On a Genocchi-Peano example

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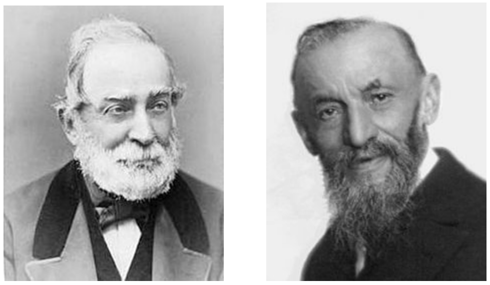
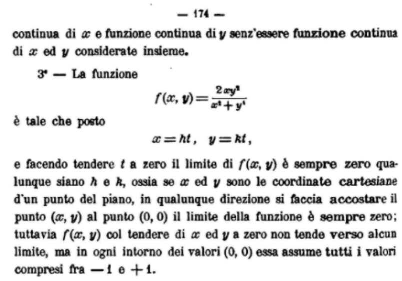
Early in a multivariable calculus class, students are asked to determine if $f(x, y)$ given by

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{when } (x,y) \neq (0,0), \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

is continuous. Although f is discontinuous (along the parabola $x = y^2$), some students are likely to think that this function is continuous since $f(0, 0)$ is equal to the limit along the x- and y-axis. Did you know that an 1821 calculus textbook of Augustin-Louis Cauchy [**1**] contains a theorem, which seems to contradict the existence of example with such properties and to agree with the naïve hypothesis, that *a two variable function is continuous if it is continuous in each variable separately*? The apparent contradiction comes from the fact that Cauchy's text is written for the set $\mathscr{R}$ of real numbers containing infinitesimals (i.e., numbers d with $0 < d < 1/n$ for every $n = 1, 2, 3, \ldots$), while the standard set $\mathbb{R}$ of real numbers does not contain such objects. The fact that Cauchy's result is false when $\mathscr{R}$ is replaced with the standard set $\mathbb{R}$ of real numbers was first observed by E. Heine and appeared in the 1870 calculus text [**6**] of J. Thomae, see [**5**]. The prominent example (1), that appears in many calculus books, comes from the 1884 treatise on calculus by A. Genocchi and G. Peanno [**4**], see Figure 1. (See [**3**] on more history related to the above mentioned Cauchy's result





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continua di x e funzione continua di y senz'essere funzione continua di x ed y considerate insieme.

3° — La funzione

$$f(x, y) = \frac{2xy^2}{x^2 + y^4}$$

è tale che posto

$$x = ht, \quad y = kt,$$

e facendo tendere t a zero il limite di $f(x, y)$ è sempre zero qualunque siano h e k, ossia se x ed y sono le coordinate cartesiane d'un punto del piano, in qualunque direzione si faccia accostare il punto (x, y) al punto $(0, 0)$ il limite della funzione è sempre zero; tuttavia $f(x, y)$ col tendere di x ed y a zero non tende verso alcun limite, ma in ogni intorno dei valori $(0, 0)$ essa assume tutti i valori compresi fra -1 e $+1$.

Figure 1. A. Genocchi (1817-1889), G. Peano (1858-1932), and (1) in [**4**]

and Genocchi-Peanno's example.) Not only (1) is continuous separately, it is also continuous when restricted to any straight line, including those passing through the origin. This article will focus on the following questions.

Q1: What are other examples of two variable functions that are discontinuous, but continuous along any straight line?

Q2: Can we generalize (1) to higher dimensions and, if so, in what sense?

Q3: What are the simplest examples of this sort?

In answering the question Q3, we will restrict our attention to the class of rational functions, one of the simplest classes containing removable discontinuities.

When generalizing (1) to higher dimensions, we need to decide whether to treat lines in $\mathbb{R}^2$ as the objects of dimension 1, or rather as hyperplanes, that is, objects of co-dimension 1. (Thus, hyperplanes in $\mathbb{R}^3$ are the standard two-dimensional planes.) In other words, do we want the functions of three or more variables to be continuous on all hyperplanes? or just on all lines? The lines option does not lead to anything truly new, as a "natural lift" of the original Genocchi-Peano function $f(x, y)$ to the higher dimensions, defined by $g(x_1, x_2, \ldots, x_n) = f(x_1, x_2)$, is clearly discontinuous, while continuous on any straight line. Therefore, in what follows, we will require our examples to be continuous on all hyperplanes. Notice, that, for $n > 2$, the above function g is not among such examples, since it is discontinuous when restricted to the hyperplane $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_3 = x_2\}$.

Generalized Genocchi-Peano examples The simplest rational functions $g\colon \mathbb{R}^n \to \mathbb{R}$ that may have a chance to lead to the examples we seek are in the form

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}} & \text{when } (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0), \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $\alpha_i, \beta_i \in \mathbb{N} = \{1, 2, 3, \ldots\}$ for all $i \in \{1, \ldots, n\}$. For the rest of the paper we will assume that every function is 0 at $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$, the origin.

We say that $g\colon \mathbb{R}^n \to \mathbb{R}$ in the form of (2) and with $n > 1$ is a ***Genocchi-Peano example*** (abbreviated *GPE*), if g is discontinuous but has a continuous restriction $g \restriction H$ to any hyperplane H in $\mathbb{R}^n$. Of course, if all β_i's are even, then the maps (2) are continuous when restricted to the hyperplanes that do not contain the origin. Thus, in such case, we will restrict our attention to the hyperplanes that contain the origin, that is, expressible via equations $\sum_{i=1}^{n} b_k x_k = 0$. One of the main goals of this article is to investigate the following general question.

For any $n > 1$, for what values of $\alpha_i, \beta_i \in \mathbb{N}$, $i \in \{1, \dots, n\}$, is the function $g(x_1, x_2, \dots, x_n)$ a Genocchi-Peano example?

It is worth noting that any GPE is, in particular, continuous on any straight line.

Clearly, the function f given by (1) is a GPE. It is also easy to see that the following function

$$h(x,y) = \frac{xy^2}{x^2 + y^6} \tag{3}$$

constitutes another such example, since it is discontinuous on the curve $x = y^3$. Actually, these two examples are essentially different: $f[\mathbb{R}^2]$ is bounded (since $|f(x,y)| = \sqrt{\frac{x^2}{x^2+y^4}}\sqrt{\frac{y^4}{x^2+y^4}} \le 1$), while $h[\mathbb{R}^2]$ is not (as $\lim_{y \to 0^+} h(y^3, y) = \infty$).

A simple GPE for $n = 3$ is given by

$$g(x_1, x_2, x_3) = \frac{x_1 x_2 x_3^2}{x_1^2 + x_2^4 + x_3^8}. \tag{4}$$

Indeed, g is discontinuous on $\{(t^4, t^2, t) \colon t \in \mathbb{R}\}$. To see that it is continuous on any hyperplane (containing the origin), notice that $|g(x_1, x_2, x_3)| = \frac{|x_1|}{d^{1/2}} \frac{|x_2|}{d^{1/4}} \left(\frac{|x_3|}{d^{1/8}}\right)^2$, where $d = x_1^2 + x_2^4 + x_3^8$. Now, each of the three quotients is bounded above by 1. Moreover, for a hyperplane $x_3 = ax_1 + bx_2$ we have $\frac{|x_3|}{d^{1/8}} \le |a|\frac{|x_1|}{d^{1/8}} + |b|\frac{|x_2|}{d^{1/8}} \le |a|\frac{d^{1/2}}{d^{1/8}} + |b|\frac{d^{1/4}}{d^{1/8}} \to 0$ as $d \to 0$. So, g is continuous at $(0,0,0)$ on this hyperplane. Similarly, g is continuous on a hyperplane $x_2 = ax_1$, since $\frac{|x_2|}{d^{1/4}} \le |a|\frac{d^{1/2}}{d^{1/4}} \to 0$ as $d \to \infty$. Hence, g is indeed a GPE.

The above argument for $g(x_1, x_2, x_3)$ well exemplifies the general argument for our main *Characterization of GPEs* result stated below. Before we state it, let us first note that none of the β_i's can be odd, if g, in the form of (2), is to be a GPE. Indeed, if β_i is odd, then g is discontinuous on any hyperplane containing a point $y = (y_1, \dots, y_n) \in (\mathbb{R} \setminus \{0\})^n$ satisfying $\sum_{i=1}^n y_i^{\beta_n} = 0$. (To see this more clearly, we can set $y_j = 1$ for $j \ne i$ and $y_i = \sqrt[\beta_i]{1-n}$.) Therefore, in the rest of the paper we will assume that all the β_is are even and, because of symmetry of the definition of g, we will also assume $\beta_1 \le \beta_2 \le \dots \le \beta_n$. Now, we are ready to state our characterization of GPEs in terms of exponents $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$.

Characterization of GPEs. *Let g be given by (2) and $\beta_1 \le \beta_2 \le \dots \le \beta_n$ be positive even numbers.*

(i) g is discontinuous iff $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} = \frac{\alpha_1}{\beta_1} + \dots + \frac{\alpha_n}{\beta_n} \le 1$.

(ii) g has a continuous restriction to every hyperplane iff

$$\sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > 1 \textit{ for every } k \in \{2, \dots, n\}. \tag{5}$$

In particular, g is a Genocchi-Peano example iff $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \le 1$ and (5) holds.

Notice, that the value of $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}$ from (5) can be calculated by replacing β_k with β_{k-1} in the expression $\frac{\alpha_1}{\beta_1} + \dots + \frac{\alpha_k}{\beta_k} + \dots + \frac{\alpha_n}{\beta_n} = \sum_{i=1}^n \frac{\alpha_i}{\beta_i}$.

It is also worthwhile to point out that, by the characterization, for a fixed sequence $\langle\beta_1,\ldots,\beta_n\rangle$, we may have only finite number of GPEs, namely those that satisfy $\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\le 1$. We will postpone the proof of the characterization to the end of this article.

Example 1. *To illustrate the power of the characterization, we will determine all GPEs of the form*

$$f(x_1,x_2)=\frac{x_1^{\alpha_1}x_2^{\alpha_2}}{x_1^6+x_2^{10}} \tag{6}$$

for $\alpha_1,\alpha_2\in\mathbb{N}$. By the characterization, such an example must satisfy the inequalities $\frac{\alpha_1}{6}+\frac{\alpha_2}{10}\le 1$ and $\frac{\alpha_1}{6}+\frac{\alpha_2}{6}>1$ or, equivalently, $5\alpha_1+3\alpha_2\le 30$ and $\alpha_1+\alpha_2>6$. It is easy to verify that there are only 23 pairs of $\langle\alpha_1,\alpha_2\rangle$ that satisfy $5\alpha_1+3\alpha_2\le 30$: $\langle 1,1\rangle$, $\langle 1,2\rangle$, $\langle 1,3\rangle$, $\langle 1,4\rangle$, $\langle 1,5\rangle$, $\langle 1,6\rangle$, $\langle 1,7\rangle$, $\langle 1,8\rangle$; $\langle 2,1\rangle$, $\langle 2,2\rangle$, $\langle 2,3\rangle$, $\langle 2,4\rangle$, $\langle 2,5\rangle$, $\langle 2,6\rangle$; $\langle 3,1\rangle$, $\langle 3,2\rangle$, $\langle 3,3\rangle$, $\langle 3,4\rangle$, $\langle 3,5\rangle$; $\langle 4,1\rangle$, $\langle 4,2\rangle$, $\langle 4,3\rangle$; and $\langle 5,1\rangle$. Among these pairs, the inequality $\alpha_1+\alpha_2>6$ has precisely eight solutions, that correspond to the following GPEs: $f(x_1,x_2)=\frac{x_1x_2^6}{x_1^6+x_2^{10}}$, $\frac{x_1x_2^7}{x_1^6+x_2^{10}}$, $\frac{x_1x_2^8}{x_1^6+x_2^{10}}$, $\frac{x_1^2x_2^5}{x_1^6+x_2^{10}}$, $\frac{x_1^2x_2^6}{x_1^6+x_2^{10}}$, $\frac{x_1^3x_2^4}{x_1^6+x_2^{10}}$, $\frac{x_1^3x_2^5}{x_1^6+x_2^{10}}$, and $\frac{x_1^4x_2^3}{x_1^6+x_2^{10}}$.

A difficulty behind using the characterization is that, for each GPE candidate function with a fixed denominator, we need to check n different inequalities: $\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\le 1$ and $(n-1)$-many of the form (5). Although, it is fairly easy to write a program in one of the common symbolic algebra systems (e.g. Mathematica, Maple, Matlab, …) which, for fixed $\beta_1,\ldots,\beta_n$, finds all α_is satisfying these inequalities (e.g., all the GPEs with a fixed denominator), finding all such α_is without computer assistance could be a challenging task. The following corollary, though more restrictive than our Characterization of GPEs result, reduces the task of checking whether a candidate is a GPE to a verification of a single equation.

Sufficient Condition for GPEs. *Let g be as in (2) and $\beta_1\le\beta_2\le\cdots\le\beta_n$.*

(i) If g is a Genocchi-Peano example, then β_is must be distinct.
(ii) If all β_is are even and $\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}=1$, then

g is a Genocchi-Peano example iff all β_is are distinct.

Moreover, the functions as in (ii) are the only GPEs with $g[\mathbb{R}^n]$ bounded.

This follows quite easily from our Characterization of GPEs. Indeed, if g is a GPE, then β_is must be distinct, since otherwise there would exist $k\in\{2,\ldots,n\}$ with $\frac{\alpha_k}{\beta_k}=\frac{\alpha_k}{\beta_{k-1}}$ and so the inequalities from parts (i) and (ii) of the characterization cannot simultaneously hold. On the other hand, if all β_is are distinct and $\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}=1$, then, for every $k\in\{2,\ldots,n\}$, $\frac{\alpha_k}{\beta_k}<\frac{\alpha_k}{\beta_{k-1}}$ and so, $\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}-\frac{\alpha_k}{\beta_k}+\frac{\alpha_k}{\beta_{k-1}}>\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}=1$. Thus, g is a GPE.

We leave the last part of the Sufficient Condition result as an exercise.

Exercise 1. Prove, using the characterization result, that the only bounded GPEs are those for which $\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}=1$. *Hint:* Follow the arguments used for h from (3) and g from (1). For general n, equation (10) might be useful.

Notice, that among the eight GPEs of the form (6) only one, $\frac{x_1^3x_2^5}{x_1^6+x_2^{10}}$, satisfies the Sufficient Condition. The Sufficient Condition implies also that each map

$$g_n(x_1,\ldots,x_n)=\frac{x_1x_2\cdots x_{n-1}x_n^2}{x_1^2+x_2^4+\cdots+x_{n-1}^{2^{n-1}}+x_n^{2^n}} \tag{7}$$

is a GPE, as $\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}+\frac{2}{2^n}=1$. (The fact that g_ns are the GPEs was first noticed, without a proof of correctness, in [**2**].) Note that the original GPE given in (1) is g_2, while g from (4) is g_3. Another general class of GPEs, each for $n>1$ variables, is given by:

$$h_n(x_1,\ldots,x_n)=\frac{x_1^2\cdots x_i^{2i}\cdots x_n^{2n}}{x_1^{2n}+\cdots+x_i^{2in}+\cdots+x_n^{2n^2}}, \tag{8}$$

where the assumptions of Sufficient Conditions hold, as $\sum_{i=1}^n \frac{2i}{2in}=n\frac{1}{n}=1$. In particular, this gives the following GPEs of 2, 3, and 4-variables, respectively: $g_2(x,y)=\frac{xy^2}{x^2+y^4}$, $h_2(x,y)=\frac{x^2y^4}{x^4+y^8}$, $g_3(x,y,z)=\frac{xyz^2}{x^2+y^4+z^8}$, $h_3(x,y,z)=\frac{x^2y^4z^6}{x^6+y^{12}+z^{18}}$, $g_4(x,y,z,t)=\frac{xyzt^2}{x^2+y^4+z^8+t^{16}}$, and $h_4(x,y,z,t)=\frac{x^2y^4z^6t^{12}}{x^8+y^{16}+z^{24}+t^{32}}$. In these examples, the degrees of the denominators of GPEs given by (7) are smaller than those given by (8). However, 2^n is bigger than $2n^2$ for large values of n, that is, this trends reverses for as $n\to\infty$. (In fact, already for any $n\geq 7$.) These observations open up a discussion of the simplest GPEs.

The simplest Genocchi-Peano examples of n-variables So far, we answered the questions Q1 and Q2 (in the class of functions of the form (2)). In this section we tackle the question Q3, on the simplest Genocchi-Peano examples. But how do you define "the simplest example," even just in the class of the Genocchi-Peano examples? We decided to express this simplicity in terms of the degree of the denominator of (2) (i.e., β_n) and declare that *the smaller β_n, the simpler associated GPE*. In general, little is known about the minimal degrees β_n for GPEs of n variables. Of course, we must have $\beta_n\geq 2n$, as all numbers β_i are even and distinct. Also, we have GPEs with $\beta_n\leq\min\{2^n,2n^2\}$, as justified by the maps g_n and h_n from (7) and (8). Thus, for all GPEs of n variables, the minimal β_n satisfy

$$2n\leq\beta_n\leq\min\{2^n,2n^2\}. \tag{9}$$

However, in general, the upper bound $\min\{2^n,2n^2\}$ is far from optimal, as we can see in the following investigation of GPEs for small values of n.

For $n=2$: The inequalities (9) immediately imply that $\beta_n=4$. In particular, the original GPE (1), which is g_2, has the denominator of minimal degree. Moreover, it is easy to see that g_2 is the only GPE of two variables with $\beta_n=4$.

For $n=3$: We know, by (9), that $6\leq\beta_n\leq 8$. Any GPE map (2) with $\beta_n<8$ would need to be of the form $\frac{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}}{x_1^2+x_2^4+x_3^6}$. Among such functions, only $\alpha_1=\alpha_2=\alpha_3=1$ gives the necessary inequality $\frac{\alpha_1}{2}+\frac{\alpha_2}{4}+\frac{\alpha_3}{6}\leq 1$. However, for this choice, (ii) of the characterization is false for $k=3$. Hence, there is no GPE of this form and so $6<\beta_n\leq 8$. In particular, $\beta_n=8$.

This means that, once again, the function g_n for $n = 3$, $g_3(x_1, x_2, x_3) = \frac{x_1 x_2 x_3^2}{x_1^2 + x_2^4 + x_3^8}$, is a GPE with the denominator of minimal degree. In fact, g_3 has also the smallest degree of numerator among all three-variable GPEs having minimal degree denominator. This is the case, since each of the functions $\frac{x_1 x_2 x_3}{x_1^2 + x_2^4 + x_3^8}$, $\frac{x_1 x_2 x_3}{x_1^2 + x_2^6 + x_3^8}$, and $\frac{x_1 x_2 x_3}{x_1^4 + x_2^6 + x_3^8}$ fails the property (5) for $k = 3$, as $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} > \frac{1}{2} + \frac{1}{6} + \frac{1}{6} > \frac{1}{4} + \frac{1}{6} + \frac{1}{6}$. Moreover, g_3 is the only GPE with $\beta_n = 8$ and the numerator of degree $\alpha_1 + \alpha_2 + \alpha_3 = 4$, since none of the following functions is a GPE: $\frac{x_1^2 x_2 x_3}{x_1^2 + x_2^4 + x_3^8}$, $\frac{x_1 x_2^2 x_3}{x_1^2 + x_2^4 + x_3^8}$, and $\frac{x_1^2 x_2 x_3}{x_1^2 + x_2^6 + x_3^8}$ (they fail (i) of the characterization); $\frac{x_1 x_2^2 x_3}{x_1^2 + x_2^6 + x_3^8}$, $\frac{x_1 x_2 x_3^2}{x_1^2 + x_2^6 + x_3^8}$, $\frac{x_1^2 x_2 x_3}{x_1^4 + x_2^6 + x_3^8}$, $\frac{x_1 x_2^2 x_3}{x_1^4 + x_2^6 + x_3^8}$, and $\frac{x_1 x_2 x_3^2}{x_1^4 + x_2^6 + x_3^8}$ (they fail (5) for $k = 3$).

Exercise 2. Prove that the maps $\frac{x_1^2 x_2 x_3}{x_1^2 + x_2^4 + x_3^8}$, $\frac{x_1 x_2^2 x_3}{x_1^2 + x_2^4 + x_3^8}$, and $\frac{x_1^2 x_2 x_3}{x_1^2 + x_2^6 + x_3^8}$ fail to be GPEs by showing explicitly (without using the characterization) that they are continuous at the origin.

Exercise 3. Check that the functions $\frac{x_1 x_2^2 x_3}{x_1^2 + x_2^6 + x_3^8}$, $\frac{x_1 x_2 x_3^2}{x_1^2 + x_2^6 + x_3^8}$, $\frac{x_1^2 x_2 x_3}{x_1^4 + x_2^6 + x_3^8}$, $\frac{x_1 x_2^2 x_3}{x_1^4 + x_2^6 + x_3^8}$, and $\frac{x_1 x_2 x_3^2}{x_1^4 + x_2^6 + x_3^8}$ indeed fail (5) with $k = 3$.

For $n = 4$**:** The inequalities (9) give bounds $8 \le \beta_n \le 16$. We will show that, in this case, the smallest possible β_n of GPE is 10, if we allow unbounded maps, and 12 otherwise.

First notice that $\beta_n > 8$, since otherwise $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \ge \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} > 1$, that is, (i) of the characterization fails. This means that $\beta_n \ge 10$. The equation here is justified by a GPE map $g(x_1, x_2, x_3, x_4) = \frac{x_1 x_2 x_3^2 x_4^3}{x_1^4 + x_2^6 + x_3^8 + x_4^{10}}$. It satisfies (i) from the characterization, since $\frac{1}{4} + \frac{1}{6} + \frac{2}{8} + \frac{3}{10} = \frac{29}{30} < 1$. At the same time the inequality $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > 1$ for every $k \in \{2, 3, 4\}$ is proved as follows:

- $k = 4$: $\frac{1}{4} + \frac{1}{6} + \frac{2}{8} + \frac{3}{8} = \frac{25}{24} > 1$,
- $k = 3$: $\frac{1}{4} + \frac{1}{6} + \frac{2}{6} + \frac{3}{10} = \frac{63}{60} > 1$, and
- $k = 2$: $\frac{1}{4} + \frac{1}{4} + \frac{2}{8} + \frac{3}{10} = \frac{21}{20} > 1$.

So, g satisfies (ii) from the characterization, showing that indeed it is a GPE.

A bounded GPE with $\beta_n = 12$ is given as $\frac{x_1 x_2 x_3 x_4}{x_1^2 + x_2^4 + x_3^6 + x_4^{12}}$ — it satisfies the Sufficient Condition as $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{12}{24} + \frac{6}{24} + \frac{4}{24} + \frac{2}{24} = 1$. Another example of this kind is $h(x_1, x_2, x_3, x_4) = \frac{x_1 x_2 x_3^2 x_4^4}{x_1^4 + x_2^6 + x_3^8 + x_4^{12}}$ — it satisfies the Sufficient Condition, since $\frac{1}{4} + \frac{1}{6} + \frac{2}{8} + \frac{4}{12} = \frac{6}{24} + \frac{4}{24} + \frac{6}{24} + \frac{8}{24} = 1$.

Finally, notice that there is no bounded GPE with $\beta_n = 10$. To see this, by Sufficient Condition, it is enough to show that $\frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{e}{10} = 1$ for no $a, b, c, d, e \in \{0, 1, 2, \ldots\}$ with precisely one of a, b, c, d being zero. Indeed, the number $\frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{e}{10} = \frac{120a + 60b + 40c + 30d + 24e}{240}$ cannot be an integer, unless e is divisible by 5. But if e is divisible by 5 and precisely one of $a, b, c, d \in \{0, 1, 2, \ldots\}$ is zero, then $\frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{e}{10} \ge \frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{5}{10} \ge \frac{0}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{5}{10} > 1$. This completes the argument.

Exercise 4. Use the Characterization of GPEs to show that $\frac{x_1x_2x_3x_4^2}{x_1^2+x_2^6+x_3^8+x_4^{10}}$ is another example of GPE with $\beta_n = 10$. Notice, that its numerator has smaller degree than the above discussed example.

Exercise 5. Use the Characterization of GPEs to show that there is no GPE of four variables with denominator of degree 10 and numerator of degree < 5.

The justification of *Characterization of GPEs* Let $\gamma = \sum_{i=1}^n \frac{\alpha_i}{\beta_i}$ and put $d = x_1^{\beta_1} + \cdots + x_n^{\beta_n}$. Then, for g of the form $\frac{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$,

$$g(x_1,\ldots,x_n) = \frac{1}{d^{1-\gamma}}\frac{x_1^{\alpha_1}}{d^{\alpha_1/\beta_1}}\cdots\frac{x_n^{\alpha_n}}{d^{\alpha_n/\beta_n}} = d^{\gamma-1}\prod_{i=1}^n \frac{x_i^{\alpha_i}}{d^{\alpha_i/\beta_i}}. \tag{10}$$

To see (i), first assume that $\gamma \le 1$. Then, for $t > 0$,

$$h(t) = g\big(t^{1/\beta_1},\ldots,t^{1/\beta_n}\big) = \frac{t^{\alpha_1/\beta_1}\cdots t^{\alpha_n/\beta_n}}{nt} = \frac{t^{\gamma-1}}{n}$$

so that h does not converge to $g(0,\ldots,0) = 0$ as $t \to 0$. Thus, g is discontinuous.

Conversely, assume that $\gamma > 1$. Since, for every $i \in \{1,\ldots,n\}$, we have $\left|\frac{x_i^{\alpha_i}}{d^{\alpha_i/\beta_i}}\right| \le \frac{|x_i|^{\alpha_i}}{\left(x_i^{\beta_i}\right)^{\alpha_i/\beta_i}} = 1$, the equation (10) implies $|g(x_1,\ldots,x_n)| \le d^{\gamma-1}$. But $\lim_{\langle x_1,\ldots,x_n\rangle\to\langle 0,\ldots,0\rangle} d^{\gamma-1} = 0$, as $\gamma - 1 > 0$. So, by the Squeeze Theorem, $\lim_{\langle x_1,\ldots,x_n\rangle\to\langle 0,\ldots,0\rangle} g(x_1,\ldots,x_n) = 0$, that is, g is continuous at the origin. So, indeed, g is continuous, as desired.

To see that (ii) holds, for $k \in \{2,\ldots,n\}$ let $\delta_k = \sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}$.

First assume that $\delta_k \le 1$ for some $k \in \{2,\ldots,n\}$ and consider the hyperplane $H = \{x \in \mathbb{R}^n \colon x_k = x_{k-1}\}$. We will show that $g \restriction H$, the restriction g to H, is discontinuous. Indeed, for every $t > 0$ and $i \in \{1,\ldots,n\}$ let

$$f_i(t) = \begin{cases} t^{1/\beta_i} & \text{if } i \ne k,\\ t^{1/\beta_{k-1}} & \text{if } i = k.\end{cases}$$

Then $\langle f_1(t),\ldots,f_n(t)\rangle \in H$. Moreover, since

$$(f_i(t))^{\alpha_i} = \begin{cases} t^{\alpha_i/\beta_i} & \text{if } i \ne k,\\ t^{\alpha_i/\beta_{k-1}} & \text{if } i = k\end{cases} \qquad \text{and} \qquad (f_i(t))^{\beta_i} = \begin{cases} t & \text{if } i \ne k,\\ t^{\beta_k/\beta_{k-1}} & \text{if } i = k\end{cases}$$

we have

$$g(f_1(t),\ldots,f_n(t)) = \frac{t^{\gamma-\frac{\alpha_k}{\beta_k}+\frac{\alpha_k}{\beta_{k-1}}}}{(n-1)t + t^{\beta_k/\beta_{k-1}}} = \frac{1}{(n-1) + t^{(\beta_k/\beta_{k-1})-1}}\,t^{\delta_k-1}.$$

Thus, $\lim_{t\to0} g(f_1(t),\ldots,f_n(t)) \ne 0$, since $\lim_{t\to0^+} t^{\delta_k-1} \ge 1$ (as $\delta_k - 1 \le 0$) and $\lim_{t\to0^+}\frac{1}{(n-1)+t^{(\beta_k/\beta_{k-1})-1}}$ is either $\frac{1}{n-1}$ (when $\beta_{k-1} < \beta_k$) or $\frac{1}{n}$ (when $\beta_{k-1} = \beta_k$). Therefore, indeed $g \restriction H$ is discontinuous at the origin.

To complete the argument, assume that $\delta_k > 1$ for every $k \in \{2,\ldots,n\}$ and let H be a hyperplane. We need to show that $g \restriction H$ is continuous. This is obvious when H does not contain the origin. So, assume that it does and that H is given by $\sum_{i=1}^{n} b_i x_i = 0$. Let $k \in \{1, \ldots, n\}$ be the largest for which $b_k \neq 0$.

If $k = 1$, then $g \restriction H \equiv 0$ is continuous. So, assume that $k > 1$. Then, the equation $\sum_{i=1}^{n} b_k x_k = 0$ can be written as $x_k = \sum_{i=1}^{k-1} a_i x_i$. In particular, since $1/\beta_i \geq 1/\beta_{k-1}$ for every $i \in \{1, \ldots, k-1\}$, for every $d \in (0, 1)$ we have

$$|x_k| = \left|\sum_{i=1}^{k-1} a_i x_i\right| \leq \sum_{i=1}^{k-1} |a_i||x_i| \leq \sum_{i=1}^{k-1} |a_i| d^{\frac{1}{\beta_i}} \leq \sum_{i=1}^{k-1} |a_i| d^{\frac{1}{\beta_{k-1}}} = A d^{\frac{1}{\beta_{k-1}}}, \quad (11)$$

where $A = \sum_{i=1}^{k-1} |a_i|$. Since $\left|\frac{x_i^{\alpha_i}}{d^{\alpha_i/\beta_i}}\right| = \left|\frac{x_i^{\beta_i}}{d}\right|^{\alpha_i/\beta_i} \leq 1$ for every $i \in \{1, \ldots, n\}$, formulas (10) and (11) imply

$$|g(x_1, \ldots, x_n)| \leq \frac{1}{d^{1-\gamma}} \frac{|x_k|^{\alpha_k}}{d^{\alpha_k/\beta_k}} \leq \frac{\left(A d^{\frac{1}{\beta_{k-1}}}\right)^{\alpha_k}}{d^{1-\gamma+\frac{\alpha_k}{\beta_k}}} = A^{\alpha_k} d^{\gamma - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} - 1} = A^{\alpha_k} d^{\delta_k - 1}.$$

But $A^{\alpha_k} d^{\delta_k - 1} \to 0 = g(0, \ldots, 0)$ as $d \to 0^+$, since $\delta_k - 1 > 0$. Hence, indeed $g \restriction H$ is continuous, being continuous at the origin.

Summary We characterize the simple rational functions of n real variables which are discontinuous but continuous when restricted to any hyperplane. The characterization is expressed by simple inequalities with respect to the exponents of each variable. In particular, the following functions have such properties: $\frac{x_1 x_2^2}{x_1^2 + x_2^4}$, $\frac{x_1 x_2 x_3^2}{x_1^2 + x_2^4 + x_3^8}$, $\frac{x_1 x_2 x_3^2 x_4^3}{x_1^4 + x_2^6 + x_3^8 + x_4^{10}}$, and $\frac{x_1 x_2 x_3 x_4}{x_1^2 + x_2^4 + x_3^6 + x_4^{12}}$. More generally, for every $n > 1$ the functions $\frac{x_1 x_2 \cdots x_{n-1} x_n^2}{x_1^2 + x_2^4 + \cdots + x_{n-1}^{2^{n-1}} + x_n^{2n}}$ and $\frac{x_1^2 \cdots x_i^{2i} \cdots x_n^{2n}}{x_1^{2n} + \cdots + x_i^{2in} + \cdots + x_n^{2n^2}}$ constitute the examples we mentioned above. Finally, the smallest degree of the denominators of such examples is also investigated.

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