Smooth extension theorems for one variable maps

Krzysztof Chris Ciesielski*

Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA
Department of Radiology, MIPG, University of Pennsylvania, Philadelphia, PA 19104, USA
KCies@math.wvu.edu

Draft of 2019.06.11

Abstract

We characterize the real valued functions f defined on perfect subsets P of $\mathbb{R}$ which admit n-times differentiable extensions $F\colon \mathbb{R} \to \mathbb{R}$. In this characterization no continuity of $F^{(n)}$ is imposed. In particular, it generalizes Jarník's Extension Theorem, according to which f admits differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ if, and only if, f is differentiable. The new characterization is also closely related to the Whitney's Extension Theorem, which characterizes partial maps f admitting n-times differentiable extensions $F\colon \mathbb{R} \to \mathbb{R}$ with *continuous* nth derivative $F^{(n)}$. We also provide an elegant description of a linear extension operator $T_n\colon C(P) \to C(\mathbb{R})$ such that $T_n(f) \in D^n(\mathbb{R})$ for every $D^n(\mathbb{R})$-extendable $f \in C(P)$ and $T_n(f) \in C^n(\mathbb{R})$ whenever $f \in C(P)$ is $C^n(\mathbb{R})$-extendable.

1 Preliminaries and background

In what follows P will always be a perfect subset of the real line $\mathbb{R}$, that is, a closed subset of $\mathbb{R}$ which equals to the set P' of its accumulation points. A function $f\colon P \to \mathbb{R}$ is *differentiable*, provided for every point $p \in P$ the following limit,

$$f'(p) := \lim_{x\to p,\ x\in P} \frac{f(x)-f(p)}{x-p}$$

exists and is finite. Of course such defined map $f'\colon P \to \mathbb{R}$ is referred to as *the derivative of* f. The above limit has no sense, unless p is an accumulation point of P. This is the reason, we restrict our attention to perfect sets, to ensure that the derivatives can be defined on the entire domain of a function.

For an $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, let $D^n(P)$ be the family of all n-times differentiable functions $f\colon P \to \mathbb{R}$ and $C^n(P)$ the family of all $f \in D^n(P)$ with

*Mathematics Subject Classification: Primary 47A57; Secondary 26A24, 46E35;
Key words: Whitney Extension Theorem; smooth extension theorems; linear extension operator; interpolation

continuous nth derivative $f^{(n)}$. Symbols $D^0(P)$ and $C^0(P)$ will stand for the class $C(P)$ of all continuous maps $f\colon P \to \mathbb{R}$.[1]

The smooth extension theorems, for real valued functions defined on closed subsets of $\mathbb{R}^k$, have been extensively studied over the past century, see e.g., [1, 3, 8–11, 15, 16, 18, 19]. However, these studies were mainly concerned with the versions of *Whitney's Extension Theorem, WET*, from 1934 papers [18, 19], in which the derivatives of all orders are to be continuous. For functions of one variable WET can be stated as follows (see, e.g., [6] or [7]), where for $f \in D^n(P)$ and $a \in P$

$$T^n_a f(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

is the n-th degree Taylor polynomial[2] of f at a and $Q^n_f\colon P^2 \to \mathbb{R}$ is defined as

$$Q^n_f(a,b) := \begin{cases} \dfrac{T^n_b f(b) - T^n_a f(b)}{(b-a)^n} & \text{for } a \neq b, \\ 0 & \text{for } a = b. \end{cases}$$

Whitney's Extension Theorem WET. *Let $P \subset \mathbb{R}$ be perfect and $n \in \mathbb{N}$. A function $f\colon P \to \mathbb{R}$ admits an extension $F \in C^n(\mathbb{R})$ if, and only if, $f \in C^n(P)$ and the map $Q^{n-i}_{f^{(i)}}\colon P^2 \to \mathbb{R}$ is continuous for every $i \leq n$.*

The problem of existence of an extension $F \in D^n(\mathbb{R})$ of $f\colon P \to \mathbb{R}$ (no continuity of $F^{(n)}$ imposed) so far was studied considerably less vigorously and only for $n = 1$. In fact, until this paper little was known in this direction beyond the following 1923 theorem of Jarník.

Jarník's Extension Theorem JET. *Let $P \subset \mathbb{R}$ be closed. A map $f\colon P \to \mathbb{R}$ admits an extension $F \in D^1(\mathbb{R})$ if, and only if, $f \in D^1(P)$.*[3]

The story behind this theorem, as well as its elementary proof, is given in details in the recent paper [4]. (See also [7]).) In short, the theorem first appeared in 1923 papers of V. Jarník: [13] in Czech and [12] in French, but with only sketch of a proof. These papers, and the theorem, were unnoticed by the mathematical community until the mid 1980's. The result was rediscovered in 1974 by G. Petruska and M. Laczkovich [17] and was further studied in 1984 paper [14] of J. Mařík. Jarník's paper [12] was rediscovered by the authors of 1985 paper [2], which discusses multivariable version of JET. Interestingly, it is shown in [2] that the theorem does not have a straightforward generalization to functions of two or more variables, since in such case the derivative of a partial

---

[1] The notation $C^n(P)$ agrees with the topological standard on what $C(P)$, our $C^0(P)$, stands for. This should not be confused with $\mathrm{C}^n(P)$, often used in the papers concerning Whitney's extension theorem (see e.g. [10]), that stands for all $f \in C^n(P)$ admitting extension $F \in C^n(\mathbb{R})$.

[2] In the literature concerning Whitney's extension theorem (see e.g. [10]) the polynomials $T^n_a f$ are often denoted as $\mathrm{J}_a(f)$ and referred to as "jets" of f at a.

[3] For closed, not necessary perfect, sets $P \subset \mathbb{R}$ we write $f \in D^1(P)$ when the finite limit $\lim_{x\to p,\ x\in P} \frac{f(x)-f(p)}{x-p}$ exists for all $p \in P'$.

function need not be of Baire class one. (At the same time, it is proved in [2] that a differentiable $f\colon P \to \mathbb{R}$, with P being a closed subset of $\mathbb{R}^k$, admits differentiable extension $F\colon \mathbb{R}^k \to \mathbb{R}$ if, and only if, the derivative of f is of Baire class one.)

The main result of this article, presented in the next section as Theorem 1, is an extension of JET to the higher order of differentiation. This constitutes a solution to a problem posed in [7, prob. 6.4].

In addition, we show in Theorem 2 how the characterization from Theorem 1 can be encompassed in WET, giving an alternative characterization in the theorem. This new characterization, which is of independent interest, allows us also to present a self-contained proof of WET, which seems to be simpler than any other proofs of WET (for one variable) in existence.

We also construct, in Section 6, the linear extension operators that associate to each $D^n(\mathbb{R})$-extendable $f \in C(P)$ its $D^n(\mathbb{R})$ extension and to each $C^n(\mathbb{R})$-extendable $f \in C(P)$ its $C^n(\mathbb{R})$ extension.

2 The theorems

Theorem 1. (Generalization of JET) *For every $n \in \mathbb{N}$ and perfect $P \subset \mathbb{R}$ the following conditions are equivalent.*

(a) $f\colon P \to \mathbb{R}$ admits an extension $F \in D^n(\mathbb{R})$.

(b) $f \in D^n(P)$ and if $n > 1$, then f' has an extension $\phi \in D^{n-1}(\mathbb{R})$ and for every such extension and the map $g \in D^n(P)$ defined for every $x \in P$ as $g(x) := f(x) - \int_0^x \phi(t)\,dt$, we have

$$\lim_{k\to\infty} \frac{g(b_k) - g(a_k)}{(b_k - a_k)^{n-1}\left(\frac{a_k+b_k}{2} - p\right)} = 0 \tag{1}$$

for every one-to-one sequence $\langle\langle a_k, b_k\rangle \in P^2 \colon k \in \mathbb{N}\rangle$ converging to a $\langle p, p\rangle \in P^2$ and such that $\emptyset \neq (a_k, b_k) \subset \mathbb{R} \setminus P$ for each $k \in \mathbb{N}$.

Since for $n = 1$ the condition (b) is just a statement $f \in D^n(P)$, Theorem 1 is clearly a generalization of JET. Note also that the part of (b) concerning f' need not be satisfied for $n = 1$, since then f' need not be continuous, in which case it clearly has no $D^{n-1}(\mathbb{R})$-extension.

We will also prove the following expanded form of WET.

Theorem 2. (Expanded form of WET) *For every $n \in \mathbb{N}$ and perfect $P \subset \mathbb{R}$ the following conditions are equivalent.*

(A) $f\colon P \to \mathbb{R}$ admits an extension $F \in C^n(\mathbb{R})$.

(B) $f \in C^n(P)$ and the map $Q^{n-i}_{f^{(i)}}\colon P^2 \to \mathbb{R}$ is continuous for every $i \le n$.

(C) $f \in C^n(P)$, *f' has an extension* $\phi \in C^{n-1}(\mathbb{R})$, *and, for every such extension map* ϕ *and function* $g \in C^n(P)$ *defined as* $g(x) := f(x) - \int_0^x \phi(t)dt$ *for every* $x \in P$, *the mapping* $q_g^n : P^2 \to \mathbb{R}$ *defined as*

$$q_g^n(a,b) := \begin{cases} \dfrac{g(b)-g(a)}{(b-a)^n} & \text{for } a \neq b, \\ 0 & \text{for } a = b \end{cases}$$

is continuous.

Of course the equivalence of (A) and (B) is just a restatement of WET. The condition (C) concerns the same function g used in (b), stressing similarity between both theorems. Beside, the property (C) seems to be a characterization of $C^n(\mathbb{R})$-extendable functions that is of independent interest, since its statement does not involve the derivatives $f^{(i)}$ for $i > 1$. In addition, the inclusion of condition (C) in the theorem allows us to present a self-contained proof of Theorem 2, which seems to be the simplest proof of WET (for one variable) in existence.

3 Canonical extensions

In this section we will describe a simple canonical extension $\tilde{g} \colon \mathbb{R} \to \mathbb{R}$ of any function g from a non-empty closed subset P of $\mathbb{R}$ into $\mathbb{R}$. For the functions g satisfying specific properties related to the properties (b) and (C) from the theorems, this extension will have the desired smoothness properties. In what follows $\psi \colon \mathbb{R} \to \mathbb{R}$ will be a fixed C^∞ non-decreasing function with $\psi \restriction (-\infty, 1/3] \equiv 0$ and $\psi \restriction [2/3, \infty) \equiv 1$. See Figure 1.

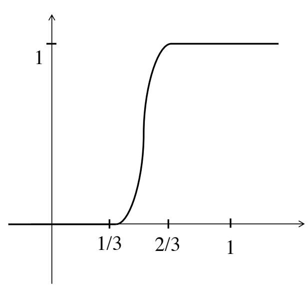

Figure 1: A graph of function ψ.

So, fix a non-empty closed $P \subset \mathbb{R}$ and a function $g \colon P \to \mathbb{R}$ to be extended. Let H be the convex hull of P, an interval, and $\{(a_j, b_j) \colon j \in J\}$ be a list of all connected components of $H \setminus P$, with no repetitions. For every $j \in J$ define the following C^∞ maps from $\mathbb{R}$ to $\mathbb{R}$:

- the linear map $\ell_j(x) := \frac{x-a_j}{b_j-a_j}$ (with $\ell_j(a_j) = 0$ and $\ell_j(b_j) = 1$);

- $\beta_j := \psi \circ \ell_j$ and $\alpha_j := 1 - \beta_j$;
- $\tilde{g}_j := \alpha_j g(a_j) + \beta_j g(b_j) = g(a_j) + [g(b_j) - g(a_j)]\beta_j$.

The desired canonical extension $\tilde{g}$ of g is defined on H as:

$$\tilde{g} \restriction P := g \quad \text{and} \quad \tilde{g} \restriction (a_j, b_j) := \tilde{g}_j \restriction (a_j, b_j) \text{ for every } j \in J. \tag{2}$$

Moreover, on any unbounded component C of $\mathbb{R} \setminus H$ we define $\tilde{g} \restriction C :\equiv g(p)$, where $p \in P$ is the only endpoint of C. See Figure 2. This completes the construction of the canonical extension $\tilde{g}$ of g.

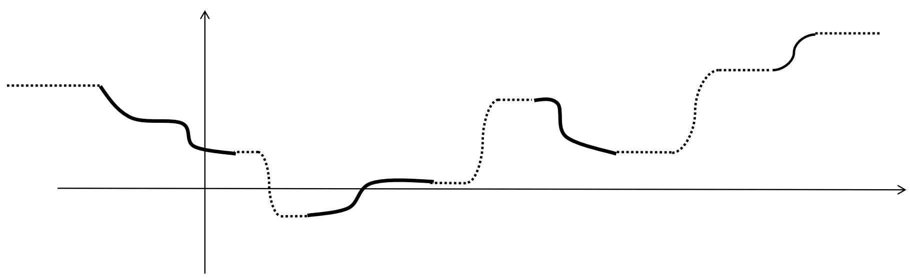

Figure 2: A graph of g (solid line) extended, by dashed lines, to $\tilde{g}$.

Fact 3. *If $\emptyset \neq P \subset \mathbb{R}$ is closed and $\tilde{g}$ is the canonical extension of $g\colon P \to \mathbb{R}$, then $\tilde{g}$ is C^∞ on $\mathbb{R} \setminus P$ and also on the closure of any connected component of $\mathbb{R} \setminus P$.*

Proof. This holds, since $\tilde{g} \restriction [a_j, b_j] = \tilde{g}_j \restriction [a_j, b_j]$ is C^∞ for every $j \in J$. □

For every $j \in J$, let $M_j = [c_j, d_j]$ be the middle third of (a_j, b_j) and put $L_j = (a_j, c_j)$ and $R_j = (d_j, b_j)$. Also, let $L = \bigcup_{j \in J} L_j$, $M = \bigcup_{j \in J} M_j$, and $R = \bigcup_{j \in J} R_j$.

Lemma 4. *Let $\emptyset \neq P \subset \mathbb{R}$ be perfect and $g \in D^1(P)$ be such that $g' \equiv 0$. If $\tilde{g}$ is the canonical extension of g, then for every $p \in P$, $s < \omega$,[4] and sequence $\langle x_k \in \mathbb{R} \setminus (M \cup P)\colon k \in \mathbb{N}\rangle$ converging to p, we have*

$$\lim_{k \to \infty} \left| \frac{\tilde{g}^{(s)}(x_k) - g^{(s)}(p)}{x_k - p} \right| = 0. \tag{3}$$

Proof. Since $\mathcal{P} = \{\mathbb{R} \setminus H, L, R\}$ is a partition of $\mathbb{R} \setminus (M \cup P)$, it is enough to show that (3) holds for any sequence $\langle x_k \colon k \in \mathbb{N}\rangle$ as in the lemma which, additionally, is strictly monotone and such that there is an $S \in \mathcal{P}$ for which $x_k \in S$ for all $k \in \mathbb{N}$.

So, let $\langle x_k \colon k \in \mathbb{N}\rangle$ be such a sequence. For simplicity, we assume that it is increasing, the other case being similar. We have the following cases.

[4]Here ω stands for the first infinite ordinal. Thus, $s < \omega$ is equivalent to $s \in \{0, 1, 2, \ldots\}$.

Case $S = \mathbb{R}\setminus H$**.** Then all but finitely many x_k belong to the same component of $\mathbb{R} \setminus H$ and so (3) follows from the fact that $\tilde{g}$ is constant on its closure.

Case $S \in \{L, R\}$**.** Then, for every $k \in \mathbb{N}$ there exists unique $j_k \in J$ such that $x_k \in (a_{j_k}, b_{j_k})$. If the set $J_0 = \{j_k \colon k \in \mathbb{N}\}$ is finite, then all but finitely many x_k belong to the same interval (a_{j_k}, b_{j_k}). Therefore, since $\tilde{g}$ is C^∞ on $[a_{j_k}, b_{j_k}]$, (3) follows. So, we can assume that J_0 is infinite. In particular,

$$a_{j_k} < x_k < b_{j_k} < p \text{ for each } k \in \mathbb{N}.$$

Now, if $S = R$, then $|x_k - p| > |b_{j_k} - p|$ and $\tilde{g}^{(s)}(x_k) = \tilde{g}_{j_k}^{(s)}(x_k) = g^{(s)}(b_{j_k})$. So, (3) holds, as, by $g' \equiv 0$,

$$\lim_{k\to\infty} \left| \frac{\tilde{g}^{(s)}(x_k) - g^{(s)}(p)}{x_k - p} \right| \le \lim_{k\to\infty} \left| \frac{g^{(s)}(b_{j_k}) - g^{(s)}(p)}{b_{j_k} - p} \right| = \left| g^{(s+1)}(p) \right| = 0.$$

Also, if $S = L$, then $|x_k - p| > \frac{1}{2}|a_{j_k} - p|$ and $\tilde{g}^{(s)}(x_k) = \tilde{g}_{j_k}^{(s)}(x_k) = g^{(s)}(a_{j_k})$. So, (3) holds, as

$$\lim_{k\to\infty} \left| \frac{\tilde{g}^{(s)}(x_k) - g^{(s)}(p)}{x_k - p} \right| \le 2 \lim_{k\to\infty} \left| \frac{g^{(s)}(a_{j_k}) - g^{(s)}(p)}{a_{j_k} - p} \right| = 2 \left| g^{(s+1)}(p) \right| = 0,$$

completing the proof. □

4 Proof of Theorem 1

First we will prove Theorem 1 under the additional assumption that the function to be extended has 0 derivative everywhere. It is stated as the following lemma.

Lemma 5. *Let $n \in \mathbb{N}$, $\emptyset \neq P \subset \mathbb{R}$ be perfect, $g \in D^1(P)$ be such that $g' \equiv 0$, and $\tilde{g}$ be the canonical extension of g.*

(i) If g satisfies (1), then $\tilde{g} \in D^n(\mathbb{R})$.

(ii) If $n > 1$ and g has an extension $\hat{g} \in D^n(\mathbb{R})$, then g satisfies (1).

Proof. (i): By Fact 3, it is enough to show that for every $s \in \{0, \ldots, n-1\}$ we have $\tilde{g}^{(s+1)}(p) = 0$ for every $p \in P$. We will prove this by induction on s. To see this, by Lemma 4 it is enough to prove that (3) holds for every monotone sequence $\langle x_k \in \mathbb{R} \colon k \in \mathbb{N}\rangle$ converging to p such that either all x_k belong to P or all of them belong to M. In the first of these cases, (3) is clearly implied by our assumption $g' \equiv 0$ and, for $s > 0$, the inductive assumption that, for every $p \in P$, $\tilde{g}^{(s)}(p) = 0$ which is equal to $g^{(s)}(p)$. So, in the rest of the argument we assume that $x_k \in M$ for all $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ let $j_k \in J$ be such that $x_k \in (a_{j_k}, b_{j_k})$. Without loss of generality we can assume that indexes j_k are distinct, that is, that the sequence

$\langle\langle a_{j_k}, b_{j_k}\rangle \in P^2 \colon k \in \mathbb{N}\rangle$ is as in the statement of (1). Notice also that $|x_k - p| > \frac{1}{3}|a_{j_k} - p|$ and $|x_k - p| > \frac{1}{3}|b_{j_k} - p|$. Now, (3) holds for $s = 0$, since then

$$\begin{aligned}
&\lim_{k\to\infty}\left|\frac{\tilde{g}^{(s)}(x_k) - \tilde{g}^{(s)}(p)}{x_k - p}\right| \\
&\qquad = \lim_{k\to\infty}\left|\frac{[\alpha_{j_k}(x_k)g(a_{j_k}) + \beta_{j_k}(x_k)g(b_{j_k})] - [\alpha_{j_k}(x_k) + \beta_{j_k}(x_k)]g(p)}{x_k - p}\right| \\
&\qquad \le \lim_{k\to\infty}\left[|\alpha_{j_k}(x_k)|\left|\frac{g(a_{j_k}) - g(p)}{x_k - p}\right| + |\beta_{j_k}(x_k)|\left|\frac{g(b_{j_k}) - g(p)}{x_k - p}\right|\right] \\
&\qquad \le 3\lim_{k\to\infty}\left|\frac{g(a_{j_k}) - g(p)}{a_{j_k} - p}\right| + 3\lim_{k\to\infty}\left|\frac{g(b_{j_k}) - g(p)}{b_{j_k} - p}\right| = 6\,|g'(p)| = 0.
\end{aligned}$$

To see that (3) holds for $s > 0$, notice that in this case $\beta^{(s)}_{j_k}(x_k) = \frac{\psi^{(s)}(\ell_{j_k}(x_k))}{(b_{j_k} - a_{j_k})^s}$ and, by the inductive assumption, $\tilde{g}^{(s)}(p) = g^{(s)}(p) = 0$. Also, $|\psi^{(s)}(\ell_{j_k}(x_k))| \le M$, where $M = \sup \psi^{(s)}[[0,1]] \in \mathbb{R}$, and $|x_k - p| > \frac{1}{2}\left|\frac{a_{j_k} + b_{j_k}}{2} - p\right|$. So

$$\begin{aligned}
&\lim_{k\to\infty}\left|\frac{\tilde{g}^{(s)}(x_k) - \tilde{g}^{(s)}(p)}{x_k - p}\right| \\
&\qquad = \lim_{k\to\infty}\left|\frac{\tilde{g}^{(s)}_{j_k}(x_k)}{x_k - p}\right| = \lim_{k\to\infty}\left|\frac{[g(b_{j_k}) - g(a_{j_k})]\beta^{(s)}_{j_k}(x_k)}{x_k - p}\right| \\
&\qquad = \lim_{k\to\infty}\left|\frac{g(b_{j_k}) - g(a_{j_k})}{x_k - p}\,\frac{\psi^{(s)}(\ell_{j_k}(x_k))}{(b_{j_k} - a_{j_k})^s}\right| \\
&\qquad \le 2M\lim_{k\to\infty}\left|\frac{g(b_{j_k}) - g(a_{j_k})}{(b_{j_k} - a_{j_k})^{n-1}\left(\frac{a_{j_k} + b_{j_k}}{2} - p\right)}\right| = 0,
\end{aligned}$$

where in the inequality we use the fact that $|b_{j_k} - a_{j_k}|^s \ge |b_{j_k} - a_{j_k}|^{n-1}$ for k large enough and the last equation is justified by (1). This completes the proof of (i).

(ii): Note that $g' \equiv 0$ implies that $\hat{g}^{(i)} \restriction P = g^{(i)} \equiv 0$ for every $i \in \mathbb{N}$. In particular, $T^{n-2}_{a_k}\hat{g}(x) = \sum_{i=0}^{n-2} \frac{\hat{g}^{(i)}(a_k)}{i!}(x - a_k)^i = \hat{g}(a_k)$ and, by the Lagrange formula for the remainder of this Taylor polynomial, for every $k \in \mathbb{N}$ there exists a $\xi_k \in (a_k, b_k)$ such that

$$\hat{g}(b_k) - \hat{g}(a_k) = \hat{g}(b_k) - T^{n-2}_{a_k}\hat{g}(b_k) = \frac{\hat{g}^{(n-1)}(\xi_k)}{(n-1)!}(b_k - a_k)^{n-1}.$$

Hence, using $\lim_{k\to\infty} \xi_k = p$ and $\hat{g}^{(n-1)}(p) = \hat{g}^{(n)}(p) = 0$, we get

$$\begin{aligned}
&\lim_{k\to\infty}\left|\frac{\hat{g}(b_k) - \hat{g}(a_k)}{(b_k - a_k)^{n-1}\left(\frac{a_k + b_k}{2} - p\right)}\right| = \lim_{k\to\infty}\left|\frac{\hat{g}^{(n-1)}(\xi_k)}{(n-1)!\left(\frac{a_k + b_k}{2} - p\right)}\right| \\
&\qquad = \lim_{k\to\infty}\left|\frac{\xi_k - p}{(n-1)!\left(\frac{a_k + b_k}{2} - p\right)}\right|\left|\frac{\hat{g}^{(n-1)}(\xi_k) - \hat{g}^{(n-1)}(p)}{\xi_k - p} - \hat{g}^{(n)}(p)\right| = 0
\end{aligned}$$

as $\left|\frac{\xi_k-p}{(n-1)!\left(\frac{a_k+b_k}{2}-p\right)}\right| \leq 2$ and $\lim_{k\to\infty}\left(\frac{\hat{g}^{(n-1)}(\xi_k)-\hat{g}^{(n-1)}(p)}{\xi_k-p}-\hat{g}^{(n)}(p)\right)=0$. □

Proof of Theorem 1. For $n = 1$ Theorem 1 is a restatement of JET. So, in what follows we assume that $n > 1$.

(b)$\Longrightarrow$(a): We will find an extension $\phi_n \in D^n(\mathbb{R})$ of f. Indeed, by (b), there is an extension $\phi_{n-1} \in D^{n-1}(\mathbb{R})$ of f' and $g_n \in D^n(P)$ defined as

$$g_n(x) := f(x) - \int_0^x \phi_{n-1}(t)\,dt \quad \text{for } x \in P, \tag{4}$$

satisfies (1). Moreover, since function $\phi_{n-1} \in D^{n-1}(\mathbb{R}) \subset D^1(\mathbb{R})$ is continuous, we have $g_n'(x) = f'(x) - \phi_{n-1}(x) = 0$ for every $x \in P$. In particular, g_n satisfies (1) and the assumptions of Lemma 5(i). Therefore, there exists an extension $\tilde{g}_n \in D^n(\mathbb{R})$ of g_n. We claim that $\phi_n \in D^n(\mathbb{R})$ given by

$$\phi_n(x) := \tilde{g}_n(x) + \int_0^x \phi_{n-1}(t)\,dt, \tag{5}$$

is as needed. Indeed, clearly it is D^n, as a sum of two such functions. Moreover, for every $x \in P$, we have $\phi_n(x) = g_n(x) + \int_0^x \phi_{n-1}(t)\,dt = f(x)$. That is, ϕ_n indeed extends f.

(a)$\Longrightarrow$(b): Let $F \in D^n(\mathbb{R})$ be an extension of f. We need to show that this implies (1). Indeed, clearly f' has an extension $\phi \in D^{n-1}(\mathbb{R})$, namely $\phi = F'$. To finish the proof of (1), fix an extension $\phi \in D^{n-1}(\mathbb{R})$ of f' and define $g \in D^n(P)$ via

$$g(x) := f(x) - \int_0^x \phi(t)\,dt \quad \text{for } x \in P.$$

We need to show that g satisfies (1). So, choose $\langle\langle a_k, b_k\rangle \in P^2 \colon k \in \mathbb{N}\rangle$ as in its statement, that is, one-to-one, converging to a $\langle p, p\rangle \in P^2$, and such that $\emptyset \neq (a_k, b_k) \subset \mathbb{R} \setminus P$ for each $k \in \mathbb{N}$.

Clearly, $\hat{g} \in D^n(\mathbb{R})$ defined as

$$\hat{g}(x) := F(x) - \int_0^x \phi(t)\,dt \quad \text{for } x \in \mathbb{R}$$

is an extension of g. Also, $\hat{g}' \restriction P \equiv 0$, as $\hat{g}'(x) = g'(x) = f'(x) - \phi(x) = 0$ for every $x \in P$. So, by Lemma 5(ii), g indeed satisfies (1). □

5 Proof of Theorem 2

To prove Theorem 2 we will also need the next lemma.

Lemma 6. *Let $n \in \mathbb{N}$, $\emptyset \neq P \subset \mathbb{R}$ be perfect, $g \in D^1(P)$ be such that $g' \equiv 0$, and $\tilde{g}$ be the canonical extension of g. If q_g^n is continuous, then $\tilde{g} \in C^n(\mathbb{R})$.*

Proof. First notice that continuity of q_g^n implies (1), since for every $p \in P$ and a sequence $\langle \langle a_k, b_k\rangle \in P^2 \colon k \in \mathbb{N}\rangle$ as in this condition, we have

$$\lim_{k\to\infty}\left|\frac{g(b_k)-g(a_k)}{(b_k-a_k)^{n-1}\left(\frac{a_k+b_k}{2}-p\right)}\right| \le \lim_{k\to\infty}\left|\frac{g(b_k)-g(a_k)}{(b_k-a_k)^{n-1}\frac{1}{2}(b_k-a_k)}\right| = 0,$$

as $\lim_{k\to\infty}\left|\frac{g(b_k)-g(a_k)}{(b_k-a_k)^{n-1}\frac{1}{2}(b_k-a_k)}\right| = 2\lim_{k\to\infty}\left|\frac{g(b_k)-g(a_k)}{(b_k-a_k)^{n}}\right| = 2q_g^n(p,p) = 0$. So, by Lemma 5(i), $\tilde{g} \in D^n(\mathbb{R})$. It remains to show that $\tilde{g}^{(n)}$ is continuous.

But, by Fact 3, it is continuous on $\mathbb{R}\setminus P$. So, we need to show that it is also continuous on P. To see this, fix $p \in P$. We need to show that for every sequence $\langle x_k \in \mathbb{R} \colon k \in \mathbb{N}\rangle$ converging to p, we have $\lim_{k\to\infty}\left(\tilde{g}^{(n)}(x_k) - \tilde{g}^{(n)}(p)\right) = 0$. But, by Lemma 4, this holds for any such sequence with every $x_k \in \mathbb{R}\setminus(M\cup P)$. Also, since $g' \equiv 0$, this holds whenever every x_k is in P. So, we can assume that every x_k is in M. In particular, for every $k \in \mathbb{N}$ there is $j_k \in J$ such that $x_k \in (a_{j_k}, b_{j_k})$. Hence, using $\beta_{j_k}^{(n)}(x_k) = \frac{\psi^{(n)}(\ell_{j_k}(x_k))}{(b_{j_k}-a_{j_k})^n}$ and for $\bar{M} = \sup \psi^{(n)}[[0,1]] \in \mathbb{R}$

$$\begin{aligned}
\lim_{k\to\infty}\left(\tilde{g}^{(n)}(x_k) - \tilde{g}^{(n)}(p)\right) &= \lim_{k\to\infty}\left|\tilde{g}_{j_k}^{(n)}(x_k)\right| \\
&= \lim_{k\to\infty}\left|[g(b_{j_k}) - g(a_{j_k})]\beta_{j_k}^{(n)}(x_k)\right| \\
&= \lim_{k\to\infty}\left|[g(b_{j_k}) - g(a_{j_k})]\frac{\psi^{(n)}(\ell_{j_k}(x_k))}{(b_{j_k}-a_{j_k})^n}\right| \\
&\le \bar{M}\lim_{k\to\infty}\left|\frac{g(b_{j_k}) - g(a_{j_k})}{(b_{j_k}-a_{j_k})^n}\right| = \bar{M} q_g^n(p,p) = 0,
\end{aligned}$$

completing the proof. □

Proof of Theorem 2. **(C)$\Longrightarrow$(A):** Note that the map g satisfies the assumptions of Lemma 6. So, $\tilde{g} \in C^n(\mathbb{R})$ and $F \colon \mathbb{R}\to\mathbb{R}$ defined as $F(x) = \tilde{g}(x) + \int_0^x \phi(t)\,dt$ is the desired $C^n(\mathbb{R})$ extension of f.

(A)$\Longrightarrow$(B): It is enough to show that for every $m < \omega$, if $h \in C^m(\mathbb{R})$, then Q_h^m is continuous. So, assume that $h \in C^m(\mathbb{R})$. Clearly Q_h^m is continuous at any point $\langle a, b\rangle \in \mathbb{R}^2$ with $a \neq b$. We need to show that Q_h^m is continuous at every $\langle a, a\rangle$. To see this, choose a sequence $\langle a_k, b_k\rangle_{k\in\mathbb{N}}$ converging to $\langle a, a\rangle$. We need to show that $\lim_{k\to\infty} Q_h^m(a_k, b_k) = 0$.

By the Lagrange formula for the remainder of Taylor polynomial, for every $k \in \mathbb{N}$ there is ξ_k between a_k and b_k with $h(b_k) - T_{a_k}^{m-1}h(b_k) = \frac{h^{(m)}(\xi_k)}{m!}(b_k-a_k)^m$. Thus, since $T_{b_k}^m h(b_k) - T_{a_k}^m h(b_k) = h(b_k) - \left(T_{a_k}^{m-1}h(b_k) + \frac{h^{(m)}(b_k)}{m!}(b_k-a_k)^m\right)$,

$$Q_h^m(a_k,b_k) = \frac{\frac{h^{(m)}(\xi_k)}{m!}(b_k-a_k)^m - \frac{h^{(m)}(b_k)}{m!}(b_k-a_k)^m}{(b_k-a_k)^m} = \frac{h^{(m)}(\xi_k) - h^{(m)}(b_k)}{m!}$$

converges to 0, as $k \to \infty$, since $h^{(m)}$ is continuous and $\langle a_k, b_k\rangle \to_{k\to\infty} \langle a, a\rangle$. Therefore,

$$\lim_{k\to\infty} Q_h^m(a_k, b_k) = 0 = Q_h^m(a, a),$$

as needed.

(B)$\Longrightarrow$(C): First we prove that for every $\phi \in C^{n-1}(\mathbb{R})$ extending f' and $g \in C^n(P)$ defined, for every $x \in P$, as $g(x) := f(x) - \int_0^x \phi(t)\,dt$ the map q_g^n is continuous. To see this, let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be given via $\Phi(x) := \int_0^x \phi(t)\,dt$. Then $\Phi \in C^n(\mathbb{R})$ and applying just proved implication (A)$\Longrightarrow$(B) to $h := \Phi \restriction P$, we see that Q_h^n is continuous. Since, by our assumption, Q_f^n is also continuous, to show continuity of q_g^n it is enough to prove that

$$q_g^n(a, b) = Q_f^n(a, b) - Q_h^n(a, b) \tag{6}$$

for all $\langle a, b\rangle \in P^2$.

Indeed, for every $p \in P$ we have $T_p^n f(b) - T_p^n h(b) = f(p) - h(p) = g(p)$ since $f^{(i)}(p) = h^{(i)}(p)$ for all $i \in \{1, \ldots, n\}$. Using this with $p = b$ and $p = a$, we get $\big(T_b^n f(b) - T_a^n f(b)\big) - \big(T_b^n h(b) - T_a^n h(b)\big) = g(b) - g(a)$. So, for $a \neq b$,

$$Q_f^n(a, b) - Q_h^n(a, b) = \frac{\big(T_b^n f(b) - T_a^n f(b)\big) - \big(T_b^n h(b) - T_a^n h(b)\big)}{(b-a)^n} = q_g^n(a, b)$$

proving (6), as it clearly holds also for $a = b$.

To finish the proof, it is enough to show that (B) implies that there is a $\phi \in C^{n-1}(\mathbb{R})$ extending f'. This is proved by induction on $n \in \mathbb{N}$.

Such ϕ clearly exists for $n = 1$. So, assume that the statement holds for some $n \in \mathbb{N}$. To see that it also holds for $n + 1$ fix an $f \in C^{n+1}(P)$ satisfying (B). Then, $f' \in C^n(P)$ also satisfies (B) and, by the inductive assumption, there is a $\phi \in C^{n-1}(\mathbb{R})$ extending f''. But this means that f' satisfies (C) and, as (C)$\Longrightarrow$(A), it satisfies also (A). Therefore, f' admits $C^n(\mathbb{R})$-extension, as needed. $\square$

6 Extensions by linear operators

For $n < \omega$ and a non-empty perfect subset P of $\mathbb{R}$ let $\mathbb{D}^n(P)$ stand for the class of all functions $f\colon P \to \mathbb{R}$ admitting D^n-extensions $F\colon \mathbb{R} \to \mathbb{R}$, that is, those characterized in Theorem 1. Similarly, $\mathbb{C}^n(P)$ will stand for the functions $f\colon P \to \mathbb{R}$ admitting C^n-extensions $F\colon \mathbb{R} \to \mathbb{R}$, that is, those characterized in Theorem 2. Also, for $f \in C(P)$ let $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ be the linear interpolation of f which on each unbounded component of $\mathbb{R} \setminus P$ (if any such component exists) is constant. See Figure 3.

The construction of the extensions presented in the proofs of the theorems can be represented in a form of linear operators that assign to each f in $\mathbb{C}^n(P)$ (or $\mathbb{D}^n(P)$) its extension F in $C^n(\mathbb{R})$ (or $D^n(\mathbb{R})$, respectively). First, we describe the operators $T_n\colon C(P) \to C(\mathbb{R})$ such that each $T_n(f)$ extends f and also

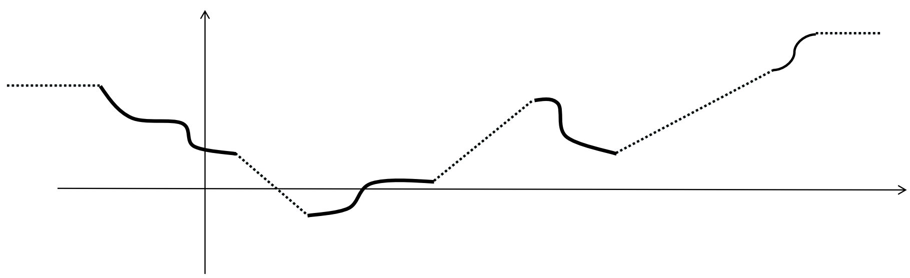

Figure 3: A graph of f (solid line) extended to its linear interpolation $\bar{f}$.

$T_n(f) \in C^n(\mathbb{R})$ whenever $f \in \mathbb{C}^n(P)$. They are defined by induction on $n < \omega$ as follows: for every $f \in C(P)$ and $x \in \mathbb{R}$ we put

- $T_0(f) = \bar{f}$,
- $T_{n+1}(f)(x) = \tilde{g}(x) + \int_0^x T_n(f')(t)\ dt$, where $g(y) = f(y) - \int_0^y T_n(f')(t)\ dt$ for every $y \in P$.

It is easy to see that each T_n is indeed a linear operator.

Theorem 7. *Let $n < \omega$, P be a perfect subset of $\mathbb{R}$, and the linear map $T_n \colon C(P) \to C(\mathbb{R})$ be the extension operator defined as above. Then T_n maps $\mathbb{D}^n(P) \cap C^1(P)$ into $D^n(\mathbb{R})$ and $\mathbb{C}^n(P)$ into $C^n(\mathbb{R})$.*

Proof. The proof is by induction on $n < \omega$. For $n = 0$ the result is obvious. So, we assume it holds for some $n < \omega$ and prove it for $n + 1$.

To see this, choose an $f \in \mathbb{D}^{n+1}(P) \cap C^1(P)$. Then $T_n(f')$ is continuous: for $n = 0$ this follows from the continuity of f' and $\bar{f}' = T_0(f')$, while for $n > 0$ by the inductive assumption, since then $f' \in \mathbb{D}^n(P) \subset C(P)$. Hence, $g'(y) = f'(y) - \frac{d}{dy}\left(\int_0^y T_n(f')(t)\ dt\right) = f'(y) - T_n(f')(y) = 0$ on P since, by the inductive assumption, $T_n(f') \restriction P = f'$. Therefore, by Theorem 1 and Lemma 5, $\tilde{g} \in D^{n+1}(\mathbb{R})$. Moreover, if f is in $\mathbb{C}^{n+1}(P)$, then so is g and, by Theorem 2 and Lemma 6 , we also have $\tilde{g} \in C^{n+1}(\mathbb{R})$.

The map $x \mapsto \int_0^x T_n(f')(t)\ dt$ is $D^{n+1}(\mathbb{R})$, since $T_n(f')$ is continuous and, by inductive assumption, $T_n(f') \in D^n(\mathbb{R})$; also it is $C^{n+1}(\mathbb{R})$ whenever $f \in \mathbb{C}^{n+1}(P)$. Hence, $T_{n+1}(f)$ is in $D^{n+1}(\mathbb{R})$ (in $C^{n+1}(\mathbb{R})$ for $f \in \mathbb{C}^{n+1}(P)$) as a sum of two such maps. Finally, for every $x \in P$ we have $T_{n+1}(f)(x) = g(x) + \int_0^x T_n(f')(t)\ dt = \left(f(x) - \int_0^x T_n(f')(t)\ dt\right) + \int_0^x T_n(f')(t)\ dt = f(x)$. So, $T_{n+1}(f) \restriction P = f$, as needed. $\square$

For an arbitrary $f \in D^1(P)$ the map $T_1(f)$ needs neither extend f nor be differentiable (everywhere), since f' may be discontinuous in which case the map $x \mapsto \int_0^x \bar{f}'(t)\ dt$ is differentiable only almost everywhere. However, a linear extension operator T_n^* from $\mathbb{D}^n(P)$ into $D^n(\mathbb{R})$ can be defined as follows.

(I) Choose a linear basis $\mathcal{B}$ of $D^1(P)$ over $\mathbb{R}$ and for every $f \in \mathcal{B}$ let $T_1^*(f) \supset f$ be a $D^1(\mathbb{R})$ map, which exists by JET. Then T_1^* on $D^1(P)$ can be defined as a unique linear map extending the map $T_1^* \restriction \mathcal{B}$.

(II) As above, we can define linear extension operators $T_n^* \colon D^n(P) \to D^n(\mathbb{R})$ by induction on $n \in \mathbb{N}$, by letting $T_{n+1}^*(f)(x) = \tilde{g}(x) + \int_0^x T_n^*(f')(t)\, dt$ for every $x \in P$, where $g(y) = f(y) - \int_0^y T_n^*(f')(t)\, dt$ for every $y \in P$.

An argument as for Theorem 7 shows that these are indeed linear extension operators from $D^n(P)$ to $D^n(\mathbb{R})$.

There has been a lot of work in literature discussing the existence of *bounded* smooth extension linear operators, that is, such as T_n—see e.g. [10] and the references cited there. In such work, the norm of an $f \in C^n(\mathbb{R})$ is defined as

$$\|f\|_{C^n(\mathbb{R})} := \max_{i \le n} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|$$

and the study is restricted to the class $C_b^n(\mathbb{R})$ of all functions $f \in C^n(\mathbb{R})$ having this norm finite. Also, the norm of an $f \in \mathbb{C}^n(P)$ is defined as

$$\|f\|_{\mathbb{C}^n(P)} := \inf\{\|F\|_{C^n(\mathbb{R})} \colon F \in C^n(\mathbb{R}) \text{ extends } f\}$$

and the study concentrates on the class $\mathbb{C}_b^n(P)$ of all $f \in \mathbb{C}^n(P)$ with finite $\|f\|_{\mathbb{C}^n(P)}$. It would be nice for the operator $T_n \restriction \mathbb{C}_b^n(P)$ to be bounded. Unfortunately, this is not the case, as the following example shows.

Example 8. There exists a perfect $P \subset \mathbb{R}$ and an $f \in \mathbb{C}_b^n(P)$ such that $T_1(f)$ is unbounded.

Construction. Let $P = \bigcup_{n \in \mathbb{N}} [2^n, 2^n + 1]$ and for every $x \in [2^n, 2^n + 1]$ define $f(x) := x - 2^n$. It is easy to see that $f \in \mathbb{C}_b^n(P)$. However, $T_1(f)$ on each interval $[2^n, \frac{4}{3}2^n]$ is still given as $x - 2^n$, so has maximum $\geq 2^n/3$. This ensures that $T_1(f)$ is unbounded. □

In spite of the difficulties that Example 8 shows, it seems quite clear that the constructions of T_n and T_n^* could be slightly modified to ensure that the modified T_n is indeed bounded on $\mathbb{C}_b^n(P)$ and similarly for T_n^*. The details of this claim will be examined in our forthcoming paper.

7 Final remarks on format of Theorem 7

One may wonder if the format of the characterization from Theorem 7 can be further simplified. We provide here some results showing that this might be hard to achieve.

We say that a function $Q \colon P^2 \to \mathbb{R}$ is continuous with respect to the first (or second) variable, provided for every $p \in P$ the map $\mathbb{R} \ni x \mapsto Q(x,p) \in \mathbb{R}$ ($\mathbb{R} \ni x \mapsto Q(p,x) \in \mathbb{R}$, respectively) is continuous. Also, Q is separately continuous, provided it is continuous with respect to both variables, first and second. The study of separately continuous functions comes back to the work of Cauchy, Heine, Peano, Baire, and Lebesgue. See recent survey [5] for more detailed history. Let us recall here only that a separately continuous $Q\colon P^2 \to \mathbb{R}$ must be of Baire class one, but need not be continuous.

The proof of the following fact is elementary, see e.g. [6, prop. 3.2(i)].

Fact 9. *Let $n \in \mathbb{N}$. If $f \in D^n(\mathbb{R})$, then $Q^n_f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous with respect to the second variable, that is, the map $\mathbb{R} \ni x \mapsto Q^n_f(a,x) \in \mathbb{R}$ is continuous for every $a \in \mathbb{R}$.*

Fact 9 and WET imply that every $D^n(\mathbb{R})$-extendable function $f\colon P \to \mathbb{R}$ satisfies

(W_n) $Q^{n-1-i}_{f^{(i)}}\colon P^2 \to \mathbb{R}$ is continuous for every $i < n$ while Q^n_f is continuous with respect to the second variable.

One may wonder, if (W_n) implies also $D^n(\mathbb{R})$-extendability of f. Although, by JET this implication indeed holds for $n = 1$, the following example shows that it does not for $n = 2$.

Example 10. There exist a perfect set $P \subset [0,1]$ and an $f\colon P \to \mathbb{R}$ such that $f \in C^2(P)$ is $C^1([0,1])$-extendable, not $D^2([0,1])$-extendable, while $Q^2_f = q^2_f$ is separately continuous. In particular, f satisfies (W_2) but is not $D^2([0,1])$-extendable.

Construction. For $n \in \mathbb{N}$ let $a_n := 2^{-n}$, $b_n := 2^{-n} + 4^{-n}$, and $J_n := (a_n, b_n)$. Let $P := [0,1] \setminus \bigcup_{n\in\mathbb{N}} J_n$ and $a_0 = 1$. We define $f(0) := 0$ and, for $n \in \mathbb{N}$, $f \restriction [b_n, a_{n-1}] \equiv 7^{-n}$. Then f is as needed.

To see that f is $C^1([0,1])$-extendable, define $f_0 \in C^0([0,1])$ by $f_0 \restriction P \equiv 0$ and for each $x \in J_n$, $n \in \mathbb{N}$, as $f_0(x) = c_n \operatorname{dist}(x, P)$, where c_n is such that $\int_{a_n}^{b_n} f_0(t)\,dt = \frac{1}{4}16^{-n}c_n$ (evaluated as an area of a triangle: $\frac{1}{2}\cdot 4^{-n}\cdot\left(\frac{1}{2}4^{-n}c_n\right)$) equals to $f(b_n) - f(a_n) = 7^{-n} - 7^{-n-1} = \frac{6}{7}7^{-n}$. In particular, $c_n = \frac{24}{7}\left(\frac{16}{7}\right)^n$ and the maximum value of f_0 on J_n, that is $f_0\left(\frac{a_n+b_n}{2}\right) = \frac{1}{2}4^{-n}c_n = \frac{12}{7}\left(\frac{4}{7}\right)^n$, converges to $0 = f_0(0)$, ensuring continuity of f_0. Therefore, the function $\bar{f}\colon [0,1] \to \mathbb{R}$ defined as $\bar{f}(x) := \int_0^x f_0(t)\,dt$ is C^1 and it extends f, since $\bar{f}(b_n) - \bar{f}(a_n) = f(b_n) - f(a_n)$ for every $n \in \mathbb{N}$.

We have $f \in C^2(P)$, as $f' \restriction P = f_0 \restriction P \equiv 0$. This also implies that $Q^2_f = q^2_f$.

To see that q^2_f is separately continuous, first notice that it is continuous at any point except possibly at $\langle 0,0\rangle$. Indeed, this is obvious at any $\langle a,b\rangle \in P^2$ with $a \neq b$, while at $\langle p,p\rangle$ with $p > 0$ this follows from WET, as $f \restriction P \cap [p/2, 1]$ has clearly $C^\infty(\mathbb{R})$-extension. The map q^2_f is separately continuous at $\langle 0,0\rangle$ since for any $b \in [b_n, a_{n-1}]$ we have $0 \le q^2_f(0,b) = \frac{f(b)}{b^2} < \frac{f(b)}{a_n^2} = \frac{7^{-n}}{4^{-n}} \to_{n\to\infty} 0 = q^2_f(0,0)$. This, with $q^2_f(b,0) = \frac{-f(b)}{b^2} = -q^2_f(0,b)$, ensures its separate continuity.

Finally, f is not $D^2([0,1])$-extendable, as it does not satisfy (1) for $\langle a_n, b_n\rangle_n$, since $\frac{f(b_n)-f(a_n)}{(b_n-a_n)\frac{a_n+b_n}{2}} = \frac{\frac{6}{7}7^{-n}}{4^{-n}(2^{-n}+\frac{1}{2}4^{-n})} = \frac{\frac{6}{7}}{(\frac{7}{8})^n + \frac{1}{2}(\frac{7}{16})^n} \to_{n\to\infty} \infty$. $\square$

It might be also natural to wonder, whether we could strengthen Fact 9 to show that, under the same assumptions, Q_f^n is also continuous with respect to the first variable. The following example shows that such strengthening of Fact 9 is false, already for $n = 2$.

Example 11. There is an $f \in D^2(\mathbb{R})$ such that $\lim_{x\to 0} Q_f^2(x,0) \neq 0 = q_f^2(0,0)$.

Proof. The statement holds for

$$f(x) := \begin{cases} x^4 \cos\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Indeed, $f \in D^2(\mathbb{R})$ with $f'(0) = f''(0) = 0$ and, for $x \neq 0$,

$$\begin{aligned} f'(x) &= 4x^3 \cos\left(x^{-1}\right) + x^2 \sin\left(x^{-1}\right) \\ f''(x) &= 12x^2 \cos\left(x^{-1}\right) + 6x \sin\left(x^{-1}\right) - \cos\left(x^{-1}\right). \end{aligned}$$

In particular, for $x_k = \frac{1}{2k\pi}$,

$$\lim_{k\to\infty} \frac{f(x_k)}{x_k^2} = \lim_{k\to\infty} \frac{f'(x_k)}{x_k} = 0 \quad \& \quad \lim_{k\to\infty} f''(x_k) = -1.$$

Therefore,

$$\begin{aligned} Q_f^2(x,0) &= \frac{T_0^2 f(0) - T_x^2 f(0)}{(0-x)^2} \\ &= \frac{f(0) - \left(f(x) + f'(x)(0-x) + \frac{1}{2} f''(x)(0-x)^2\right)}{(0-x)^2} \\ &= -\frac{f(x)}{x^2} + \frac{f'(x)}{x} - \frac{1}{2} f''(x) \end{aligned}$$

does not converge to 0, as $x \to 0$, since $\lim_{k\to\infty} Q_f^2(x_k, 0) = \frac{1}{2}$. $\square$

It would be interesting to find a version of Theorem 1 for the functions of more than one variable. However, a simple-minded generalization of Theorem 1 is not valid in such setting since, as we mentioned earlier, this is already the case for JET, as proved in [2].

Acknowledgements. The author likes to express his gratitude to an anonymous referee for extremely careful reading of a submitted draft, catching a large number of typos, and helping in improving exposition of the presented material.

References

[1] J.-C. Archer and E. Le Gruyer, *A constructive proof of a Whitney extension theorem in one variable*, J. Approx. Theory **71** (1992), no. 3, 312–328, DOI 10.1016/0021-9045(92)90122-5.

[2] V. Aversa, M. Laczkovich, and D. Preiss, *Extension of differentiable functions*, Comment. Math. Univ. Carolin. **26** (1985), no. 3, 597–609.

[3] E. Bierstone, *Differentiable functions*, Bol. Soc. Brasil. Mat. **11** (1980), no. 2, 139–189.

[4] M. Ciesielska and K. C. Ciesielski, *Differentiable extension theorem: a lost proof of V. Jarník*, J. Math. Anal. Appl. **454** (2017), no. 2, 883–890, DOI 10.1016/j.jmaa.2017.05.032.

[5] K. C. Ciesielski and D. Miller, *A continuous tale on continuous and separately continuous functions*, Real Anal. Exchange **41** (2016), no. 1, 19–54, DOI 10.14321/realanalexch.41.1.0019.

[6] K. C. Ciesielski and J. B. Seoane-Sepúlveda, *Simultaneous small coverings by smooth functions under the covering property axiom*, Real Anal. Exchange **43** (2018), no. 2, 359–386, DOI 10.14321/realanalexch.43.2.0359.

[7] ———, *Differentiability versus continuity: Restriction and extension theorems and monstrous examples*, Bull. Amer. Math. Soc. **56** (2019), no. 2, 211–260, DOI 10.1090/bull/1635.

[8] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[9] C. Fefferman, *Whitney's extension problem for C^m*, Ann. of Math. (2) **164** (2006), no. 1, 313–359, DOI 10.4007/annals.2006.164.313.

[10] ———, *Whitney's extension problems and interpolation of data*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 2, 207–220, DOI 10.1090/S0273-0979-08-01240-8.

[11] M. Koc and L. Zajíček, *A joint generalization of Whitney's C^1 extension theorem and Aversa-Laczkovich-Preiss' extension theorem*, J. Math. Anal. Appl. **388** (2012), no. 2, 1027–1037, DOI 10.1016/j.jmaa.2011.10.049.

[12] V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dérivabilité de la fonction*, Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

[13] ———, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce [On the extension of the domain of a function preserving differentiability of the function]*, Rozpravy Čes. akademie, II. tř. **XXXII** (1923), no. 15, 1–5 (Czech).

[14] J. Mařík, *Derivatives and closed sets*, Acta Math. Hungar. **43** (1984), no. 1-2, 25–29, DOI 10.1007/BF01951320.

[15] J. Merrien, *Prolongateurs de fonctions différentiables d'une variable réelle*, J. Math. Pures Appl. (9) **45** (1966), 291–309 (French).

[16] W. Pawłucki, *Examples of functions $\mathcal{C}^k$-extendable for each k finite, but not $\mathcal{C}^\infty$-extendable*, Singularities Symposium—Łojasiewicz 70 (Kraków, 1996; Warsaw, 1996), Banach Center Publ., vol. 44, Polish Acad. Sci. Inst. Math., Warsaw, 1998, pp. 183–187.

[17] G. Petruska and M. Laczkovich, *Baire* 1 *functions, approximately continuous functions and derivatives*, Acta Math. Acad. Sci. Hungar. **25** (1974), 189–212, DOI 10.1007/BF01901760.

[18] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89, DOI 10.2307/1989708.

[19] ———, *Differentiable functions defined in closed sets. I*, Trans. Amer. Math. Soc. **36** (1934), no. 2, 369–387, DOI 10.2307/1989844.