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Sierpiński's topological characterization of $\mathbb{Q}$*

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In a 1920 paper [**5**] Wacław Sierpiński proved the following theorem characterizing the space $\mathbb{Q}$ of rational numbers considered with the standard topology:

ST. *Any countable metric space $\langle X, d\rangle$ without isolated points is homeomorphic to $\mathbb{Q}$.*

Its simple and natural form seems to indicate, that its proof could be included in an undergraduate topology curriculum as soon as a notion of homeomorphism is introduced. This result can help to illuminate the difference between the standard topologies on $\mathbb{R}$ and on $\mathbb{Q}$: according to ST, $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}^2$ and to $\mathbb{Q}_\ell$ (i.e., $\mathbb{Q}$ with the "Sorgenfrey topology," generated by all left closed intervals $[p, q)$); in contrast, their real counterparts $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}_\ell$—obtained by replacing $\mathbb{Q}$ with $\mathbb{R}$ in their respective definitions—are mutually topologically different. Also, ST implies that the Furstenberg topology on the integers $\mathbb{Z}$ used to prove the infinitude of primes (see [**4**] or [**1**]) is actually homeomorphic to the standard topology on $\mathbb{Q}$.

Nevertheless, so far ST could not have been included early in topological education for a simple reason—all proofs of ST published so were too complicated for such purpose. True, the proofs of ST presented in [**3**] and [**2**] are relatively simple; however, they both are considerably longer than our proof of ST provided below and are not self contained, since they both rely on Cantor's characterization of the linear structure of $\mathbb{Q}$: *Any linearly ordered dense set with neither smallest nor greatest element is order-isomorphic to* $(\mathbb{Q}, \leq)$.

Proof of ST. Let S be the set of all infinite sequences $s = \langle s_1, s_2, \dots\rangle$ of natural numbers that are eventually zero, that is, such that $0 = s_n = s_{n+1} = \cdots$ for some $n \in \mathbb{N} := \{1, 2, 3, \dots\}$. Notice that S is countable and so is the set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers. Consider S with a topology τ generated by a basis formed by all sets $[t]$, with $t \in \mathbb{N}^{<\omega}$, defined as

$$[t] := \{s \in S \colon t \subset s\},$$

where "$t \subset s$" means "the sequence s extends t."* To finish the proof it is enough to show that

$$\text{there is a homeomorphism } h \colon X \to S, \tag{1}$$

since then there exists also a homeomorphism $H \colon \mathbb{Q} \to S$ and so $H^{-1} \circ h \colon X \to \mathbb{Q}$ is a homeomorphism proving ST.

To see (1), let $\{x_n \colon n < \omega\}$ be an enumeration, with no repetitions, of the set X and let $D := \{d(x, y) \colon x, y \in X\}$ be the set of all distances between the elements in X. Notice that D is countable. Also, for any $r > 0$ with $r \notin D$, the open ball $B_d(c, r) := \{x \in X \colon d(c, x) < r\}$ centered in $c \in X$ and with radius r is also a closed set in X, since it is equal to $B_d[c, r] := \{x \in X \colon d(c, x) \leq r\}$. All open balls in X considered below will be with radiuses *not* in D. So, they will be clopen sets.

In the construction of h we will repeatedly use the following simple fact.

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*Of course, $\langle S, \tau\rangle$ is actually a subspace of $\mathbb{N}^{\mathbb{N}}$ considered with the product topology.

($*$) For every $k \in \mathbb{N}$ and nonempty open subset U of X there exists a sequence $S_k(U) := \langle B_i \colon i \in \mathbb{N}\rangle$ of pairwise disjoint clopen balls contained in U, each of radius $\leq 2^{-k}$, such that $U = \bigcup_{i\in\mathbb{N}} B_i$. Moreover, we assume that B_0 contains $x_{n(U)}$, where $n(U) := \min\{i < \omega \colon x_i \in U\}$.

The balls are chosen by induction on $i \in \mathbb{N}$: each B_i is centered at the point $x_{n(U_i)}$, where $U_i := U \setminus \bigcup_{j<i} B_j$, and has radius $r_i \in (0, 2^{-k}) \setminus D$ small enough so that $B_i \subsetneq U \setminus \bigcup_{j<i} B_j$. More specifically, each U_i is open, as a difference of open U and a finite union of balls, each of which is a closed set according to our choice of balls of radiuses not in D. Each nonempty U_{i-1}, including $U_0 = U$, has more than one point, since X has no isolated points. This allows us to choose each r_i small enough, so that $U_i = U_{i-1} \setminus B_i$ is nonempty, as long as $U_{i-1} \neq \emptyset$. Finally, each $x_k \in U$ belongs to $\bigcup_{j\leq i} B_j$, according to our rule of choosing the center of each B_i with the smallest possible available index.

Next, we construct the family $\{B_s \colon s \in \mathbb{N}^{<\omega}\}$ of nonempty clopen sets in X. The construction is by induction on the length of sequences s. Thus, for the sequence $\emptyset$ of length 0 we put $B_\emptyset := X$. Also, if $\mathbb{N}^k$ is the set of all sequences in $\mathbb{N}^{<\omega}$ of length k (possibly 0) and for every $s \in \mathbb{N}^k$ and $i \in \mathbb{N}$ the symbol $s\hat{\ }i$ denotes the sequence s extended by one more term with value i, then we define $\langle B_{s\hat{\ }i} \colon i \in \mathbb{N}\rangle := S_k(B_s)$.

Notice that

$$\text{for every } x \in X \text{ and } k \in \mathbb{N} \text{ there exists a unique } s \in \mathbb{N}^k \text{ so that } x \in B_s. \tag{2}$$

This is justified by an easy inductive argument. For $k = 1$ this holds, since the sets $\{B_{\emptyset\hat{\ }i} \colon i \in \mathbb{N}\}$ form a partition of $B_\emptyset = X$. Also, if $x \in B_s$ for some $s \in \mathbb{N}^k$, then there is precisely $t \in \mathbb{N}^{k+1}$, which must be of the form $t = s\hat{\ }i$, for which $x \in B_{s\hat{\ }i}$, as the sets $\{B_{s\hat{\ }i} \colon i \in \mathbb{N}\}$ form a partition of B_s.

Notice that, by (2), for every $x \in X$ there is a unique sequence $s = s_x \in \mathbb{N}^\mathbb{N}$ such that $x \in \bigcap_{k\in\mathbb{N}} B_{s\restriction k}$, where $s \restriction k$ is the restriction of s to its first k elements. Define $h \colon X \to \mathbb{N}^\mathbb{N}$ by letting $h(x) := s_x$ for every $x \in X$. This is a homeomorphism from (1).

Indeed, clearly h is one-to-one. To see that h is onto S, first notice that $h[X] \subset S$. Indeed, by ($*$), for every $x_j \in X$ we have $h(x_j) \restriction k = 0$ for every $k > n$, where n is such that $d(x_i, x_j) > 2^{-n}$ for every $i < j$. Thus, $h(x_j) \in S$, since $h(x_j)$ is eventually 0. Also, h is onto S, since for every $s \in S$ there exists a $k \in \mathbb{N}$ such that $s_m = 0$ for all $m \geq k$. Let $j = n(B_{s\restriction k})$. Then, by ($*$), $x_j \in B_{s\restriction m}$ for all $m \geq k$ and so $h(x_j) = s$.

Finally, we need to show that both h and h^{-1} are continuous. Indeed, h is continuous, since for every basic open set $[s]$ in S, $s \in \mathbb{N}^{<\omega}$, we have $h^{-1}([s]) = B_s$ is open in X. Also, h^{-1} is continuous, since $\{B_s \colon s \in \mathbb{N}^{<\omega}\}$ is a basis for X and, for every $s \in \mathbb{N}^{<\omega}$, $(h^{-1})^{-1}(B_s) = h(B_s) = [s]$ is open in S.

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Summary. In a 1920 paper Sierpiński proved the following theorem characterizing the space $\mathbb{Q}$ of rational numbers considered with the standard topology: *Any countable* *metric space* $\langle X, d\rangle$ *without isolated points is homeomorphic to* $\mathbb{Q}$. In this note we provide a simple proof of this result, that requires only basic topological background. As such, it can be incorporated into an undergraduate topology curriculum.

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