TYPICAL BAD DIFFERENTIABLE EXTENSIONS

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Abstract. By a 1923 result of V. Jarník, every differentiable map φ from a closed subset P of $\mathbb{R}$ into $\mathbb{R}$ has a differentiable extension $f\colon \mathbb{R} \to \mathbb{R}$. It has been recently proved, by the authors, that among such differentiable extensions of φ there is always one that is nowhere monotone on $\mathbb{R} \setminus P$. In particular, the family $E^1_\varphi(\mathbb{R})$ of "bad" differentiable extensions $f\colon \mathbb{R} \to \mathbb{R}$ of φ, for which the set $[f' = 0] := \{x \in \mathbb{R}\colon f'(x) = 0\}$ is dense in $\mathbb{R} \setminus P$, is nonempty.

We notice here that $E^1_\varphi(\mathbb{R})$ with a natural distance is a complete metric space and prove that actually a typical function in $E^1_\varphi(\mathbb{R})$ is nowhere monotone on $\mathbb{R} \setminus P$. At the same time, the set $M_\varphi(\mathbb{R})$, of functions $f \in E^1_\varphi(\mathbb{R})$ which are monotone on some nonempty subinterval of every nonempty open $U \subset \mathbb{R} \setminus P$, is dense in $E^1_\varphi(\mathbb{R})$. This last statement remains true, when the term "monotone" is replaced with either of the following three terms: "strictly increasing," "strictly decreasing," or "constant."

1. Background

In 1923, V. Jarník proved that every differentiable map φ from a closed subset P of $\mathbb{R}$ into $\mathbb{R}$ has a differentiable extension $f\colon \mathbb{R} \to \mathbb{R}$. An interesting story of this result being forgotten and rediscovered is described in details in [3]. (Compare also [6].) In short, Jarník's full paper with this result [9], written in Czech, and its announcement [8] in French, with a sketch of construction, were published in rather obscure journals. So, the theorem was unnoticed by the mathematical community until the mid 1980's, when it was cited in [1]. In the meantime, the theorem was rediscovered in 1974 by G. Petruska and M. Laczkovich [12] and further elaborated on in 1984 by J. Mařík [11].

In a 2012 paper [10] M. Koc and L. Zajíček proved a version of Jarník's Extension Theorem showing that an extension f of φ can be, on the set $\mathbb{R} \setminus P$, as good as possible, that is, C^∞. In the opposite direction, the authors proved in [5] that this f can also be, on $\mathbb{R} \setminus P$, as bad as possible, that is, nowhere monotone. The goal of this paper is to extend the above-mentioned result from [5] by showing that, within a complete metric space $E^1_\varphi(\mathbb{R})$ defined in the next section, a typical differentiable extension of φ is nowhere monotone on $\mathbb{R} \setminus P$. Here we use the term *typical* to say that the set of all such functions is *residual*, that is, contains a dense G_δ-set.

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2. The space of Differentiable Functions

Let $D^1(\mathbb{R})$ be the family of all differentiable functions from $\mathbb{R}$ into $\mathbb{R}$ and let $C^1(\mathbb{R})$ stand for all $f \in D^1(\mathbb{R})$ with derivative f' being continuous. It is well known (see, e.g., [7, example 5.4]) that the subspace of $C^1(\mathbb{R})$, consisting of all functions $f \in C^1(\mathbb{R})$ for which their C^1-norm

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$$

is finite, forms a Banach space. Of course $\|\cdot\|_\infty$ is the supremum norm. This follows immediately from the following well-known theorem, see e.g. [14, theorem 9.37].

Fact 1. *If a sequence $\langle f_n \in D^1(\mathbb{R})\colon n \in \mathbb{N}\rangle$ converges uniformly to an $f \in C(\mathbb{R})$ and the sequence $\langle f_n'\colon n \in \mathbb{N}\rangle$ is Cauchy with respect to the uniform convergence, then f is differentiable and $\lim_{n\to\infty} f_n' = f'$.*

Fact 1 immediately implies also that:

Proposition 1. *$D^1(\mathbb{R})$ is a complete metric with respect to metric*

$$\rho(f,g) := \min\{1, \|f-g\|_{C^1}\}. \tag{1}$$

In what follows, P will always denote a closed subset of $\mathbb{R}$ (possibly empty) and φ a differentiable function from P into $\mathbb{R}$, that is, such that for any non-isolated $p \in P$ the limit

$$\varphi'(p) := \lim_{x\to p,\ x\in P} \frac{\varphi(x)-\varphi(p)}{x-p}$$

exists and is finite. For $f \in D^1(\mathbb{R})$, let $[f'=0] := \{x \in \mathbb{R}\colon f'(x)=0\}$ and define

$$D^1_\varphi(\mathbb{R}) := \{f \in D^1(\mathbb{R})\colon \varphi \subset f\}$$
$$E^1_\varphi(\mathbb{R}) := \{f \in D^1(\mathbb{R})\colon \varphi \subset f \ \&\ [f'=0] \text{ is dense in } \mathbb{R}\setminus P\}.$$

We will write $E^1(\mathbb{R})$ for $E^1_\emptyset(\mathbb{R})$ (i.e., for $E^1_\varphi(\mathbb{R})$, where the domain of φ is empty).

Lemma 1. *$E^1_\varphi(\mathbb{R})$ is a nonempty closed subspace of $D^1(\mathbb{R})$ considered with the metric ρ. In particular, $E^1_\varphi(\mathbb{R})$ is a complete metric space.*

Proof. It has been proved by the authors in [5] that there is an $f \in D^1_\varphi(\mathbb{R})$ which is nowhere monotone on $\mathbb{R}\setminus P$. Such f belongs to $E^1_\varphi(\mathbb{R})$ since the derivative f' must have both positive and negative values on every nonempty $(a,b) \subset \mathbb{R}\setminus P$ as f is monotone nowhere on $\mathbb{R}\setminus P$. Thus, it must also have value 0, as any derivative has the Darboux property (i.e., satisfies the conclusion of Intermediate Value Theorem), see [6, theorem 2.1] or [14, theorem 7.31]. This shows that $E^1_\varphi(\mathbb{R})$ is nonempty.

Clearly, $D^1_\varphi(\mathbb{R})$ is closed in $D^1(\mathbb{R})$. Since $E^1_\varphi(\mathbb{R}) = D^1_\varphi(\mathbb{R}) \cap E^1(\mathbb{R})$, it remains to show that $E^1(\mathbb{R})$ is closed in $D^1(\mathbb{R})$. Our argument for this comes from a paper [15] of C. Weil.

To see this, assume that a sequence $\langle f_n \in E^1(\mathbb{R})\colon n \in \mathbb{N}\rangle$ converges to an $f \in D^1(\mathbb{R})$ with respect to ρ. We need to show that $f \in E^1(\mathbb{R})$. Indeed, for every $n \in \mathbb{N}$, the set $G_n := [f_n' = 0]$ is dense in $\mathbb{R} \setminus P$. It is also G_δ in $\mathbb{R} \setminus P$, since this is true for all derivatives, see e.g. [6]. Hence $f' = \lim_{n\to\infty} f_n'$ has value 0 on the set $\bigcap_{n\in\mathbb{N}} G_n$, which is also dense G_δ in $\mathbb{R} \setminus P$, as $\mathbb{R} \setminus P$ is a Baire space (it is locally compact Hausdorff). $\square$

Notice, that we proved also that $E^1(\mathbb{R})$ is nonempty and closed in $D^1(\mathbb{R})$.

Remark 1. *If $P \neq \mathbb{R}$, then the space $E^1_\varphi(\mathbb{R})$, with the metric ρ, has a closed discrete subset of cardinality continuum. In particular, neither $E^1_\varphi(\mathbb{R})$ nor $D^1(\mathbb{R})$ is separable.*

Proof. Let $a < b$ be such that $[a, b] \cap P = \emptyset$ and let $f \in E^1_\varphi(\mathbb{R})$.

Choose a sequence $b = b_1 > a_1 > b_2 > a_2 > \cdots$ converging to a such that a is a Lebesgue density point of the complement of $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$. For every $n \in \mathbb{N}$, let $g_n\colon \mathbb{R} \to \mathbb{R}$ be a map as in Lemma 2 with support in $[a_n, b_n]$ and vertically rescaled so that $\|g_n'\|_\infty = 1$. For every $s\colon \mathbb{N} \to \{-1, 1\}$, let $h_s = \sum_{n\in\mathbb{N}} s(n)g_n'$. It is bounded and approximately continuous—this is ensured at $x = a$ by the Lebesgue density requirement.

This implies that each $H_s(x) = \int_0^x h_s(t)\,dt$ is in $E^1(\mathbb{R})$, since $H_s' = h_s$, see e.g. [2, theorem 7.36, p. 317]. In particular, $H_s + f \in E^1_\varphi(\mathbb{R})$. Also, for every distinct $s, t\colon \mathbb{N} \to \{-1, 1\}$, we have $\rho(H_s + f, H_t + f) = \min\{1, \|H_s - H_t\|_{C^1}\} \geq \min\{1, \|H_s' - H_t'\|_\infty\} = 1$. That is, we indeed found a closed discrete subset $\{H_s + f\colon s\colon \mathbb{N} \to \{-1, 1\}\}$ of $E^1_\varphi(\mathbb{R})$ of cardinality continuum. $\square$

3. The main theorem

Theorem 1. *If $P \subset \mathbb{R}$ is closed and $\varphi\colon P \to \mathbb{R}$ is differentiable, then a typical function in $E^1_\varphi(\mathbb{R})$ is nowhere monotone on $\mathbb{R} \setminus P$.*

The key step in the proof of Theorem 1 is the following lemma, which is a bit similar to [5, lemma 5].

Lemma 2. *For every $a < b < c < d$, there exists a "bump" map $g \in E^1(\mathbb{R})$ strictly increasing on (a, c), strictly decreasing on (c, d) such that $g'(b) > 0$, $\|g\|_{C^1} < \infty$, and with $g(x) = 0$ for any $x \in \mathbb{R} \setminus (a, d)$.*

Proof. Let $a < b < c < d$. We first construct a strictly increasing differentiable $g_0 = g \restriction [a, c]$ with bounded C^1-norm, $[g_0' = 0]$ dense in $[a, c]$, $g_0'(b) > 0$, and $g_0(a) = g_0'(a) = g_0'(c) = 0$.

For this, let $h\colon [0, 1] \to \mathbb{R}$ be a Pompeiu function, that is, strictly increasing and differentiable with $[h' = 0]$ dense in $[0, 1]$. Its construction can be found in a 1907 paper [13] of D. Pompeiu, as well as in more contemporary works [14, sec. 9.7] and [4]. In addition, h has bounded C^1-norm, since it is an inverse of a function defined as

$\gamma(x) = \sum_{i=1}^{\infty} A_i(x-q_i)^{1/3}$, where $\{q_i : i \in \mathbb{N}\}$ is dense in $[0,1]$, and the numbers A_i are positive with $\sum_{i=1}^{\infty} A_i < \infty$. (As such, $\gamma'(x)$ is bounded away from 0, so that the derivative of $h = \gamma^{-1}$ is bounded.)

Let $p < q < r$ be in $[0,1]$ such that $h'(p) = h'(r) = 0$ and $h'(q) > 0$ and let β be a C^1 map with strictly positive derivative that maps $[a,c]$ onto $[p,r]$ and with $\beta(b) = q$. Then, for $L(t) = t - h(p)$, the composition $g_0 = L \circ h \circ \beta$ is as needed.

To finish the proof, it is enough to notice that if $\ell\colon [c,d] \to \mathbb{R}$ is a linear function with $\ell(c) = c$ and $\ell(d) = a$, then $g_0 \cup (g_0 \circ \ell)$ is the desired function g on $[a,d]$, which can be uniquely extended to the required g. $\square$

Proof of Theorem 1. For an open nonempty interval $J \subset \mathbb{R} \setminus P$, let

$$U_J^+ = \{f \in E_\varphi^1(\mathbb{R})\colon (\exists x \in J) f'(x) > 0\}$$

and

$$U_J^- = \{f \in E_\varphi^1(\mathbb{R})\colon (\exists x \in J) f'(x) < 0\}.$$

We claim that

($\star$) the sets U_J^+ and U_J^- are open and dense in $E_\varphi^1(\mathbb{R})$.

Indeed, they are open, since for every $f \in U_J^-$ (or $f \in U_J^+$) and $x \in J$ for which $f'(x) < 0$ ($f'(x) > 0$, respectively), the ρ-ball centered at f and of radius $|f'(x)|$ is contained in U_J^- (U_J^+, respectively).

To see that $U_J^+ \cap U_J^-$ is dense in $E_\varphi^1(\mathbb{R})$, choose an arbitrary $f \in E_\varphi^1(\mathbb{R})$ and $\varepsilon \in (0,1)$. It is enough to find a $g \in D^1(\mathbb{R})$ with $\|g\|_{C^1} < \varepsilon$ such that $f+g \in U_J^+ \cap U_J^-$. To find such g choose $a_0 < b_0 < d_0 < a_1 < b_1 < d_1$ in J with f' equal 0 at each of these points. Let g_0 and g_1 be as in Lemma 2 applied to numbers $a_0 < b_0 < d_0$ and $a_1 < b_1 < d_1$, respectively. Multiplying these functions by a small enough constant, if necessary, we can also assume that $\|g_0\|_{C^1} < \varepsilon$ and $\|g_1\|_{C^1} < \varepsilon$. Then the function $g = g_0 - g_1$ is as needed, completing the proof of ($\star$).

Finally, let $\mathcal{J}$ be the countable family of all nonempty intervals $J \subset \mathbb{R} \setminus P$ with rational endpoints. Then $G = \bigcap_{J \in \mathcal{J}} (U_J^+ \cap U_J^-)$ is a dense G_δ set in $E_\varphi^1(\mathbb{R})$, and every function in G is nowhere monotone on $\mathbb{R} \setminus P$. $\square$

4. Functions that are nowhere nowhere-monotone

We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *increasing at a point* $x \in \mathbb{R}$ provided there exists an interval (a,b) containing x on which f is strictly increasing. Let $M_\varphi^{\nearrow}(\mathbb{R})$ be the family of all $f \in E_\varphi^1(\mathbb{R})$ for which the set of points at which f is increasing is dense in $\mathbb{R} \setminus P$.

Similarly, we say that $f\colon \mathbb{R} \to \mathbb{R}$ is *monotone* (*constant* or *decreasing*) *at a point* $x \in \mathbb{R}$ provided there exists an interval (a,b) containing x on which f is monotone, constant, or strictly decreasing, respectively. With each of these notions we associate their respective families

$M_\varphi(\mathbb{R})$, $M_\varphi^{\rightarrow}(\mathbb{R})$, and $M_\varphi^{\searrow}(\mathbb{R})$, in a way in which $M_\varphi^{\nearrow}(\mathbb{R})$ is associated with the notion of "increasing at a point."

Clearly, $M_\varphi^{\nearrow}(\mathbb{R})$, $M_\varphi^{\rightarrow}(\mathbb{R})$, and $M_\varphi^{\searrow}(\mathbb{R})$ are disjoint and contained in $M_\varphi(\mathbb{R})$ which, by Theorem 1, is first category in $E_\varphi^1(\mathbb{R})$. The goal of this section is to show that each of these first category sets is dense in $E_\varphi^1(\mathbb{R})$.

Theorem 2. *Each of the sets $M_\varphi^{\nearrow}(\mathbb{R})$, $M_\varphi^{\rightarrow}(\mathbb{R})$, and $M_\varphi^{\searrow}(\mathbb{R})$ is dense in $E_\varphi^1(\mathbb{R})$.*

The main step in the proof of Theorem 2 is the following Lemma 3.

Lemma 3. *For every $f \in E_\varphi^1(\mathbb{R})$, $\varepsilon \in (0,1)$, and a nonempty open set $U \subset \mathbb{R} \setminus P$, there exist $f_1 \in E_\varphi^1(\mathbb{R})$ and $p < q < r < s$ with $(p,s) \subset U$ such that $f_1 = f$ on $\mathbb{R} \setminus (p,s)$, $\rho(f_1, f) < \varepsilon$, $f_1 \restriction (p,q)$ is strictly increasing, $f_1 \restriction (q,r)$ is constant, and $f_1 \restriction (r,s)$ is strictly decreasing.*

Proof. Let g be as in Lemma 2 with $a = 0$, $c = 1$, and $d = 2$. Multiplying it by a constant, if necessary, we can also assume that $g(c) = 1$. Let $M = \|g\|_{C^1}$ and notice that $M \geq 1$.

Notice that for every $\delta > 0$ the set $[|f'| \geq \delta] := \{x \in \mathbb{R} \colon |f'(x)| \geq \delta\}$ cannot be dense in U. Indeed, as $[|f'| \geq \delta]$ is a G_δ-set, its density in U would imply it intersects $[f' = 0]$, which is not possible.

Let $\delta = \frac{\varepsilon}{12M}$ and choose a nonempty interval $(p,s) \subset U$ disjoint from $[|f'| \geq \delta]$ and with $s - p \leq 1$. Decreasing this interval, if necessary, we can also assume that $f'(p) = f'(s) = 0$ and that $|f(x) - f(y)| \leq \varepsilon/4$ for every $x, y \in [p,s]$. Let $[q,r]$ be the middle third of $[p,s]$.

For some $\xi \in (p,s)$, we have $\left|\frac{f(s)-f(p)}{q-p}\right| = 3\left|\frac{f(s)-f(p)}{s-p}\right| = 3|f'(\xi)| < 3\delta$. Thus, there exists a $v > \max\{f(s), f(p)\}$ such that $\frac{v-f(p)}{q-p} < 3\delta$ and $\frac{v-f(s)}{s-r} < 3\delta$. Consider the following two linear surjections: increasing $\ell_1 \colon [p,q] \to [0,1]$ and decreasing $\ell_2 \colon [r,s] \to [0,1]$. Notice, that ℓ_1 has the slope $m := \frac{1}{q-p}$, while ℓ_2 has the slope $\frac{1}{r-s} = -m$. Define $g_1 = (v - f(p)) \cdot g \circ \ell_1$ and notice that we have $g_1(p) = 0$ and $g_1(q) = v - f(p)$. Also, $\|g_1'\|_\infty = (v - f(p))\frac{1}{q-p}\|g'\|_\infty \leq \frac{v-f(p)}{q-p}M < 3\delta M = \varepsilon/4$ and $\|g_1\|_\infty = (v - f(p))\|g\|_\infty \leq \frac{v-f(p)}{q-p}M < \varepsilon/4$. Therefore, we have $\|g_1\|_{C^1} = \|g_1\|_\infty + \|g_1'\|_\infty < \varepsilon/2$. Similarly, if $g_2 = (v - f(s)) \cdot g \circ \ell_2$, then $\|g_2\|_{C^1} < \varepsilon/2$, $g_2(s) = 0$, and $g_2(r) = v - f(s)$. Define

$$f_1(x) = \begin{cases} f(p) + g_1(x) & \text{for } x \in [p,q] \\ v & \text{for } x \in [q,r] \\ f(s) + g_2(x) & \text{for } x \in [r,s] \\ f(x) & \text{for } x \in \mathbb{R} \setminus (p,s). \end{cases}$$

and notice that it is as needed.

Indeed, it is easy to see that f_1 is well defined differentiable function with $f_1'(p) = f_1'(q) = f_1'(r) = f_1'(s) = 0$. All other requirements for f_1

are clearly satisfied, except possibly for $\rho(f_1, f) < \varepsilon$. To see this, it is enough to prove that $\|(f_1 - f) \restriction J\|_{C^1} < \varepsilon$ for J being $[p, q]$, $[q, r]$, and $[r, s]$.

But on $J = [p, q]$ we have

$$\begin{aligned} \|f_1 - f\|_{C^1} &\leq \|g_1\|_{C^1} + \|f - f(p)\|_{C^1} \\ &< \varepsilon/2 + \|f - f(p)\|_\infty + \|(f - f(p))'\|_\infty \\ &< \varepsilon/2 + \varepsilon/4 + \delta < \varepsilon. \end{aligned}$$

Similarly, $\|f_1 - f\|_{C^1} < \varepsilon$ on $J = [r, s]$. Finally, on $J = [q, r]$,

$$\|f_1 - f\|_{C^1} \leq \|v - f(p)\|_{C^1} + \|f - f(p)\|_{C^1} \leq \varepsilon/4 + \varepsilon/4 + \delta < \varepsilon,$$

as needed. □

Proof of Theorem 2. We prove only the density of $M_\varphi^\nearrow(\mathbb{R})$, the proof for the other two cases being essentially the same. Fix an $f_0 \in E_\varphi^1(\mathbb{R})$ and an $\varepsilon \in (0, 1)$. We need to find $g \in M_\varphi^\nearrow(\mathbb{R})$ with $\|f_0 - g\|_{C^1} \leq \varepsilon$.

Let $\{B_n : n \in \mathbb{N}\}$ be the intervals forming a basis for $\mathbb{R} \setminus P$. Define $J_0 = \emptyset$. By induction on $n \in \mathbb{N}$ we will construct the sequences $\langle J_n \subset \mathbb{R} \setminus P : n \in \mathbb{N}\rangle$ of pairwise disjoint, possibly empty, open intervals and $\langle f_n \in E_\varphi^1(\mathbb{R}) : n \in \mathbb{N}\rangle$ such that the following inductive properties hold for every $n \in \mathbb{N}$.

(i) $B_n \cap \bigcup_{i \leq n} J_i \neq \emptyset$,
(ii) f_n is strictly increasing on each J_i with $i \leq n$, $f_{n-1} = f_n$ on $\mathbb{R} \setminus \bigcup_{i<n} J_i$, and $\rho(f_{n-1}, f_n) < 2^{-n}\varepsilon$.

The inductive step is facilitated by Lemma 3. Specifically, we let U to be the interior of $B_n \setminus \bigcup_{i<n} J_i$. If $U = \emptyset$, we let $f_n = f_{n-1}$ and $J_n = \emptyset$. Otherwise we choose $p < q < r < s$ and f_n by applying Lemma 3 to $f_{n-1} \in E_\varphi^1(\mathbb{R})$, $2^{-n}\varepsilon > 0$, and just chosen U. Let $J_n = (p, q)$. This choice of f_n and J_n ensures properties (i) and (ii).

To finish the proof, notice that the sequence $\langle f_n \in E_\varphi^1(\mathbb{R}) : n \in \mathbb{N}\rangle$ is Cauchy with respect to ρ, so that the limit $g = \lim_{n\to\infty} f_n$ exists and belongs to $E_\varphi^1(\mathbb{R})$. By (ii), $\rho(f_0, f_n) < (1 - 2^{-n})\varepsilon$ for every $n \in \mathbb{N}$. Therefore, $\rho(f_0, g) \leq \varepsilon$.

Finally, to see that $g \in M_\varphi^\nearrow(\mathbb{R})$ notice that, by (i), for every $n \in \mathbb{N}$ there exists an $i \leq n$ such that $B_n \cap J_i \neq \emptyset$. Moreover, by (ii), the restriction $g \restriction (B_n \cap J_i) = f_i \restriction (B_n \cap J_i)$ is strictly increasing. So, indeed $g \in M_\varphi^\nearrow(\mathbb{R})$. □

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