \hat{f}(v)) \in \mathrm{lin}_{\mathbb{Q}} \left(\{f(x_\zeta) : \zeta < \xi\} \cup \bigcup_{\zeta \leq \xi} g_\zeta[\mathrm{lin}_{\mathbb{Q}}(\{x_\zeta : \zeta \leq \xi\})] \right),$$

contradicting (4.2). □

The proof of the second theorem is considerably more intricate and will depend on the following lemmas and notation. For $A \subset \mathbb{R}^2$ we let $\mathrm{dom}(A)$ stand for the projection of A onto the fist coordinate. Let $\mathcal{G}$ be the family of all continuous functions from G_δ subsets of $\mathbb{R}$ into $\mathbb{R}$. Also, let

$$\begin{aligned} \hat{\mathcal{G}}_0 &:= \{g \in \mathcal{G} : \ \mathrm{cl}(\mathrm{dom}(g)) \text{ is a non-trivial interval}\}, \\ \hat{\mathcal{G}}_1 &:= \{g \in \mathcal{G} : \ \mathrm{cl}(\mathrm{dom}(g)) \text{ is nowhere dense}\}, \end{aligned}$$

and put $\hat{\mathcal{G}} := \hat{\mathcal{G}}_0 \cup \hat{\mathcal{G}}_1$.

Lemma 4.7. *Let $f \in \mathbb{R}^{\mathbb{R}}$ and assume that for every $\hat{g} \in \hat{\mathcal{G}}_0$ there exist a $g \in \hat{\mathcal{G}}_0$ and a non-trivial interval J such that $J \cap \mathrm{dom}(f \cap g) \neq \emptyset$ and $\mathrm{dom}(\hat{g} \cap g)$ is dense in J. Then $f \in \mathrm{AC}$.*

Proof. Let $\mathbb{B}$ be the family of all *blocking sets*, where by blocking set we mean a closed $B \subset \mathbb{R}^2$ that meets the graph of every continuous function and is disjoint with some arbitrary function. It follows immediately from the definition of AC that if $f \in \mathbb{R}^{\mathbb{R}}$ intersects every blocking set, then $f \in \mathrm{AC}$. Recall, also, that every $B \in \mathbb{B}$ contains the graph of some $g \in \hat{\mathcal{G}}_0$, see [74, lemma 1] and the proof of [6, theorem 1].[14]

To see that $f \in \mathrm{AC}$, fix $B \in \mathbb{B}$. It suffices to show that $f \cap B \neq \emptyset$. Indeed, B contains some $\hat{g} \in \hat{\mathcal{G}}_0$ and, by our assumption, there exist $g \in \hat{\mathcal{G}}_0$ and a non-trivial interval J such that $J \cap \mathrm{dom}(f \cap g) \neq \emptyset$ and $D := \mathrm{dom}(\hat{g} \cap g) \cap J$ is dense in J. In particular, since g is continuous, $\mathrm{cl}(g \restriction J) \subset \mathrm{cl}(\hat{g} \cap g \restriction J)$[15] and

$$\emptyset \neq f \cap g \restriction J \subset \mathrm{cl}(g \restriction J) \subset \mathrm{cl}(\hat{g} \cap g \restriction J) \subset \mathrm{cl}(\hat{g}) \subset B.$$

Thus, $\emptyset \neq f \cap g \restriction J \subset f \cap B$, as needed. □

To ensure that our function belongs to $\mathrm{Add} \cap \mathrm{SZ}$ we shall also need the following lemma.

Lemma 4.8. *Let $g \in \mathcal{G}$, $V \subset \mathbb{R}$ be a $\mathbb{Q}$–vector space, $x \in \mathbb{R} \setminus V$, and define $W := \mathrm{lin}_{\mathbb{Q}}(V \cup \{x\})$. If $f : W \to \mathbb{R}$ is additive, then $[f = g] \subset V$ if, and only if,*

$$f(x) \neq q^{-1}(g(qx+v) - f(v)) \ \text{ for every } v \in V \text{ and } 0 \neq q \in \mathbb{Q} \text{ with } qx + v \in \mathrm{dom}(g).$$

[14]By [74, lemma 1], $\mathrm{dom}(B)$ has nonempty interior. Thus, by the Baire category theorem, there exists $n \in \mathbb{N}$ for which the same is true for the set $B_n := B \cap (\mathbb{R} \times [-n, n])$. If J is a nonempty interval contained in $\mathrm{dom}(B_n)$ and $h : J \to \mathbb{R}$ is defined via $h(x) = \inf\{y : \langle x, y \rangle \in B_n\}$, then h is of Baire class 1. Thus, $g := h \restriction C(h)$ is as needed.

[15]The map γ from $X := J \cap \mathrm{dom}(g)$ into $g \restriction J \subset \mathbb{R}^2$, given as $\gamma(x) := \langle x, g(x) \rangle$, is continuous and so $g \restriction J = \gamma[J \cap \mathrm{dom}(g)] = \gamma[\mathrm{cl}_X(D)] \subset \mathrm{cl}_{\mathbb{R}}(\gamma[D]) = \mathrm{cl}_{\mathbb{R}}(\hat{g} \cap g \restriction J)$.

Proof. Since $W\setminus V=\{qx+v\colon q\in\mathbb{Q}\setminus\{0\}\ \&\ v\in V\}$, then $[f=g]\not\subset V$ if, and only if, there are $q\in\mathbb{Q}\setminus\{0\}$ and $v\in V$ such that $qx+v\in\mathrm{dom}(g)$ and

$$qf(x)+f(v)=f(qx+v)=g(qx+v)$$

or, equivalently, that $f(x)=q^{-1}(g(qx+v)-f(v))$, as needed. □

Proof of Theorem 4.5. Let the sets $\{H_n\subset H\colon n<\omega\}$ be as in the above proof of Theorem 4.6(i), for every $n<\omega$ let $\{H^n_\xi\in\mathrm{Perf}\colon \xi<\mathfrak{c}\}$ be a partition of H_n and let $\mathcal{H}:=\{\bigcup_{n<\omega}H^n_\xi\colon \xi<\mathfrak{c}\}$. Thus, $\mathcal{H}$ is a partition of a linearly independent meager set $M:=\bigcup_{n<\omega}H_n$.

Let $\{g_\xi\colon \xi<\mathfrak{c}\}$ and $\{P_\xi\colon\xi<\mathfrak{c}\}$ be enumerations of $\mathcal{G}$ and Perf, respectively, and choose a sequence $\langle r_\xi\in\mathbb{R}\colon \xi<\mathfrak{c}\rangle$ such that every $r\in\mathbb{R}$ appears in it $\mathfrak{c}$-many times. We construct, by induction, a sequence $\langle\langle f_\xi,H_\xi\rangle\colon\xi<\mathfrak{c}\rangle$ such that, for every $\xi<\mathfrak{c}$, f_ξ is a function from at most countable $D_\xi\subset\mathbb{R}$ into $\mathbb{R}$ and $H_\xi\in\mathcal{H}$. Moreover, each initial segment $\langle\langle f_\zeta,H_\zeta\rangle\colon\zeta<\xi\rangle$ satisfies the following inductive conditions for every $\eta<\xi$.

(i) $D_\alpha\cap D_\beta=\emptyset$ for every $\alpha<\beta$; for every $\delta\le\xi$ the set $T_\delta:=\bigcup_{\zeta<\delta}D_\zeta$ is linearly independent; thus, the map $\bigcup_{\zeta<\delta}f_\zeta$ has a unique extension to an additive function $\hat f_\delta$ from $V_\delta:=\mathrm{lin}_{\mathbb{Q}}(T_\delta)$ into $\mathbb{R}$.

(ii) Let $Z_\eta:=\bigcup\{Z^{q,v}_{\eta,\zeta}\colon q\in\mathbb{Q}\setminus\{0\}\ \&\ v\in V_\eta\ \&\ \zeta<\eta\}$, where

$$Z^{q,v}_{\eta,\zeta}:=\mathrm{dom}\left(g_\eta\cap\left\{\langle x,q^{-1}(g_\zeta(qx+v)-\hat f_\eta(v))\rangle\colon qx+v\in\mathrm{dom}(g_\zeta)\right\}\right).$$

If U_η is the maximal open, possibly empty, subset of $\mathbb{R}$ such that each set $\mathrm{dom}(g_\eta)\setminus Z^{q,v}_{\eta,\zeta}$ is residual in U_η, then $E_\eta:=\mathrm{dom}(g_\eta\cap f_\eta)$ is a countable dense subset of $U_\eta\cap\mathrm{dom}(g_\eta)\setminus Z_\eta$; moreover, $f_\eta\restriction E_\eta=g_\eta\restriction E_\eta$.

(iii) If $r_\eta\notin\mathrm{lin}_{\mathbb{Q}}\left(E_\eta\cup\bigcup_{\zeta<\eta}(H_\zeta\cup D_\zeta)\right)$ or $r_\eta\in\bigcup_{\zeta<\eta}H_\zeta\setminus\mathrm{lin}_{\mathbb{Q}}\left(E_\eta\cup\bigcup_{\zeta<\eta}D_\zeta\right)$, then $D_\eta:=E_\eta\cup\{r_\eta\}$; otherwise $D_\eta:=E_\eta$.

(iv) If $r_\eta\in D_\eta\setminus E_\eta$, then $f_\eta(r_\eta)\notin\mathrm{lin}_{\mathbb{Q}}\left(\hat f_\eta[T_\eta\cup E_\eta]\cup\bigcup_{\zeta\le\eta}g_\zeta[T_\eta\cup E_\eta]\right)$; moreover, if $r_\eta\in H_\zeta$ for a $\zeta<\eta$, then $f_\eta(r_\eta)\in P_\zeta$.

(v) $H_\eta\in\mathcal{H}$ is disjoint with $\mathrm{lin}_{\mathbb{Q}}\left(D_\eta\cup\bigcup_{\zeta<\eta}(H_\zeta\cup D_\zeta)\right)$.

(vi) $\bigcup_{\zeta\le\eta}(H_\zeta\cup D_\zeta)$ is linearly independent over $\mathbb{Q}$.

(vii) For every $\zeta\le\eta$ we have $\mathrm{dom}(\hat f_\eta\cap g_\zeta)\subset V_{\zeta+1}$.

To construct such a sequence $\langle\langle f_\xi,H_\xi\rangle\colon\xi<\mathfrak{c}\rangle$, assume that for some $\xi<\mathfrak{c}$ its initial segment $\langle\langle f_\zeta,H_\zeta\rangle\colon\zeta<\xi\rangle$ satisfying conditions (i)–(vii) is already constructed. We need to choose a pair $\langle f_\xi,H_\xi\rangle$ so that the sequence $\langle\langle f_\zeta,H_\zeta\rangle\colon\zeta<\xi+1\rangle$ still satisfies the properties (i)–(vii).

We start this by defining the set E_ξ. If $U_\xi=\emptyset$, then we put $E_\xi:=\emptyset$. Otherwise, let $\{G_n\ne\emptyset\colon n<\omega\}$ be a countable basis of U_ξ and define $E_\xi:=\{x_n\colon n<\omega\}$, where the points x_n are chosen by induction on $n<\omega$ subject to the following condition, where $W^n_\xi:=\mathrm{lin}_{\mathbb{Q}}\left(\{x_i\colon i<n\}\cup\bigcup_{\zeta<\xi}(H_\zeta\cup D_\zeta)\right)$:

$$x_n\in(G_n\cap\mathrm{dom}(g_\xi))\setminus\left(W^n_\xi\cup\bigcup\{Z^{q,v}_{\xi,\zeta}\colon q\in\mathbb{Q}\setminus\{0\}\ \&\ v\in W^n_\xi\ \&\ \zeta<\xi\}\right).\tag{4.3}$$

Such a choice is possible by the assumption that $\mathrm{cov}_{\mathcal{M}}=\mathfrak{c}$, since $G_n\cap\mathrm{dom}(g_\xi)$ is comeager in G_n while, by (ii),

$$W^n_\xi\cup\bigcup\{Z^{q,v}_{\xi,\zeta}\cap G_n\cap\mathrm{dom}(g_\xi)\colon q\in\mathbb{Q}\setminus\{0\}\ \&\ v\in W^n_\xi\ \&\ \zeta<\xi\}$$

is a union of $< \mathfrak{c}$ meager sets. The set E_ξ is extended to D_ξ according to (iii) and H_ξ is chosen according to (v). The inductive assumption and (4.3) ensure that the set $E_\xi \cup \bigcup_{\zeta<\xi}(H_\zeta \cup D_\zeta)$ is linearly independent over $\mathbb{Q}$ and, thus, (vi) holds. Of course we define f_ξ on D_ξ according to (ii) and (iv).

This finishes the inductive construction.

The construction clearly preserves properties (i)–(vi). To see that (vii) is also preserved, first notice that for every $\zeta < \xi$ and $n < \omega$, we have

$$\mathrm{dom}(\hat{f}_{\xi+1} \cap g_\zeta) \cap \mathrm{lin}_{\mathbb{Q}}\left(\{x_i \colon i < n\} \cup V_\xi\right) \subset V_{\zeta+1}.$$

For $n = 0$ this is ensured by the inductive assumption (vii) while, for $n > 0$, this is proved by induction using Lemma 4.8 (with $V = \mathrm{lin}_{\mathbb{Q}}\left(\{x_i \colon i < n\} \cup V_\xi\right)$ and $x = x_n$) and the restrictions imposed by (4.3). Thus,

$$\mathrm{dom}(\hat{f}_{\xi+1} \cap g_\zeta) \cap \mathrm{lin}_{\mathbb{Q}}\left(E_\xi \cup V_\xi\right) \subset V_{\zeta+1}.$$

This clearly ensures that (vii) is preserved for $\zeta < \xi$ in the case when $D_\xi = E_\xi$. But otherwise, $D_\xi \setminus E_\xi = \{r_\xi\}$ and, in this case, (vii) for $\zeta < \xi$ holds by (iv) and an another use of Lemma 4.8. Finally (vii) holds for $\zeta = \xi$, since he have $\mathrm{dom}(\hat{f}_{\xi+1} \cap g_\xi) \subset \mathrm{dom}(\hat{f}_{\xi+1}) \subset V_{\xi+1}$.

Next, notice that $\tilde{H} := \bigcup_{\xi<\mathfrak{c}} D_\xi$ spans $\mathbb{R}$, that is, that it is a Hamel basis. To see this, first notice that $\bigcup_{\zeta<\mathfrak{c}} H_\zeta \subset \tilde{H}$: this is ensured by (iii) and the fact that every $r \in H_\zeta$, with $\zeta < \mathfrak{c}$, there is $\xi > \zeta$ with $r_\xi = r$. To see that $\mathbb{R} \subset \mathrm{lin}_{\mathbb{Q}}(\tilde{H})$ fix an $r \in \mathbb{R}$ and let $\xi < \mathfrak{c}$ be such that $r = r_\xi$. It is enough to show that $r_\xi \in \mathrm{lin}_{\mathbb{Q}}(\tilde{H})$. Indeed, by (iii), either $r_\xi \notin \mathrm{lin}_{\mathbb{Q}}\left(E_\xi \cup \bigcup_{\zeta<\xi}(H_\zeta \cup D_\zeta)\right)$, so $r_\xi \in \bigcup_{\zeta\le\xi} D_\xi \subset \mathrm{lin}_{\mathbb{Q}}(\tilde{H})$, or else $r_\xi \in \mathrm{lin}_{\mathbb{Q}}\left(E_\xi \cup \bigcup_{\zeta<\xi}(H_\zeta \cup D_\zeta)\right) \subset \mathrm{lin}_{\mathbb{Q}}\left(\bigcup_{\zeta\le\xi} D_\xi \cup \bigcup_{\zeta<\mathfrak{c}} H_\zeta\right) \subset \mathrm{lin}_{\mathbb{Q}}(\tilde{H})$, as needed.

This means that $\bigcup_{\xi<\mathfrak{c}} f_\xi$ has a unique extension $\hat{f} \in \mathrm{Add}$. We claim that it is as desired, that is, that $\hat{f} \in \mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{CIVP}$. Indeed, since $\hat{f} = \bigcup_{\xi<\mathfrak{c}} \hat{f}_\xi$, Lemma 2.5, and (vii) imply that $\hat{f} \in \mathrm{SZ}$.

To see that $\hat{f} \in \mathrm{CIVP}$, notice that for every $K \in \mathrm{Perf}$ there exists a $\zeta < \mathfrak{c}$ such that $K = P_\zeta$, while conditions (iii) and (iv) imply that $\hat{f}$ maps H_ζ into $P_\zeta = K$. Since for every $a < b$ the set $(a,b) \cap H_\zeta$ contains a perfect set, it follows that $\hat{f} \in \mathrm{CIVP}$.

Finally, we will show that $\hat{f} \in \mathrm{AC}$. In order to see this, fix $\hat{g} \in \hat{\mathcal{G}}_0$. By Lemma 4.7, it is enough to find $g \in \hat{\mathcal{G}}_0$ such that both sets $\mathrm{dom}(\hat{g} \cap g)$ and $\mathrm{dom}(\hat{f} \cap g)$ are dense in some non-trivial interval.

Thus, let $\xi < \mathfrak{c}$ be such that $\hat{g} = g_\xi$ and let $\eta \le \xi$ be the smallest such that, for some $v \in V_\xi$ and $q \in \mathbb{Q} \setminus \{0\}$, the set $Z^{q,v}_{\xi,\eta}$ is somewhere dense. Next, let S be the largest open set in $\mathbb{R}$ in which $Z^{q,v}_{\xi,\eta}$ is dense and let $U := qS + v$. Then $S \cap Z^{q,v}_{\xi,\eta} \subset S \cap \mathrm{dom}(g_\xi)$ is a dense G_δ subset of S and $U \cap \mathrm{dom}(g_\eta) = (qS+v) \cap \mathrm{dom}(g_\eta)$ is a dense G_δ subset of U. Notice, that the minimality of η implies that

$$U \subset U_\eta. \tag{4.4}$$

We shall see (4.4) by way of contradiction. Thus, let us assume that this is not the case. By $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$, this can happen only when there are $\zeta < \eta$, $w \in V_\eta$, and nonzero $p \in \mathbb{Q}$ such that $Z^{p,w}_{\eta,\zeta}$ is dense in some non-trivial interval $I \subset U$. Then, $J := q^{-1}(I - v)$ is a non-trivial interval contained in S and so $J \cap Z^{q,v}_{\xi,\eta}$

is a dense G_δ subset of J. Moreover, $J \cap q^{-1}(Z^{p,w}_{\eta,\zeta} - v) = q^{-1}(I \cap Z^{p,w}_{\eta,\zeta} - v)$ is a dense G_δ subset of $J \subset S$, as $I \cap Z^{p,w}_{\eta,\zeta}$ is a dense G_δ subset of I. Therefore, $T := J \cap q^{-1}(Z^{p,w}_{\eta,\zeta} - v) \cap Z^{q,v}_{\xi,\eta} \subset \mathrm{dom}(g_\xi)$ is a dense G_δ subset of J and for any $x \in T$ we have

$$g_\xi(x) = q^{-1}(g_\eta(qx+v) - \hat{f}(v))$$

and, also,

$$g_\eta(qx+v) = p^{-1}(g_\zeta(p(qx+v)+w) - \hat{f}(w))$$

since $qx + v \in Z^{p,w}_{\eta,\zeta}$. In particular, if $s := pq \in \mathbb{Q} \setminus \{0\}$ and $u := pv + w \in V_\xi$, then

$$\begin{aligned} g_\xi(x) &= q^{-1}(g_\eta(qx+v) - \hat{f}(v)) \\ &= q^{-1}(p^{-1}(g_\zeta(p(qx+v)+w) - \hat{f}(w)) - \hat{f}(v)) \\ &= s^{-1}(g_\zeta(sx+u) - \hat{f}(u)). \end{aligned}$$

In other words, we would have that $T \subset Z^{s,u}_{\xi,\zeta}$, contradicting the minimality of η, as $T \subset \mathrm{dom}(g_\xi)$ is a dense subset of J. So, indeed $U \subset U_\eta$ and (4.4) is proved.

Now, let I_ξ be a nontrivial interval contained in S. Then, by (4.4), the interval $I_\eta := qI_\xi + v$ is contained in $qS + v = U \subset U_\eta$ so that $I_\eta \cap \mathrm{dom}(g_\eta)$ is a dense G_δ subset of I_η. Therefore,

$$G := I_\xi \cap q^{-1}(\mathrm{dom}(g_\eta) - v) = q^{-1}((\mathrm{dom}(g_\eta) \cap I_\eta) - v)$$

is a dense G_δ subset of $I_\xi = q^{-1}(I_\eta - v)$.

Define $g\colon G \to \mathbb{R}$ via the formula

$$g(x) = q^{-1}(g_\eta(qx+v) - \hat{f}(v)).$$

Then $g \in \hat{\mathcal{G}}_0$. To finish the proof it is enough to show that both sets $\mathrm{dom}(\hat{g} \cap g)$ and $\mathrm{dom}(\hat{f} \cap g)$ are dense in I_ξ. However, $\mathrm{dom}(\hat{f} \cap g)$ is dense in I_ξ since it contains a dense set $E_\xi \cap I_\xi \subset I_\xi \cap Z^{q,v}_{\xi,\eta}$. At the same time $\mathrm{dom}(\hat{g} \cap g) = \mathrm{dom}(g_\xi \cap g)$ is dense in I_ξ since it contains a dense set $G \cap \mathrm{dom}(g_\xi) \subset Z^{q,v}_{\xi,\eta}$. $\square$

4.3. SZ **and differences of Darboux-like classes.** Using the inclusions indicated in Fig. 6 the algebra $A(\mathbb{U})$ of subsets of $\mathbb{R}^\mathbb{R}$ generated by the classes in $\mathbb{U}$ has 4 atoms: $\{\mathrm{AC}, \mathrm{Conn} \cap \neg\, \mathrm{AC}, \mathscr{D} \cap \neg\, \mathrm{Conn}, \neg\, \mathscr{D}\}$. Similarly, $A(\mathbb{L})$ generated by the classes in $\mathbb{L}$ has atoms: $\{\mathrm{CIVP}, \mathrm{PR} \cap \neg\, \mathrm{CIVP}, \mathrm{PC} \cap \neg\, \mathrm{PR}, \neg\, \mathrm{PC}\}$. This means that the algebra $A(\mathbb{U} \cup \mathbb{L})$ has theoretically 16 atoms, the intersections $L \cap U$, where $L \in A(\mathbb{L})$ and $U \in A(\mathbb{U})$. However, since $\mathscr{D} \subset \mathrm{PC}$ three of these potential atoms are empty:

$$\mathrm{AC} \cap \neg\, \mathrm{PC} = \mathrm{Conn} \cap \neg\, \mathrm{AC} \cap \neg\, \mathrm{PC} = \mathscr{D} \cap \neg\, \mathrm{Conn} \cap \neg\, \mathrm{PC} = \emptyset.$$

Moreover, the nonempty atom $\neg\, \mathscr{D} \cap \neg\, \mathrm{PC}$ is not contained in any of Darboux-like classes of functions, so not of interest for us. This leads to 12 interesting atoms presented in the following Table 1.

	CIVP	PR $\setminus$ CIVP	PC $\setminus$ PR
AC	AC $\cap$ CIVP	AC $\cap$ PR $\setminus$ CIVP	AC $\setminus$ PR
Conn $\setminus$ AC	Conn $\cap$ CIVP $\setminus$ AC	Conn $\cap$ PR $\setminus$(AC $\cup$ CIVP)	Conn $\setminus$(AC $\cup$ PR)
$\mathscr{D} \setminus$ Conn	$\mathscr{D} \cap$ CIVP $\setminus$ Conn	$\mathscr{D} \cap$ PR $\setminus$(Conn $\cup$ CIVP)	$\mathscr{D} \setminus$(Conn $\cup$ PR)
$\neg\, \mathscr{D}$	CIVP $\setminus \mathscr{D}$	PR $\setminus(\mathscr{D} \cup$ CIVP)	PC $\setminus(\mathscr{D} \cup$ PR)

Table 1. Nonempty atoms of $A(\mathbb{U} \cup \mathbb{L})$ contained in PC

One may naturally wonder, if all these atoms intersect (at least consistently) SZ class. What is known about this is listed in Table 2.

	CIVP	PR $\setminus$ CIVP	PC $\setminus$ PR
AC	Thm 4.10 Add: Thm 4.5	Thm 4.10 Add: Thm 4.9	Thm 4.10 Add: Pr 4.11
Conn $\setminus$ AC	Thm 4.10 Add: Thm 4.12, Pr 4.14	Thm 4.10 Add: Thm 4.12, Pr 4.14	Thm 4.10 Add: Thm 4.12, Pr 4.14
$\mathscr{D}\setminus$ Conn	Thm 4.10 Add: Thm 4.13, Pr 4.15	Thm 4.10 Add: Thm 4.13, Pr 4.15	Thm 4.10 Add: Thm 4.13, Pr 4.15
$\neg\mathscr{D}$	Add: Thm 4.6	Add: Thm 4.6	Add: Thm 4.6

TABLE 2. Indication on nonempty intersections with SZ (and Add if indicated). The results in the last row are in ZFC. The other under the assumption that $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$, unless otherwise specified.

Here are the additional results and problems supporting Table 2. The next theorem comes from a 2004 paper [91] of T. Natkaniec and H. Rosen.

Theorem 4.9. *If* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$*, then* $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{AC}\cap\mathrm{PR}\setminus\mathrm{CIVP}\neq\emptyset$*.*

The following result has recently been proved by K. C. Ciesielski and Cheng-Han Pan[16] [40].

Theorem 4.10. *If* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$*, then the intersection of* SZ *with any of the atoms from Table 1 is non-empty.*

Problem 4.11. *Does* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$ *imply that* $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{AC}\setminus\mathrm{PR}\neq\emptyset$*?*

The next two theorems come from [92, examples 9 and 8].

Theorem 4.12. *If CH holds, then* $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{Conn}\setminus\mathrm{AC}\neq\emptyset$*.*

Theorem 4.13. *If* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$*, then* $\mathrm{Add}\cap\mathrm{SZ}\cap\mathscr{D}\setminus\mathrm{Conn}\neq\emptyset$*.*

Problem 4.14. *Can Theorem 4.12 be proved assuming only that* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$*? Is any of the following 3 properties consistent with ZFC:*

- $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{Conn}\cap\mathrm{CIVP}\setminus\mathrm{AC}\neq\emptyset$*;*
- $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{Conn}\cap\mathrm{PR}\setminus(\mathrm{AC}\cup\mathrm{CIVP})\neq\emptyset$*;*
- $\mathrm{Add}\cap\mathrm{SZ}\cap\mathrm{Conn}\setminus(\mathrm{AC}\cup\mathrm{PR})\neq\emptyset$*?*

Does any of this follow from $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$ *?*

Problem 4.15. *Is any of the following 3 properties consistent with ZFC:*

- $\mathrm{Add}\cap\mathrm{SZ}\cap\mathscr{D}\cap\mathrm{CIVP}\setminus\mathrm{Conn}\neq\emptyset$*;*
- $\mathrm{Add}\cap\mathrm{SZ}\cap\mathscr{D}\cap\mathrm{PR}\setminus(\mathrm{Conn}\cup\mathrm{CIVP})\neq\emptyset$*;*
- $\mathrm{Add}\cap\mathrm{SZ}\cap\mathscr{D}\setminus(\mathrm{Conn}\cup\mathrm{PR})\neq\emptyset$*?*

Does any of this follow from $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$ *?*

A preliminary work towards [40] indicates, that all parts of Problems 4.14 and 4.15 that do not involve class Add have positive answer.

[16] Current Ph.D. student of K. C. Ciesielski.

4.4. **Additivity and lineability coefficients.** There is actually relatively little known about these coefficients for the classes of SZ maps that are also Darboux-like.

The following theorem was proved in 2015 paper [96] of K. Płotka under the assumption of CH. It was noticed, in a 2017 paper [32] of K. C. Ciesielski, J. L. Gámez-Merino, Lucian Mazza, and J. B. Seoane-Sepúlveda that the argument from [96] actually provides a stronger result: Theorem 4.16. We omit the proof here due to both, its length and technicality.

Theorem 4.16. *If* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$ *holds, then* $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) > \mathfrak{c}^+$.

Let us mention that, assuming GCH, the previous result implies that $\mathrm{AC} \cap \mathrm{SZ}$ is $2^{\mathfrak{c}}$-lineable and, therefore, $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) = \mathcal{L}(\mathrm{SZ})$. However, taking into account Theorem 4.2, from [96], the following corollary and corresponding natural open question are posed.

Corollary 4.17.

(i) It is consistent with ZFC, follows from GCH, that

$$\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) = \mathcal{L}(\mathrm{Conn} \cap \mathrm{SZ}) = \mathcal{L}(\mathscr{D} \cap \mathrm{SZ}) = \mathcal{L}(\mathrm{SZ}).$$

(ii) It is consistent with ZFC, follows from $\mathrm{CPA} + 2^{\omega_1} = \omega_2$*, that*

$$\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) = \mathcal{L}(\mathrm{Conn} \cap \mathrm{SZ}) = \mathcal{L}(\mathscr{D} \cap \mathrm{SZ}) = 1 < \mathfrak{c}^+ < \mathcal{L}(\mathrm{SZ}).$$

Problem 4.18. *Is it consistent for any* $F \in \{\mathrm{AC}, \mathrm{Conn}, \mathscr{D}\}$ *that* $F \cap \mathrm{SZ} \neq \emptyset$ *while also* $\mathcal{L}(F \cap \mathrm{SZ}) < \mathcal{L}(\mathrm{SZ})$*?*

The results presented in Table 3 constitute the current state of knowledge on the additivity and lineability coefficients for $\mathrm{SZ} \cap \mathcal{F}$ for some $\mathcal{F} \in A(\mathbb{U} \cup \mathbb{L})$. The symbol $\mathfrak{c}_-$ used in the table is defined:

$$\mathfrak{c}_- := \begin{cases} \kappa & \text{when } \mathfrak{c} = \kappa^+ \text{ and } \mathrm{cof}(\kappa) > \omega, \\ \mathfrak{c} & \text{otherwise.} \end{cases}$$

CLASS	$\mathcal{A}$	source	$\mathcal{L}$	source
SZ	$d_{\mathfrak{c}}$	Theorem 3.6	$> d_{\mathfrak{c}}$	Cor 3.8 & Thm 3.2
$\mathrm{SZ} \cap \mathrm{PR}$	$\mathfrak{c}^+$	[32, thm 2.6]	$> \mathfrak{c}^+$	Cor 3.8
$\mathrm{SZ} \cap \mathrm{CIVP}$	$\mathfrak{c}^+$	[32, thm 2.6]	$> \mathfrak{c}^+$	Cor 3.8
$\mathrm{SZ} \cap \mathrm{CIVP} \setminus \mathscr{D}$	$\mathfrak{c}^+$	[46]	$> \mathfrak{c}^+$	Cor 3.8
$\mathrm{SZ} \cap \mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathscr{D})$	$\mathfrak{c}^+$	[46]	$> \mathfrak{c}^+$	Cor 3.8
$\mathrm{SZ} \cap \mathrm{PC}$	$d_{\mathfrak{c}}$	[32, thm 2.6]	$> d_{\mathfrak{c}}$	Cor 3.8 & Thm 3.2
$\mathrm{SZ} \cap \mathrm{AC}$	$\leq \mathfrak{c}$	under $\mathrm{cof}(\mathfrak{c}) = \mathfrak{c}$, [32, thm 2.11]	$> \mathfrak{c}^+$	under $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$, Thm 4.16
$\mathrm{SZ} \cap \mathrm{Conn}$	$\leq \mathfrak{c}$	under $\mathrm{cof}(\mathfrak{c}) = \mathfrak{c}$, [32, thm 2.11]	$> \mathfrak{c}^+$	under $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$, Thm 4.16
$\mathrm{SZ} \cap \mathscr{D}$	$\leq \mathfrak{c}$	under $\mathrm{cof}(\mathfrak{c}) = \mathfrak{c}$, [32, thm 2.11]	$> \mathfrak{c}^+$	under $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$, Thm 4.16
$\mathrm{SZ} \cap \mathscr{D} \setminus \mathrm{Conn}$	$\leq \mathfrak{c}_-$	under $2^{\mathfrak{c}_-} = \mathfrak{c}$, [46]		

TABLE 3. Summary of what is currently known on the additivity and lineability coefficients for classes considered in this section.

Problem 4.19. *Are any among the coefficients* $\mathcal{A}(\mathrm{AC}\cap\mathrm{SZ})$*,* $\mathcal{A}(\mathrm{Conn}\cap\mathrm{SZ})$*, and* $\mathcal{A}(\mathscr{D}\cap\mathrm{SZ})$ *provably equal (in ZFC)? What about* $\mathcal{L}(\mathrm{AC}\cap\mathrm{SZ})$*,* $\mathcal{L}(\mathrm{Conn}\cap\mathrm{SZ})$*, and* $\mathcal{L}(\mathscr{D}\cap\mathrm{SZ})$*?*

Problem 4.20. *Does the assumption* $\mathrm{SZ}\cap\mathscr{D}\neq\emptyset$ *imply that* $\mathrm{SZ}\cap\mathscr{D}$ *is* $\mathfrak{c}^+$*-lineable? Does it imply that* $\mathrm{SZ}\cap\mathscr{D}$ *is* κ*-lineable whenever* SZ *is* κ*-lineable?*

4.5. **Generalized additivity.** All results presented in this section come from 2002 paper [94] of K. Płotka and were a part of his Ph.D. thesis. For the families $\mathcal{F}_1,\mathcal{F}_2\subset\mathbb{R}^{\mathbb{R}}$, the additivity of $\mathcal{F}_1$ over $\mathcal{F}_2$ is defined as the following cardinal number:

$$\mathcal{A}(\mathcal{F}_1,\mathcal{F}_2)=\min(\{|F|\colon F\subset\mathbb{R}^{\mathbb{R}}, h+F\not\subseteq\mathcal{F}_2,\forall h\in\mathcal{F}_1\}\cup\{(2^{\mathfrak{c}})^+\}).$$

Notice that $\mathcal{A}(\mathcal{F})=\mathcal{A}(\mathbb{R}^{\mathbb{R}},\mathcal{F})$.

Proposition 4.21. *Let* $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\mathbb{R}^{\mathbb{R}}$ *and* $\mathcal{F}\subseteq\mathbb{R}^{\mathbb{R}}$*.*
(1) $\mathcal{A}(\mathcal{F}_1,\mathcal{F})\leq\mathcal{A}(\mathcal{F}_2,\mathcal{F})$
(2) $\mathcal{A}(\mathcal{F},\mathcal{F}_1)\leq\mathcal{A}(\mathcal{F},\mathcal{F}_2)$
(3) $\mathcal{A}(\mathcal{F}_1,\mathcal{F}_2)\geq 2$ *if, and only if,* $\mathbb{R}^{\mathbb{R}}=\mathcal{F}_2-\mathcal{F}_1$*.*
(4) If $\mathcal{A}(\mathcal{F}_1,\mathcal{F}_2)\geq 2$*, then* $\mathcal{F}_1\cap\mathcal{F}_2\neq\emptyset$*.*
(5) $\mathcal{A}(\mathcal{F})=\mathcal{A}(\mathcal{F},\mathcal{F})+1$*. In particular, if* $\mathcal{A}(\mathcal{F})\geq\omega$ *then* $\mathcal{A}(\mathcal{F},\mathcal{F})=\mathcal{A}(\mathcal{F})$*.*

In relation to the family SZ we have the following results.

Theorem 4.22.
(i) If Martin's Axiom holds, then $\mathcal{A}(\mathscr{D},\mathrm{SZ})\geq\mathcal{A}(\mathrm{AC},\mathrm{SZ})\geq\omega$*.*
(ii) If Martin's Axiom holds, then $\mathcal{A}(\mathrm{SZ},\mathrm{AC})=\mathcal{A}(\mathrm{SZ},\mathscr{D})=\mathfrak{c}$*.*
(iii) If the theory "ZFC + $\exists$ *measurable cardinal" is consistent, then so is the theory "ZFC +* $\mathcal{A}(\mathrm{AC},\mathrm{SZ})>\mathfrak{c}>\omega_1$*".*
(iv) $\mathcal{A}(\mathrm{PC},\mathrm{SZ})=\mathcal{A}(\mathrm{SZ})$ *and* $\mathcal{A}(\mathrm{SZ},\mathrm{PC})=2^{\mathfrak{c}}$*.*

Since $\mathrm{SZ}=\{-f\colon f\in\mathrm{SZ}\}$, Proposition 4.21 and Theorem 4.22(ii) immediately imply the following result.

Corollary 4.23. *If Martin's Axiom holds, then* $\mathbb{R}^{\mathbb{R}}=\mathrm{AC}+\mathrm{SZ}=\mathscr{D}+\mathrm{SZ}$*.*

Notice also that $\mathrm{SZ}\cap\mathscr{D}=\emptyset$ implies $\mathrm{SZ}\cap\mathrm{AC}=\emptyset$ and so, by parts (3) and (4) of Proposition 4.21, that $\mathrm{AC}+\mathrm{SZ}\neq\mathbb{R}^{\mathbb{R}}\neq\mathscr{D}+\mathrm{SZ}$. Hence, by Theorem 4.2,

Corollary 4.24. *The equalities* $\mathbb{R}^{\mathbb{R}}=\mathrm{AC}+\mathrm{SZ}$ *and* $\mathbb{R}^{\mathbb{R}}=\mathscr{D}+\mathrm{SZ}$ *are independent of ZFC.*

More on the generalized additivity, including some results related to SZ can be found in the 2008 paper [95] by K. Płotka. For instance, let HF stand for the class of Hamel functions, that is, functions in $\mathbb{R}^{\mathbb{R}}$ whose graph is a Hamel basis for $\mathbb{R}^2$. These maps are as far from being additive as possible. In [95] we can find that

$$\mathcal{A}(\mathrm{SZ},\mathrm{HF})=\mathcal{A}(\mathrm{HF})\text{ and }\mathcal{A}(\mathrm{HF},\mathrm{SZ})>\mathfrak{c}.$$

However, it is unknown whether, in ZFC, we have

$$\mathcal{A}(\mathrm{HF},\mathrm{SZ})=\mathcal{A}(\mathrm{SZ}).$$

We also have, [95, Remark 5], that

$$\mathcal{A}(\mathrm{Add},\mathrm{SZ})\leq\mathcal{A}(\mathrm{HF},\mathrm{SZ}).$$

5. Inverses, products, and compositions of SZ–functions.

5.1. **Sierpiński-Zygmund functions and their inverses.** Let an SZ-map $f \in \mathbb{R}^{\mathbb{R}}$ be one-to-one. The question we are interested in here is the following:

Can its inverse f^{-1} be an SZ-map?

Up to this moment, all our SZ-maps were defined on the entire real line. If we continue to keep this requirement, then f^{-1} could be an SZ-map only when the original f is surjective. However, according to Theorem 4.3, no ZFC example of SZ bijection exists. Nevertheless, it is consistent with ZFC that such an example can be constructed, as follows from the following 1997 result of K. C. Ciesielski and T. Natkaniec, see [39, theorem 7 and corollary 6].

Theorem 5.1. *Assume that* $\mathrm{cov}_{\mathcal{M}} = \mathfrak{c}$ *holds. Then*

(i) there exists an SZ *bijection* $f \in \mathbb{R}^{\mathbb{R}}$ *such that* $f^{-1} = f$*;*
(ii) there exists an SZ *bijection* $f \in \mathbb{R}^{\mathbb{R}}$ *such that* $f^{-1} \notin \mathrm{SZ}$*.*

This theorem was (partially) generalized in 2005 by T. Natkaniec and H. Rosen [92, examples 13 and 12]. They proved, under the same set-theoretical assumption, that:

- There is an additive SZ bijection $f \in \mathbb{R}^{\mathbb{R}}$ such that $f^{-1} \in \mathrm{SZ}$. (However, the constructed map need not be its own inverse.)
- There exists an additive SZ bijection $f \in \mathbb{R}^{\mathbb{R}}$ such that $f^{-1} \notin \mathrm{SZ}$.

This is all that can be proved in this direction about SZ bijections. But what if we assume only that an SZ-map $f \in \mathbb{R}^{\mathbb{R}}$ is one-to-one? Then the inverse function f^{-1} is defined only on $f[\mathbb{R}]$, which can be a proper subset of $\mathbb{R}$. Luckily, the definition of SZ functions can be naturally extended to partial maps defined on $X \subset \mathbb{R}$ (of cardinality $\mathfrak{c}$):

an $f\colon X \to \mathbb{R}$ is an SZ-*map* provided $f \restriction S$ is discontinuous for every $S \subset X$ of cardinality $\mathfrak{c}$.

In this setting, allowing partial SZ functions, considerably more can be proved. Nevertheless, perhaps surprisingly, only one kind of such an example can be constructed in ZFC. It comes from [39, theorem 2].

Theorem 5.2. *There exists an* SZ *injection* $f \in \mathbb{R}^{\mathbb{R}}$ *such that* f^{-1} *is continuous (and, thus, not* SZ*).*

Sketch of proof. This is a small variation of the proof of Theorem 2.6. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuous surjection such that $h^{-1}(y)$ has cardinality $\mathfrak{c}$ for every $y \in \mathbb{R}$.

Let $\{x_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration, with no repetition, of $\mathbb{R}$ and let $\{g_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of $\mathscr{B}$. For every $\xi < \mathfrak{c}$ choose

$$f(x_\xi) \in h^{-1}(x_\xi) \setminus (\{g_\zeta(x_\xi)\colon \zeta < \xi\} \cup \{f(x_\zeta)\colon \zeta < \xi\}).$$

Then $f \in \mathrm{SZ}$ and f^{-1} is continuous, as its graph is contained in the graph of h. $\square$

A variant of Theorem 5.2 is proved also in [92, example 11], where the authors construct an additive SZ injection $f \in \mathbb{R}^{\mathbb{R}}$ with $f^{-1} \notin \mathrm{SZ}$. However, in this case, f^{-1} is not continuous. Also, in [83, theorem 5], Zbigniew Lipecki showed the following result for topological vector spaces.

Theorem 5.3. *Let X be a topological linear space of dimension $\mathfrak{c}$ and let Y be a Hausdorff topological linear space with $dim(Y) \geq \mathfrak{c}$. Then there exists an injective operator $T : X \to Y$ such that no restriction of T to a separable subspace of X of dimension $\mathfrak{c}$ is continuous.*

Finally, it is natural to ask if one can construct a ZFC example of an SZ injection $f \in \mathbb{R}^{\mathbb{R}}$ such that f^{-1} is also SZ. At a first glance this should be possible, since Theorem 4.3 does not seem to prevent it. Nevertheless, the existence of such a function is still independent of ZFC. Specifically, its consistent existence follows from Theorem 5.1(i). The existence of a model of ZFC without such a function follows from the next result that comes from [39, theorem 8 and corollary 9].

Theorem 5.4. *The following properties are equivalent.*

- *(i) There is no* SZ *injection from an $X \subset \mathbb{R}$ of cardinality $\mathfrak{c}$ into $\mathbb{R}$ such that f^{-1} is* SZ.
- *(ii) There exists a family $\mathcal{H}$ of continuous functions from $X \subset \mathbb{R}$ into $\mathbb{R}$ such that $\mathcal{H}$ has cardinality $< \mathfrak{c}$ and that $\mathbb{R}^2$ is covered by the graphs of $h \in \mathcal{H}$ and their inverses.*

In particular, since (ii) is consistent with ZFC—it follows from CPA*—so is (i).*

The fact that (ii) holds in the iterated perfect set (Sacks) model was proved in 1999 paper [106] of Juris Steprāns. The fact that it follows from CPA was proved by K. C. Ciesielski and J. Pawlikowski, see [42, chapter 4] and [43]. Compare also [47] for related results.

5.2. **Products.** In this section we will examine for which functions $f \in \mathbb{R}^{\mathbb{R}}$ there exists an $h \in \mathbb{R}^{\mathbb{R}}$ such that $hf \in \mathrm{SZ}$. All results presented here come from 1997 paper [38] of K. C. Ciesielski and T. Natkaniec.

First of all, observe that if $|[f = 0]| = \mathfrak{c}$, then $hf \in \mathrm{SZ}$ for no $h \in \mathbb{R}^{\mathbb{R}}$. Thus, we restrict here our attention to the family

$$\mathcal{R}_0 := \left\{ f \in \mathbb{R}^{\mathbb{R}} \colon\ |[f = 0]| < \mathfrak{c} \right\}.$$

Theorem 5.5. *For every $\mathcal{F} \subset \mathcal{R}_0$ of cardinality $\leq \mathfrak{c}$ there exists an $h \colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $hf \in \mathrm{SZ}$ for each $f \in \mathcal{F}$.*

Theorem 5.5 allows us to conclude the following characterization of functions in $\mathbb{R}^{\mathbb{R}}$ can be expressed as the product of two Sierpiński-Zygmund functions.

Corollary 5.6. *For every function $f \in \mathbb{R}^{\mathbb{R}}$ the following conditions are equivalent:*

- *(i) $|[f = 0]| < \mathfrak{c}$, that is, $f \in \mathcal{R}_0$;*
- *(ii) f is the product of two* SZ*-functions.*

One can also define the following multiplicative analogue of the additivity cardinal coefficient $\mathcal{A}(\mathrm{SZ})$:

$$m(\mathrm{SZ}) := \min\left(\{|F| \colon F \subset \mathcal{R}_0 \text{ and } \varphi \cdot F \not\subset \mathrm{SZ} \text{ for every } \varphi \in \mathbb{R}^{\mathbb{R}}\} \cup \{(2^{\mathfrak{c}})^+\}\right).$$

Then, we have the following result [38, theorem 3.3]:

Theorem 5.7. $m(\mathrm{SZ}) = \mathcal{A}(\mathrm{SZ})$.

5.3. **Compositions.** In this section we are going to present some results regarding the composition with SZ-functions and when such composition is also a function in SZ. Again, all these results come from [38].

The following notation shall be crucial in what follows

$$\mathcal{M}_{out}(\mathrm{SZ}) = \left\{ f \in \mathbb{R}^{\mathbb{R}} : f \circ h \in \mathrm{SZ} \ \text{ for each } h \in \mathrm{SZ} \right\},$$

$$\mathcal{M}_{in}(\mathrm{SZ}) = \left\{ f \in \mathbb{R}^{\mathbb{R}} : h \circ f \in \mathrm{SZ} \ \text{ for each } h \in \mathrm{SZ} \right\}.$$

Theorem 5.8. *If $\mathfrak{c}$ is a regular cardinal, then $\mathcal{M}_{out}(\mathrm{SZ}) = \mathcal{M}_{in}(\mathrm{SZ})$. However, if $\mathfrak{c}$ is singular, then $\mathcal{M}_{in}(\mathrm{SZ}) \not\subset \mathcal{M}_{out}(\mathrm{SZ})$.*

The following open question comes out naturally after the previous result.

Problem 5.9. *Can the inclusion $\mathcal{M}_{out}(\mathrm{SZ}) \subset \mathcal{M}_{in}(\mathrm{SZ})$ be proved without assuming that $\mathfrak{c}$ is regular?*

Theorem 5.10. *For each $f \in \mathbb{R}^{\mathbb{R}}$ the following conditions are equivalent:*

(i) there exists an $h \in \mathrm{SZ} \cap \mathbb{R}^{\mathbb{R}}$ such that $h \circ f \in \mathrm{SZ}$;
(ii) there exists an $h \in \mathbb{R}^{\mathbb{R}}$ such that $h \circ f \in \mathrm{SZ}$;
(iii) $|f^{-1}(y)| < \mathfrak{c}$ for each $y \in \mathbb{R}$.

Theorem 5.11. *Assume that $\mathfrak{c}$ is a regular cardinal. For each $f \in \mathbb{R}^{\mathbb{R}}$ the following conditions are equivalent:*

(i) there exists an $h \in \mathrm{SZ} \cap \mathbb{R}^{\mathbb{R}}$ such that $f \circ h \in \mathrm{SZ}$;
(ii) there exists an $h \in \mathbb{R}^{\mathbb{R}}$ such that $f \circ h \in \mathrm{SZ}$;
(iii) $|f[\mathbb{R}]| = \mathfrak{c}$.

6. Closing remarks and comments

In the above text we have presented a comprehensive overview of the current state of knowledge related to the question of *how much continuity an arbitrary function from $\mathbb{R}$ into $\mathbb{R}$ must have.* From the seminal 1922 theorem of H. Blumberg [18] and 1923 example of W. Sierpiński and A. Zygmund [104], we discuss almost 100 years of history of the research in this subject. Actually during the first 70 years after the publication of the above-mentioned two papers, there were no publications directly related to SZ-functions and relatively few related to (generalizations of) Blumberg theorem, namely (in chronological order, from 1954 to 1984) papers [61], [19], [21], [103], [22], [27], and [84]. The situation drastically changed in early 1990's with a sudden "explosion" of papers generalizing the results of both [18] and [104]. In addition to these generalizations, algebraic properties of SZ functions have also been studied, as well as the relation of the class SZ to the classes of generalized continuous functions, mainly Darboux-like. This increased interest was certainly sparked by a dynamic development of set-theoretical tools in the 1970's and the 1980's, which constitutes an integral part of this renewed interest.

At the closure of this work, we would like to emphasize that the research around the presented topics is far from being finished. This is witnessed, for example, by the long list of open problems we explicitly stated in this text: 4.11, 4.14, 4.15, 4.18, 4.19, 4.20, and 5.9. Solving these questions would certainly deepen our understanding of the interactions of Darboux-like functions with the class SZ, together with their corresponding additivity and lineability coefficients. The solutions of many of these problems will certainly require a good understanding of the tools that come from real analysis, set theory, and algebra. The authors expect to keep researching this class and obtain further results in the future.

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