Fixed point theorems for maps with local and pointwise contraction properties

Krzysztof Chris Ciesielski and Jakub Jasiński*

Abstract

The paper constitutes a comprehensive study of ten classes of self-maps on metric spaces $\langle X, d\rangle$ with the local and pointwise (a.k.a. local radial) contraction properties. Each of those classes appeared previously in the literature in the context of fixed point theorems.

We begin with presenting an overview of these fixed point results, including concise self contained sketches of their proofs. Then, we proceed with a discussion of the relations among the ten classes of self-maps with domains $\langle X, d\rangle$ having various topological properties which often appear in the theory of fixed point theorems: completeness, compactness, (path) connectedness, rectifiable path connectedness, and d-convexity. The bulk of the results presented in this part consists of examples of maps that show non-reversibility of the previously established inclusions between theses classes. Among these examples, the most striking is a differentiable auto-homeomorphism f of a compact perfect subset X of $\mathbb{R}$ with $f' \equiv 0$, which constitutes also a minimal dynamical system. We finish with discussing a few remaining open problems on weather the maps with specific pointwise contraction properties must have the fixed points.

*Mathematics Subject Classification: Primary 54H25; Secondary 37C25.
Key words: Fixed point; Periodic point; Contractive maps; Locally contractive maps; Pointwise contractive maps; Radially contractive maps; Rectifiably path connected space; d-convex; Geodesics; Remetrization; Contraction mapping principle.

Contents

1 Background

The famous 1922 Banach Fixed Point Theorem states that every self-map f of a complete metric space $\langle X, d\rangle$ must have a *fixed point* (i.e., an $\xi \in X$ with $f(\xi) = \xi$) provided there exists a constant $\lambda \in [0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Maps like this are called *contractions*, and Banach's theorem is also known as the *Contraction Principle*.

The Contraction Principle has a multitude of generalizations, where the contraction assumption on f is weakened. Among them are two 1962 results of Edelstein that f must have a fix point provided: (i) X is compact and function f is *shrinking*, that is, $d(f(x), f(y)) < d(x, y)$ for all distinct $x, y \in X$; (ii) X is compact connected and f is *locally shrinking*, that is, when for every $x \in X$ there exists an $\varepsilon > 0$ such that f restricted to the *open ball* $B(x, \varepsilon)$, centered at x and of radius ε, is shrinking. In particular, (ii) implies that any

f on a compact connected space X must have fixed point provided f is *locally contractive*, that is, when for every $x \in X$ there exist an $\varepsilon > 0$ and a $\lambda \in [0,1)$ such that $d(f(y), f(z)) \leq \lambda d(y,z)$ for all $y, z \in B(x,\varepsilon)$. Yet another group of generalizations involves the functions f that are *pointwise contractive*, that is, such that for every $x \in X$ there exist an $\varepsilon > 0$ and a $\lambda \in [0,1)$ for which $d(f(x), f(y)) \leq \lambda d(x,y)$ as long as $y \in B(x,\varepsilon)$. Notice that this notion is closely related to that of a derivative, see Remark 2.3. Here we have the following results: (iii) 1978 theorem of Hu and Kirk, with proof corrected in 1982 by Jungck, that f must have a fixed point, provided X is rectifiably path connected and f is *uniformly pointwise contractive*, that is, pointwise contractive but such that λ is the same for all $x \in X$; and (iv) 2016 theorem of the authors of this paper, that f must have a fixed point, provided X is compact rectifiably path connected and f is pointwise contractive [11]. Also, another 2016 paper [10] of the authors gives a paradoxically-looking example providing a key insight into a possible behavior of the pointwise contractive maps.

The four local and pointwise notions of contractive and shrinking maps mentioned above, together with their several uniform versions, lead to twelve classes of mappings, of which only ten are distinct, precisely defined in Section 2. The goal of these paper is to fully discuss the fixed and periodic point theorems available for these mappings (see Section 3), as well as the inclusions among these classes of functions (see Section 4). In this work we restrict our attention to the mappings defined in Section 3. In particular, the multitude of other contraction-like notions that appear in literature (see e.g. 1977 paper [31] of Rhoades comparing 125 different global contraction-like conditions, most of which involve distances of the form $d(x, f(x))$, or the more recent work [23, 24, 5]) fall outside of the scope of presented material. Section 5 contains the remetrization results on which the generalizations of Fixed Point Theorem are based.

The relations between the considered classes of maps depend on the topological properties of the space $\langle X, d\rangle$ on which the maps act. We will restrict our attention to the topological properties that already appeared in the context of the fixed point theorems. These include: completeness, compactness, connectedness and path connectedness, rectifiable path connectedness, and, so called, d-convexity, which encompasses convexity in the Banach spaces. There are eight different topological classes that can be defined in terms of the aforementioned properties. In Section 6, using diagrams, we summarize the inclusions between the ten classes of maps we consider for the eight classes of topological spaces mentioned above. We reference examples showing that no implication between the classes exist, unless the diagrams force the implication. These examples are described in Section 7. All examples are with no periodic and/or fixed points, unless their existence is implied by an appropriate fix/periodic point theorem. The last section discusses the few remaining open problems.

We should mention that a large portion of fixed point theory (including locally contractive maps) is developed in metric spaces with additional algebraic structure, like Banach spaces, partially ordered sets, complete lattices, and many other. Such topics are not discussed in this paper and we refer interested readers to the monographs [19], and more recent [3], [7] and [24].

2 Dozen notions of contractive maps

In what follows, all self maps we consider are defined on the complete metric spaces, with the space usually denoted by X and the metric by d. However, the notions defined below are valid also for maps $f : X \to Y$ between arbitrary metric spaces X and Y.

Definition 2.1. Let X be a metric space and let $f : X \to X$. The following properties are also identified with the corresponding classes of functions.

Global notions:

(C) f is *contractive* (with a contraction constant λ), provided there exists a $\lambda \in [0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for every $x, y \in X$;

(S) f is *shrinking*, provided $d(f(x), f(y)) < d(x, y)$ for every distinct $x, y \in X$.

Local notions:

(LC) f is *locally contractive*, provided for every $y \in X$ there exists an open $U \ni y$ such that $f \restriction U$ is contractive;

(uLC) f is *locally contractive with the same contraction constant*, provided there exists a $\lambda \in [0, 1)$ such that for every $y \in X$ there exists an open $U \ni y$ for which $f \restriction U$ is contractive with the contraction constant λ; occasionally we will use an abbreviation (λ)-(uLC) when we like to stress that (uLC) is satisfied with a constant λ;

(ULC) f is *uniformly locally contractive*, provided there exist $\varepsilon > 0$ and $\lambda \in [0, 1)$ such that for every $y \in X$ the restriction $f \restriction B(y, \varepsilon)$ is contractive with a contraction constant λ; we will occasionally use an abbreviation (ε, λ)-(ULC) when we like to stress that (ULC) is satisfied with the constants ε and λ;

(LS) f is *locally shrinking*, provided for every $y \in X$ there exists an open $U \ni y$ such that $f \restriction U$ is shrinking;

(ULS) f is *uniformly locally shrinking*, provided there exists an $\varepsilon > 0$ such that $f \restriction B(y, \varepsilon)$ is shrinking for every $y \in X$; occasionally we will use notation (ε)-(ULS) to stress that (ULS) is satisfied with a radius ε;

Pointwise notions:

(PC) f is *pointwise contractive*, if for every $x \in X$ there exist an open $U \ni x$ and a $\lambda \in [0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $y \in U$;[1]

[1] The notions in this group are often named *local radial contractions*, see e.g, [20] or [21]. We feel that the term *pointwise contraction* better describes the nature of these functions, see [18] or [12].

(uPC) f is *pointwise contractive with the same contraction constant*, if there exists a $\lambda \in [0,1)$ such that for every $x \in X$ there is an open set $U \ni x$ for which $d(f(x), f(y)) \leq \lambda d(x,y)$ for all $y \in U$;

(UPC) f is *uniformly pointwise contractive*, if there exist a $\lambda \in [0,1)$ and an $\varepsilon > 0$ such that for every $x \in X$, $d(f(x), f(y)) \leq \lambda d(x,y)$ for all $y \in B(x,\varepsilon)$;

(PS) f is *pointwise shrinking*, if there for every $x \in X$ there exists open $U \ni x$ such that $d(f(x), f(y)) < d(x,y)$ for all $y \in U$;

(UPS) f is *uniformly pointwise shrinking*, if there exists an $\varepsilon > 0$ such that for every $x \in X$ we have $d(f(x), f(y)) < d(x,y)$ for all $y \in B(x,\varepsilon)$.

The obvious relations among the defined properties, plus those indicated by Remark 2.2, are shown in Figure 1. We have included notions (UPC) and (UPS) in Definition 2.1 for symmetry. However, as they are redundant, we will drop them from further considerations. (Compare also Figure 3.)

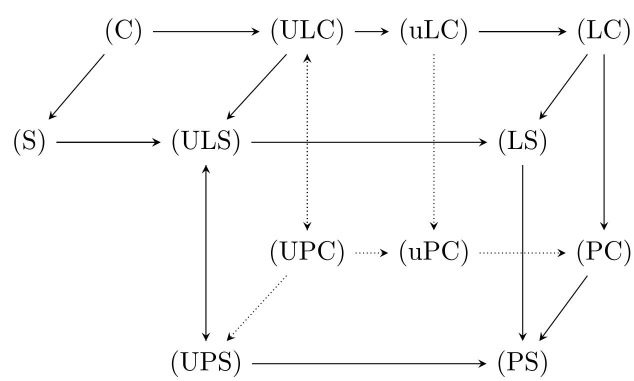

Figure 1: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a complete metric space. The upward arrows are justified by Remark 2.2. No other implications in the figure exist, see Theorem 6.1.

Figure 1, as well as the similar figures and the associated theorems for the maps defined on the spaces X with other topological properties, will be discussed in detail in Section 6.

Remark 2.2. (UPC) is equivalent to (ULC) and (UPS) is equivalent to (ULS).

Proof. Clearly (ULC) implies (UPC). Now assume that $f\colon X \to X$ satisfies (UPC) with some $\varepsilon > 0$ and $\lambda \in [0,1)$. Let $x \in X$ and suppose that $y, z \in B\left(x, \frac{\varepsilon}{2}\right)$. Then $d(y,z) \leq d(y,x) + d(x,z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ so $z \in B(y,\varepsilon)$. By the (UPC) property $d(f(y), f(z)) \leq \lambda d(y,z)$ which shows that f is (ULC) with the same λ and $\frac{\varepsilon}{2}$. The argument for (UPS) $\Longrightarrow$ (ULS) is similar. ∎

For a self-map f on $\langle X, d\rangle$ and a limit point $x \in X$, let

$$D^*f(x) = \limsup_{y\to x} \tfrac{d(f(x),f(y))}{d(x,y)},$$

and for isolated point x we set $D^*f(x) = 0$. In particular, if X is a subset of $\mathbb{R}$ (considered with the standard metric), X has no isolated points, and f is differentiable, then $D^*f(x) = |f'(x)|$. The (PC) and (uPC) properties can be expressed in terms of this notion as follows.

Remark 2.3. For every $f\colon X \to X$, the (uPC) property simply says that $\sup\{D^*f(x)\colon x \in X\} < 1$; (PC) is equivalent to $D^*f(x) < 1$ for all $x \in X$.

PROOF. (uPC) gives us a number $\lambda < 1$ such that $D^*f(x) \le \lambda$ for all limit $x \in X$ because we have $\frac{d(f(x),f(y))}{d(x,y)} \le \lambda$ for y sufficiently close but not equal to x. Inversely, if $\sup\{D^*f(x)\colon x \in X\} = \eta < 1$, then f is (uPC) with any λ such that $\eta < \lambda < 1$. The other equivalence follows similarly. ■

In the next two sections we review the fixed/periodic points theorems utilizing the above defined terminology and further discuss how the classes are related.

3 Fixed and periodic point theorems

For $f\colon X \to X$ and a number $n \in \omega = \{0, 1, 2, \ldots\}$, the n-th iteration $f^{(n)}$ of f is defined as $f \circ \cdots \circ f$, the composition of n instances of f. In particular, $f^{(1)} = f$ and $f^{(0)}$ is the identity function.

Nearly a century old theorem of Banach [4] states that

Theorem 3.1. (Banach 1922) *If X is a complete metric space and $f\colon X \to X$ is* (C)*, then f has a unique fixed point.*

PROOF. Fix an $x \in X$ and notice that $\langle f^{(n)}(x)\colon n < \omega\rangle$ is a Cauchy sequence, since the series formed by the distances $d\left(f^{(n)}(x), f^{(n+1)}(x)\right) \le \lambda^n d(x, f(x))$ is is convergent as it is bounded by the geometric series $\sum_{n=0}^{\infty} d(x, f(x))\lambda^n$, where $\lambda \in [0, 1)$ is a contraction constant for f. So, the sequence converges to a point $\xi \in X$, which is a fixed point, since

$$\begin{aligned} d(\xi, f(\xi)) &= \lim_{n\to\infty} d\big(f^{(n)}(x), f(f^{(n)}(x))\big) \\ &= \lim_{n\to\infty} d\big(f^{(n)}(x), f^{(n+1)}(x)\big) = d(\xi, \xi) = 0, \end{aligned}$$

implying that $f(\xi) = \xi$. Property (C) also implies the uniqueness of ξ. ■

This theorem, often called the Banach Contraction Principle, was studied in great detail, see for example [31, 23, 24]. Here we focus solely on the fixed and periodic point theorems for the mappings f, with properties from Definition 2.1.

An $x \in X$ is a *periodic point of* $f\colon X \to X$ provided $f^{(n)}(x) = x$ for some $n > 0$. In particular, $x \in X$ is a fixed point of f if, and only if, it is a periodic point of f with period 1, that is, $f^{(1)}(x) = x$. The (forward) *orbit* of an $x \in X$ is defined as $O(x) = \{f^{(n)}(x) \colon n < \omega\}$. Very significant contributions in the study of shrinking and locally shrinking maps are due to Edelstein [16], see also [15].

Theorem 3.2. (Edelstein 1962) *Let $\langle X, d\rangle$ be compact and let $f \colon X \to X$.*

(i) If f is (S)*, then f has a unique fixed point.*

(ii) If f is (LS)*, then f has a periodic point.*

(iii) If f is (LS) *and X is connected, then f has a unique fixed point.*

Proof. In case (i), notice that $X \ni x \mapsto d(x, f(x)) \in \mathbb{R}$ is a continuous mapping on a compact space. Thus, it attains its minimum at some $x \in X$, which, by (S), must be a fixed point. This fixed point is clearly unique.

In case (ii), first notice (see Theorem 4.2) that f is actually (ULS) with some constant $\varepsilon > 0$. Fix an $x_0 \in X$ and notice that there exist $i < j < \omega$ such that $d(f^{(i)}(x_0), f^{(j)}(x_0)) < \varepsilon$. Put $n = j - i > 0$ and notice that $\hat{f} = f^{(n)}$ still satisfies (ULS) with the constant ε.

Fix a $\xi \in X$ at which the mapping $X \ni x \mapsto d(x, \hat{f}(x)) \in \mathbb{R}$ achieves the minimum. So, $d(\xi, \hat{f}(\xi)) \leq d(f^{(i)}(x_0), \hat{f}(f^{(i)}(x_0))) = d(f^{(i)}(x_0), f^{(j)}(x_0)) < \varepsilon$ and we must have $d(\xi, \hat{f}(\xi)) = 0$, since otherwise $d(\hat{f}(\xi), \hat{f}(\hat{f}(\xi))) < d(\xi, \hat{f}(\xi))$, contradicting the choice of ξ. Hence, $f^{(n)}(\xi) = \hat{f}(\xi) = \xi$, that is, that ξ is a periodic point of f.

In case (iii), notice that, by Proposition 5.2(ii), there exists a complete metric D on X topologically equivalent to d, such that f is (S) with respect to this metric. Thus, by (i), f has a unique fixed point. ■

It is worth noting that in [14, theorem 2.6 p. 796] Ding and Nadler generalize items (ii) and (iii) of Theorem 3.2 to locally compact spaces X. Theorem 3.2 is also discussed in [13].

To state our next theorem, of Hu and Kirk [21] with a proof corrected by Jungck [22] (see discussion in [2, p 66]), we need the following definitions. A metric space X is *rectifiably path connected* provided any two points $x, y \in X$ can be connected in X by a path $p \colon [0, 1] \to X$ of finite length $\ell(p)$, that is, by a continuous map p satisfying $p(0) = x$ and $p(1) = y$, and having a finite *length* $\ell(p)$ defined as the supremum over all numbers:

$$\sum_{i=1}^{n} d(p(t_i), p(t_{i-1})) \text{ with } 0 < n < \omega \text{ and } 0 = t_0 < t_1 < \cdots < t_n = 1.$$

Theorem 3.3. (Hu and Kirk 1978; Jungck 1982) *If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f \colon X \to X$ is* (uPC)*, then f has a unique fixed point.*

Proof. The assumptions on $\langle X, d\rangle$ and f imply (see Proposition 5.5(iii)) that there exists a complete metric D on X such that f is (C), when X is considered with the metric D. So, by Theorem 3.1, f has a unique fixed point, see also [2, theorem 6 p 66]. ■

It is worth noting that Rakotch, in 1962 paper [30], and Marjanović, in 1976 paper [25], proved earlier Theorem 3.3 under the stronger assumption that f is (uLC). The next theorem is very recent and comes from authors paper [11]. It generalizes Theorem 3.3 to the (PC) maps, no uniformity assumption, at the expense of requiring that the domain of f is compact.

Theorem 3.4. (Ciesielski and Jasinski 2016) *Assume that $\langle X, d\rangle$ is compact rectifiably path connected metric space. If $f: X \to X$ is* (PC)*, then f has a unique fixed point.*

Proof. Let D is the distance from Proposition 5.5. By part (iv) of the proposition, $f: \langle X, D\rangle \to \langle X, D\rangle$ is (S). Let $M = \inf\{D(x, f(x)) \colon x \in X\}$. Then, by Corollary 5.4(ii), there exists an $\bar{x} \in X$ such that $D(\bar{x}, f(\bar{x})) = M$ (which is not completely obvious, since $\langle X, D\rangle$ need not be compact, see the footnote to Proposition 5.5).

To finish the proof it is enough to notice that M must be equal 0, since otherwise $D(f(\bar{x}), f(f(\bar{x}))) < D(\bar{x}, f(\bar{x}))$, contradicting minimality of M. Thus, $D(\bar{x}, f(\bar{x})) = 0$ and $f(\bar{x}) = \bar{x}$, as required. The uniqueness of the fixed point is ensured by the fact that $f: \langle X, D\rangle \to \langle X, D\rangle$ is (S). ■

The main part of the following theorem has been proved in 1961 paper [15] of Edelstein. The proof the entire theorem can be also found in the authors paper [11]. (We missed this 1961 result of Edelstein when writing [11].)

Theorem 3.5. *Assume that $\langle X, d\rangle$ is complete and that $f: X \to X$ is* (ULC)*.*

- *(i)* **(Edelstein 1961)** *If X is connected, then f has a unique fixed point.*
- *(ii)* **(Ciesielski and Jasinski 2016)** *If X has a finite number of components, then f has a periodic point.*

Proof. To see (i), let $\varepsilon > 0$ and $\lambda \in [0, 1)$ be such that f (ε, λ)-(ULC). By Remark 5.1 and Proposition 5.2(i), there exists a metric $\hat{D}$ on X topologically equivalent to d such that $f: \langle X, \hat{D}\rangle \to \langle X, \hat{D}\rangle$ is (C) with constant λ. Hence, by the Banach Contraction Principle, f has a unique fixed point.

To see (ii), let $C_1, \ldots, C_m$ be the connected components of X. Since $f^{(n)}[C_1]$ is connected, there must exist $i < i+k$ with $f^{(i)}[C_1]$ and $f^{(i+k)}[C_1]$ intersecting the same component of X, call it C. Then $f^{(k)}[C] \subset C$. Applying (i) to $f^{(k)} \restriction C \colon C \to C$, we can find an $x \in C$ with $f^{(k)}(x) = x$. So, x is a periodic point of f. ■

Notice that Theorems 3.3 and 3.5 are reduced to the Banach Contraction Principle by using appropriate (but different) remetrization results. Similar (but different) connections between the Theorem 3.2 and the Banach Contraction Principle were also discussed in 1975 paper [32] of Rosenholtz.

4 Implications between the contractive notions

Following Jungck [22], we say that $\langle X, d\rangle$ is *d-convex* provided for any distinct points $x, y \in X$ there exists a path $p\colon [0,1] \to X$ from x to y such that $d(p(t_1), p(t_3)) = d(p(t_1), p(t_2)) + d(p(t_2), p(t_3))$ whenever $0 \le t_1 < t_2 < t_3 \le 1$. Clearly, every d-convex space is rectifiably path connect. On the other hand, any convex subset of a Banach space is d-convex. In particular, such is any interval considered with the standard distance.

The part (i) of the next theorem is a particular case of [22, theorem p. 503] of Jungck.

Theorem 4.1. *Let $\langle X, d\rangle$ be d-convex and let $f\colon X \to X$.*

(i) **(Jungck 1982)** *If f is* (uPC) *with a constant λ, then it is* (C) *with the same constant.*

(ii) If f is (PS)*, then it is* (S).

Proof. First notice, that for every distinct $y, z \in X$

($\bullet$) if $L = \frac{d(f(y),f(z)}{d(y,z)}$, then there exist $x \in X$ and a sequence $\langle x_n \neq x\colon n < \omega\rangle$ in X converging to x such that $\frac{d(f(x),f(x_n)}{d(x,x_n)} \ge L$ for all $n < \omega$.

Indeed, let $p\colon [0,1] \to X$ be a path from y to z from the definition of d-convexity. Define a nested sequence $\langle [s_n, t_n]\colon n < \omega\rangle$ of intervals in $[0,1]$ such that, for every $n < \omega$, $[s_n, t_n]$ has length 2^{-n} and $\frac{d(f(p(s_n)),f(p(t_n))}{d(p(s_n),p(t_n))} \ge L$: we start with $[s_0, t_0] = [0,1]$ and, having $[s_n, t_n]$, at lest one of the halfs of $[s_n, t_n]$ can be chosen as $[s_{n+1}, t_{n+1}]$. Let $\{t\} = \bigcap_{n<\omega}[s_n, t_n]$. Then $x = p(t)$ is as desired, since for every $n < \omega$, there is $u_n \in \{s_n, t_n\}$ for which $x_n = p(u_n) \neq x$ and satisfies $\frac{d(f(x),f(x_n))}{d(x,x_n)} \ge L$.

Now, to see (i), notice that if $L = \frac{d(f(y),f(z)}{d(y,z)}$ for some distinct $y, z \in X$ then, by ($\bullet$), $L \le D^*f(x) \le \lambda$.

To see (ii), assume that f is not (S). Then there exist distinct $y, z \in X$ with $L = \frac{d(f(y),f(z)}{d(y,z)} \ge 1$. Let x be as in ($\bullet$) for this pair. Then f is not ((PS)) at x. ∎

For the compact spaces we have the following implications. (See [11, proposition 4.3]. Compare also [14, theorem 4.2].)

Theorem 4.2. (LC)$\Longrightarrow$(ULC) *and* (LS)$\Longrightarrow$(ULS) *for maps $f\colon X \to X$ with compact X.*

Proof. Suppose X is compact. To see that (LC) implies (ULC), for each $y \in X$ find an open set $U_y \ni y$ such that $f \restriction U_y$ is Lipschitz with a constant $\lambda_y \in [0,1)$. By compactness of X, there is a finite $X_0 \subset X$ such that $\mathcal{U}_0 = \{U_y\colon y \in X_0\}$ covers X. Let $\delta > 0$ be a Lebesgue number for the cover $\mathcal{U}_0$ of X. (See e.g. [27, lemma 27.5].) Then $\varepsilon = \delta/2$ satisfies (ULC) with the contraction constant $\lambda = \max\{\lambda_y\colon y \in X_0\}$.

The argument for (LS)$\Longrightarrow$(ULS), when X is compact, is similar. ■

The next result seems to be never published before.

Theorem 4.3. (S)&(ULC) $\Rightarrow$ (C) *for maps $f\colon X \to X$ with compact X.*

Proof. Let $\varepsilon > 0$ and $\lambda \in [0,1)$ be such that $d(f(x), f(y)) \le \lambda d(x,y)$ for any $x, y \in X$ with $d(x,y) < \varepsilon$. Let $Z = \{\langle x,y\rangle \in X^2 \colon d(x,y) \ge \varepsilon\}$ and define g on Z by $g(x,y) = \left|\frac{d(f(x),f(y))}{d(x,y)}\right|$. Since Z is compact, g attains its maximum value λ_1 on Z. We must have $\lambda_1 < 1$, since f is (S). Thus, f is (C) with a contraction constant $\max\{\lambda, \lambda_1\} < 1$. ■

5 Geodesics and remetrization results

For $\varepsilon > 0$, we say that X is *ε-chainable*, provided for every $p, q \in X$ there exists a finite sequence $s = \langle x_0, x_1, \ldots, x_n\rangle$, referred to as an *$\varepsilon$-chain from p to q*, such that $x_0 = p$, $x_n = q$, and $d(x_i, x_{i+1}) \le \varepsilon$ for every $i < n$. The length of the ε-chain s is defined as $\mathfrak{l}(s) = \sum_{i<n} d(x_{i+1}, x_i)$.

Remark 5.1. All connected spaces are ε-chainable for any $\varepsilon > 0$.

Proof. (See Engelking [17, Exercise 6.1.D(a) p 359]) Fix $x, y \in X$ and $\varepsilon > 0$. Define, by induction on $n < \omega$, a sequence $\langle B_n \subset X \colon n < \omega\rangle$ as $B_0 = \{x\}$ and $B_{n+1} = \{z \in X \colon \exists b \in B_n (d(z,b) < \varepsilon)\}$. The union $\bigcup_{n<\omega} B_n \neq \emptyset$ is a clopen, so by, connectedness of the space X, we have $\bigcup_{n<\omega} B_n = X$. Thus, $y \in B_n$ for some $n < \omega$ and so, there exists an ε-chain, with $n+1$ terms, from x to y. ■

The part (ii) of Proposition 5.2 can be found (with slightly different proof) in Rosenholtz [32]. The proposition resembles also the results of Jungck from [22] and of Hu and Kirk from [21]. See also [2, Lemma 2 p. 70].

Proposition 5.2. *Let $\varepsilon > 0$ and assume that $\langle X, d\rangle$ is connected or, more generally, ε-chainable. Then the map $\hat{D}\colon X^2 \to [0,\infty)$ given as*

$$\hat{D}(x,y) = \inf\{\mathfrak{l}(s) \colon s \text{ is an } \varepsilon\text{-chain from } x \text{ to } y\}$$

is a metric on X topologically equivalent to d. If $\langle X, d\rangle$ is complete, then so is $\langle X, \hat{D}\rangle$. Moreover,

(i) If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (η, λ)-(ULC) for some $\eta > \varepsilon$, then $f\colon \langle X, \hat{D}\rangle \to \langle X, \hat{D}\rangle$ is (C) with constant λ.

(ii) If $\langle X, d\rangle$ is compact and $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (ULS) with a constant $\eta > \varepsilon$, then $f\colon \langle X, \hat{D}\rangle \to \langle X, \hat{D}\rangle$ is (S).

Proof. To see that $\hat{D}$ is a metric on X it is enough to show that $\hat{D}$ satisfies the triangle inequality. So, fix $x, y, z \in X$ and $\delta > 0$. Then, there exist the ε-chains $s = \langle x_0, \ldots, x_n\rangle$ from x to y and $t = \langle y_0, \ldots, y_m\rangle$ from y to z with

$\hat{D}(x,y) \geq \mathfrak{l}(s) - \delta$ and $\hat{D}(y,z) \geq \mathfrak{l}(t) - \delta$. Since $u = \langle x_0, \dots, x_n, y_0, \dots, y_m \rangle$ is an ε-chain from x to z with $\mathfrak{l}(u) = \mathfrak{l}(s) + \mathfrak{l}(t)$, we have

$$\hat{D}(x,y) + \hat{D}(y,z) \geq \mathfrak{l}(s) - \delta + \mathfrak{l}(t) - \delta = \mathfrak{l}(u) - 2\delta \geq \hat{D}(x,z) - 2\delta.$$

Since the constant $\delta > 0$ was arbitrary, we obtain the desired triangle inequality $\hat{D}(x,y) + \hat{D}(y,z) \geq \hat{D}(x,z)$.

Also if $d(x,y) \leq \varepsilon$, then we have $\hat{D}(x,y) = d(x,y)$ (since then $d(x,y) \leq \hat{D}(x,y) \leq \mathfrak{l}(\langle x,y \rangle) = d(x,y)$). This implies topological equivalence and completeness statements, finishing the proof of the main part of the proposition.

To see (i), fix $x, y \in X$. We need to show that $\hat{D}(f(x), f(y)) \leq \lambda \hat{D}(x,y)$. For this, fix a $\delta > 0$ and let $s = \langle x_0, \dots, x_n \rangle$ be an ε-chain from x to y with $\hat{D}(x,y) \geq \mathfrak{l}(s) - \delta$. Notice that, by (η,λ)-(ULC), for every $i < n$ we have $d(f(x_{i+1}), f(x_i)) \leq \lambda d(x_{i+1}, x_i)$. In particular, $t = \langle f(x_0), \dots, f(x_n) \rangle$ is an ε-chain and $\mathfrak{l}(t) = \sum_{i<n} d(f(x_{i+1}), f(x_i)) \leq \sum_{i<n} \lambda d(x_{i+1}, x_i) = \lambda \mathfrak{l}(s)$. Hence, $\hat{D}(f(x), f(y)) \leq \mathfrak{l}(t) \leq \lambda \mathfrak{l}(s) \leq \lambda(\hat{D}(x,y) + \delta)$. Since $\delta > 0$ was arbitrary, we obtain the desired inequality $\hat{D}(f(x), f(y)) \leq \lambda \hat{D}(x,y)$.

To see (ii), choose distinct $x, y \in X$. We need to show that $\hat{D}(f(x), f(y)) < \hat{D}(x,y)$. So, let $\{U_k \colon k < n\}$ be a cover of X by open sets of d-diameter less than ε. Notice that if $s = \langle x_0, x_1, \dots, x_m \rangle$ is an ε-chain from p to q and $i < j \leq m$ are such that x_i and x_j belong to the same U_k, then $t = \langle x_0, \dots, x_i, x_j, \dots, x_m \rangle$ is also an ε-chain from p to q for which $\mathfrak{l}(t) \leq \mathfrak{l}(s)$. In particular, for any ε-chain s from p to q there exists an ε-chain t from p to q such that any U_k contains at most two of the terms in t. In particular,

$$\hat{D}(x,y) = \inf\{\mathfrak{l}(s) \colon s \text{ is an } \varepsilon\text{-chain from } x \text{ to } y \text{ containing } 2n \text{ terms}\}.$$

In other words, if $Z \subset X^{2n}$ is the set of all ε-chains $\langle x_0, \dots, x_{2n-1} \rangle$ from x to y, then Z is compact (as a closed subset of X^{2n}) and $\hat{D}(x,y) = \inf\{\mathfrak{l}(s) \colon s \in Z\}$. Therefore, the Extreme Value Theorem implies that there exists an ε-chain $s = \langle x_0, \dots, x_{2n-1} \rangle$ from x to y with $\hat{D}(x,y) = \mathfrak{l}(s)$. To finish the proof, it is enough to notice that, by the (η)-(ULS) assumption, $\langle f(x_0), \dots, f(x_{2n-1}) \rangle$ is an ε-chain from $f(x)$ to $f(y)$ and so

$$\hat{D}(f(x), f(y)) \leq \sum_{i<2n-1} d(f(x_{i+1}), f(x_i)) < \sum_{i<2n-1} d(x_{i+1}, x_i) = \mathfrak{l}(s) = \hat{D}(x,y)$$

as needed. ∎

In what follows we will use the following 1945 result of Myers [29, page 219]. For reader's convenience, we include its short self-contained proof.

Lemma 5.3. *Let $\langle X, d \rangle$ be a compact metric space and, for any $n < \omega$, let $p_n \colon [0,1] \to X$ be a rectifiable path such that $\ell(p_n \restriction [0,t]) = t\ell(p_n)$ for any $t \in [0,1]$. If $L = \liminf_{n\to\infty} \ell(p_n) < \infty$, then there exists a subsequence $\langle p_{n_k} \colon k < \omega \rangle$ converging uniformly to a rectifiable path $p \colon [0,1] \to X$ with $\ell(p) \leq L$.*

Proof. Select a countable dense subset $U = \{u_m : m < \omega\}$ of $[0,1]$. By compactness of X, it is possible to find a subsequence $\langle p_{n_k} : k < \omega\rangle$ with $\ell(p_{n_k}) \to_{k\to\infty} L$ such that $\lim_{k\to\infty} p_{n_k}(u_m) = p(u_m)$ for all $m < \omega$. Then, maps $\{p_{n_k} : k < \omega\}$ converge uniformly to continuous $p\colon [0,1] \to X$ with $\ell(p) \le L$.

Indeed, let $m \in \{n_k : k < \omega\}$ be such that $\ell(p_{n_k}) \le \ell(p_m)$ for all $k < \omega$ and notice that for every $0 \le s \le t \le 1$ we have

$$d(p_{n_k}(s), p_{n_k}(t)) \le \ell(p_{n_k} \restriction [s,t]) = (t-s)\ell(p_{n_k}) \le (t-s)\ell(p_m).$$

To see that the maps p_{n_k} form uniformly converging sequence, choose an $\varepsilon > 0$. It is enough to show that there exists an N such that

$$d(p_{n_j}(s), p_{n_k}(s)) < \varepsilon \text{ for all } s \in [0,1] \text{ and } j,k > N.$$

So, let $\mathcal{J}$ be a finite cover of $[0,1]$ by open intervals each of length not exceeding $\delta = \frac{\varepsilon}{4\ell(p_m)}$. For every $J \in \mathcal{J}$ choose a $u \in U \cap J$ and an N_J such that $d(p_{n_k}(u), p(u)) < \varepsilon/4$ for all $k > N_J$. Then, for every $s \in J$ and $j,k > N_J$ we have $d(p_{n_k}(s), p(u)) \le d(p_{n_k}(s), p_{n_k}(u)) + d(p_{n_k}(u), p(u)) < |s-u|\ell(p_m) + \frac{\varepsilon}{4} \le \delta\ell(p_m) + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$ and $d(p_{n_k}(s), p_{n_j}(s)) \le d(p_{n_k}(s), p(u)) + d(p(u), p_{n_j}(s)) < \varepsilon$. Hence, p_{n_k}s converge uniformly to a continuous path p.

To see that $\ell(p) \le L$ notice that for every $0 = t_0 < t_1 < \cdots < t_n = 1$

$$\begin{array}{rcl} \sum_{i=1}^n d(p(t_i), p(t_{i-1})) & = & \lim_{k\to\infty} \sum_{i=1}^n d(p_{n_k}(t_i), p_{n_k}(t_{i-1})) \\ & \le & \lim_{k\to\infty} \sum_{i=1}^n (t_i - t_{i-1})\ell(p_m) = \ell(p_m), \end{array}$$

so that $\ell(p) \le \ell(p_m)$.

Finally, since for every $k_0 < \omega$, by removing from a sequence $\langle p_{n_k} : k < \omega\rangle$ a finite number of elements we can ensure that that $\ell(p_m) = \ell(p_{n_{k_0}})$, we have that $\ell(p) \le \ell(p_{n_k})$ for every $k < \omega$, that is, $\ell(p) \le \lim_{k\to\infty} \ell(p_{n_k}) = L$. ■

Recall that a rectifiable path $p\colon [0,1] \to X$ from x to y is a *geodesic* provided $\ell(p) \le \ell(q)$ for any other path $q\colon [0,1] \to X$ from x to y. From Lemma 5.3 it is easy to deduce the following corollary, the first part of which is a 1930 theorem of Menger [26].

Corollary 5.4. *Let $\langle X, d\rangle$ be compact metric space.*

(i) **(Menger 1930)** *If there is a rectifiable path in X from x to y, then there is a geodesic in X from x to y.*

(ii) If $\langle X, d\rangle$ is compact rectifiably path connected, then for every continuous $f\colon X \to X$ there exist an $\bar{x} \in X$ and a path p from $\bar{x}$ to $f(\bar{x})$ such that $\ell(p) \le \ell(q)$ for any path q from any $x \in X$ to $f(x)$.

Proof. First notice that any rectifiable path $p\colon [0,1] \to X$ admits a reparametrization $\bar{p}\colon [0,1] \to X$ (i.e., a path with the same range and same length) satisfying the condition from Lemma 5.3: $\ell(\bar{p} \restriction [0,t]) = t\ell(\bar{p})$ for any $t \in [0,1]$. Indeed, the map $\bar{p} = \left\{\left\langle \frac{\ell(p\restriction[0,t])}{\ell(p)}, p(t)\right\rangle : t \in [0,1]\right\}$ is as required, since for any $s = \frac{\ell(p\restriction[0,t])}{\ell(p)} \in [0,1]$, we have $\ell(\bar{p} \restriction [0,s]) = \ell(p \restriction [0,t]) = s\ell(p) = s\ell(\bar{p})$.

To see (i), assume that $x, y \in X$ can be joint by a rectifiable path. Let L be the infimum of the lengths of all such paths and choose rectifiable paths $p_n\colon [0,1] \to X$ from x to y such that $\lim_{n\to\infty} \ell(p_n) = L$. Application of Lemma 5.3 to the sequence $\langle \bar{p}_n\colon n < \omega\rangle$ gives a path $p\colon [0,1] \to X$ from x to y with $\ell(p) = L$.

To see (ii), let $L = \inf\{\ell(q)\colon\ q \text{ is a path from } x \in X \text{ to } f(x)\}$. Then, there exists paths $p_n\colon [0,1] \to X$ from x_n to $f(x_n)$ such that $\lim_{n\to\infty} \ell(p_n) = L$. Application of Lemma 5.3 to $\langle \bar{p}_n\colon n < \omega\rangle$ gives subsequence $\langle \bar{p}_{n_k}\colon k < \omega\rangle$ converging uniformly to a rectifiable path $p\colon [0,1] \to X$ with $\ell(p) \le L$. If $\bar{x} = p(0) = \lim_{k\to\infty} p_{n_k}(0)$, then p is from $\bar{x} = p(0)$ to $p(1) = \lim_{k\to\infty} p_{n_k}(1) = \lim_{k\to\infty} f(p_{n_k}(0)) = f(\bar{x})$. So, $\bar{x}$ and p are as needed. ■

The following proposition is an elaboration of the results from [21] and [22], see also [2, Theorem 6 p. 66].

Proposition 5.5. *If $\langle X, d\rangle$ is a rectifiably path connected metric space, then the map $D\colon X^2 \to [0,\infty)$ given as*

$$D(x,y) = \inf\{\ell(p)\colon p \text{ is a rectifiable path from } x \text{ to } y\}$$

is a metric on X. If $\langle X, d\rangle$ is complete, then so is $\langle X, D\rangle$.[2]

(i) If P is the range of a rectifiable path p in X, $\lambda \ge 0$, and for every $x \in X$, $D^* f(x) \le \lambda$ with respect to the metric d, then $\ell(f \circ p) \le \lambda \ell(p)$.

(ii) If $\lambda \ge 0$ and, for every $x \in X$, $D^* f(x) \le \lambda$ with respect to the metric d, then $f\colon \langle X, D\rangle \to \langle X, D\rangle$ is Lipschitz with the constant λ. In particular, if $0 \le \lambda < 1$ and $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (λ)-(uPC), *then $f\colon \langle X, D\rangle \to \langle X, D\rangle$ is* (C) *with the contraction constant λ.*

(iii) If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (LC), *then $f\colon \langle X, D\rangle \to \langle X, D\rangle$ is* (S).

(iv) If X is compact and $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PC), *then $f\colon \langle X, D\rangle \to \langle X, D\rangle$ is* (S).

Proof. The main part is straightforward and seems to be well known. For a proof see, for example, Hu and Kirk [21] or Mycielski [28].

To show (i), fix an $\varepsilon > 0$. First notice that

$$d(f(p(t)), f(p(s))) \le (\lambda + \varepsilon)\ell(p \restriction [s,t]) \text{ for every } 0 \le s < t \le 1. \tag{1}$$

Indeed, for every $x \in [s,t]$ we have $D^*(f \restriction P)(x) \le \lambda$, so there exists a proper open interval $U_x = (x - \delta_x, x + \delta_x)$ such that

$$d(f(p(x)), f(p(x'))) \le (\lambda + \varepsilon) d(p(x), p(x')) \text{ for every } x' \in U_x \cap [s,t]. \tag{2}$$

[2]Notice that the metrics d and D do not need to be topologically equivalent. For example, let X be union of the "topologist's sine curve" (see Munkres [27, p. 156]) and a semi-circular curve connecting one end of the vertical segment with the "end" of the sine curve. If d is the standard metric on $\mathbb{R}^2$, then $\langle X, d\rangle$ is compact rectifiably path connected, while $\langle X, D\rangle$ is not compact – it is homeomorphic to $[0,\infty)$.

Let J be a finite subset of $[s,t]$ such that $\mathcal{U}=\{U_x\colon x\in J\}$ is a cover of $[s,t]$ containing no proper subcover. Let $\langle x_1,x_3,\ldots,x_{2n-1}\rangle$ be a list of elements of J in the increasing order. Then, by minimality of $\mathcal{U}$, for every $0<i<n$ there exists an $x_{2i}\in U_{x_{2i-1}}\cap U_{x_{2i+1}}\cap(x_{2i-1},x_{2i+1})$. Moreover, $x_0=s\in U_{x_1}$ and $x_{2n}=t\in U_{x_{2n-1}}$. In particular, $s=x_0\le x_1<x_2<\cdots<x_{2n-1}\le x_{2n}=t$ and $x_{2i},x_{2i+2}\in U_{x_{2i+1}}$ for every $i<n$. Therefore, by (2),

$$\begin{array}{rcl} d(f(p(t)),f(p(s))) & \le & \sum_{k<2n} d(f(p(x_k)),f(p(x_{k+1}))) \\ & \le & \sum_{k<2n}(\lambda+\varepsilon)d(p(x_k),p(x_{k+1}))\le(\lambda+\varepsilon)\ell(p\restriction[s,t]), \end{array}$$

justifying (1).

To finish the argument for (i) choose the numbers $0=t_0<t_1<\cdots<t_n=1$ such that $\ell(f\circ p)\le\sum_{i<n} d(f(p(t_{i+1})),f(p(t_i)))+\varepsilon$. Then, by (1),

$$\begin{array}{rcl} \ell(f\circ p) & \le & \sum_{i<n} d(f(p(t_{i+1})),f(p(t_i)))+\varepsilon \\ & \le & \sum_{i<n}(\lambda+\varepsilon)\ell(p\restriction[t_{i-1},t_i])+\varepsilon=(\lambda+\varepsilon)\ell(p)+\varepsilon. \end{array}$$

Since this holds with any $\varepsilon>0$, the desired inequality, $\ell(f\circ p)\le\lambda\ell(p)$, follows.

Item (ii) follows from (i). Indeed, for every $a,b\in X$ and $\varepsilon>0$ there is a rectifiable path p from a to b such that $\ell(p)<D(a,b)+\varepsilon$. Then, by (i),

$$\begin{array}{rcl} D(f(a),f(b)) & = & \inf\{\ell(q)\colon q \text{ is a rectifiable path from } f(a) \text{ to } f(b)\} \\ & \le & \ell(f\circ p)\le\lambda\ell(p)\le\lambda(D(a,b)+\varepsilon). \end{array}$$

Since ε was arbitrary, we get $D(f(a),f(b))\le\lambda D(a,b)$ for every $a,b\in X$.

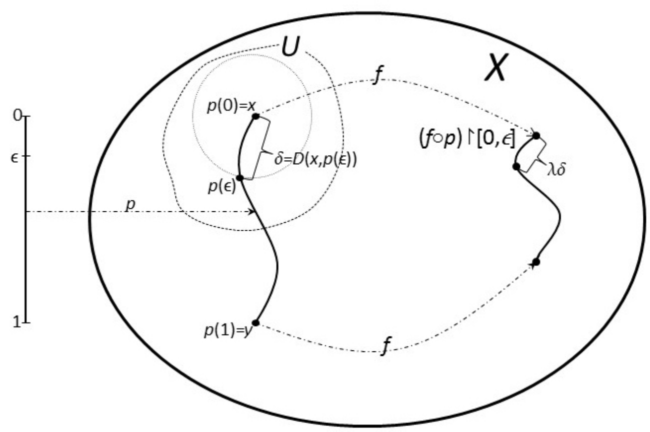

Figure 2: In a rectifiably path connected space X, if f is (LC) with metric d, then f is (S) in metric $D=\inf\{l(p)\colon P$ is a rectifiable path from x to $y\}$.

To prove the property (iii), take distinct $x,y\in X$. We need to show that $D(f(x),f(y))<D(x,y)$. Notice, that, by (ii), $f\colon\langle X,D\rangle\to\langle X,D\rangle$ is Lipschitz with the constant 1. Also, by (LC), there exists an open $U\ni x$ such that $f\restriction U$ is d-contractive with a constant $\lambda\in[0,1)$.

Choose a $\delta \in (0, D(x,y))$ such that $z \in U$ whenever $d(x,z) \leq \delta$, a rectifiable path $p\colon [0,1] \to X$ from x to y with $\ell(p) < D(x,y) + (1-\lambda)\delta$, and pick the smallest $\varepsilon \in (0,1)$ with $D(x, p(\varepsilon)) = \delta$ see Figure 2. Then, $p(t) \in U$ for every $t \in [0,\varepsilon]$, since, for such t, $d(x,p(t)) \leq D(x,p(t)) \leq D(x,p(\varepsilon)) = \delta$. Therefore, by (ii), $\ell(f \circ p \restriction [0,\varepsilon]) \leq \lambda \ell(p \restriction [0,\varepsilon])$. Hence

$$\begin{aligned}
D(f(x),f(y)) &\leq D(f(x), f(p(\varepsilon))) + D(f(p(\varepsilon)), f(y)) \\
&\leq \ell(f \circ p \restriction [0,\varepsilon]) + D(p(\varepsilon), y) \\
&\leq \lambda \ell(p \restriction [0,\varepsilon]) + \ell(p \restriction [\varepsilon, 1]) \\
&= -(1-\lambda)\ell(p \restriction [0,\varepsilon]) + \ell(p) \\
&\leq -(1-\lambda) D(x, p(\varepsilon)) + \ell(p) \\
&= -(1-\lambda)\delta + \ell(p) < D(x,y),
\end{aligned}$$

as required.

To see (iv), fix distinct $x, y \in X$. We need to show $D(f(x), f(y)) < D(x,y)$. By Corollary 5.4(i), there exists a path $p\colon [0,1] \to X$ from x to y with $D(x,y) = \ell(p)$. Since $\ell(f \circ p) \geq D(f(x), f(y))$, it is enough to prove that $\ell(p) > \ell(f \circ p)$.

To see this, let $Y = p[[0,1]]$. It is easy to see that for every $n < \omega$ the set

$$K_n = \{x \in Y\colon d(f(x), f(x')) \leq \tfrac{n}{n+1} d(x,x') \text{ for all } x' \in Y \text{ with } d(x,x') < \tfrac{1}{n+1}\}$$

is closed in Y. Since f is (PC), we have $Y = \bigcup_{n<\omega} K_n$. So, by Baire category theorem, there is an $n < \omega$ such that the interior $\mathrm{int}_Y K_n$ of K_n in Y is non-empty. Thus, there exist $a < b$ such that $[a,b] \subset p^{-1}(\mathrm{int}_Y K_n)$. In particular, $D^* f(x) \leq \frac{n}{n+1}$ for every $x \in p[[a,b]]$ and, by (i), $\ell(f \circ p \restriction [a,b]) \leq \frac{n}{n+1} \ell(p \restriction [a,b])$. Moreover, property (S) implies that $D^* f(x) \leq 1$ for every $x \in Y$ so, again by (i), $\ell(f \circ p \restriction [c,d]) \leq \ell(p \restriction [c,d])$ for every $0 \leq c \leq d \leq 1$. Thus,

$$\begin{aligned}
\ell(p) &= \ell(p \restriction [0,a]) + \ell(p \restriction [a,b]) + \ell(p \restriction [b,1]) \\
&> \ell(p \restriction [0,a]) + \frac{n}{n+1} \ell(p \restriction [a,b]) + \ell(p \restriction [b,1]) \\
&\geq \ell(f \circ p \restriction [0,a]) + \ell(f \circ p \restriction [a,b]) + \ell(f \circ p \restriction [b,1]) \\
&= \ell(f \circ p),
\end{aligned}$$

as needed. ∎

6 Discussion of the relations between the contractive classes

In all theorems in this section we present the examples of maps which, if possible, have no fixed and/or periodic points.

6.1 Complete spaces

$$\begin{tikzcd}
 & (\mathrm{C})^{\mathbf{F}}_{3.1} \arrow[r] \arrow[dl] & (\mathrm{ULC}) \arrow[r] \arrow[d] & (\mathrm{uLC}) \arrow[rr] \arrow[dd, dotted] & & (\mathrm{LC}) \arrow[dl] \arrow[dd] \\
(\mathrm{S}) \arrow[rr] & & (\mathrm{ULS}) \arrow[rr] \arrow[ddrr] & & (\mathrm{LS}) \arrow[dd] & \\
 & & & (\mathrm{uPC}) \arrow[rr, dotted] & & (\mathrm{PC}) \arrow[dl] \\
 & & & & (\mathrm{PS}) &
\end{tikzcd}$$

Figure 3: The relations between the local contractive and shrinking properties for the maps $f : X \to X$, with X being an arbitrary complete metric space. Maps from (C) are indicated as $(\mathrm{C})^{\mathbf{F}}_{3.1}$, to denote that they have the fixed point property, **F**, according to Theorem 3.1. The maps from the other classes need not have periodic points, the existence of which will later be denoted by a superscript **P**. No other implications in the figure exist, see Theorem 6.1.

Theorem 6.1. *No combination of any of the properties shown in Figure 3 imply any other property, unless the graph forces such implication. In particular, for the classes in the figure, listed by rows, we have:*

(C): (C) $\not\Leftarrow$ (S)&(ULC) – *Example 27, with no periodic point;*

(ULC): (ULC) $\not\Leftarrow$ (S)&(uLC) – *Example 16, with no periodic point;*

(uLC): (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC) – *Example 19, with no periodic point;*

(LC): (LC) $\not\Leftarrow$ (S)&(uPC) – *Example 20, with no periodic point;*

(S): (S) $\not\Leftarrow$ (ULC) – *Example 24, with no periodic point;*

(ULS): (ULS) $\not\Leftarrow$ (uLC) – *Example 18, with no periodic point;*

(LS): (LS) $\not\Leftarrow$ (uPC) – *Examples 28 and 21, with no periodic point;*

(uPC): (uPC) $\not\Leftarrow$ (S)&(LC) – *Example 4, with no periodic point;*

(PC): (PC) $\not\Leftarrow$ (S) – *Example 3, with no periodic point.*

6.2 Connected and path connected spaces

$$\begin{tikzcd}
 & (\mathrm{C})^{\mathbf{F}}_{3.1} \arrow[rr] \arrow[dl] & & (\mathrm{ULC})^{\mathbf{F}}_{3.5} \arrow[r] \arrow[dl] \arrow[ll, bend right, dashed, red] & (\mathrm{uLC}) \arrow[rr] \arrow[dd, dotted] & & (\mathrm{LC}) \arrow[dl] \arrow[dd] \\
(\mathrm{S}) \arrow[rr] & & (\mathrm{ULS}) \arrow[rrr] \arrow[ddrrr] & & & (\mathrm{LS}) \arrow[dd] & \\
 & & & & (\mathrm{uPC}) \arrow[rr, dotted] & & (\mathrm{PC}) \arrow[dl] \\
 & & & & & (\mathrm{PS}) &
\end{tikzcd}$$

Figure 4: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a complete metric space which is either connected or path connected. The left dashed arrow indicates that, by Proposition 5.2(i), there exists equivalent metric for which any map that is (ULC) in the old metric becomes (C). No other implications in the figure exist, see Theorem 6.2.

Theorem 6.2. *No combination of any of the properties shown in Figure 4 imply any other property, unless the graph forces such implication. In particular, for the classes in the figure, listed by rows, we have:*

(C): (C) $\not\Leftarrow$ (S)&(ULC) – *Example 10, with fixed point;*
(C) $\not\Leftarrow$ (S)&(uLC), *with no periodic point – see below* (ULC) $\not\Leftarrow$ (S)&(uLC)*;*

(ULC): (ULC) $\not\Leftarrow$ (S)&(uLC) – *Example 16, with no periodic point;*

(uLC): (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC) – *Example 19, with no periodic point;*

(LC): (LC) $\not\Leftarrow$ (S)&(uPC) – *Example 20, with no periodic point;*

(S): (S) $\not\Leftarrow$ (ULC) – *Example 6, with fixed point;*
(S) $\not\Leftarrow$ (ULS)&(uLC) – *Example 17, with no periodic point;*

(ULS): (ULS) $\not\Leftarrow$ (uLC) – *Example 18, with no periodic point;*

(LS): (LS) $\not\Leftarrow$ (uPC) – *Example 21, with no periodic point;*

(uPC): (uPC) $\not\Leftarrow$ (S)&(LC) – *Example 4, with no periodic point;*

(PC): (PC) $\not\Leftarrow$ (S) – *Example 3, with no periodic point.*

6.3 Rectifiably path connected spaces

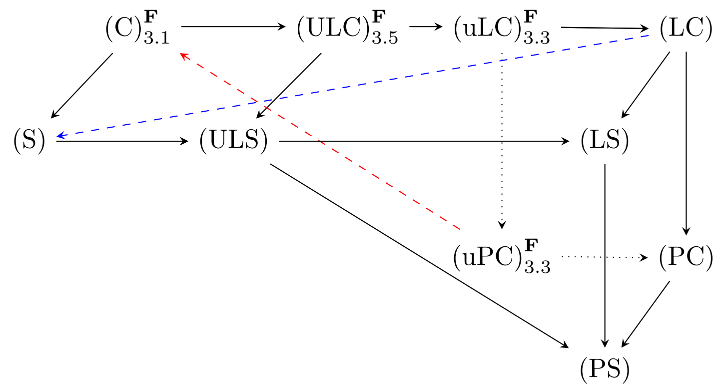

Figure 5: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a complete metric space which is rectifiably path connected. Each of the dashed (not dotted) arrows indicates that, by Proposition 5.5, there exists another complete rectifiably path connected metric on X that makes any map from the bigger class to belong to the smaller class. No other implications in the figure exist, see Theorem 6.3.

Theorem 6.3. *No combination of any of the properties shown in Figure 5 imply any other property, unless the graph forces such implication. In particular, for the classes in the figure, listed by rows, we have:*

(C): (C) $\not\Leftarrow$ (S)&(ULC) – *Example 10, with fixed point;*
(C) $\not\Leftarrow$ (S)&(LC), *with no periodic point – see below* (uPC) $\not\Leftarrow$ (S)&(LC)*;*

(ULC): (ULC) $\not\Leftarrow$ (S)&(uLC) – *Example 11, with fixed point;*
(ULC) $\not\Leftarrow$ (S)&(LC), *with no periodic point – see* (uPC) $\not\Leftarrow$ (S)&(LC)*;*

(uLC): (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC) – *Example 13, with fixed point;*
(uLC) $\not\Leftarrow$ (S)&(LC), *with no periodic point – see* (uPC) $\not\Leftarrow$ (S)&(LC)*;*

(LC): (LC) $\not\Leftarrow$ (S)&(uPC) – *Example 14, with fixed point;*
(LC) $\not\Leftarrow$ (S)&(PC) – *Example 5, with no periodic point;*

(S): (S) $\not\Leftarrow$ (ULC) – *Example 6, with fixed point;*
(S) $\not\Leftarrow$ (ULS)&(LC) – *Example 7, with no periodic point;*

(ULS): (ULS) $\not\Leftarrow$ (uLC) – *Example 12, with fixed point;*
(ULS) $\not\Leftarrow$ (LC) – *Example 15, with no periodic point;*

(LS): (LS) $\not\Leftarrow$ (uPC) – *Example 9, with fixed point;*
(LS) $\not\Leftarrow$ (PC) – *Example 8, with no periodic point;*

(uPC): (uPC) $\not\Leftarrow$ (S)&(LC) – *Example 4, with no periodic point;*

(PC): (PC) $\not\Leftarrow$ (S) – *Example 3, with no periodic point.*

6.4 d-convex spaces

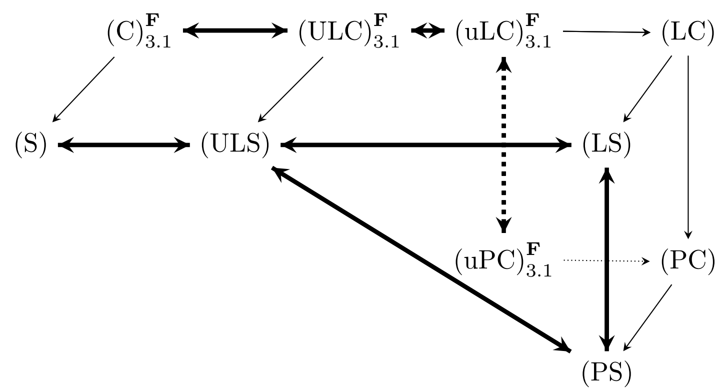

Figure 6: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a d-convex metric space. The left and upper portions of the equivalences $\longleftrightarrow$ follow from Theorem 4.1. No other implications in the figure exist, see Theorem 6.4.

Theorem 6.4. *No combination of any of the properties shown in Figure 6 imply any other property, unless the graph forces such implication. In particular, for the classes in the figure, listed by rows, we have:*

(C): (C) $\not\Leftarrow$ (S)&(LC), *see below the example for* (uPC)*;*

(ULC): (ULC) $\not\Leftarrow$ (S)&(LC), *see below the example for* (uPC)*;*

(uLC): (uLC) $\not\Leftarrow$ (S)&(LC) *see below the example for* (uPC)*;*

(LC): (LC) $\not\Leftarrow$ (S)&(PC) – *Example 5, with no periodic point;*

(uPC): (uPC) $\not\Leftarrow$ (S)&(LC) – *Example 4, with no periodic point;*

(PC): (PC) $\not\Leftarrow$ (S) – *Example 3, with no periodic point.*

6.5 Compact spaces

$$\begin{tikzcd}
 & (\mathrm{C})^{\mathbf{F}}_{3.1} \arrow[rr] \arrow[dl] & & (\mathrm{ULC})^{\mathbf{P}}_{3.2} \arrow[r, leftrightarrow, thick] \arrow[dl] & (\mathrm{uLC})^{\mathbf{P}}_{3.2} \arrow[rr, leftrightarrow, thick] \arrow[dd, dotted] & & (\mathrm{LC})^{\mathbf{P}}_{3.2} \arrow[dl] \arrow[dd] \\
(\mathrm{S})^{\mathbf{F}}_{3.2} \arrow[rr] & & (\mathrm{ULS})^{\mathbf{P}}_{3.2} \arrow[rrr, leftrightarrow, thick] \arrow[ddrrr] & & & (\mathrm{LS})^{\mathbf{P}}_{3.2} \arrow[dd] & \\
 & & & & (\mathrm{uPC}) \arrow[rr, dotted] & & (\mathrm{PC}) \arrow[dl] \\
 & & & & & (\mathrm{PS}) &
\end{tikzcd}$$

Figure 7: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a compact metric space. The left portions of the equivalences $\longleftrightarrow$ follow from Theorem 4.2. Moreover, we also have implication (S)&(ULC) $\Rightarrow$ (C), see Theorem 4.3. No other implications in the figure exist, see Theorem 6.5.

Theorem 6.5. *No combination of any of the properties shown in Figure 7 imply any other property, unless the graph forces such implication, with the exception of the implication* (S)&(ULC) $\Rightarrow$ (C)*. In particular, for the classes in the figure, listed by rows, we have:*

(C): (C) $\Leftarrow$ (S)&(ULC) *– see Theorem 4.3;*
(C) $\not\Leftarrow$ (S)&(uPC), *with fixed point – see below for* (LC) $\not\Leftarrow$ (S)&(uPC)*;*
(C) $\not\Leftarrow$ (ULC), *with periodic but not fixed point – see below* (S) $\not\Leftarrow$ (ULC)*;*
(C) $\not\Leftarrow$ (uPC), *with no periodic point – see below* (LS) $\not\Leftarrow$ (uPC)*;*

(ULC): *see below the examples for* (LC)*;*
(uLC): *see below the examples for* (LC)*;*

(LC): (LC) $\not\Leftarrow$ (S)&(uPC) *– Example 14, with fixed point;*
(LC) $\not\Leftarrow$ (ULS)&(uPC) *– Example 26, with periodic but not fixed point;*
(LC) $\not\Leftarrow$ (uPC), *with no periodic point – see example for* (LS) $\not\Leftarrow$ (uPC)*;*

(S): (S) $\not\Leftarrow$ (ULC) *– Example 23, with periodic but not fixed point;*
(S) $\not\Leftarrow$ (uPC), *with no periodic point – see example for* (LS) $\not\Leftarrow$ (uPC)*;*

(ULS): *see below the example for* (LS)*;*

(LS): (LS) $\not\Leftarrow$ (uPC) *– Example 28, with no periodic point;*

(uPC): (uPC) $\not\Leftarrow$ (S)&(PC) *– Example 2, with fixed point;*
(uPC) $\not\Leftarrow$ (ULS)&(PC) *– Example 25, with periodic but not fixed point;*
(uPC) $\not\Leftarrow$ (PC) *– Example 30, with no periodic point;*

(PC): (PC) $\not\Leftarrow$ (S) *– Example 1, with fixed point;*
(PC) $\not\Leftarrow$ (ULS) *– Example 22, with periodic but not fixed point;*
(PC) $\not\Leftarrow$ (PS) *– Example 29, with no periodic point.*

6.6 Compact (path) connected spaces

$$\begin{tikzcd}
 & (\mathrm{C})^{\mathbf{F}}_{3.1} \arrow[rr] \arrow[dl] & & (\mathrm{ULC})^{\mathbf{F}}_{3.2} \arrow[ll, dashed, red, bend right] \arrow[r, leftrightarrow, thick] \arrow[dl] & (\mathrm{uLC})^{\mathbf{F}}_{3.2} \arrow[rr, leftrightarrow, thick] \arrow[dd, dotted] & & (\mathrm{LC})^{\mathbf{F}}_{3.2} \arrow[dl] \arrow[dd] \\
(\mathrm{S})^{\mathbf{F}}_{3.2} \arrow[rr] & & (\mathrm{ULS})^{\mathbf{F}}_{3.2} \arrow[ll, dashed, red, bend right] \arrow[rrr, leftrightarrow, thick] \arrow[ddrrr] & & & (\mathrm{LS})^{\mathbf{F}}_{3.2} \arrow[dd] & \\
 & & & & (\mathrm{uPC})^{\textbf{?}}_{8.1} \arrow[rr, dotted] & & (\mathrm{PC})^{\textbf{?}}_{8.1} \arrow[dl] \\
 & & & & & (\mathrm{PS})^{\textbf{?}}_{8.1} &
\end{tikzcd}$$

Figure 8: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a compact and either connected or path connected metric space. The dashed arrows indicate that, by Proposition 5.2, there exists a complete metric on X topologically equivalent to the original, that makes any map from the bigger class to belong to the smaller class. The left portions of the equivalences $\leftrightarrow$ follow from Theorem 4.2. Moreover, we also have implication (S)&(ULC) $\Rightarrow$ (C), see Theorem 4.3. No other implications in the figure exist, see Theorem 6.6. The question marks in the graph refer to open problems.

Theorem 6.6. *No combination of any of the properties shown in Figure 8 imply any other property, unless the graph forces such implication. In particular, for the classes in the figure, listed by rows, we have:*

(C): (C) $\Leftarrow$ (S)&(ULC) – *see Theorem 4.3;*
(C) $\not\Leftarrow$ (S)&(uPC), *with fixed point – see example for* (LC) $\not\Leftarrow$ (S)&(uPC)*;*
(C) $\not\Leftarrow$ (ULC), *with fixed point – see below* (S) $\not\Leftarrow$ (ULC)*;*

(ULC): see below the example for (LC)*;*

(uLC): see below the example for (LC)*;*

(LC): (LC) $\not\Leftarrow$ (S)&(uPC) – *Example 14, with fixed point;*

(S): (S) $\not\Leftarrow$ (ULC) – *Example 6, with fixed point;*

(ULS): see below the example for (LS)*;*

(LS): (LS) $\not\Leftarrow$ (uPC) – *Example 9, with fixed point;*

(uPC): (uPC) $\not\Leftarrow$ (S)&(PC) – *Example 2, with fixed point;*

(PC): (PC) $\not\Leftarrow$ (S) – *Example 1, with fixed point.*

6.7 Compact rectifiably path connected spaces

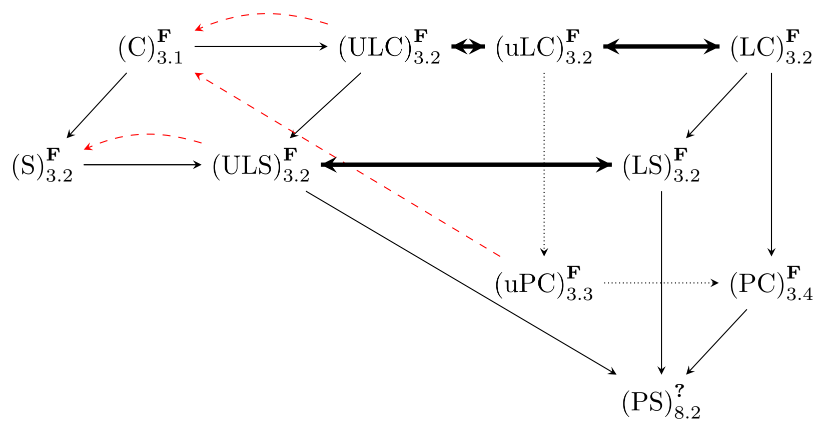

Figure 9: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a compact and rectifiably path connected metric space. The dashed arrows indicates that, by Propositions 5.2 and 5.5 there exists equivalent metric that makes any map from the bigger class to belong to the smaller class. The left portions of the equivalences $\leftrightarrow$ follow from Theorem 4.2. Moreover, we also have implication (S)&(ULC) $\Rightarrow$ (C), see Theorem 4.3. No other implications in the figure exist, see Theorem 6.7. The question mark in the graph refers to an open problem.

Theorem 6.7. *No combination of any of the properties shown in Figure 9 imply any other property, unless the graph forces such implication. In particular, for the classes in the figure, listed by rows, we have:*

- *(C):* (C) $\Leftarrow$ (S)&(ULC) *– see Theorem 4.3;*
  (C) $\not\Leftarrow$ (S)&(uPC), *with fixed point – see example for* (LC) $\not\Leftarrow$ (S)&(uPC);
  (C) $\not\Leftarrow$ (ULC), *with fixed point – see below* (S) $\not\Leftarrow$ (ULC)*;*
- *(ULC): see below the example for* (LC)*;*
- *(uLC): see below the example for* (LC)*;*
- *(LC):* (LC) $\not\Leftarrow$ (S)&(uPC) *– Example 14, with fixed point;*
- *(S):* (S) $\not\Leftarrow$ (ULC) *– Example 6, with fixed point;*
- *(ULS): see below the example for* (LS)*;*
- *(LS):* (LS) $\not\Leftarrow$ (uPC) *– Example 9, with fixed point;*
- *(uPC):* (uPC) $\not\Leftarrow$ (S)&(PC) *– Example 2, with fixed point;*
- *(PC):* (PC) $\not\Leftarrow$ (S) *– Example 1, with fixed point.*

6.8 Compact d-convex spaces

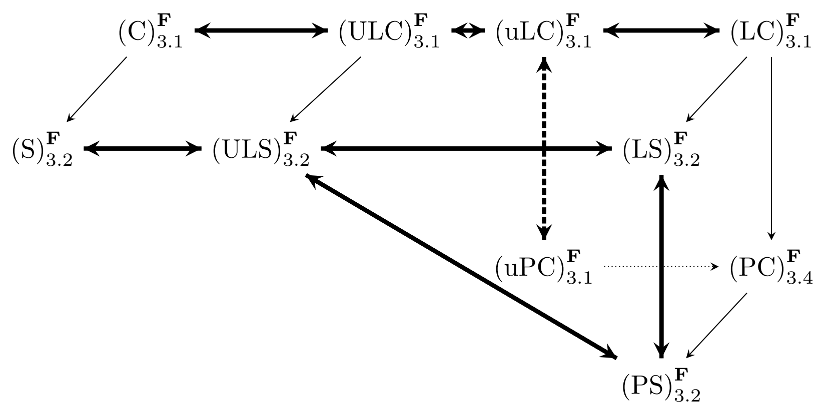

Figure 10: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being a compact d-convex metric space. The left and upper portions of the equivalences $\longleftrightarrow$ follow from Theorem 4.1 and Theorem 4.2. No other implications in the figure exist, see Theorem 6.8.

Theorem 6.8. *No combination of any of the properties shown in Figure 10 imply any other property, unless the graph forces such implication. In particular,*

(C): *see below the example for* (uPC);

(ULC): *see below the example for* (uPC);

(uLC): *see below the example for* (uPC);

(LC): *see below the example for* (uPC);

(uPC): (uPC) $\not\Leftarrow$ (S)&(PC) – *Example 2, with fixed point;*

(PC): (PC) $\not\Leftarrow$ (S) – *Example 1, with fixed point.*

7 The examples

All examples presented in this section consist of the self-maps of *complete* metric spaces. Actually, all these metric spaces will be subsets of $\mathbb{R}$ considered with the standard topology; however, the metric we use will often be a non-standard metric.

7.1 Examples on intervals with standard metric

Notice that, by Theorem 3.2(i), if X is compact, than any map $f\colon X \to X$ as in Example 1 must have a fixed point.

Example 1. *The map* $f\colon [0,1] \to [0,1]$, $f(x) = \arctan x$, *is from* (S)&¬(PC). *It has a fixed point, as* $f(0) = 0$.

PROOF. It is not (PC) by Remark 2.3, as $f'(0) = 1$. It is (S) by the Mean Value Theorem, since $f'(x) \in (0,1)$ for every $x > 0$. ■

Notice that, by Theorem 3.2(i), if X is compact, than any map $f\colon X \to X$ as in Example 2 must have a fixed point. Note also that although the map in Example 2 is differentiable, it cannot be continuously differentiable since for $C^1[0,1]$ maps (PC)⇒(uPC).

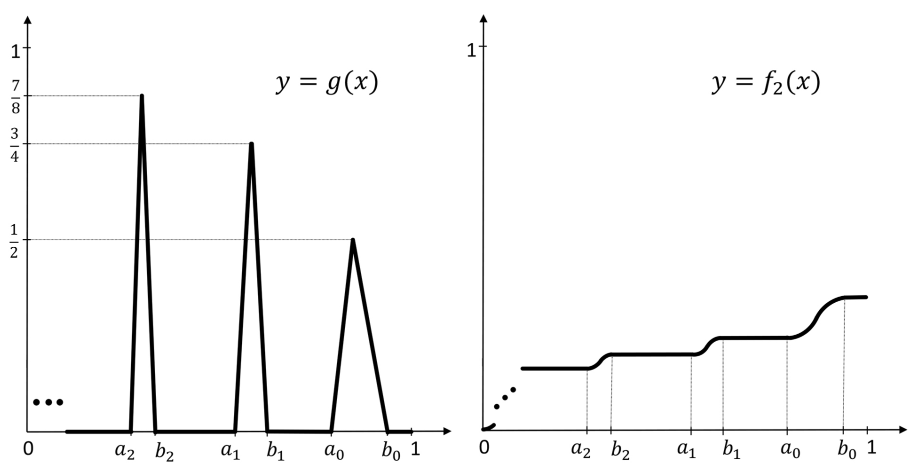

Figure 11: The graph of $g\colon [0,1] \to [0,1]$ from Example 2 for which the map $f_2(x) = \int_0^x g(t)\,dt$ is (S) and (PC) but not (uPC).

Example 2. *There exists a map* $f_2\colon [0,1] \to [0,1]$ *from* (S)&(PC)&¬(uPC). *It has a fixed point, as* $f_2(0) = 0$.

CONSTRUCTION. Choose a sequence $b_0 > a_0 > b_1 > a_1 > \cdots$ converging to 0 such that 0 is a Lebesgue density point of the complement of $\bigcup_{n<\omega}(a_n, b_n)$.

For every $n < \omega$ let g_n be a map from $[0,1]$ onto $[0, 1-2^{-n-1}]$ with support in $[a_n, b_n]$. For example, take a tent-like $g_n(x) = 2\frac{1-2^{-n-1}}{b_n - a_n}\operatorname{dist}(x, \mathbb{R} \setminus [a_n, b_n])$. Then $g = \sum_{n<\omega} g_n$, see Figure 11, is from $[0,1]$ onto $[0,1)$ which is approximately continuous—it is ensured at $x = 0$ by the Lebesgue density requirement.

This implies that $f_2(x) = \int_0^x g(t)\,dt$ is differentiable from $[0,1]$ to $[0,1]$ with $f_2'(x) = g$, see e.g. [6, Theorem 7.36, p. 317]. By Remark 2.3, this property of f_2' implies that f_2 is (PC) and not (uPC). Also, by the Mean Value Theorem, f_2 is (S). ■

Notice that, by Theorem 3.2(i), if $f\colon X \to X$ is as in Example 3, then X cannot be compact.

Example 3. *The map* $f\colon [0,\infty) \to [0,\infty)$, $f(x) = x + e^{-x^2}$, *is from* (S)&¬(PC) *and has no periodic point.*

Proof. Notice that $f'(x) = 1 - 2xe^{-x^2}$. Thus, $f'(0) = 1$ and so, by Remark 2.3, f is not (PC). Moreover, $f'[(0,\infty)] \subseteq (0,1)$ so that f is (S) since, for any distinct $x, y \in [0,\infty)$, the inequality $|f(x) - f(y)| < |x - y|$ follows from the Mean Value Theorem. Finally, f has no periodic point, since $f(x) > x$ for every $x \geq 0$. This inequality implies also that f is indeed into $[0,\infty)$. ■

The next example comes from Munkres [27, p. 182]. Notice that, by Theorem 3.2(i), if $f\colon X \to X$ is as in Example 4, then X cannot be compact.

Example 4. *The map* $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)$, *is from the class* (S)&(LC)&¬(uPC) *and has no periodic point.*

Proof. Notice that $f'(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$. Therefore, for any $a \in \mathbb{R}$, $f'[(-\infty, a]] = (0, c]$ for some $c \in (0,1)$. Thus, the Mean Value Theorem implies that f is (S)&(LC). On the other hand $\lim_{x\to\infty} f'(x) = 1$ so, by Remark 2.3, f is not (uPC). Finally, f has no periodic point since $f(x) > x$ all $x \in \mathbb{R}$. ■

Notice that, by Theorem 3.2(i), if $f\colon X \to X$ is as in Example 5, then X cannot be compact.

Example 5. *There exists a map* $f\colon [0,\infty) \to [0,\infty)$ *from* (S)&(PC)&¬(LC) *having no periodic point.*

Construction. Let $f_2\colon [0,1] \to [0,1]$ be as in Example 2 and let $r = f_2(1)$. Define $g_r(x) = \frac{1}{2}\left(x + \sqrt{x^2 + 4r + 4r^2}\right)$. This is a modification of the Munkres' function from Example 4. It has the property that $g_r(1) = r + 1 = f_2(1) + 1$. Define

$$f(x) = \begin{cases} f_2(x) + 1 & \text{for } x \in [0,1], \\ g_r(x) & \text{for } x \in [1,\infty). \end{cases}$$

Clearly, f is continuous and $f(x) > x$ for all $x \in [0,\infty)$, so f has no periodic points. The restriction $f \restriction [0,1]$ is (S)&(PC)&¬(uPC), as these are the properties of f_2 and both functions have the same derivative. In particular, $f \restriction [0,1]$ cannot be (LC), since otherwise, by Theorem 4.2, $f \restriction [0,1]$

would be (uLC) and, therefore, (uPC). Hence, f is not (LC). Notice also that $g_r'(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{x^2+4r+4r^2}}\right)$ so, as in Example 4, $f \restriction [1,\infty) = g_r$ is both (S) and (PC). Since $f \restriction [0,1]$ has the same two properties, it is easy to verify that the same is true for f, compare discussion in [1]. ■

7.2 Examples on intervals with non-standard distances

In all examples $f\colon X \to X$ presented in Section 7.2 the space X is an interval. However, in none of these examples X can be equipped with the standard metric on $\mathbb{R}$, as justified by Theorem 4.1 and further elaborated before each example. Nevertheless, all metrics we use here are complete and topologically equivalent to the standard metric on $\mathbb{R}$ (i.e., each X can be treated as a path in $\mathbb{R}^2$). The only property about these spaces that we did not fully investigated is a question of which of the spaces can be isometric to the subset of $\mathbb{R}^2$ or, more generally, of $\mathbb{R}^n$ with $n > 1$.

7.2.1 Using simple rectifiably path connected metrics

Notice that, by Theorem 3.2(i), if X is compact, than any map $f\colon X \to X$ as in Example 6 must have a fixed point. Also, by Theorem 4.1, the metric $\bar\rho$ in the example cannot be the standard metric (as, in such case, (ULC)⇒(uPC) ⇒(C)⇒(S)).

Example 6. *There exists a function $f\colon \langle [0,4], \bar\rho\rangle \to \langle [0,4], \bar\rho\rangle$ from the class* (ULC)&¬(S)*, where $\bar\rho$ is a rectifiably path connected metric on $[0,4]$ topologically equivalent to the standard metric. Clearly f has a fixed point.*

CONSTRUCTION. Define $\bar\rho$ via formula $\bar\rho(x,y) = \min\{|x-y|, 1\}$ and put $f(x) = x/2$. It is easy to see that f is $\left(\frac{1}{2}, \frac{1}{2}\right)$-(ULC). It is not (S), since $\bar\rho(f(0), f(4)) = \bar\rho(0,2) = 1 = \bar\rho(0,4)$. ■

Notice that, by Theorem 3.2(i), if $f\colon X \to X$ is as in the Example 7, then X cannot be compact. Also, by Theorem 4.1, the metric $\bar\rho$ in the example cannot be the standard metric (as, in such case, (ULS)⇒(PS)⇒(S)).

Example 7. *There exists a map $f\colon \langle [0,\infty), \bar\rho\rangle \to \langle [0,\infty), \bar\rho\rangle$ from the class* (ULS)&(LC)&¬(S) *having no periodic point, where $\langle [0,\infty), \bar\rho\rangle$ is a rectifiable path connected and topologically equivalent to $[0,\infty)$ with the standard metric.*

CONSTRUCTION. The function $f\colon [0,\infty) \to [0,\infty)$ from Example 4 has the desired properties, when the metric is defined as $\bar\rho(x,y) = \min\{|x-y|, 1\}$. Indeed, the properties (LC) and (ULC) are not affected by this metric change. However, f is not (S), since there are $0 < a < b$ for which $b - a > f(b) - f(a) > 1$ and we obtain $\bar\rho(a,b) = 1 = \bar\rho(f(a), f(b))$. ■

Notice that, by Theorem 3.4, if $f\colon X \to X$ is as in Example 8, then X cannot be compact. Also, by Theorem 4.1, the metric ρ in the example cannot be the standard metric (as, in such case, (PC)⇒(PS)⇒(S)⇒(LS)). However, we actually prove that there is a map $h\colon [0,\infty) \to \mathbb{R}$ for which the domain of f is the graph of h considered as a subset of $\mathbb{R}^2$ with the standard distance.

Example 8. *There exists a map $f\colon \langle [0,\infty),\rho\rangle \to \langle [0,\infty),\rho\rangle$ from the class* (PC)&¬(LS) *having no periodic point, where $\langle [0,\infty),\rho\rangle$ is a rectifiable path connected and topological equivalent to $[0,\infty)$ with the standard metric.*

Construction. For $n<\omega$ let $h_n\colon [n,n+1]\to\mathbb{R}$ be defined by a formula $h_n(x) = \frac{1}{n+1}\mathrm{dist}(x,\{n,n+1\})$. Also, let $h=\bigcup_{n<\omega} h_n$, see Figure 12, and define ρ as

$$\rho(x,y) = \|\langle x,h(x)\rangle - \langle y,h(y)\rangle\|.$$

Let $f(x)=x+1$.

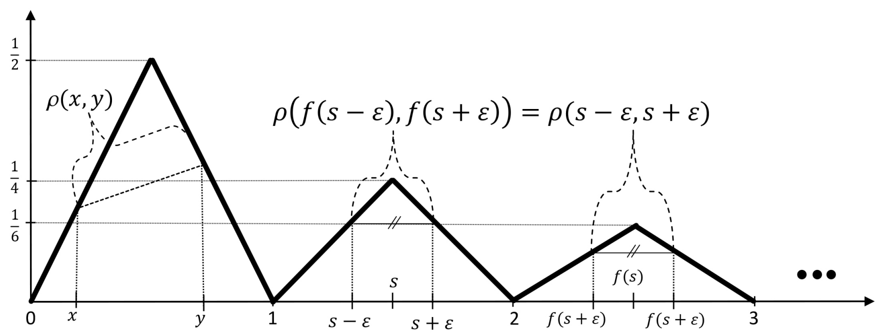

Figure 12: The graph of $h\colon [0,\infty]\to\mathbb{R}$ from Example 8 for which $f(x)=x+1$ is (PC) but not (LS).

Clearly, $\langle [0,\infty),\rho\rangle$ is rectifiable path connected and f has no periodic points. To see that f is (PC), it is enough to prove that so is its restriction to any of the intervals $[n,n+0.5]$ and $[n+0.5,n+1]$. Indeed, if x and y are distinct points of such an interval, then

$$\frac{\rho(f(x),f(y))}{\rho(x,y)} = \frac{\sqrt{1+\frac{1}{(n+2)^2}}|x-y|}{\sqrt{1+\frac{1}{(n+1)^2}}|x-y|} = \frac{\sqrt{1+\frac{1}{(n+2)^2}}}{\sqrt{1+\frac{1}{(n+1)^2}}} < 1.$$

The function f is not (LS) at any point $s=n+\frac{1}{2}$ since for any $\varepsilon\in(0,1/2)$ we have $\rho(f(s-\varepsilon),f(s+\varepsilon)) = 2\varepsilon = \rho(s-\varepsilon,s+\varepsilon)$. ■

Notice that, by Theorem 3.3, if $f\colon X\to X$ is as in Example 9 and X is compact, then f must have a fixed point. Also, by Theorem 4.1, the metric ρ in the example cannot be the standard metric (as, in such case, (uPC)⇒(C)⇒(LS)). However, we actually prove that there is a map $h\colon [0,2]\to\mathbb{R}$ for which the domain of f is the graph of h considered as a subset of $\mathbb{R}^2$ with the standard distance.

Example 9. *There exists a map $f\colon \langle [0,2],\rho\rangle \to \langle [0,2],\rho\rangle$ from* (uPC)&¬(LS), *where $\langle [0,2],\rho\rangle$ is a rectifiable path connected and topologically equivalent to $[0,2]$ with the standard metric. Clearly, f has a fixed point, as $f(2)=2$.*

Construction. Let ρ be the restriction to $[0,2]$ of the metric from Example 8 and define $f(x)=\min\{2,x+1\}$. The restriction $f\restriction[0,1]$ is identical for this example and Example 8. So, as in Example 8, it is not (LS) at $x=\frac{1}{2}$, while it is (uPC). This ensures that our f is not (LS). However, it is (uPC), since so are $f\restriction[0,1]$ and $f\restriction[1,2]$ (with $f\restriction[1,2]$ being a constant map). ■

Notice that, by Theorem 4.3, if $f\colon X\to X$ is as in Example 10, then X cannot be compact. Also, by Theorem 3.4, the map in the example must have a fixed point. Moreover, by Theorem 4.1, the metric d in the example cannot be the standard metric (as, in such case, (ULC)⇒(uPC)⇒(C)). However, we actually prove that there is a map $h\colon[0,\infty)\to\mathbb{R}$ for which the domain of f is the graph of h considered as a subset of $\mathbb{R}^2$ with the standard distance.

Example 10. *There exists a function $f\colon\langle[0,\infty),d\rangle\to\langle[0,\infty),d\rangle$ from the class* (S)&(ULC)&¬(C), *where d is a rectifiable path connected complete metric on $[0,\infty)$ topologically equivalent to the standard metric. The map f has a fixed point, $f(0)=0$.*

Construction. Define d by a formula $d(x,y)=\ln(1+|x-y|)$. It is a metric, since the map $[0,\infty)\ni t\mapsto\ln(1+t)\in[0,\infty)$ is concave down on $[0,\infty)$. It is easy to see that the metric d is complete and topologically equivalent to the standard metric. It is rectifiably path connected since the inequality $\ln(1+t)\le t$ implies that the length, with respect to the metric d, of a path from x to y is at most $|x-y|$.

Define $f\colon[0,\infty)\to[0,\infty)$ by $f(x)=x/2$. For any $x\in[0,\infty)$ and $z>0$ we have $\frac{d(f(x),f(x+z))}{d(x,x+z)}=\frac{\ln(1+z/2)}{\ln(1+z)}<1$. Therefore, f is (S). The map f is not (C), since, by l'Hôspital's Rule, $\lim_{z\to\infty}\frac{d(f(x),f(x+z))}{d(x,x+z)}=\lim_{z\to\infty}\frac{1+z}{2+z}=1$. On the other hand, $\lim_{z\to 0}\frac{d(f(x),f(x+z))}{d(x,x+z)}=\lim_{z\to 0}\frac{1+z}{2+z}=\frac{1}{2}$ so, there exists an $\varepsilon>0$ such that $\frac{d(f(x),f(x+z))}{d(x,x+z)}<\frac{3}{4}$ for every $x\ge 0$ and $z\in(0,\varepsilon)$. But this means that f is $\left(\frac{\varepsilon}{2},\frac{3}{4}\right)$-(ULC). ■

Notice that, by Theorem 4.2, if $\hat f\colon X\to X$ is as in Example 11, then X cannot be compact. Also, by Theorem 3.3, the map $\hat f$ must have a fixed point. Moreover, by Theorem 4.1, the metric ρ in the example cannot be the standard metric (as, in such case, (uLC)⇒(uPC)⇒(C)⇒(ULC)). We actually prove that there is a subset $X\subseteq\mathbb{R}^2$ homeomorphic with $\mathbb{R}$ and a function $f\colon X\to X$ which has the desired properties when X is taken with the standard metric.

Example 11. *There exists a map $f\colon\langle\mathbb{R},\rho\rangle\to\langle\mathbb{R},\rho\rangle$ from* (S)&(uLC)&¬(ULC), *where ρ is a rectifiable path connected complete metric on $\mathbb{R}$ topologically equivalent to the standard metric. The map f has a fixed point.*

Construction. Choose numbers $\cdots<a_{-2}<a_{-1}<a_0<a_1<a_2<\cdots$ such that $a_0=0$ and if $I_n=[a_n,a_{n+1}]$ for every $n\in\mathbb{Z}$, then for every $k<\omega$:

$$|I_{-(k+1)}|=\tfrac{1}{k+2},\ |I_{2k}|=\tfrac{1}{k^2+1},\ \text{and}\ |I_{2k+1}|=\tfrac{1}{k+1}.$$

Notice that this choice ensures that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n$ and

$$\lim_{k\to\infty} |I_k| = 0 \text{ and } \lim_{k\to\infty} \tfrac{|I_{-k-2}|+|I_{-k-1}|}{|I_{2k+1}|+|I_{2k+2}|+|I_{2k+3}|}. \tag{3}$$

For every $k < \omega$ define: $g \restriction I_{-(k+1)}$ and $g \restriction I_{4k}$ as a constant 0 function; $g \restriction I_{4k+2}$ as a constant 1 function; $g \restriction I_{4k+1}$ as a linear increasing map onto $[0,1]$, and $g \restriction I_{4k+3}$ as a linear decreasing map onto $[0,1]$. Define the metric ρ on $\mathbb{R}$ via formula $\rho(x,y) = \|\langle x, g(x)\rangle - \langle y, g(y)\rangle\|$. Also, define $f\colon \mathbb{R} \to \mathbb{R}$ as follows. Put $f(x) = 0$ for every $x \le 0$ and, for every $k < \omega$, let f map each interval I_{2k+1} decreasingly and linearly onto $I_{-k-1} = [a_{-k-1}, a_{-k}]$ and each interval I_{2k} onto the singleton $\{a_{-k}\}$, see Figure 13. We claim that f is as needed.

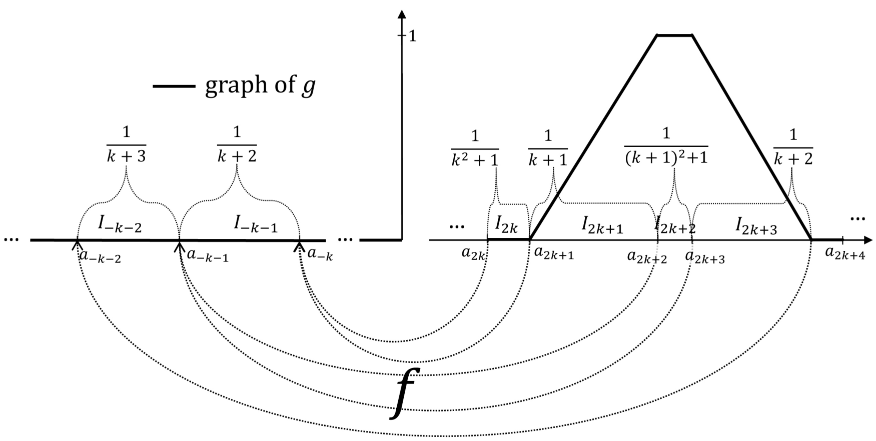

Figure 13: Illustration of g and $f \restriction [a_{2k}, a_{2k+4}]$ from Example 11.

Indeed, f is not (ULC) because, for every $k < \omega$,

$$\frac{\rho(f(a_{4k+1}), f(a_{4k+4}))}{\rho(a_{4k+1}, a_{4k+4})} = \frac{|a_{-2k} - a_{-2k-2}|}{|a_{4k+1} - a_{4k+4}|} = \frac{|I_{-2k-2}| + |I_{-2k-1}|}{|I_{4k+1}| + |I_{4k+2}| + |I_{4k+3}|}$$

which, by (3), converges to 1, as $k \to \infty$, and $\lim_{k\to\infty} \rho(a_{4k+1}, a_{4k+4}) = 0$.

To see that the function f is (S), choose $x < y$ in $\mathbb{R}$. We need to show the inequality $\rho(f(x), f(y)) < \rho(x,y)$. But $\rho(f(x), f(y)) = |f(x) - f(y)|$, since $f(x), f(y) \in (-\infty, 0]$ and ρ on $(-\infty, 0]$ is the standard metric. Moreover, $\rho(x,y) = \|\langle x, g(x)\rangle - \langle y, g(y)\rangle\| \ge |x - y|$. Hence, it is enough to show that $|f(x) - f(y)| < |x - y|$, that is, that f is (S) when considered with the standard metric. However, if ρ_n is the standard metric on I_n, then the metric $\bar{\rho}$ on $\mathbb{R}$ induced by these metrics as in Lemma 7.1 is the standard metric. Clearly, with respect to the standard metric $\bar{\rho}$, $f \restriction I_n$ is (S) since it is linear and $|f[I_n]| < |I_n|$. Hence, by Lemma 7.1, f is (S) when considered the standard metric $\bar{\rho}$, finishing the argument.

Finally, to see that f is (uLC), it is enough to prove that, for every $n \in \mathbb{Z}$,

$$f \restriction (I_n \cup I_{n+1}) \text{ is } (\tfrac{1}{2})\text{-(C)}. \tag{4}$$

Clearly, this is true when $n < 0$, since then $f \restriction (I_n \cup I_{n+1})$ is constant. So, assume that $n \geq 0$. We will assume also that n is an odd number $2k + 1$, the even case being essentially identical. Thus, to see (4), fix $x < y$ from $I_{2k+1} \cup I_{2k+2}$. We need to show that $\rho(f(x), f(y)) \leq \frac{1}{2}\rho(x, y)$.

This inequality is obvious when $x, y \in I_{2k+2}$, since $f \restriction I_{2k+2}$ is constant. For $x, y \in I_{2k+1}$ this follows from the fact that $f \restriction I_{2k+1}$ is linear with the slope $\frac{\rho(f(a_{2k+1}),f(a_{2k+2}))}{\rho(a_{2k+1},a_{2k+2})} \leq \rho(f(a_{2k+1}), f(a_{2k+2})) = |I_{-k-1}| = \frac{1}{k+2} \leq \frac{1}{2}$. The remaining case is when $x < a_{2k+2} < y$. Then the inequality holds, since $\rho(f(x), f(y)) = \rho(f(x), f(a_{2k+2})) \leq \frac{1}{2}\rho(x, a_{2k+2}) \leq \frac{1}{2}\rho(x, y)$, where the last inequality is justified by the fact that the angle between the segments $g \restriction I_{2k+1}$ and $g \restriction I_{2k+2}$ is obtuse. ■

7.2.2 Using more involved rectifiably path connected metrics

All remaining examples presented in Section 7.2 will be based on the next lemma. It will be primarily used for the families $\mathcal{J}$ of the form $\{(a_n, a_{n+1}) \colon n < \omega\}$, with $0 = a_0 < a_1 < a_2 < \cdots$. However, in several examples, it will be used also with the families $\mathcal{J}$ of more complex format.

Lemma 7.1. *Let $\mathcal{J} = \{(a_t, b_t) \colon t \in T\}$ be a family of pairwise disjoint non-empty open bounded intervals in $\mathbb{R}$ and let J be a closed interval in $\mathbb{R}$ containing $U = \bigcup_{t\in T}(a_t, b_t)$. Let d be the standard metric on $U^c = J \setminus U$ and, for every $t \in T$, let ρ_t be a metric on $[a_t, b_t]$ such that $\rho_t(a_t, b_t) = |a_t - b_t|$. Extend the function $\delta = d \cup \bigcup_{t\in T} \rho_t$ to the metric $\rho \colon J^2 \to \mathbb{R}$ by putting, for every $x \leq y$ from J,*

$$\rho(x, y) = \rho(y, x) = \delta(x, x^+) + \delta(x^+, y^-) + \delta(y^-, y),$$

where $x^+ = \inf U^c \cap [x, \infty)$ and $y^- = \sup U^c \cap (-\infty, y]$. Then ρ is a metric on J. It is complete and topologically equivalent to the standard metric, provided so is every ρ_t. Moreover, for every mapping f from $\langle J, \rho\rangle$ into a metric space $\langle Y, \eta\rangle$ the following hold.

- *(S): f is* (S) *provided all maps $f \restriction [a_t, b_t]$ are* (S) *and U^c is discrete.*
- *(uLC): f is* (uLC) *with a constant $\lambda \in [0, 1)$ provided all maps $f \restriction [a_t, b_t]$ are* (uLC) *with constant λ and U^c is discrete.*
- *(uPC): f is* (uPC) *with a constant $\lambda \in [0, 1)$ providedall maps $f \restriction [a_t, b_t]$ are* (uPC) *with constant λ and U^c is discrete.*
- *(LC): f is* (LC) *providedall maps $f \restriction [a_t, b_t]$ are* (LC) *with constant λ and U^c is discrete.*
- *(C): f is* (C) *with a constant $\lambda \in [0, 1)$ provided $f \restriction U^c$ as well as all maps $f \restriction [a_t, b_t]$ are* (C) *with constant λ.*

Proof. It is easy to see that ρ is a metric on J and that it is complete and topologically equivalent to the standard metric, when every ρ_t is such.

To see (S), choose $x < y$ in J. Since U^c is discrete, there exists a finite sequence $x = x_0 < \cdots < x_n = y$ such that: for all $i < n$, the pair $\{x_i, x_{i+1}\}$ is contained in one of the intervals $[a_{t_i}, b_{t_i}]$; and $x_j \in U^c$ for all $0 < j < n$. Then, $\rho(x,y) = \sum_{i<n} \rho_{t_i}(x_i, x_{i+1}) > \sum_{i<n} \eta(f(x_i), f(x_{i+1})) \geq \eta(f(x), f(y))$ as needed, where the equation is ensured by the definition of ρ, while the strict inequality by the assumption on maps $f \restriction [a_t, b_t]$.

To see (uLC), choose $z \in J$. We need to find open neighborhood $U \subset J$ of z such that $f \restriction U$ is (C) with the constant λ. If there is an open neighborhood $W \subset J$ of z contained in a single interval $[a_t, b_t]$, then, by our assumption on $f \restriction [a_t, b_t]$, there is a $U \subset W$ as needed. Otherwise, there are distinct $s, t \in T$ such that $z = b_s = a_t$. Then, by our assumption, there are $p \in (a_s, b_s)$ and $q \in (a_t, b_t)$ such that both $f \restriction (p, z]$ and $f \restriction [z, q)$ are (C) with constant λ. Then $U = (p, q)$ is as needed. To see this, take $x < y$ from (p, q). We need to show that $\eta(f(x), f(y)) \leq \lambda\rho(x, y)$. If $z \notin (x, y)$, then this holds by what we know about $f \restriction (p, z]$ and $f \restriction [z, q)$. Otherwise, $z = x^+ = y^-$ and $\lambda\rho(x,y) = \lambda\rho(x,z) + \lambda\rho(z,y) \geq \eta(f(x), f(z)) + \eta(f(z), f(y)) \geq \eta(f(x), f(y))$, as needed.

The proofs of parts (uPC) and (LC) are straightforward variations of that for (uLC).

To see (C), notice that for every $x \leq y$ from J we have

$$\begin{aligned}
\lambda\rho(x,y) &= \lambda\rho(x,x^+) + \lambda\rho(x^+,y^-) + \lambda\rho(y^-,y) \\
&\geq \eta(f(x), f(x^+)) + \eta(f(x^+), f(y^-)) + \eta(f(y^-), f(y)) \\
&\geq \lambda\eta(f(x), f(y)),
\end{aligned}$$

as needed. ∎

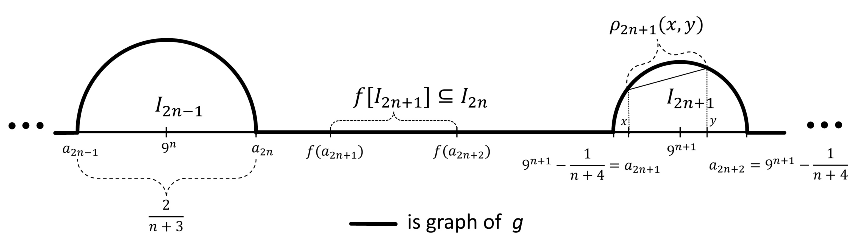

Figure 14: Illustration of the graph of g from Example 12 for which the map f is (uLC) but not (ULS). Notice that $f[I_{2n+1}] \subseteq I_{2n}$.

By Theorem 3.3, if the map $f : X \to X$ is as in Example 12 then it must have a fixed point. By Theorem 4.1, the metric ρ on X cannot be the standard metric (as, in such case, (uLC)⇒(uPC)⇒(C)⇒(ULS)). Also, by Theorem 3.2(i), X cannot be compact since, by Theorem 4.2, in such case any (uLC) map is also (ULC), so it is (ULS).

Example 12. *There exists a map $f \colon \langle [0,\infty), \rho\rangle \to \langle [0,\infty), \rho\rangle$ from the class* (uLC)&¬(ULS), *where $\langle [0,\infty), \rho\rangle$ is rectifiable path connected and topologically equivalent to $[0,\infty)$ with the standard metric. It has a fixed point, $f(0) = 0$.*

Construction. Choose a sequence $0 = a_0 < a_1 < \cdots$ such that if, for every $n < \omega$, we put $I_n = [a_n, a_{n+1}]$, then each interval I_{2n+1} is centered at 9^{n+1} and of length $\frac{2}{n+4}$. Let $g\colon [0,\infty) \to [0,1]$ be such that $g \restriction I_n \equiv 0$ for every even $n < \omega$, while, for every odd $n < \omega$, the graph of $g \restriction I_n$ is an upper semicircle of radius $\frac{1}{n+4}$ and centered at $\langle 9^{n+1}, 0\rangle$. See Figure 14.

For each $n < \omega$ let ρ_n be a metric on I_n defined as

$$\rho_n(x,y) = \|\langle x, g_n(x)\rangle - \langle y, g_n(y)\rangle\|$$

and let ρ be the metric on $[0,\infty)$ from Lemma 7.1 associated with metrics $\{\rho_n \colon n < \omega\}$. Then, ρ is complete and, clearly, rectifiable path connected.

Define an increasing bijection $L\colon [0,\infty) \to [0,\infty)$ as $L(x) = \ell(g \restriction [0,x])$ and let $f\colon [0,\infty) \to [0,\infty)$ be defined as $f(x) = L^{-1}\left(\frac{2}{3}L(x)\right)$. In other words, $f(x)$ is the unique point $r \in [0,\infty)$ such that $\ell(g \restriction [0,r]) = \frac{2}{3}\ell(g \restriction [0,x])$ or equivalently,

$$L(f(x)) = \tfrac{2}{3}L(x). \tag{5}$$

We claim that f is as needed.

To see that f is (uLC), choose an arbitrary $\eta \in (1, 3/2)$. By Lemma 7.1(uLC), it is enough to show that, for every $n < \omega$, $f \restriction I_n$ is (uLC) with constant $\lambda = \frac{2}{3}\eta$. Indeed, if n is odd, then, for any $x \in I_n$, there is an open subset V of I_n containing x such that for any $y, z \in V$, we have $|L(y) - L(z)| \le \eta\rho(y,z)$ and, by (5),

$$\rho(f(y), f(z)) \le |L(f(y)) - L(f(z))| = \tfrac{2}{3}|L(y) - L(z)| \le \tfrac{2}{3}\eta\rho(y,z) = \lambda\rho(y,z).$$

On the other hand, if n is even, then the following holds for every $y, z \in I_n$:

$$\rho(f(y), f(z)) \le |L(f(y)) - L(f(z))| = \tfrac{2}{3}|L(y) - L(z)| = \tfrac{2}{3}\rho(y,z) \le \lambda\rho(y,z).$$

To see that f is not (ULS), first notice that

$$f[I_{2n+1}] \subseteq I_{2n} \quad \text{for any } n < \omega. \tag{6}$$

Indeed, since $\ell(g \restriction I_{2k+1}) = \pi\frac{1}{k+4} < 1$ for every $k < \omega$, for every $x \in [0, a_{2n+1}]$ we have $x \le L(x) \le x + n$. In particular, $L[I_{2n+1}] \subseteq [a_{2n+1}, a_{2n+2} + (n+1)]$ and $[a_{2n} + n, a_{2n+1}] \subseteq L[I_{2n}]$, since $I_{2n+1} \subset [0, a_{2(n+1)+1}]$ and $I_{2n} \subset [0, a_{2n+1}]$. Hence,

$$\begin{aligned}
\tfrac{2}{3}L[I_{2n+1}] &\subset \tfrac{2}{3}[a_{2n+1}, a_{2n+2} + (n+1)] \\
&= \tfrac{2}{3}\left[9^{n+1} - \tfrac{1}{n+4}, 9^{n+1} + \tfrac{1}{n+4} + (n+1)\right] \\
&\subseteq \left[9^n + \tfrac{1}{n+3} + n, 9^{n+1} - \tfrac{1}{n+4}\right] \qquad (7)\\
&\subseteq [a_{2n} + n, a_{2n+1}] \subseteq L[I_{2n}],
\end{aligned}$$

where (7) is justified by the inequalities as $9^n + \frac{1}{n+3} + n \le \frac{2}{3}\left(9^{n+1} - \frac{1}{n+4}\right)$ and $\frac{2}{3}\left(9^{n+1} + \frac{1}{n+4} + (n+1)\right) \le 9^{n+1} - \frac{1}{n+4}$, which hold for any $n < \omega$. Therefore, $f[I_{2n+1}] = L^{-1}\left[\frac{2}{3}L[I_{2n+1}]\right] \subseteq L^{-1}\left[L[I_{2n}]\right] = I_{2n}$ and (6) is proved.

Now, using (6), we can see that f is not (ULS). Indeed, for the endpoints y and z of I_{2n+1}, we have $\rho(y,z)=|y-z|=\frac{2}{n+4}\to 0$, as $n\to\infty$, and

$$\rho(f(y),f(z))=|L(f(y))-L(f(z))|=\tfrac{2}{3}|L(y)-L(z)|=\tfrac{2}{3}\pi\tfrac{|y-z|}{2}>\rho(y,z),$$

finishing the argument. ∎

By Theorem 3.3, the map from Example 13 must have fixed point while, by Theorem 4.1, the metric ρ in the example cannot be the standard metric (as, in such case, (uPC)⇒(C)⇒(uLC)). Also, by Theorem 3.2(i), if $f\colon X\to X$ is as in the example, then X cannot compact.

Example 13. *There exists a map $f\colon\langle[0,\infty),\rho\rangle\to\langle[0,\infty),\rho\rangle$ from the class* (S)&(LC)&(uPC)&¬(uLC)*, where $\langle[0,\infty),\rho\rangle$ is rectifiable path connected and topological equivalent to $[0,\infty)$ with the standard metric. The map has a fixed point, $f(0)=0$.*

Construction. Choose a sequence $0=a_0<a_1<\cdots$ such that if, for every $n<\omega$, we put $I_n=[a_n,a_{n+1}]$, then each interval I_{2n+1} has length 2. Moreover, the centers c_n of intervals I_{2n+1} are chosen to ensure

$$[a_{2k+1},a_{2k+2}+2(k+1)]\cap\tfrac{1}{2}[a_{2n+1},a_{2n+2}+2(n+1)]=\emptyset \text{ for every } k,n<\omega. \quad (8)$$

For example, (8) is satisfied when pick $c_n=9^{n+1}$.

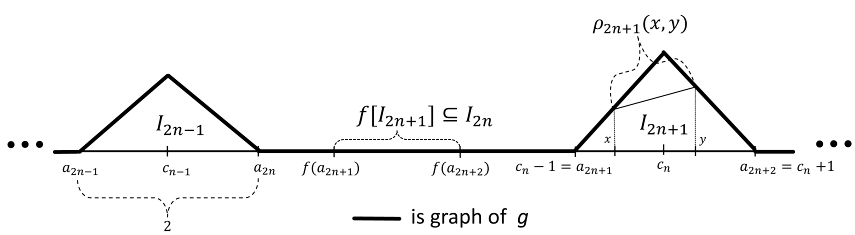

Figure 15: Illustration of the graph of g from Example 13 for which the map f is is (S), (uPC), and (LC) but not (uLC). Notice that $f[I_{2n+1}]\subseteq I_{2n}$.

Choose an increasing sequence $\langle m_n\colon n<\omega\rangle$ of positive numbers for which

$$\lim_{n\to\infty}\sqrt{1+m_n^2}=2 \quad (9)$$

and define $g\colon[0,\infty)\to\mathbb{R}$ via formula (see Figure 15)

$$g(x)=\begin{cases} m_n\operatorname{dist}(x,\{a_{2n+1},a_{2n+2}\}) & \text{when } x\in I_{2n+1} \text{ for some } n<\omega,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that the segments forming the graph of $g\restriction I_{2n+1}$ are the sides of isosceles triangles (with basis of length 2) which are approaching the sides of an equilateral triangle, as $n\to\infty$.

For each $n < \omega$ let ρ_n be a metric on I_n defined as

$$\rho_n(x, y) = \|\langle x, g(x)\rangle - \langle y, g(y)\rangle\|$$

and let ρ be the metric on $[0, \infty)$ from Lemma 7.1 associated with metrics $\{\rho_n : n < \omega\}$. Then, ρ is complete and, clearly, rectifiable path connected.

Define an increasing bijection $L: [0, \infty) \to [0, \infty)$ as $L(x) = \ell(g \restriction [0, x])$. Then, for every $n < \omega$,

$$L[I_{2n+1}] \subset [a_{2n+1}, a_{2n+2} + 2(n + 1)] \tag{10}$$

as $\ell(g \restriction I_{2n+1}) < 4$. Let $f: [0, \infty) \to [0, \infty)$ be defined as $f(x) = L^{-1}\big(\frac{1}{2}L(x)\big)$. In other words, $f(x)$ is the unique point $r \in [0, \infty)$ such that $\ell(g \restriction [0, r]) = \frac{1}{2}\ell(g \restriction [0, x])$. We have

$$L(f(x)) = \frac{1}{2}L(x). \tag{11}$$

We claim that f is as needed.

To see this, first notice that, by (8) and (10), for every $k, n < \omega$,

$$\begin{aligned} f[I_{2n+1}] \cap I_{2k+1} &= L^{-1}\left[\tfrac{1}{2}L[I_{2n+1}] \cap L[I_{2k+1}]\right] \\ &\subseteq L^{-1}\left[\tfrac{1}{2}[a_{2n+1}, a_{2n+2} + 2(n + 1)] \cap [a_{2k+1}, a_{2k+2} + 2(k + 1)]\right] = \emptyset. \end{aligned}$$

In particular, for all $n < \omega$,

$$f[I_{2n+1}] \cap \textstyle\bigcup_{k<\omega} I_{2k+1} = \emptyset. \tag{12}$$

The key fact for this construction is the following property.

($*$) For every $y \in (0, 1]$ the mapping $[0, 1] \ni x \stackrel{\eta}{\mapsto} \frac{|L(c_n - x) - L(c_n + y)|}{\rho_n(c_n - x, c_n + y)}$ achieves its maximum value $\sqrt{1 + m_n^2}$ for $x = y$.

(For the proof, put $m_n = m$. Then $\eta(x) = \frac{\sqrt{1+m^2}(x+y)}{\sqrt{(x+y)^2+m^2(x-y)^2}}$ and η has only one critical point, at $x = y$, as $\eta'(x) = \frac{2m^2\sqrt{1+m^2}y(y-x)}{((x+y)^2+m^2(x-y)^2)^{3/2}}$.[3] This is the maximum by the First Derivative Test.)

Now, for any $p, q \in I_{2n+1}$, $\rho(f(p), f(q)) = |f(p) - f(q)| = |L(f(p)) - L(f(q))|$, since, by (12), $f(p), f(q) \in f[I_{2n+1}] \subset [0, \infty) \setminus \bigcup_{k<\omega} I_{2k+1}$. From this, (11) and ($*$) we conclude that, for any $p, q \in I_{2n+1}$,

$$\rho(f(p), f(q)) = |L(f(p)) - L(f(q))| = \tfrac{1}{2}|L(p) - L(q)| \le \tfrac{\sqrt{1+m_n^2}}{2}\rho(p, q), \tag{13}$$

[3]Here is the computation:

$$\begin{aligned} \eta'(x) &= \sqrt{1+m^2}\,\frac{\sqrt{(x+y)^2+m^2(x-y)^2} - \frac{2(x+y)+2m^2(x-y)}{2\sqrt{(x+y)^2+m^2(x-y)^2}}(x+y)}{(x+y)^2+m^2(x-y)^2} \\ &= \frac{\sqrt{1+m^2}\left([(x+y)^2+m^2(x-y)^2] - [(x+y)^2+m^2(x^2-y^2)]\right)}{((x+y)^2+m^2(x-y)^2)^{3/2}} \\ &= \frac{\sqrt{1+m^2}m^2[(x-y)^2-(x^2-y^2)]}{((x+y)^2+m^2(x-y)^2)^{3/2}} = \frac{2m^2\sqrt{1+m^2}y(y-x)}{((x+y)^2+m^2(x-y)^2)^{3/2}}. \end{aligned}$$

with equation holding when p and q are symmetric with respect to the point $x = c_n$. This clearly shows that f is not (uLC), as $\frac{\sqrt{1+m_n^2}}{2} \to_{n\to\infty} 1$.

Notice also that (13) implies that $f \restriction I_{2n+1}$ is (C) for any $n < \omega$. Moreover, for any $n < \omega$, $f \restriction I_{2n}$ is (C) with the constant $\frac{1}{2}$, since for any numbers $y, z \in I_{2n}$,

$$\rho(f(z), f(y)) \leq |L(f(z)) - L(f(y))| = \tfrac{1}{2}|L(z) - L(y)| = \tfrac{1}{2}\rho(z, y). \tag{14}$$

Therefore, by parts (S) and (LC) of Lemma 7.1, f is (S) and (LC).

Finally, notice that, for any $n < \omega$, each $f \restriction [a_{2n+1}, c_n]$ and $f \restriction [c_n, a_{2n+2}]$ is (C) with the constant $\frac{1}{2}$ since for y and z belonging to one of these intervals, the formula (14) holds. In particular, for any $n < \omega$, $f \restriction I_{2n+1}$ is (uPC) with the constant $\frac{1}{2}$. So, by (14) and Lemma 7.1(uPC), f is (uPC). ■

Notice that, by Theorem 4.1, the metric ρ in Example 14 cannot be the standard metric (as, in such case, (uPC)⇒(C)⇒(LC)).

Example 14. *There exist an $a > 0$ and a map $f : \langle [0, a], \rho\rangle \to \langle [0, a], \rho\rangle$ from* (uPC)&(S)&¬(LC)*, where $\langle [0, a], \rho\rangle$ is a rectifiable path connected and topological equivalent to $[0, a]$ with the standard metric. The map f has a fixed point, $f(a) = a$.*

Construction. Choose a sequence $0 = a_0 < a_1 < \cdots$ such that if, for every $n < \omega$, we put $I_n = [a_n, a_{n+1}]$, then each interval I_{2k} has length $|I_{2k}| = 2^{-2k}$ and each interval I_{2k+1} has length $\frac{k+8}{k+9}2^{-2k}$. In particular, $a = \lim_{n\to\infty} a_n$ is finite.

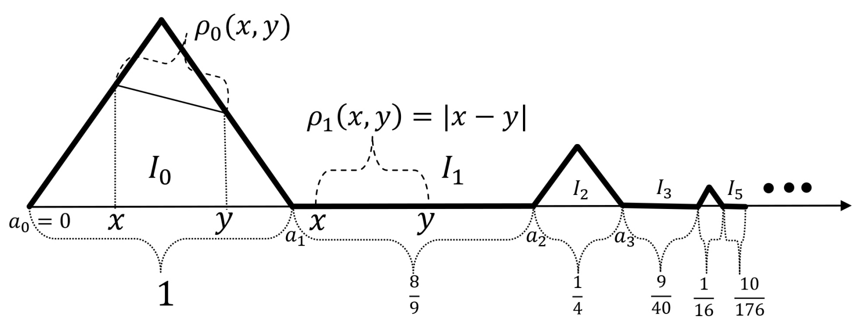

Figure 16: Illustration of the graph of g from Example 14 for which the map f is (S) and (uPC) but not (LC).

Define function $g : [0, a] \to \mathbb{R}$ for every $x \in [0, a]$ as

$$g(x) = \begin{cases} \sqrt{3}\,\mathrm{dist}(x, \{a_{2k}, a_{2k+1}\}) & \text{if } x \in I_{2k} \text{ for some } k < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 16. Thus, the two segments forming $g \restriction I_{2k}$ constitute the sides of a equilateral triangle and so

$$\ell(g \restriction [x, y]) = 2|x - y| \quad \text{for every } x < y \text{ from } I_{2k}. \tag{15}$$

For each $n < \omega$ let ρ_n be a metric on I_n defined as

$$\rho_n(x, y) = \|\langle x, g(x)\rangle - \langle y, g(y)\rangle\|$$

and let ρ be the metric on $[0, a]$ from Lemma 7.1 associated with the metrics $\{\rho_n : n < \omega\}$. Then, ρ is complete and, clearly, rectifiable path connected.

Define $f : [0, a] \to [0, a]$ as an increasing function mapping linearly each interval I_n onto I_{n+1}. (So, $f(a) = a$.) We claim that this f is as required.

Indeed, f is not (LC), as it is not (C) on any open neighborhood of a, since

$$\frac{\rho(f(a_{2k+1}), f(a_{2k}))}{\rho(a_{2k+1}, a_{2k})} = \frac{a_{2k+2} - a_{2k+1}}{a_{2k+1} - a_{2k}} = \frac{k+8}{k+9} \to_{k\to\infty} 1.$$

To see the property (S) notice that $f \restriction I_n$ is (S), even (C), for every $n < \omega$. Indeed, by (15) and equation $\frac{a_{2k+3}-a_{2k+2}}{a_{2k+2}-a_{2k+1}} = \frac{2^{-2(k+1)}}{\frac{k+8}{k+9}2^{-2k}} = \frac{1}{4}\frac{k+9}{k+8}$, for every $x < y$ from I_{2k+1} we have

$$\begin{aligned} \rho(f(x), f(y)) \le \ell(g \restriction [f(x), f(y)]) = 2|f(x) - f(y)| \\ = \tfrac{1}{2}\tfrac{k+9}{k+8}|x - y| \le \tfrac{9}{16}|x - y| = \tfrac{9}{16}\rho(x, y), \quad (16) \end{aligned}$$

while for every $x < y$ from I_{2k},

$$\rho(f(x), f(y)) = |f(x) - f(y)| = \tfrac{k+8}{k+9}|x - y| \le \tfrac{8}{9}\rho(x, y).$$

In particular, by Lemma 7.1, f is (S) on every interval $[0, a_n]$ and so, on their union $[0, a)$. To finish the argument, it is enough to notice that this implies that f is (S) on the entire $[0, a]$. Indeed, choose an $x \in [0, a)$. To see that $\rho(x, a) > \rho(f(x), f(a))$ choose an $n < \omega$ such that $x < a_n$. For every $m > n$ we have $\rho(a_n, a_m) > \rho(f(a_n), f(a_m))$ so, taking limit as $m \to \infty$, we get $\rho(a_n, a) \ge \rho(f(a_n), f(a))$. So,

$$\rho(x, a) = \rho(x, a_n) + \rho(a_n, a) > \rho(f(x), f(a_n)) + \rho(f(a_n), f(a)) \ge \rho(f(x), f(a))$$

as needed.

To see that f is (uPC), first notice that the maps $f \restriction I_n$ are (uPC) with the same constant: for odd n with constant $\frac{9}{16}$, as follows from (16); for even $n = 2k$ with constant $\frac{8}{18}$, since for every $x < y$ from the same half of I_{2k},

$$\rho(f(x), f(y)) = |f(x) - f(y)| = \tfrac{k+8}{k+9}|x - y| = \tfrac{k+8}{k+9}\tfrac{1}{2}\ell(g \restriction [x, y]) \le \tfrac{8}{18}\rho(x, y).$$

Thus, by Lemma 7.1, $f \restriction [0, a)$ is (uPC). To finish the proof, it is enough to prove that f is (PC) at a, which will be achieved by finding a $\lambda \in [0, 1)$ such that $\frac{\rho(f(x),f(a))}{\rho(x,a)} \le \lambda$ for all $x \in [0, a)$.

For this, fix an $x \in I_n \subset I_{2k} \cup I_{2k+1}$ and notice that

$$\rho(a, x) = \rho(a, a_{n+1}) + \rho(a_{n+1}, x) \le \rho(a, a_{n+1}) + |a_{n+1} - a_n| = a - a_n \le a - a_{2k}.$$

Hence, $\rho(a, x) \le a - a_{2k} \le 2\sum_{i=k}^{\infty} 4^{-i} = \frac{8}{3}4^{-k}$.

Next, we will show that there exists an $\alpha > 0$, independent of k, such that

$$N(x) = \rho(a, x) - \rho(a, f(x)) \geq \alpha 4^{-k} \quad \text{for every } x \in I_{2k} \cup I_{2k+1}. \tag{17}$$

Indeed, if $x \in I_{2k}$, then we have $N(x) = \rho(a_{2k+1}, x) + \rho(a_{2k+1}, f(x)) \geq |a_{2k+1} - x| + |a_{2k+1} - f(x)| = f(x) - x \geq |I_{2k+1}| = \frac{k+8}{k+9} 2^{-2k} \geq \frac{8}{9} 4^{-k}$ indicating that any $\alpha \leq \frac{8}{9}$ works for this case. On the other hand, if $x = a_{2k+2} - t|I_{2k+1}| \in I_{2k+1}$ for some $t \in [0,1]$, then

$$\begin{aligned} N(x) &\geq t|I_{2k+1}| + |I_{2k+2}| - \rho(f(x), a_{2k+3}) \\ &\geq t|I_{2k+1}| + |I_{2k+2}| - \ell(g \restriction [f(x), a_{2k+3}]) \\ &= t\tfrac{k+8}{k+9} 4^{-k} + \tfrac{1}{4} 4^{-k} - 2t\tfrac{1}{4} 4^{-k} \geq t\tfrac{8}{9} 4^{-k} + \tfrac{1}{4} 4^{-k} - t\tfrac{1}{2} 4^{-k} \geq \tfrac{1}{4} 4^{-k}, \end{aligned}$$

showing that (17) holds with the constant $\alpha = \frac{1}{4}$.

Now, to finish the proof of (uPC) for f notice that, by (17),

$$\frac{\rho(f(a), f(x))}{\rho(a, x)} = 1 - \frac{\rho(a, x) - \rho(f(a), f(x))}{\rho(a, x)} \leq 1 - \frac{\alpha 4^{-k}}{\frac{8}{3} 4^{-k}} = 1 - \frac{3\alpha}{4}.$$

So, $\lambda = 1 - \frac{3\alpha}{8}$, with $\alpha = \frac{1}{4}$, is as needed. ■

All remaining examples presented in Section 7.2 will be constructed on the space $\langle [0, \infty), \rho \rangle$ with ρ obtained using Lemma 7.1 with the family $\mathcal{J} = \{(a_n, a_{n+1}) \colon n < \omega\}$, where $0 = a_0 < a_1 < a_2 < \cdots$ and $a_n \to_{n \to \infty} \infty$. Moreover, the constructed mappings f will be non-decreasing and mapping each interval $I_n = [a_n, a_{n+1}]$ onto I_{n+1}.

Notice that, by Theorem 3.2(i), if $f \colon X \to X$ is as in Example 15, then X cannot compact. Also, by Theorem 4.1, the metric ρ in the example cannot be the standard metric (as, in such case, (LC)⇒(uPC)⇒(C)⇒(ULS)).

Example 15. *There exists a map $f \colon \langle [0, \infty), \rho \rangle \to \langle [0, \infty), \rho \rangle$ from the class* (LC)&¬(ULS) *having no periodic point, where $\langle [0, \infty), \rho \rangle$ is rectifiable path connected and topological equivalent to $[0, \infty)$ with the standard metric.*

Construction. Choose a sequence $0 = a_0 < a_1 < \cdots$ such that if, for every $n < \omega$, we put $I_n = [a_n, a_{n+1}]$, then each interval I_n has the length $\frac{1}{n+1}$ when n is even and the length $\frac{2}{\pi} \frac{1}{n+1}$ when n is odd. This ensures that $a_n \to_{n \to \infty} \infty$. Let $g \colon [0, \infty) \to [0, 1]$ be such that $g \restriction I_n \equiv 0$ for every even $n < \omega$, while, for every odd $n < \omega$, the graph of $g \restriction I_n$ is an an upper semicircle with its diameter coinciding with I_n, see Figure 17. Notice that our choice ensures that $\ell(g \restriction I_n) = \frac{1}{n+1}$ for every $n < \omega$.

For each $n < \omega$ let ρ_n be a metric on I_n defined as

$$\rho_n(x, y) = \| \langle x, g(x) \rangle - \langle y, g(y) \rangle \|$$

and let ρ be the metric on $[0, \infty)$ from Lemma 7.1 associated with the metrics $\{\rho_n \colon n < \omega\}$. Then, ρ is complete and, clearly, rectifiable path connected.

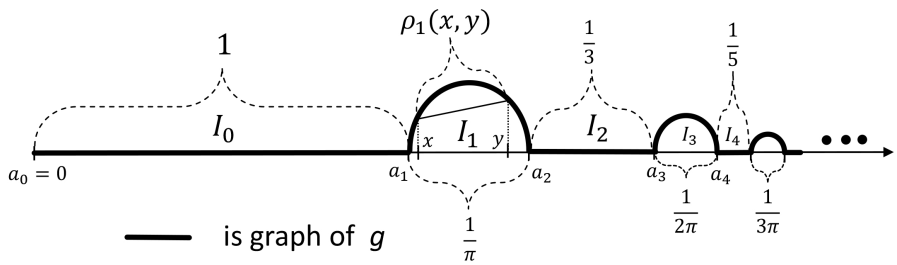

Figure 17: Illustration of the graph of g from Example 15 for which the map f is (LC) but not (ULS).

For every $n<\omega$ and $x\in I_n$ let $f(x)$ be the unique point $r\in I_{n+1}$ such that

$$\ell(g\restriction[a_{n+1},r])=\tfrac{\ell(g\restriction I_{n+1})}{\ell(g\restriction I_n)}\ell(g\restriction[a_n,x])=\tfrac{n+1}{n+2}\ell(g\restriction[a_n,x]).$$

In other words $f\colon[0,\infty)\to[0,\infty)$ maps each I_n onto I_{n+1} linearly, according to the length ℓ of $g\restriction I_n$ and $g\restriction I_{n+1}$. We claim that this f is as required.

Clearly f has no periodic point as $f(x)>x$ for all $x\geq 0$. Notice that the equation above implies that for any $x<y$ from I_n,

$$\ell(g\restriction[f(x),f(y)])=\frac{n+1}{n+2}\ell(g\restriction[x,y]). \tag{18}$$

To prove that f is (LC), by Lemma 7.1(LC), it is enough to prove that $f\restriction I_n$ is (LC) for every $n<\omega$. The argument depends on the parity of n. If $n<\omega$ is even, then $f\restriction I_n$ is (C) with $\lambda=\frac{n+1}{n+2}$. This follows from the fact that, in this case, $g\restriction I_n\equiv 0$ and, for any $x,y\in I_n$ with $x<y$, by (18) we have

$$\begin{array}{rcl}\rho(f(x),f(y)) & = & \|\langle f(x),g(f(x))\rangle-\langle f(y),g(f(y))\rangle\|\\ & < & \ell(g\restriction[f(x),f(y)])\\ & = & \frac{n+1}{n+2}\ell(g\restriction[x,y])=\frac{n+1}{n+2}\rho(x,y).\end{array}$$

So, turn to the case when $n<\omega$ is odd. We need to refine the argument above, as, in this case, $f\restriction I_n$ is only (ULC). To see this choose an $\eta\in(1,1.5)$ such that $\frac{n+1}{n+2}\eta<1$. Notice that $g\restriction I_n$ is a semicircle of length $\frac{1}{n+1}$ and that there exists an $\alpha\in\left(0,\frac{1}{n+1}\right)$ such that $\rho(x,y)\geq\eta^{-1}\ell(g\restriction[x,y])$ whenever $x\leq y$ are from I_n and such that $\ell(g\restriction[x,y])\leq\alpha$ (as $\frac{\ell(g\restriction[x,y])}{\rho(x,y)}\to 1$ when $\rho(x,y)\to 0$). Moreover, there exists an $\varepsilon>0$ such that $\ell(g\restriction[x,y])\leq\alpha$ whenever $x\leq y$ are from I_n and such that $|x-y|\leq\varepsilon$.[4] Then $f\restriction I_n$ is $\left(\frac{\varepsilon}{2},\frac{n+1}{n+2}\eta\right)$-(ULC): for all

[4]If K is the family of all arcs on $g\restriction I_n$ of length α and, for every $\kappa\in K$, $p(\kappa)$ is the projection of κ on the x-axis, then the number ε the minimizer of the values of the continuous mapping $K\ni\kappa\mapsto\ell(p(\kappa))\in(0,\infty)$ defined on the compact space K.

$x \le y$ from I_n with $|x-y| \le \varepsilon$ we have $\eta\, \rho(x,y) \ge \ell(g \restriction [x,y])$ and

$$\begin{aligned}\rho(f(x),f(y)) &= \|\langle f(x), g(f(x))\rangle - \langle f(y), g(f(y))\rangle\| \\ &= \ell(g \restriction [f(x),f(y)]) = \frac{n+1}{n+2}\,\ell(g \restriction [x,y]) \le \frac{n+1}{n+2}\,\eta\, \rho(x,y).\end{aligned}$$

The function f is not (ULS) since, for all odd indices $n > \frac{\pi+4}{\pi-2}$,

$$\rho(f(a_n), f(a_{n+1})) = \rho(a_{n+1}, a_{n+2}) = \frac{1}{n+2} > \frac{2}{\pi(n+1)} = \rho(a_n, a_{n+1})$$

and, at the same time, $\rho(a_n, a_{n+1}) = \frac{2}{\pi(n+1)} \to 0$ as $n \to \infty$. ■

7.2.3 Using non-rectifiably path connected metrics

The remaining examples on connected spaces will be constructed with the use of the following lemma, which is extracted from an example given by Hu and Kirk in [21, p. 123]. It is not difficult to see that the metrics from this lemma are not rectifiably path connected. In what follows the length of an interval I is denoted as $|I|$.

Lemma 7.2. *Let $0 < \beta_0 < \beta_1 < 1$ and let f be a linear function from $I_0 = [a_0, b_0]$ onto $I_1 = [a_1, b_1]$. For each $i < 2$ let $\rho_i \colon I_i \to \mathbb{R}$ be defined by a formula*

$$\rho_i(x,y) = |I_i| \left(\frac{|x-y|}{|I_i|}\right)^{\beta_i}.$$

Then ρ_i is a complete metric on I_i topologically equivalent to the standard metric. The map $f \colon \langle I_0, \rho_0\rangle \to \langle I_1, \rho_1\rangle$ is Lipschitz with the constant $L = |I_1|/|I_0|$. It is also (ULC) *with each constant $\lambda \in (0,1)$.*

Proof. Clearly a sequence in I_i is Cauchy with respect to the metric ρ_i if, and only if, it is Cauchy with respect to the standard metric on I_i. Thus, indeed ρ_i is a complete metric on I_i topologically equivalent to the standard metric.

To see the second part notice that for every $x, y \in I_0$, the linearity of f implies that $\frac{|f(x)-f(y)|}{|I_1|} = \frac{|x-y|}{|I_0|}$. Hence

$$\rho_1(f(x),f(y)) = |I_1| \left(\tfrac{|x-y|}{|I_0|}\right)^{\beta_1} = \tfrac{|I_1|}{|I_0|}|I_0| \left(\tfrac{|x-y|}{|I_0|}\right)^{\beta_1} = \tfrac{|I_1|}{|I_0|} \left(\tfrac{|x-y|}{|I_0|}\right)^{\beta_1-\beta_0} \rho_0(x,y).$$

Thus, the inequality $\frac{|I_1|}{|I_0|}\left(\frac{|x-y|}{|I_0|}\right)^{\beta_1-\beta_0} \le \frac{|I_1|}{|I_0|}$ implies the Lipschitz condition statement. Also, for every $\lambda \in (0,1)$, we have $\frac{|I_1|}{|I_0|}\left(\frac{|x-y|}{|I_0|}\right)^{\beta_1-\beta_0} \le \lambda$ if, and only if, $\rho_0(x,y) = |I_0|\left(\frac{|x-y|}{|I_0|}\right)^{\beta_0} \le |I_0|\left(\frac{|I_0|}{|I_1|}\lambda\right)^{\frac{\beta_0}{\beta_1-\beta_0}}$. Therefore, f is (ULC) with a constant λ for $\varepsilon = \frac{1}{2}|I_0|\left(\frac{|I_0|}{|I_1|}\lambda\right)^{\frac{\beta_0}{\beta_1-\beta_0}}$. ■

Notice that, by Theorems 3.2(i) and 3.3, if $f\colon X \to X$ is as in Example 16, then X can be neither compact nor rectifiably path connected. An example of a periodic free mapping $f\colon X \to X$ from the class (uLC)&¬(ULC) is also given in [30, example 1], where X is a (non-rectifiable, non-compact) curve of $\mathbb{R}^2$ considered with the standard metric. However, this example is not (S), since for every $n<\omega$ it maps $(n,0)\in X$ to $(n+1,0)\in X$.

Example 16. *There exists a map* $f\colon \langle [0,\infty),\rho\rangle \to \langle [0,\infty),\rho\rangle$ *from the class* (S)&(uLC)&¬(ULC) *having no periodic point, where* $\langle [0,\infty),\rho\rangle$ *is a complete metric topologically equivalent to the standard metric. Moreover, f satisfies* (uLC) *with every contraction constant* $\lambda\in(0,1)$.

Construction. Choose strictly increasing sequence $\langle \beta_n \in (0,1)\colon n<\omega\rangle$ and let $0=a_0<a_1<\cdots$ be such that each interval $I_n=[a_n,a_{n+1}]$ has the length $\frac{1}{n+1}$. For every $n<\omega$, let ρ_n be a metric on I_n defined by formula $\rho_n(x,y)=|I_n|\left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ and let ρ be the metric on $[0,\infty)$ from Lemma 7.1 associated with the metrics $\{\rho_n\colon n<\omega\}$. Then, ρ is complete and, clearly, path connected. On each interval I_n define f as a linear increasing map onto I_{n+1}. Then f is as needed.

Indeed, by Lemma 7.2, for every $n<\omega$ the restriction $f\restriction I_n$ is (C) with a constant $\frac{|I_{n+1}|}{|I_n|}=\frac{n}{n+1}$, so it is (S). Hence, by Lemma 7.1, f is (S).

Next fix a $\lambda\in(0,1)$. Then, by Lemma 7.2, for every $n<\omega$ the restriction $f\restriction I_n$ is (ULC) with constant λ, so it also (uLC) with the same constant. Hence, by Lemma 7.1, f is (uLC).

Finally, f is not (ULC) since for every $\lambda\in(0,1)$ and $\varepsilon>0$ there is an $n<\omega$ with $\frac{\rho(f(a_n),f(a_{n+1}))}{\rho(a_n,a_{n+1})}=\frac{|I_{n+1}|}{|I_n|}=\frac{n+1}{n+2}>\lambda$ and $\rho(a_n,a_{n+1})=|I_n|=\frac{1}{n+1}<\varepsilon$. Clearly, f has no periodic points. ■

Notice that, by Theorem 3.3, if $f\colon X\to X$ is as in Example 17, then X cannot be rectifiably path connected. Also, by Theorem 4.2, X cannot compact.

Example 17. *There exists a map* $f\colon \langle [0,\infty),\rho\rangle \to \langle [0,\infty),\rho\rangle$ *from the class* (ULS)&(uLC)&¬(S) *having no periodic point, where* $\langle [0,\infty),\rho\rangle$ *is a complete metric topologically equivalent to the standard metric. Moreover, f satisfies* (uLC) *with every contraction constant* $\lambda\in(0,1)$.

Construction. Choose strictly increasing sequence $\langle \beta_n \in (0,1)\colon n<\omega\rangle$ and let $0=a_0<a_1<\cdots$ be such that if, for every $n<\omega$, we put $I_n=[a_n,a_{n+1}]$, then $|I_0|=1$ and $|I_n|=\frac{1}{n}$ for every $0<n<\omega$. For every $n<\omega$, let ρ_n be a metric on I_n defined by formula $\rho_n(x,y)=|I_n|\left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ and let ρ be the metric on $[0,\infty)$ from Lemma 7.1 associated with the metrics $\{\rho_n\colon n<\omega\}$. Then, ρ is complete and, clearly, path connected. On each interval I_n define f as a linear increasing map onto I_{n+1}. Then f is as needed.

To see that f is (ULS) first notice that, by Lemma 7.2, for every $n>0$ the map $f\restriction I_n$ is (S), as it is (C) with constant $\frac{n}{n+1}$. So, by Lemma 7.1, $f\restriction[1,\infty)$

is (S). Also, by Lemma 7.2, $f \restriction I_0$ is (ULC), so also (ULS) with some constant radius ε. Thus, f is (ULS) with the same radius.

To see (uLC), fix $\lambda \in (0,1)$ and notice that, by Lemma 7.2, for every $n < \omega$ the restriction $f \restriction I_n$ is (ULC) with constant λ, so it also (uLC) with the same constant. Hence, by Lemma 7.1, f is (uLC).

Finally, f is not (S), as $\rho(f(a_0), f(a_1)) = \rho(a_1, a_2) = |I_1| = 1 = \rho(a_0, a_1)$. Clearly f has no periodic points. ■

Notice that, by Theorem 3.3, if $f\colon X \to X$ is as in Example 18, then X cannot be rectifiably path connected. Also, by Theorem 4.2 (that any (LS) map on a compact space is also (ULS)), X cannot compact.

Example 18. *There exists a map $f\colon \langle[0,\infty),\rho\rangle \to \langle[0,\infty),\rho\rangle$ from the class* (uLC)&¬(ULS) *having no periodic point, where $\langle[0,\infty),\rho\rangle$ is a complete metric topologically equivalent to the standard metric. Moreover, f satisfies* (uLC) *with every contraction constant $\lambda \in (0,1)$.*

Construction. Choose strictly increasing sequence $\langle \beta_n \in (0,1)\colon n < \omega\rangle$ and let $0 = a_0 < a_1 < \cdots$ be such that if, for every $n < \omega$, we put $I_n = [a_n, a_{n+1}]$, then the intervals I_{2k} and I_{2k+1} have length $\frac{1}{k+1}$ for every $k < \omega$. For every $n < \omega$, let ρ_n be a metric on I_n defined by formula $\rho_n(x,y) = |I_n| \left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ and let ρ be the metric on $[0,\infty)$ from Lemma 7.1 associated with the metrics $\{\rho_n\colon n < \omega\}$. Then, ρ is complete and, clearly, path connected. On each interval I_n define f as a linear increasing map onto I_{n+1}. Then f is as needed.

To see this, fix $\lambda \in (0,1)$ and notice that, by Lemma 7.2, for every $n < \omega$ the restriction $f \restriction I_n$ is (ULC) with constant λ, so it also (uLC) with the same constant. Hence, by Lemma 7.1, f is (uLC).

At the same time, f is not (ULS) since for every $\varepsilon > 0$ there is a $k < \omega$ with $\frac{\rho(f(a_{2k}), f(a_{2k+1}))}{\rho(a_{2k}, a_{2k+1})} = \frac{|I_{2k+1}|}{|I_{2k}|} = 1$ and $\rho(a_{2k}, a_{2k+1}) = |I_{2k}| = \frac{1}{k+1} < \varepsilon$. Clearly f has no periodic points. ■

Notice that, by Theorem 3.3, if $f\colon X \to X$ is as in Example 19, then X cannot be rectifiably path connected. Also, by Theorem 3.2(i), X cannot compact.

Example 19. *There is a map f from* (S)&(LC)&(uPC)&¬(uLC) *having no periodic point, where $f\colon \langle[0,\infty),\rho\rangle \to \langle[0,\infty),\rho\rangle$ and ρ is a complete metric on $[0,\infty)$ topologically equivalent to the standard metric. Moreover, f satisfies* (uPC) *with an arbitrary constant $\lambda \in (0,1)$. Also, there exists a perfect unbounded $X \subset \mathbb{R}$ such that ρ on X is the standard metric on $\mathbb{R}$ and $f \restriction X$ belongs to the same class.*

Construction. Choose a sequence $0 = a_0 < a_1 < \cdots$ such that each interval $I_n = [a_n, a_{n+1}]$ has length $\frac{1}{n+1}$. Define a function $h\colon [0,\infty) \to [0,\infty)$, approximating f, by putting, for every $x \in I_n$,

$$h(x) = \frac{1}{n+2}\left[(n+1)(x-a_n)\right]^{\frac{n+1.5}{n+1}} + a_{n+1}.$$

Notice that h is strictly increasing and maps every I_n onto I_{n+1}. Moreover, the maps $h_n = h \restriction I_n$ are convex, differentiable, and with derivatives $h_n'(a_n) = 0$ and $s_n = h_n'(a_{n+1}) = \frac{n+1.5}{n+2}$. It is important for the construction that

$$s_n = h_n'(a_{n+1}) \nearrow 1 \text{ as } n \to \infty.$$

Choose a sequence $a_1 = c_0^0 > c_1^0 > \cdots$ converging to $a_0 = 0$ such that $\frac{c_{k+1}^0}{c_k^0} \to_{k\to\infty} 1$, e.g., $c_k^0 = \frac{a_1}{k+1}$, and, for every $n, k < \omega$, let $c_k^n = h^{(n)}(c_k^0)$, where $h^{(n)} = h \circ \cdots \circ h$ is the nth iteration of h. Since, as an easy induction on $0 < n < \omega$ can show, $h^{(n)} \restriction I_0$ is given, for every $x \in I_0$, by a formula

$$h^{(n)}(x) = \tfrac{1}{n+1} x^{\alpha_n} + a_n, \quad \text{ where } \alpha_n = \textstyle\prod_{i=1}^n \frac{n+0.5}{n},$$

we have

$$\frac{c_{k+1}^n - a_n}{c_k^n - a_n} \to_{k\to\infty} 1 \quad \text{ for every } n < \omega. \tag{19}$$

Indeed, for $n = 0$ this is ensured by our choice of numbers c_k^0, while, for $n > 0$, we have $\frac{c_{k+1}^n - a_n}{c_k^n - a_n} = \frac{\frac{1}{n+1}(c_{k+1}^0)^{\alpha_n}}{\frac{1}{n+1}(c_k^0)^{\alpha_n}} = \left(\frac{c_{k+1}^0}{c_k^0}\right)^{\alpha_n} \to_{k\to\infty} 1$.

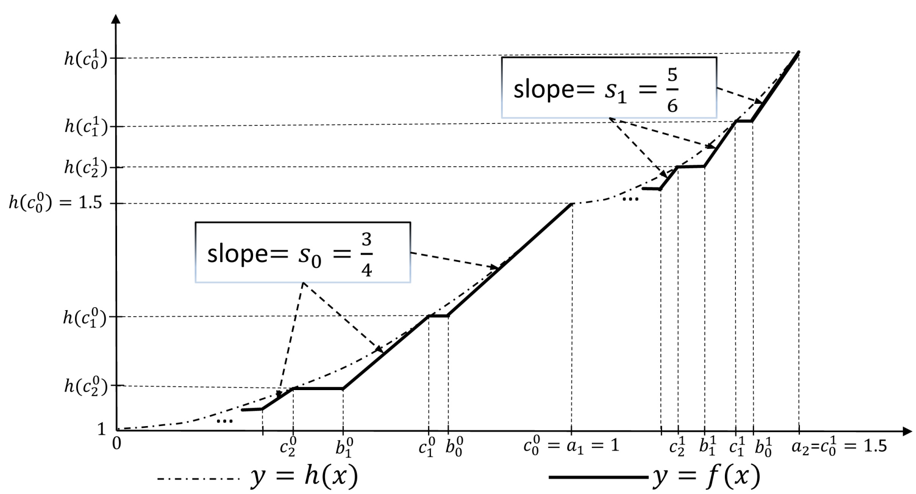

Figure 18: Functions h and f from Example 19 restricted to the interval $[a_0, a_2]$.

For every $n, k < \omega$, choose unique $b_k^n \in (c_{k+1}^n, c_k^n)$ such that the slope of the segment joining points $\langle b_k^n, h(c_{k+1}^n)\rangle$ and $\langle c_k^n, h(c_k^n)\rangle$ is equal to s_n. The map f is defined as

$$f(x) = \begin{cases} h(x) & \text{for } x = c_k^n \text{ for some } k, n < \omega \\ h(c_{k+1}^n) & \text{for } x \in [c_{k+1}^n, b_k^n] \text{ for some } k, n < \omega \end{cases}$$

and as a linear function on each interval $[b_k^n, c_k^n]$, see Figure 18.

To define metric ρ, choose an increasing sequence $\langle \beta_n \in (0,1) \colon n<\omega\rangle$. For every $k,n<\omega$ define metric ρ^n_k on the closure of the interval $J^n_k=(b^n_k,c^n_k)$ by a formula

$$\rho^n_k(x,y)=|J^n_k|\left(\frac{|x-y|}{|J^n_k|}\right)^{\beta_n}$$

and notice that $\rho^n_k(b^n_k,c^n_k)=|J^n_k|=|b^n_k-c^n_k|$. Let ρ be the metric on $[0,\infty)$ from Lemma 7.1 associated with the metrics $\{\rho^n_k \colon k,n<\omega\}$. We claim that f has the desired properties as a self-mapping of $\langle [0,\infty),\rho\rangle$.

To see this, let $U=\bigcup_{k,n<\omega} J^n_k$ and $X=[0,\infty)\setminus U$. Notice that f maps X into X and that ρ on X is the standard distance. First we prove that $f\restriction X$ is not (uLC). Indeed, for every $\lambda\in[0,1)$ there exists an $n<\omega$ such that $\lambda<s_n$ and on no open neighborhood V of a_n in X the map f is (C) with constant λ, since every such V contains points b^n_k,c^n_k for some $k<\omega$ while $\frac{|f(b^n_k)-f(c^n_k)|}{|b^n_k-c^n_k|}=s_n>\lambda$. So, neither $f\restriction X$ nor f is (uLC).

Next, we will prove that,

$$\text{for every } n<\omega,\ f\restriction I_n \text{ is (C) with a constant } \lambda=s_n\in(0,1). \tag{20}$$

Indeed, for every $k<\omega$, both $f\restriction[c^n_{k+1},b^n_k]$ and $f\restriction[b^n_k,c^n_k]$ are (C) with constant s_n: the first being constant, the second by Lemma 7.2. Hence, by Lemma 7.1, $f\restriction[b^n_k,a_{n+1}]$ is (C) with constant s_n for every $k<\omega$, and thus, so is $f\restriction(a_n,a_{n+1}]$. This and continuity of f imply (20).

Clearly (20) implies that, for every $n<\omega$, $f\restriction I_n$ is both (S) and (LC). Hence, by Lemma 7.1, f is (S) and (LC).

To finish the proof, choose a $\lambda\in(0,1)$. We need to show that f is (uPC) with constant λ. By Lemma 7.1, it is enough to show that, for every n, $f\restriction I_n$ is (uPC) with constant λ. So, fix an $n<\omega$ and notice that, for every $k<\omega$, both $f\restriction[c^n_{k+1},b^n_k]$ and $f\restriction[b^n_k,c^n_k]$ are (uPC) with constant λ: the first being constant, the second by Lemma 7.2. Hence, by Lemma 7.1, $f\restriction[b^n_k,a_{n+1}]$ is (uPC) with constant λ for every $k<\omega$, and thus, so is $f\restriction(a_n,a_{n+1}]$. Therefore, to finish the proof it is enough to show that there exist an open $V\ni a_n$ in I_n such that $\frac{\rho(f(a_n),f(x))}{\rho(a_n,x)}\le\lambda$ for every $x\in V$, $x\neq a_n$. But, for every $x\in[c^n_{k+1},c^n_k]$,

$$\frac{\rho(f(a_n),f(x))}{\rho(a_n,x)}\le\frac{\rho(f(a_n),f(c^n_k))}{\rho(a_n,c^n_{k+1})}=\frac{c^{n+1}_k-a_{n+1}}{c^n_k-a_n}\,\frac{c^n_k-a_n}{c^n_{k+1}-a_n}\to_{k\to\infty}0,$$

since $\frac{c^{n+1}_k-a_{n+1}}{c^n_k-a_n}=\frac{h_n(c^n_k)-h_n(a_n)}{c^n_k-a_n}\to_{k\to\infty}h'_n(a_n)=0$ and, by property (19), $\frac{c^n_k-a_n}{c^n_{k+1}-a_n}\to_{k\to\infty}1$. Therefore, there exists a $k_0<\omega$ such that $\frac{\rho(f(a_n),f(c^n_k))}{\rho(a_n,c^n_{k+1})}<\lambda$ for every $k\ge k_0$, implying that $V=[a_n,c^n_{k_0})$ is as needed.

Clearly f has no periodic points. ■

Notice that, by Theorem 3.3, if $f\colon X\to X$ is as in Example 20, then X cannot be rectifiably path connected. Also, by Theorem 3.2(i), X cannot compact.

Example 20. *There is a map f from* (S)&(uPC)&¬(LC) *having no periodic point, where $f\colon \langle[0,\infty),\rho\rangle \to \langle[0,\infty),\rho\rangle$ and ρ is a complete metric on $[0,\infty)$ topologically equivalent to the standard metric. Moreover, f satisfies* (uPC) *with an arbitrary constant $\lambda\in(0,1)$. Also, there exists a perfect unbounded $X\subset\mathbb{R}$ such that ρ on X is the standard metric on $\mathbb{R}$ and $f\restriction X$ belongs to the same class.*

Construction. The example is obtained by a slight modification of one described as Example 19. Specifically, the modification is obtained by choosing a sequence $s_0=t_0<t_1<\cdots$ converging to 1 and then, for every $k<\omega$, choosing the unique $b^0_k\in(c^0_{k+1},c^0_k)$ such that the slope of the segment joining points $(b^0_k,h(c^0_{k+1}))$ and $(c^0_k,h(c^0_k))$ is equal to t_k. All other parts of the construction from Example 19, including the choice of points b^n_k for $n>0$, remain unchanged. We claim that f has the desired properties as a self-mapping of $\langle[0,\infty),\rho\rangle$.

Indeed, as before, we let $U=\bigcup_{k,n<\omega} J^n_k$ and $X=[0,\infty)\setminus U$. Once again, f maps X into X and ρ on X is the standard distance. To finish the proof it is enough to show that $f\restriction X$ is not (LC) and that f has the remaining two properties.

To see that $f\restriction X$ is not (LC) notice that for every $\lambda\in(0,1)$ and every open V containing 0, there exist $k<\omega$ such that $\lambda<t_k$ and $b^0_k,c^0_k\in V$. Then $\frac{|f(b^0_k)-f(c^0_k)|}{|b^0_k-c^0_k|}=t_k>\lambda$. So, indeed, $f\restriction X$ is not (LC).

By Lemma 7.1, to finish the proof it is enough to show that, for every $\lambda\in(0,1)$ and $n<\omega$,

$$f\restriction I_n \text{ is (S) and (uPC) with a constant } \lambda=s_n\in(0,1). \tag{21}$$

Indeed, for every $k<\omega$, both $f\restriction[c^n_{k+1},b^n_k]$ and $f\restriction[b^n_k,c^n_k]$ are (S) and (uPC) with a constant λ: the first being constant, the second by Lemma 7.2. Hence, by Lemma 7.1, for every $k<\omega$ the map $f\restriction[b^n_k,a_{n+1}]$ is (S) and (uPC) with a constant λ. Therefore, $f\restriction(a_n,a_{n+1}]$ has the same property.

To see that $f\restriction[a_n,a_{n+1}]$ is (S) first notice that the continuity of f imply that, for every $y\in(a_n,a_{n+1}]$ we have $\rho(f(a_n),f(y))\le\rho(a_n,y)$. Therefore, for every $x\in(a_n,a_{n+1}]$, if $y=c^n_k=x^-$ for some $k<\omega$, then $f(y)=f(x)^-$ and we have

$$\rho(f(a_n),f(x))=\rho(f(a_n),f(y))+\rho(f(y),f(x))<\rho(a_n,y)+\rho(y,x)=\rho(a_n,x)$$

proving (S) of $f\restriction I_n$.

Since $f\restriction(a_n,a_{n+1}]$ is (uPC) with a constant λ, to finish the proof it is enough to show that there exists an open $V\ni a_n$ in I_n such that $\frac{\rho(f(a_n),f(x))}{\rho(a_n,x)}\le\lambda$ for every $x\in V$, $x\ne a_n$. But, for every $x\in[c^n_{k+1},c^n_k]$,

$$\frac{\rho(f(a_n),f(x))}{\rho(a_n,x)}\le\frac{\rho(f(a_n),f(c^n_k))}{\rho(a_n,c^n_{k+1})}=\frac{c^{n+1}_k-a_{n+1}}{c^n_k-a_n}\,\frac{c^n_k-a_n}{c^n_{k+1}-a_n}\to_{k\to\infty}0,$$

since $\frac{c^{n+1}_k-a_{n+1}}{c^n_k-a_n}=\frac{h_n(c^n_k)-h_n(a_n)}{c^n_k-a_n}\to_{k\to\infty}h'_n(a_n)=0$ and, by property (19), $\frac{c^n_k-a_n}{c^n_{k+1}-a_n}\to_{k\to\infty}1$. Therefore, there exists a $k_0<\omega$ such that $\frac{\rho(f(a_n),f(c^n_k))}{\rho(a_n,c^n_{k+1})}<\lambda$

for every $k \geq k_0$, implying that $V = [a_n, c^n_{k_0})$ is as needed. Clearly f has no periodic points. ■

Notice that, by Theorem 3.3, the space X from Example 21 cannot be rectifiably path connected. It is shown in Example 28 that such X can be path connected. However, it is not clear if X in such example can be simultaneously compact and connected, see Problem 8.1.

Example 21. *There exists a map $f\colon \langle [0,\infty),\rho\rangle \to \langle [0,\infty),\rho\rangle$ from the class* (uPC)&¬(LS) *having no periodic point, where ρ is a complete metric on $[0,\infty)$ topologically equivalent to the standard metric. Moreover, f satisfies* (uPC) *with every contraction constant $\lambda \in (0,1)$.*

Construction. Choose strictly increasing sequence $\langle \beta_n \in (0,1)\colon n<\omega\rangle$ and let $0 = a_0 < a_1 < \cdots$ be such that each interval $I_n = [a_n, a_{n+1}]$ has the length $\frac{1}{n+1}$. For every $n<\omega$, let ρ_n be a metric on I_n defined by formula $\rho_n(x,y) = |I_n|\left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ and let ρ be the metric on $[0,\infty)$ from Lemma 7.1 associated with the metrics $\{\rho_n\colon n<\omega\}$. Then, ρ is complete and, clearly, path connected.

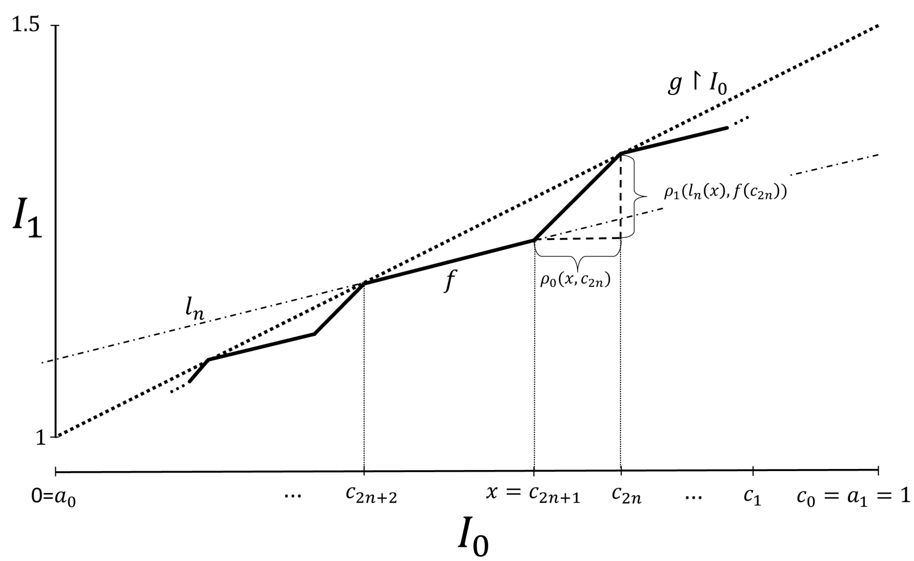

Figure 19: Illustration of g and f on $[c_{2n+2}, c_{2n}]$ from Example 21.

Define increasing function $g\colon [0,\infty) \to [0,\infty)$ such that it maps each interval I_n onto I_{n+1} linearly (with respect to the standard metric). The map f is a modification of g: it coincides with g on $[a_1,\infty)$, while on I_0 is defined as follows. Choose a sequence $a_1 = c_0 > c_2 > c_4 > \cdots$ converging to $a_0 = 0$. For every $n<\omega$, put $f(c_{2n}) = g(c_{2n})$ and let $\ell_n\colon \mathbb{R}\to\mathbb{R}$ be a line through point $(c_{2n+2}, f(c_{2n+2}))$ having slope which is half of the slope of $g \restriction I_0$. Let $c_{2n+1} \in (c_{2n+2}, c_{2n})$ be a solution of the equation $\rho(\ell_n(x), f(c_{2n})) = \rho(x, c_{2n})$, see Figure 19. Such a solution exists by the Intermediate Value Theorem, as $\lim_{x\to c_{2n}} \frac{\rho(\ell_n(x),f(c_{2n}))}{\rho(x,c_{2n})} = \infty$ and $\frac{\rho(\ell_n(c_{2n+2}),f(c_{2n}))}{\rho(c_{2n+2},c_{2n})} < \frac{|I_1|}{|I_0|} < 1$. Define $f(c_{2n+1}) = \ell_n(c_{2n+1})$ and on each interval $[c_{n+1}, c_n]$ extend f linearly. The function f is as desired.

Indeed, f is not (LS), since any open neighborhood V of 0 contains, for some $n < \omega$, the points c_{2n+1} and c_{2n} which satisfy the the following equation $\rho(f(c_{2n+1}), f(c_{2n})) = \rho(\ell_n(x), f(c_{2n})) = \rho(x, c_{2n}) = \rho(c_{2n+1}, c_{2n})$.

To see (uPC), choose $\lambda \in (0, 1)$. By Lemma 7.2, f is (uPC) with constant λ on each interval I_n, for $0 < n < \omega$, and $[c_{n+1}, c_n]$ for every $n < \omega$. Therefore, by Lemma 7.1, f is (uPC) with constant λ on $(0,\infty)$. Finally, it is (uPC) with constant λ at point 0, since $\frac{\rho(f(x),f(0))}{\rho(x,0)} \leq \frac{\rho(g(x),g(0))}{\rho(x,0)} < \lambda$ for small enough x, since, by Lemma 7.2, $D^*g(0) = 0$. ■

7.3 Examples on disconnected $X \subset \mathbb{R}$ with standard distance

Notice that, by Theorem 3.2(iii), if $f: X \to X$ is as in Example 22, then X cannot be connected. Also, by Theorem 3.2(ii), such map must have a periodic point.

Example 22. *There exists a compact set $X \subset \mathbb{R}$ and a map $f: X \to X$ from* (ULS)&¬(PC) *having no fixed point. It has a periodic point, as $f^{(2)}(0) = 0$.*

Construction. Let $X = [0, 1] \cup [2, 3]$ and define: $f(x) = 2 + \arctan x$ for $x \in [0, 1]$ and $f(x) = 0$ for $x \in [2, 3]$, compare Example 1. Such f is as needed. ■

Notice that, by Theorem 3.2(iii), if $f: X \to X$ is as in Example 23, then X cannot be connected. Also, by Theorem 3.2(ii), such map must have a periodic point.

Example 23. *Let $X = [-2, -1] \cup [1, 2]$ and let map $f: X \to X$ be defined as $f(x) = -\frac{x}{|x|}$. Then f is* (ULC)&¬(S) *having no fixed point. It has a periodic point, as $f^{(2)}(1) = 1$.* ■

Notice that, by Theorems 3.2(i) and 3.5, the space X in Example 24 cannot be compact and it must have infinitely many components.

Example 24. *There exists a map $f: X \to X$ from* (ULC)&¬(S) *having no periodic point, where X is an unbounded perfect subset of $\mathbb{R}$.*

Construction. Let $X = \bigcup_{n<\omega}[2n, 2n + 1]$ and define f as $f(x) = 2(n + 1)$ for $x \in [2n, 2n + 1]$. Clearly, f satisfies (ULC) with $\lambda = 0$. It is not (S), as $f(2) - f(0) = 4 - 2 = 2 - 0$. It has no periodic points since $f(x) > x$ for every $x \in X$. ■

Notice that, by Theorem 3.2(iii), if $f: X \to X$ is as in Example 25, then X cannot be connected. Also, by Theorem 3.2(ii), such map must have periodic point.

Example 25. *For $X = [0,1] \cup [2,3]$ there exists a map $f\colon X \to X$ from* (ULS)&(PC)&¬(uPC) *having no fixed point. It has a periodic point, $f^{(2)}(0) = 0$.*

Construction. Let $f_2\colon [0,1] \to [0,1]$ be a map from (S)&(PC)&¬(uPC) constructed in Example 2. We define $f(x) = f_2(x) + 2$ for $x \in [0,1]$ and $f(x) = f_2(x-2)$ for $x \in [2,3]$. Such f is as needed. ∎

Notice that, by Theorem 3.2(iii), if $f\colon X \to X$ is as in Example 26, then X cannot be connected. Also, by Theorem 3.2(ii), such map must have periodic point.

Example 26. *There exists a compact perfect set $X \subset \mathbb{R}$ and a map $f\colon X \to X$ from* (ULS)&(uPC)&¬(LC) *having no fixed point. Such map must have periodic point.*

Construction. For $n < \omega$ let $a_n = 2^{-2^n}$ so that $a_{n+1} = a_n^2$ and $a_n \searrow 0$. Let $b_0 = 1$ and, for $0 < n < \omega$, let $b_n \in (a_n, a_{n-1})$ be such that the slope of the segment joining points $\langle b_n, a_n^2\rangle$ and $\langle a_{n-1}, a_{n-1}^2\rangle$ is $1 - 5^{-n}$. Then, the set $Y = \{0\} \cup_{n<\omega} [a_n, b_n]$ is perfect. Define $g\colon Y \to Y$ by putting $g(0) = 0$ and $g(x) = a_n^2$ for every $x \in [a_n, b_n]$. See Figure 20.

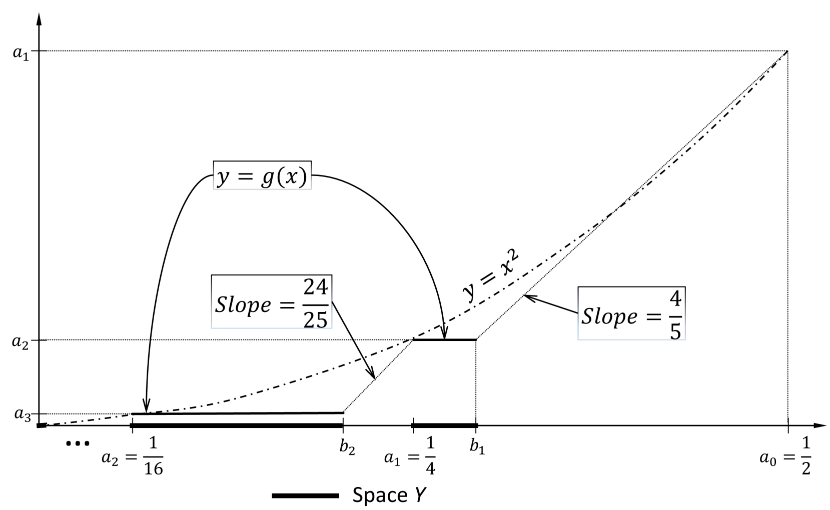

Figure 20: Function $g : Y \to Y$ for Example 26.

Notice that g (uPC) with any constant $\lambda \in (0,1)$. It is easy to verify that g is (S). Also, g is not (LC), since for any $\lambda \in (0,1)$ and any open $V \ni 0$ in Y there exists anon-zero $n < \omega$ such that $b_n, a_{n-1} \in V$ and $1 - 5^{-n} > \lambda$, giving $\frac{|g(b_n)-g(a_{n-1})|}{|b_n-a_{n-1}|} = \frac{|a_n^2-a_{n-1}^2|}{|b_n-a_{n-1}|} = 1 - 5^{-n} > \lambda$.

Let $X = Y \cup (2+Y)$ and define $f\colon X \to X$ by putting $f(x) = g(x) + 2$ for $x \in Y$ and $f(x) = g(x-2)$ for $x \in (2+Y)$. It is easy to see that such f is as needed. ∎

Notice that, by Theorems 3.2(i) and 3.5, the space X in Example 27 cannot be compact and it must have infinitely many components.

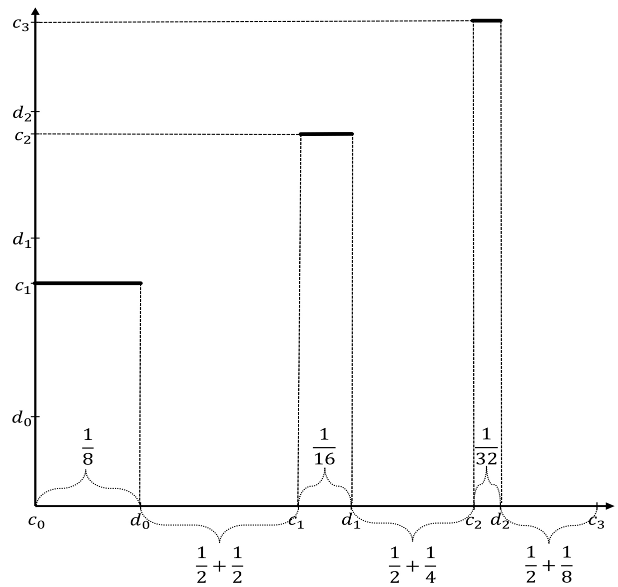

Figure 21: Function $f : X \to X$ for Example 27.

Example 27. *There exists a map* $f\colon X \to X$ *from* (S)&(ULC)&¬(C) *having no periodic point, where* X *is an unbounded perfect subset of* $\mathbb{R}$.

Construction. Let $X = \bigcup_{n<\omega}[c_n, d_n]$, where we define, by induction, $c_0 = 0$, $d_n = c_n + 2^{-(n+3)}$, and $c_{n+1} = d_n + \frac{1}{2} + 2^{-(n+1)} = c_n + 2^{-(n+3)} + \frac{1}{2} + 2^{-(n+1)}$. The space X is complete, since $c_n \geq \frac{n}{2} \nearrow \infty$ as $n \to \infty$. Put $f(u) = c_{n+1}$ for $u \in [c_n, d_n]$ and $n < \omega$, see Figure 21. Clearly f has no periodic point and is (ULC) with any $\lambda > 0$ and $0 < \varepsilon < \frac{1}{2}$ (since the length of any $[c_n, d_n]$ is $2^{-(n+3)} < \frac{1}{2}$). To see that f satisfies (S), choose $u < v$ from X. We need to show that $\left|\frac{f(v)-f(u)}{v-u}\right| < 1$. This is obvious, when u and v belong to the same interval $[c_n, d_n]$. So, assume that $u \in [c_n, d_n]$ and $v \in [c_{n+k}, d_{n+k}]$ for some $k \geq 1$. Then $\left|\frac{f(v)-f(u)}{v-u}\right| \leq \left|\frac{c_{n+k+1}-c_{n+1}}{c_{n+k}-d_n}\right| = \left|\frac{(c_{n+k}+2^{-(n+k+3)}+\frac{1}{2}+2^{-(n+k+1)})-(d_n+\frac{1}{2}+2^{-(n+1)})}{c_{n+k}-d_n}\right| < 1$, since $2^{-(n+k+3)} + 2^{-(n+k+1)} - 2^{-(n+1)} < 0$, completing the argument. ■

Notice that, by Theorem 3.3, the space X from Example 28 cannot be rectifiably path connected. It is shown in Example 21 that such X can be path connected. However, it is not clear if X in such example can be simultaneously compact and connected, see Problem 8.1.

Example 28. *There exists a compact perfect* $X \subset \mathbb{R}$ *and a map* $f\colon X \to X$ *from* (uPC)&¬(LS) *having no periodic point that satisfies* (uPC) *with every contraction constant* $\lambda \in (0, 1)$.

Construction. In [10, theorem 1] the authors present a perfect compact set $X \subset \mathbb{R}$ and a differentiable homeomorphism $f\colon X \to X$ which is (uPC) with any $\lambda > 0$. All orbits of the map f are dense in X, so f has no periodic points. Hence, by Theorem 3.2(i), f is not (LS). ■

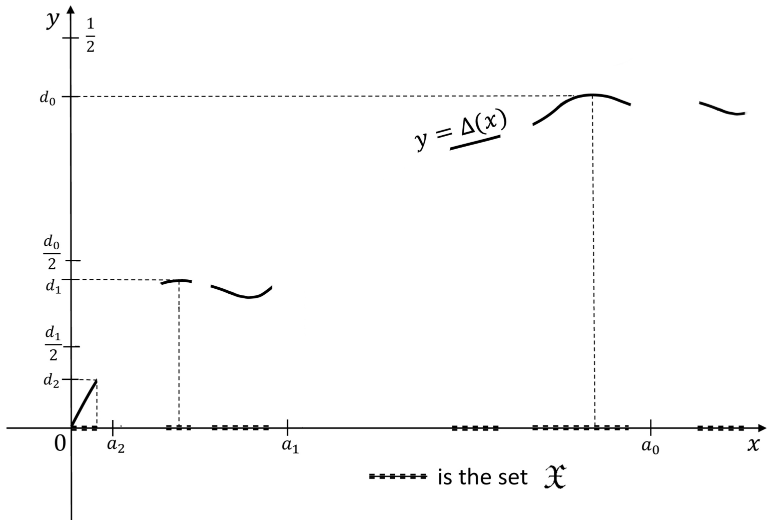

Figure 22: Relation between sequences $\langle a_n : n < \omega\rangle$, $\langle d_n : n < \omega\rangle$, and the set $\mathfrak{X}$.

It is not clear if the space X from Example 29 can be simultaneously compact and connected, see Problem 8.1. However, X can be $[0,\infty)$ with the standard metric, as shown by Example 3.

Example 29. *There exists a bijection $f: X \to X$ from* (PS)&¬(PC) *having no periodic point, where X is a compact perfect subset of $\mathbb{R}$ considered with the standard metric. Moreover, we will have $f'(x) = 0$ for all but one $x \in X$.*

Construction. Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{X}$ be as in Theorem 1 of [10], that is, $\mathfrak{f}$ is a periodic point free differentiable auto-homeomorphism of a perfect compact nowhere dense $\mathfrak{X} \subseteq \mathbb{R}$ such that $\mathfrak{f}'(x) = 0$ for all $x \in \mathfrak{X}$. We will construct an appropriate increasing bijection $g: \mathbb{R} \to \mathbb{R}$ for which $X = g[\mathfrak{X}]$ and $f = g \circ \mathfrak{f} \circ g^{-1}: X \to X$ will be as needed.

Translating $\mathfrak{X}$, if necessary, we can assume that $\min \mathfrak{X} = 0$. Now, since $\mathfrak{f}'(0) = 0$, the function $\Delta: \mathfrak{X} \to \mathbb{R}$ defined as

$$\Delta(x) = \begin{cases} \frac{|\mathfrak{f}(x) - \mathfrak{f}(0)|}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is continuous. In particular, it is an easy task to choose a strictly decreasing sequence $\langle a_n \in (0,\infty) \setminus \mathfrak{X} : n < \omega\rangle$ converging to 0 and associated numbers $d_n = \sup \Delta[[0,a_n] \cap \mathfrak{X}]$ such that:

- $\mathfrak{f}[\mathfrak{X} \cap [0,a_0]]$ is disjoint with $[0,a_0]$ (possible, as $\mathfrak{f}(0) > 0$ and $\mathfrak{f}$ is continuous),
- $d_0 < 1/2$, and
- $d_{n+1} \leq \frac{1}{2} d_n$ for all $n < \omega$. See Figure 22.

Let $\langle b_n \in (a_n, a_{n-1})\colon 0 < n < \omega\rangle$ be a decreasing sequence such that $[b_n, a_{n-1}] \cap \mathfrak{X} = \emptyset$. Define function g as the identity on the complement of $(0, a_0)$, while, for every $0 < n < \omega$, put

$$g(x) = \frac{(n+1)d_n}{n} x \text{ on } [a_n, b_n]$$

and extend it linearly on $[b_n, a_{n-1}]$, see Figure 23.

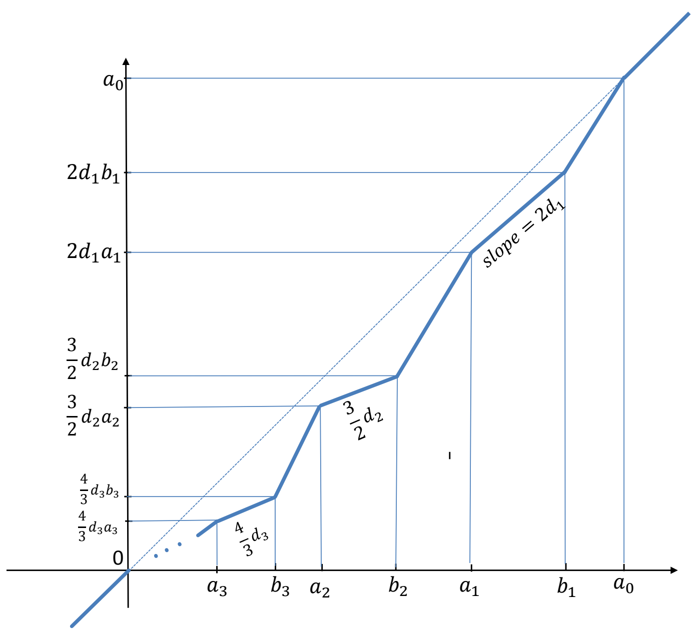

Figure 23: The thick line is the graph of g from Example 29.

Notice that g is indeed increasing, since $\frac{(n+2)d_{n+1}}{n+1} \leq \frac{(n+2)}{n+1}\frac{1}{2}d_n < \frac{(n+1)d_n}{n}$ for all $0 < n < \omega$. Moreover,

$$g(x) \leq x \text{ for all } x \geq 0, \tag{22}$$

since $\frac{(n+1)d_n}{n} \leq 1$ for all $0 < n < \omega$.

To see that f is as desired, first notice that

$$f'(x) = 0 \text{ for every nonzero } x \in X. \tag{23}$$

Indeed, for any such x there exists a nonzero $s \in \mathfrak{X}$ with $x = g(s)$. Choose $c \in (0, x)$ with $c < f(x)$ in case when $f(x) \neq 0$. Then the graph of $g \restriction [c, \infty)$ is a union of finite number of segments of positive slope and so, $g \restriction [c, \infty)$ is bi-Lipschitz with some constant $M > 0$, that is,

$$M^{-1}|a - b| \leq |g(a) - g(b)| \leq M|a - b|$$

whenever $a, b \in [c, \infty)$. Choose $y \in X \setminus \{x\}$ and $t \in \mathfrak{X}$ with $y = g(t)$.

Now, if $f(x) \neq 0$, then for every $d < f(x)$ and y close enough to x we have $s, t, \mathfrak{f}(s), \mathfrak{f}(t) \in [d, \infty)$, so that

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{|g(\mathfrak{f}(s)) - g(\mathfrak{f}(t))|}{|g(s) - g(t)|} \leq \frac{M|\mathfrak{f}(s) - \mathfrak{f}(t)|}{M^{-1}|s - t|} \to_{t \to s} 0,$$

as needed for (23). Similarly, for $f(x) = 0$ and y close enough to x, using (22) we obtain

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{g(\mathfrak{f}(t))}{|g(s) - g(t)|} \leq \frac{\mathfrak{f}(t)}{M^{-1}|s - t|} = \frac{|\mathfrak{f}(s) - \mathfrak{f}(t)|}{M^{-1}|s - t|} \to_{t \to s} 0,$$

completing the proof of (23).

Clearly, (23) implies that f is (PS) at every $x \in X \setminus \{0\}$. To see that f is (PS) notice that every $y \in X \cap (0, a_0)$, there exists a $t \in \mathfrak{X} \cap (0, a_0)$ such that $y = g(t)$. Moreover, $t \in [a_n, b_n]$ for some $0 < n < \omega$ and $g(\mathfrak{f}(t)) = \mathfrak{f}(t)$, as g is the identity on $[a_0, \infty) \ni \mathfrak{f}(t)$. Thus

$$\frac{|f(y) - f(0)|}{|y - 0|} = \frac{|g(\mathfrak{f}(t)) - g(\mathfrak{f}(0))|}{|g(t)|} = \frac{|\mathfrak{f}(t) - \mathfrak{f}(0)|}{\frac{(n+1)d_n}{n} t} = \frac{\frac{|\mathfrak{f}(t) - \mathfrak{f}(0)|}{t}}{d_n} \frac{n}{n+1} \leq \frac{n}{n+1},$$

ensuring (PS) of f at 0.

Finally, to see that f is not (PC) at 0, it is enough to notice that, by the definition of numbers d_n, the inequality $\leq$ in the last display becomes equation for some $t \in [a_n, b_n]$. (The maximum d_n of Δ on $[0, a_n] \cap \mathfrak{X}$ must be be attained on $[a_n, b_n]$, since for any $s \in [0, a_{n+1}] \cap \mathfrak{X}$ we have $\Delta(s) \leq d_{n+1} < d_n$.) ■

It is not clear if the space X from Example 30 can be simultaneously compact and connected, see Problem 8.1. However, X can be $[0, \infty)$ with the standard metric, as shown by Example 4.

Example 30. *There exists a bijection $f : X \to X$ from* (PC)&¬(uPC) *having no periodic point, where X is a compact perfect subset of $\mathbb{R}$ considered with the standard metric. Moreover, $f'(x) = 0$ for all but countably many $x \in X$.*

Construction. The construction is a variation of one used in Example 29. A difficulty here is that, instead of having just one point $x \in X$ with $D^* f(x) \neq 0$, we will need to have a sequence of points $\langle a_n \in X : n < \omega \rangle$, with $D^* f(a_n) \nearrow 1$ as $n \to \infty$.

As before, we start with the function $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ from [10, theorem 1], so that $\mathfrak{f}$ is a periodic point free differentiable auto-homeomorphism of a perfect compact nowhere dense $\mathfrak{X} \subseteq \mathbb{R}$ such that $\mathfrak{f}'(x) = 0$ for all $x \in \mathfrak{X}$. We also assume that $\min \mathfrak{X} = 0$. We will construct an increasing bijection $g : \mathbb{R} \to \mathbb{R}$ for which $X = g[\mathfrak{X}]$ and $f = g \circ \mathfrak{f} \circ g^{-1} : X \to X$ are as needed.

Since $\mathfrak{f}(0) > 0$, by continuity of $\mathfrak{f}$ we can find an $a_{-1} \in \mathfrak{X}$ such that $\mathfrak{f}(x) > a_{-1}$ for every $x \in [0, a_{-1}] \cap \mathfrak{X}$. Choose a sequence $\mathfrak{f}(0) > a_{-1} > a_0 > a_1 > \cdots$ in $\mathfrak{X}$ converging to 0 and such that for every $-2 < n < \omega$ there exists a $c_n < a_n$ with $(c_n, a_n) \cap \mathfrak{X} = \emptyset$ (so, a_ns are isolated from the left, but not from the right).

Notice that, for every $n < \omega$, the function $\Delta_n \colon \mathfrak{X} \to \mathbb{R}$ defined as

$$\Delta_n(x) = \begin{cases} \frac{|\mathfrak{f}(x)-\mathfrak{f}(a_n)|}{|x-a_n|} & \text{for } x \neq a_n \\ 0 & \text{for } x = a_n \end{cases}$$

is continuous, as $\mathfrak{f}'(a_n) = 0$. By induction on $k < \omega$ choose a strictly decreasing sequence $\langle b_k^n \in (a_n, a_{n-1}) \setminus \mathfrak{X} \colon k < \omega \rangle$ converging to a_n and the associated numbers $d_k^n = \max \Delta_n\left[[a_n, b_k^n] \cap \mathfrak{X}\right]$ such that:

- $b_0^n < 2a_n$, $d_0^n \leq \frac{1}{2}$, and $d_{k+1}^n < d_k^n$ for all $k < \omega$.

Let $\langle c_k^n \in (b_k^n, b_{k-1}^n) \colon 0 < k < \omega \rangle$ be such that $[c_k^n, b_{k-1}^n] \cap \mathfrak{X} = \emptyset$. Define function g as the identity (i.e., via $g(x) = x$) on $\mathbb{R} \setminus \bigcup_{n<\omega}(a_n, b_0^n)$, for every $n < \omega$ and $0 < k < \omega$ put

$$g(x) = a_n + \frac{n+2}{n+1} d_k^n (x - a_n) \text{ for } x \in [b_k^n, c_k^n],$$

and extend it linearly on each interval $[c_k^n, b_{k-1}^n]$, see Figure 24.

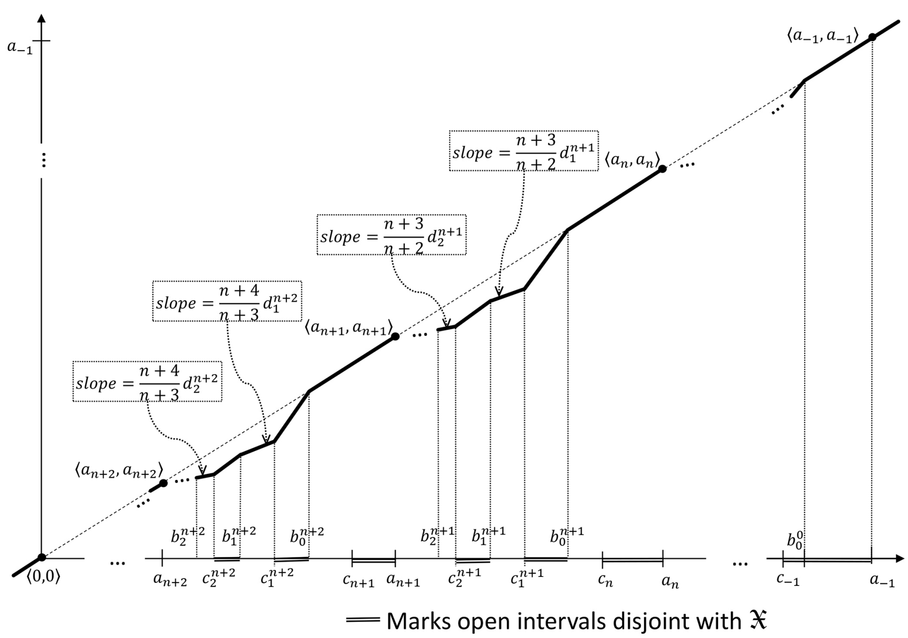

Figure 24: The graph of $y = g(x)$ between a_{n+2} and a_n.

Notice that g is indeed strictly increasing, since so is $g \restriction [a_n, b_0^n]$ for every $n < \omega$: every line ℓ_k^n containing $g \restriction [b_k^n, c_k^n]$ passes through the point $\langle a_n, a_n \rangle$ and the slope $\frac{n+2}{n+1} d_{k+1}^n$ of ℓ_{k+1}^n does not exceed the slope $\frac{n+2}{n+1} d_k^n \leq 1$ of ℓ_k^n.

Let $A = \{0\} \cup \{a_n \colon n < \omega\}$. Notice that, for every $n < \omega$, we have $g[[a_n, b_0^n]] = [a_n, b_0^n]$. This, the slopes of lines ℓ_k^n being less than one, and that fact that

$g(x) = x$ for all $x \in \mathbb{R} \setminus \bigcup_{n<\omega}(a_n, b_0^n)$, imply that

$$|g(t) - g(a)| \le |t - a| \text{ for every } a \in A \text{ and } t \ge a. \tag{24}$$

To see that g, $X = g[\mathfrak{X}]$, and $f = g \circ \mathfrak{f} \circ g^{-1} \colon X \to X$ are as desired first notice that, for every $n < \omega$, $a_n = g(a_n) \in X$ and $D^* f(a_n) = \frac{n+1}{n+2}$. Indeed, if $y \in X$ and $|y - a_n| < \delta_n = \min\{b_0^n - a_n, a_n - c_n\}$, then there exists a non-zero $k < \omega$ such that $y = g(t)$ for some $t \in [b_k^n, c_k^n]$. Then, by the definition of g on $[b_k^n, c_k^n]$, $|g(t) - a_n| = \frac{n+2}{n+1} d_k^n |t - a_n|$. Therefore, as $g(\mathfrak{f}(x)) = \mathfrak{f}(x)$ for any $x \in [0, a_{-1}] \cap \mathfrak{X}$,

$$\frac{|f(y)-f(a_n)|}{|y-a_n|} = \frac{|g(\mathfrak{f}(t))-g(\mathfrak{f}(a_n))|}{|g(t)-a_n|} = \frac{|\mathfrak{f}(t)-\mathfrak{f}(a_n)|}{\frac{n+2}{n+1} d_k^n |t-a_n|} = \frac{\frac{|\mathfrak{f}(t)-\mathfrak{f}(a_n)|}{|t-a_n|}}{d_k^n} \frac{n+1}{n+2} \le \frac{n+1}{n+2},$$

since $\frac{|\mathfrak{f}(t)-\mathfrak{f}(a_n)|}{|t-a_n|} \le d_k^n$ by the definition of d_k^n. Hence, $D^* f(a_n) \le \frac{n+1}{n+2}$. Moreover, the equality holds, since there exists an $s \in [a_n, c_k^n]$ with $d_k^n = \frac{|\mathfrak{f}(s)-\mathfrak{f}(a_n)|}{|s-a_n|}$. Also, $s \notin [a_n, c_{k+1}^n]$, since $d_{k+1}^n < d_k^n$. Hence, $s \in [b_k^n, c_k^n]$ and $\frac{|f(g(s))-f(a_n)|}{|g(s)-a_n|} = \frac{n+1}{n+2}$, proving that $D^* f(a_n) \ge \frac{n+1}{n+2}$.

The equation $D^* f(a_n) = \frac{n+1}{n+2}$ proves that f is not (uPC) and that it is (PC) at every $x = a_n$. So, to finish the proof, it is enough to show that $f'(x) = 0$ for any $x \in X \setminus \{a_n \colon n < \omega\}$.

So, choose such x. We consider the following three cases.

Case 1: $x = 0$**.** Then $x = 0 = g(0)$ and, for every $y = g(t) \in X$ close enough to x, we have $f(y) = g(\mathfrak{f}(t)) = \mathfrak{f}(t)$. Notice that having $b_0^n < 2a_n$ for all $n < \omega$ gives us $g(t) \ge \frac{1}{2}t$ for all $t \ge 0$. So, for $y \ne 0$, we obtain

$$\frac{|f(0) - f(y)|}{|0 - y|} = \frac{|\mathfrak{f}(0) - \mathfrak{f}(t)|}{g(t)} \le \frac{|\mathfrak{f}(0) - \mathfrak{f}(t)|}{\frac{1}{2}t} = 2\frac{|\mathfrak{f}(0) - \mathfrak{f}(t)|}{|0 - t|} \to_{t\to 0} 2\,\mathfrak{f}'(0) = 0,$$

giving required $f'(0) = 0$.

Case 2: $x, f(x) \notin A$**.** Then, neither $g^{-1}(x)$ nor $\mathfrak{f}(g^{-1}(x))$ belongs to A, as $g(a) = a$ for every $a \in A$. It is easy to see that every $z \in \mathbb{R} \setminus A$ admits an open neighborhood $U \ni z$ for which the graph of $g \restriction U$ is a union of at most two non-constant linear functions. In particular, $g \restriction U$ is bi-Lipschitz, that is, there exists an $L > 0$ such that

$$L^{-1}|a - b| \le |g(a) - g(b)| \le L|a - b| \text{ for all } a, b \in U. \tag{25}$$

Let U_0 and U_1 be the neighborhoods of $g^{-1}(x)$ and $\mathfrak{f}(g^{-1}(x))$, respectively, satisfying (25). Since g and $\mathfrak{f}$ are homeomorphisms, we can find an open neighborhood V of x in X such that $g^{-1}(V) \subset U_0$ and $\mathfrak{f}(g^{-1}(V)) \subset U_1$. Then, for every $y \in V$, $y \ne x$, we have

$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{|g(\mathfrak{f}(g^{-1}(x)))-g(\mathfrak{f}(g^{-1}(y)))|}{|g(g^{-1}(x))-g(g^{-1}(y))|} \le \frac{L|\mathfrak{f}(g^{-1}(x))-\mathfrak{f}(g^{-1}(y))|}{L^{-1}|g^{-1}(x)-g^{-1}(y)|} \to_{y\to x} L^2\mathfrak{f}'(g^{-1}(x)).$$

Since $\mathfrak{f}'(g^{-1}(x)) = 0$, we obtain required $f'(0) = L^2\mathfrak{f}'(g^{-1}(x)) = 0$.

Case 3: $x \notin A$ **and** $f(x) \in A$**.** Let $a \in A$ be such that $f(x) = a$ and notice that $\mathfrak{f}(g^{-1}(x)) = g^{-1}(f(x)) = g^{-1}(a) = a$. Since a is isolated from the left, there exists an open neighborhood U_1 of $a = \mathfrak{f}(g^{-1}(x))$ in $\mathfrak{X}$ such that $U_1 \subset [a, \infty)$ and so (24) holds for every $t \in U_1$. Moreover, since $x \notin A$, we have also $g^{-1}(x) \notin A$ and so, there exists an open neighborhood U_0 of $g^{-1}(x)$ in $\mathfrak{X}$ satisfying (25). Now, as in the previous case, we can find an open neighborhood V of x in X such that $g^{-1}(V) \subset U_0$ and $\mathfrak{f}(g^{-1}(V)) \subset U_1$. Then, for every $y \in V$, $y \neq x$, we have

$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{|g(a)-g(\mathfrak{f}(g^{-1}(y)))|}{|g(g^{-1}(x))-g(g^{-1}(y))|} \le \frac{|a-\mathfrak{f}(g^{-1}(y))|}{L^{-1}|g^{-1}(x)-g^{-1}(y)|}$$

$$= \frac{|\mathfrak{f}(g^{-1}(x))-\mathfrak{f}(g^{-1}(y))|}{L^{-1}|g^{-1}(x)-g^{-1}(y)|} \to_{y\to x} L\mathfrak{f}'(g^{-1}(x)).$$

Since $\mathfrak{f}'(g^{-1}(x)) = 0$, we obtain required $f'(0) = L\mathfrak{f}'(g^{-1}(x)) = 0$. ■

8 Remaining open problems and remarks

The in-depth analysis of this article, for the most part, presents a clear picture of the place of fixed and periodic point theorems among classes of functions described in Definition 2.1, considered in various topological configurations. However, there remain a few cases, indicated in the problems below, which "locally" cloud this image. In particular, the first of this problems, seems to be particularly intriguing, especially for the classes (PC) and (uPC).

Problem 8.1. Assume that $\langle X, d\rangle$ is compact and either connected or path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), must f have either fix or periodic point? What if f is (PC)? or (uPC)?

Note that, for the class (PC), the answer to Problem 8.1 (and Problem 8.2) is affirmative when the space $\langle X, d\rangle$ is rectifiably path connected, see Theorem 3.4.

Problem 8.2. Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), must f have either fix or periodic point?

Notice also that a large number of examples of functions we discussed are defined on spaces $\langle X, \rho\rangle$, where X is an interval and ρ cannot be the standard metric from $\mathbb{R}$. However, in all such cases with the exception of Example 11, it seems to be unknown if in these examples the space $\langle X, \rho\rangle$ can be isometric to a subset of $\mathbb{R}^n$ for some $n > 1$. We believe that in the cases when $\langle X, \rho\rangle$ can be rectifiably path connected — that is, in Examples 6, 7, 10, 12, 13, 14, and 15 — it is indeed possible to find the examples with $\langle X, \rho\rangle$ being isometric to subsets of $\mathbb{R}^3$. Verifying this conjecture might be an interesting project. In the cases when $\langle X, \rho\rangle$ cannot be rectifiably path connected — that is, in Examples 16, 17, 18, 19, and 20 — the possibility of finding the examples on the subsets of $\mathbb{R}^n$ seems still possible, but it is less clear to us. Somewhat encouraging is [30, example 1], see also our comment preceding Example 16.

We like to finish here with few words on what brought us to pursue the work on this project since, perhaps surprisingly, it was not our interest in the fixed point theorems. Instead, it stemmed from examining differentiability of the Peano-like maps g from the subsets X of $\mathbb{R}$ onto X^2, see [8, 9, 10]. It is easy to see that the differentiability of such g implies that X has Lebesgue measure 0. But, in [8], we gave an example of an infinitely many times differentiable function $g\colon \mathbb{R} \to \mathbb{R}^2$ which maps an unbounded perfect set $X \subset \mathbb{R}$ (clearly of measure 0) onto X^2. We also showed that for every continuously differentiable function $g\colon \mathbb{R} \to \mathbb{R}^2$, $X^2 \not\subset g[X]$ for every compact perfect set $X \subset \mathbb{R}$. However, the following problem from [8] remains open.

Problem 8.3. Let $X \subset \mathbb{R}$ be compact perfect and let g be a function from X onto X^2. Can g be differentiable? continuously differentiable?

If such a $g = \langle f, h \rangle$ exists, then f maps X onto X and, as we remarked in [8, lemma 3.2], $f'(x) = 0$ for all $x \in X$ except possible of a first category subset of X.

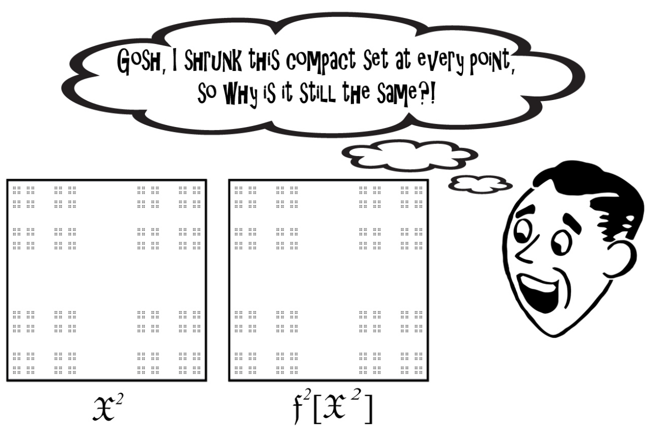

Figure 25: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

Can surjection with such properties exist? What if $f'(x) = 0$ for all (rather than "almost" all) $x \in X$? Our (false) intuition was that f with this last property (i.e., being a map from compact perfect $X \subset \mathbb{R}$ onto X with $f' \equiv 0$) cannot exist. In our attempt to show such claim, we proved (see [10, theorem 9]) that for any such f there exists a perfect $\mathfrak{X} \subset X$ such that $\mathfrak{f} = f \restriction \mathfrak{X}$ has no periodic points, bringing us to the realm of fixed point theorems and (uPC) maps. Of course, we eventually discovered (see [10, theorem 9] and Example 28) that such paradoxically behaving function $\mathfrak{f}$ (see Figure 25) indeed exists. So one may say that this entire study stems from Example 28.

Finally, notice that the existence of map $\mathfrak{f}$ seems to indicate, that the answer to Problem 8.3 is affirmative. However, the delicate construction of $\mathfrak{f}$ so far defied any attempts to transform it to the example confirming this indication.

References

[1] D. Acosta and T. Lawson, *Weak contractions and fixed points*, College Math. J. **46**(1) (2015), 35–41.

[2] M.A. Alghamdi, W.A. Kirk, and N. Shahzad, *Locally nonexpansive mappings in geodesic and length spaces*, Topology Appl. **173** (2014), 59–73.

[3] S. Almezel, Q.H. Ansari and M.A. Khamsi - Editors, *Topics in Fixed Point Theory*, Springer, Switzerland 2014.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[5] T.Q. Binh, *Some results on locally contractive mappings*, Nonlinear Funct. Anal. Appl. **11**(3) (2006), 371–383.

[6] A.M. Bruckner, J.B. Bruckner and B.S. Thomson, *Real Analysis*, Prentice-Hall, New Jersey 1997.

[7] S. Carl and S. Heikkilä, *Fixed Point Theory in Ordered Sets and Applications*, Springer, Springer New York Dordrecht Heidelberg London, 2011.

[8] K.C. Ciesielski and J. Jasinski, *Smooth Peano functions for perfect subsets of the real line*, Real Analysis Exchange **39**(1) (2014), 57–72.

[9] K.C. Ciesielski and J. Jasinski, *On closed subsets of R and of R2 admitting Peano functions*, Real Anal. Exchange **40**(2) (2015), 309–317.

[10] K.C. Ciesielski and J. Jasinski, *An auto-homeomorphism of a Cantor set with derivative zero everywhere*, J. Math. Anal. Appl. **434** (2016), 1267–1280.

[11] K.C. Ciesielski and J. Jasinski, *On fixed points of locally and pointwise contracting maps*, Topology Appl. **204** (2016), 70–78.

[12] F.H. Clarke, *Pointwise contraction criteria for the existence of fixed points*, Canad. Math. Bull. **21**(1) (1978), 7–11.

[13] P.Z. Daffer and H. Kaneko, *A new proof of a fixed point theorem of Edelstein*, Scientiae Mathematicae **1** (1998), 97–101.

[14] C. Ding and S.B. Nadler, *The periodic points and the invariant set of an ε-contractive map*, Applied Mathematics Letters **15**(7) (2002), 793–801.

[15] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7–10.

[16] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

[17] R. Engelking, *General Topology*, Heldermann, Berlin 1989.

[18] R. Espínola, E.S. Kim, and W.A. Kirk, *Fixed point properties of mappings satisfying local contractive conditions*, Nonlinear analysis and its applications, (St. Johnís, NF, 1999). Nonlinear Anal. Forum **6**(1) (2001), 103–111.

[19] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York 2003.

[20] R.D. Holmes, *Fixed points for local pointwise contractions*, Proc. Seminar on Fixed Points Theory and Its Appl., (Dalhousie Univ., 1975), Academic Press New York, 78–89.

[21] T. Hu and W.A. Kirk, *Local contractions in metric spaces*, Proc. Amer. Math. Soc. **68** (1978), 121–124.

[22] G. Jungck, *Local radial contractions — a counter-example*, Houst. J. Math. **8** (1982), 501–506.

[23] W.A. Kirk, *Contracting Mappings and Extensions*, in *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001, pp 1-34.

[24] W.A. Kirk and N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, Switzerland, 2014.

[25] M.M. Marjanović, *Fixed points of local contractions*, Publications de l'Institut Mathmatique **34** (1976), 185–190.

[26] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. **103**(1) (1930), 466–501.

[27] J.R. Munkres, *Topology*, Prentice Hall, 2nd ed., 2000.

[28] J. Mycielski, *On the existence of a shortest arc between two points of a metric space*, Houston J. Math. **20**(3) (1994), 491–494.

[29] S.B. Myers, *Arcs and Geodesics in Metric Spaces*, Transactions of the American Mathematical Society **57** no(2), (1945), 217–227.

[30] E. Rakotch, A note on α-locally contractive mappings, *Bull. Res. Council Israel* **40** (1962), 188–191.

[31] B.E. Rhoades, *A Comparison of Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.

[32] I. Rosenholtz, *Evidence of a conspiracy among fixed point theorems*, Proc. Amer. Math. Soc. **53**(1) (1975), 213–218.

Krzysztof Chris Ciesielski, *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310* and *Department of Radiology, MIPG, University of Pennsylvania, Philadelphia, PA 19104-6021*, e-mail: KCies@math.wvu.edu

Jakub Jasinski, *Mathematics Department, University of Scranton, Scranton, PA 18510,* e-mail: Jakub.Jasinski@scranton.edu