CARDINAL COEFFICIENTS RELATED TO SURJECTIVITY, DARBOUX, AND SIERPIŃSKI-ZYGMUND MAPS

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Abstract. We investigate the additivity A and lineability $\mathcal{L}$ cardinal coefficients for the following classes of functions: $\mathrm{ES}\setminus\mathrm{SES}$ of everywhere surjective functions that are not strongly everywhere surjective, Darboux-like, Sierpiński-Zygmund, surjective, and their corresponding intersections. The classes SES and ES have been shown to be $2^{\mathfrak{c}}$-lineable. In contrast, although we prove here that $\mathrm{ES}\setminus\mathrm{SES}$ is $\mathfrak{c}^+$-lineable, it is still unclear whether it can be proved in ZFC that $\mathrm{ES}\setminus\mathrm{SES}$ is $2^{\mathfrak{c}}$-lineable. Moreover, we prove that if $\mathfrak{c}$ is a regular cardinal number, then $A(\mathrm{ES}\setminus\mathrm{SES})\le\mathfrak{c}$. This shows that, for the class $\mathrm{ES}\setminus\mathrm{SES}$, there is an unusual big gap between the numbers A and $\mathcal{L}$.

1. Preliminaries

Since the beginning of the 21st century many authors have become interested in the study of linearity within the non-linear settings and searched for linear structures in the mathematical objects enjoying certain special or *unexpected* properties. Vector spaces and linear algebras are elegant mathematical structures which, at first glance, seem to be "forbidden" in the families of "strange" objects. In this line of research one typically starts with an example of a function having some special (often referred to as) "pathological" property, like the classical example of a continuous nowhere differentiable function, also known as Weierstrass' monster. Can a class of all such examples admit a large subclass with a linear structure? Since, typically, coming up with a first single concrete example of such a function is difficult, there is a natural tendency to think that there cannot be too many functions of such kind. So, it *seems* unlikely, that such a class of examples could contain a subclass forming an infinitely dimensional vector space. However, in recent years, this intuition has been repeatedly proven incorrect: "large" linear spaces and algebras have been found within the classes of "strange" objects (usually functions) that come from the multitude of mathematical areas: from Linear Chaos to Real and Complex Analysis [5, 6, 12, 13, 23],

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passing through Set Theory [27] and Linear and Multilinear Algebra, and within Operator Theory, Topology, Measure Theory, Abstract Algebra, and Probability Theory. For a complete modern state of the art of this area of research see [2, 14].

The notion of a large linear structure within a given class, intuitively discussed above, is nowadays typically expressed in the following more precise terminology: given a (finite or infinite) cardinal number κ, a subset M of a vector space X is called *κ-lineable* in X if there exists a linear space $Y \subset M \cup \{0\}$ of dimension κ (see, e.g., [2–5, 7, 14, 25, 34].) Intuitively, lineability research seeks for a linear structure within $M \cup \{0\}$ of the highest possible dimension. However, there exist sets M, with no linear substructures of highest dimension. (For a simple example of M that admits a linear subspace of any finite dimension but is not ω-lineable see, e.g., [5].) Therefore, the intuition of the "maximal lineability number" is best expressed as the lineability coefficient $\mathcal{L}$ defined, see below, as the least cardinal for which there is *no* linear substructure of that cardinality. (See [16].)

From this point on, we assume that all the structures M we consider are the classes $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ (that is, of functions from $\mathbb{R}$ to $\mathbb{R}$), where $\mathbb{R}^{\mathbb{R}}$ is considered as a linear space over $\mathbb{R}$.

Definition 1.1. The *lineability coefficient* of a class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is defined as

$$\mathcal{L}(\mathcal{F}) = \min\{\kappa\colon \text{there is no } \kappa\text{-dimensional vector space } V \text{ with } V \subset \mathcal{F} \cup \{0\}\}.$$

Notice that $\mathcal{F}$ admits the maximal lineability number if, and only if, $\mathcal{L}(\mathcal{F})$ is a cardinal successor, that is, $\mathcal{L}(\mathcal{F})$ is of the form κ^+.

Lately, and since the appearance of the work [26] (see, also, [10, 16]), the notion of lineability has been linked (see Proposition 1.3) to that of the additivity coefficient A, which was introduced by Natkaniec in [30, 31] and thoroughly studied by the first named author [15, 17–19, 22] and Jordan [28].

Definition 1.2. Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. The *additivity* of $\mathcal{F}$ is defined as the following cardinal number:

$$A(\mathcal{F}) = \min\Big(\big\{|F|\colon F \subset \mathbb{R}^{\mathbb{R}} \ \&\ \big(\forall g \in \mathbb{R}^{\mathbb{R}}\big)(g + F \not\subset \mathcal{F})\big\} \cup \big\{(2^{\mathfrak{c}})^+\big\}\Big),$$

where $(2^{\mathfrak{c}})^+$ stands for the successor cardinal of $2^{\mathfrak{c}}$.

The above definition gives us, roughly, the *biggest* cardinal number κ for which every family $\mathcal{G} \subset \mathbb{R}^{\mathbb{R}}$, with $|\mathcal{G}| < \kappa$, can be translated into $\mathcal{F}$.

Notice that the operators A and $\mathcal{L}$ are clearly monotone, in a sense that

$$\text{if } \mathcal{F} \subset \mathcal{G} \subset \mathbb{R}^{\mathbb{R}}, \text{ then } A(\mathcal{F}) \le A(\mathcal{G}) \text{ and } \mathcal{L}(\mathcal{F}) \le \mathcal{L}(\mathcal{G}).$$

$$\begin{tikzcd}
\mathrm{PES} \arrow[r] & \mathrm{SES} \arrow[r] \arrow[drr] & \mathrm{ES} \arrow[r] & \mathrm{D} \arrow[d, dashed] \\
& & & \mathbb{R}^{\mathbb{R}} \setminus \mathrm{SZ}
\end{tikzcd}$$

FIGURE 1. The inclusions between Darboux and Sierpiński-Zygmund classes of functions, indicated by arrows. The dashed arrow indicates the implication that is consistent with (follows from CPA) but independent of the ZFC axioms of set theory.

To state the next proposition, linking A and $\mathcal{L}$, we need also the following notation

$$\mathrm{st}(\mathcal{F}) = \{f \in \mathcal{F} \colon rf \in \mathcal{F} \text{ for every non-zero } r \in \mathbb{R}\}.$$

Notice that all the classes $\mathcal{F}$ defined below satisfy $\mathrm{st}(\mathcal{F}) = \mathcal{F}$.

The following result comes from [16].

Proposition 1.3. *If $\mathcal{F} \subsetneq \mathbb{R}^{\mathbb{R}}$ and $\mathrm{st}(\mathcal{F}) = \mathcal{F}$, then $A(\mathcal{F}) > \mathfrak{c}$ implies that $A(\mathcal{F}) < \mathcal{L}(\mathcal{F})$.* □

The results presented in this paper constitute research on the coefficients A and $\mathcal{L}$ for several classes of real functions and some of their algebraic combinations. For completeness sake, we provide below the full definitions of these classes.

Definition 1.4. Given a function $f \colon \mathbb{R} \to \mathbb{R}$, we say (see, e.g., [2, 16]) that:

(I) f is *surjective* ($f \in \mathcal{S}$) if $f[\mathbb{R}] = \mathbb{R}$.

(II) f is *everywhere surjective* ($f \in \mathrm{ES}$) if $f[G] = \mathbb{R}$ for every nonempty open set $G \subset \mathbb{R}$.

(III) f is *strongly everywhere surjective* ($f \in \mathrm{SES}$) if $f^{-1}(y) \cap G$ has cardinality $\mathfrak{c}$ for every $y \in \mathbb{R}$ and every nonempty open set $G \subset \mathbb{R}$; this class was also studied in [18] (under the name of $\mathfrak{c}$-strongly Darboux functions).

(IV) f is *perfectly everywhere surjective* ($f \in \mathrm{PES}$) if $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}$.

(V) f is *Sierpiński-Zygmund* ($f \in \mathrm{SZ}$) if $f \restriction X$ is discontinuous for every $X \in [\mathbb{R}]^{\mathfrak{c}}$ (i.e., a subset X of $\mathbb{R}$ of cardinality continuum $\mathfrak{c}$).

(VI) $f \in F_{<\mathfrak{c}}$ if $f^{-1}(y)$ has cardinality smaller than $\mathfrak{c}$ for every $y \in \mathbb{R}$.

(VII) f is *Darboux* ($f \in \mathrm{D}$) if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected $K \subset \mathbb{R}$.

Remark 1.5. *The inclusions between some of these classes are shown in Figure 1. In particular,* $\mathrm{SZ} \cap \mathrm{SES} = \emptyset$ *and* $\mathrm{ES} \cap \mathrm{SZ} \subset \mathrm{ES} \setminus \mathrm{SES}$.

$$\begin{tikzcd}
 & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathcal{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] &
\end{tikzcd}$$

FIGURE 2. Classes of Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$, $\mathcal{C}$ denotes the class of continuous functions. (The arrows indicate strict inclusions.)

The maps defined below are commonly known as Darboux-like functions. The relations within these classes of functions are represented in Figure 2. (See, e.g., [16].)

Definition 1.6. Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we say that:

(I) f has the *Cantor intermediate value property* ($f \in$ CIVP) if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$.

(II) f has the *strong Cantor intermediate value property* ($f \in$ SCIVP) if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous.

(III) f has *perfect roads* ($f \in$ PR) if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral (i.e., two-sided) limit point for which $f \restriction P$ is continuous at x.

(IV) $f : X \to \mathbb{R}$ is *almost continuous* ($f \in$ AC) *in the sense of Stallings* if each open subset of $X \times \mathbb{R}$ containing the graph of f contains also the graph of a continuous function from X to $\mathbb{R}$;

(V) for a topological space X, $g\colon X \to \mathbb{R}$ is a *connectivity* function ($f \in \mathrm{Conn}(X)$) if the graph of $g \restriction Z$ is connected in $Z \times \mathbb{R}$ for any connected subset Z of X; we write Conn for $\mathrm{Conn}(\mathbb{R})$;

(VI) f is *extendable* ($f \in$ Ext) provided that there exists a connectivity function $F\colon \mathbb{R}\times[0,1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in \mathbb{R}$;

(VII) f is *peripherally continuous* ($f \in$ PC) if for every $x \in \mathbb{R}$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$; note that any function f with a graph dense in $\mathbb{R}^2$ is PC. Here, $\mathrm{bd}(W)$ denotes the boundary of W.

In the rest of this section we briefly summarize what is known about all these classes in terms of inclusions and coefficients A and $\mathcal{L}$.

We must remark that all of these classes coincide when we restrict ourselves to functions in the first class of Baire. In contrast,

Proposition 1.7. *Within the class* SES *of strongly everywhere surjective functions, the inclusions presented in Figure 2 remain strict.*

Proof. The inclusion AC $\subset$ Conn is implicitly shown in [33]. For the other inclusions, see the examples of additive functions described in [17]. (In general such functions must be ES but not necessarily SES. However, the examples given in [17] are SES as well.) $\square$

Proposition 1.8.

(I) $A(\mathrm{PC}) = 2^{\mathfrak{c}}$;
(II) $A(\mathrm{Ext}) = A(\mathrm{SCIVP}) = A(\mathrm{CIVP}) = A(\mathrm{PR}) = \mathfrak{c}^+$;
(III) $\mathfrak{c}^+ \le A(\mathrm{AC}) = A(\mathrm{Conn}) = A(\mathrm{D}) = A(\mathrm{ES}) = A(\mathrm{SES}) = A(\mathrm{PES}) \le 2^{\mathfrak{c}}$ *and this is all that can be proved in ZFC.*

Proof. The results on all coefficients in (III) except for $A(\mathrm{PES})$ are proved in [18]. The value of $A(\mathrm{PES})$ is obtained in [26]. All other results are proved in [22]. (See also [15].) $\square$

Proposition 1.9.

(I) $\mathcal{L}(\mathcal{F} \cap \mathcal{G}) = (2^{\mathfrak{c}})^+$ *for the families* $\mathcal{F} \in \{\mathrm{SES}, \mathrm{ES}, \mathcal{S}, \mathbb{R}^{\mathbb{R}}\}$ *and* $\mathcal{G} \in \{\mathrm{Ext}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}, \mathbb{R}^{\mathbb{R}}\}$.
(II) $\mathcal{L}(\mathcal{C} \cap \mathcal{S}) = \mathfrak{c}^+$, *while* $\mathcal{L}(\mathcal{C} \cap \mathrm{ES}) = \mathcal{L}(\mathcal{C} \cap \mathrm{SES}) = 1$.
(III) $\mathcal{L}(\mathcal{G} \cap \mathrm{PES}) = (2^{\mathfrak{c}})^+$ *for any* $\mathcal{G} \in \{\mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}, \mathbb{R}^{\mathbb{R}}\}$.
(IV) $\mathcal{L}(\mathrm{PES} \cap \mathcal{G}) = 1$ *for any* $\mathcal{G} \in \{\mathrm{Ext}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}\}$.

Proof. (I): This is an immediate consequence of [16, Prop. 3.2] and the monotonicity property of the operator $\mathcal{L}$.

(II): For the first part see, e.g., [1, 34]. The second part follows from the fact that $\mathcal{C} \cap \mathrm{ES} = \emptyset$.

(III): Let J be the class of Jones functions, as defined in [26]:

$$J = \{f \in \mathbb{R}^{\mathbb{R}} \colon\ C \cap f \neq \emptyset \text{ for every closed } C \subset \mathbb{R}^2 \text{ with } |\mathrm{dom}\, C| = \mathfrak{c}\}.$$

Then $J \subset \mathrm{AC} \cap \mathrm{PES}$. (The inclusion $J \subset \mathrm{PES}$ is obvious, while $J \subset \mathrm{AC}$ is proved in [29].) Thus, (III) follows from the monotonicity property of the operator $\mathcal{L}$ and the equation $\mathcal{L}(J) = (2^{\mathfrak{c}})^+$, which is proved in [24].

(IV): This follows from the fact that $\mathrm{PR} \cap \mathrm{PES} = \emptyset$. $\square$

After this preliminary section and first cycle of ideas and notions, our main goal in what follows is to give a thorough study of the additivity and lineability numbers of the class $\mathrm{ES} \setminus \mathrm{SES}$, the classes related to it, and some of the intersections between them that have not been discussed above. In particular, the problem of the value $\mathcal{L}$ for $\mathrm{ES} \setminus \mathrm{SES}$, and related classes, have, lately, attracted the attention of several authors. (See, e.g., [9, 25, 32].) So far, and since the arrows in Remark 1.5 are all strict inclusions, the class SES (and thus ES) has been shown to be $2^{\mathfrak{c}}$-lineable. However, the ZFC value of $\mathcal{L}(\mathrm{ES} \setminus \mathrm{SES})$ remains, still, a mystery.

2. New results on A and $\mathcal{L}$ for the classes defined above

The following theorem generalizes Proposition 1.8 by giving the values of A for the classes not covered there.

Theorem 2.1.

(I) *For every* $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}\}$, $\mathcal{G} \in \{\mathrm{SES}, \mathrm{ES}, \mathcal{S}, \mathbb{R}^{\mathbb{R}}\}$, *and* $\mathcal{H} \in \{\mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}, \mathbb{R}^{\mathbb{R}}\}$ *we have* $A(\mathcal{F} \cap \mathcal{G} \cap \mathcal{H}) = \mathfrak{c}^+$.

(II) *For every* $\mathcal{F} \in \{\mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}\}$ *and* $\mathcal{G} \in \{\mathrm{PES}, \mathrm{SES}, \mathrm{ES}, \mathcal{S}\}$ *we have* $\mathfrak{c}^+ \leq A(\mathcal{F} \cap \mathcal{G}) = A(\mathcal{G}) = A(\mathrm{AC}) = A(\mathrm{Conn}) = A(\mathrm{D}) \leq 2^{\mathfrak{c}}$ *and this is all that can be proved in ZFC.*

Proof. (I): Clearly we have

$$A(\mathrm{Ext} \cap \mathrm{SES}) \leq A(\mathcal{F} \cap \mathcal{G} \cap \mathcal{H}) \leq A(\mathrm{PR}) = \mathfrak{c}^+,$$

where $A(\mathrm{PR}) = \mathfrak{c}^+$ follows from Proposition 1.8. Thus, it is enough to prove that $A(\mathrm{Ext} \cap \mathrm{SES}) \geq \mathfrak{c}^+$. To see this, fix $F \subset \mathbb{R}^{\mathbb{R}}$ with $|F| = \mathfrak{c}$. There exists $g \in \mathbb{R}^{\mathbb{R}}$ with $g + F \subset \mathrm{Ext} \cap \mathrm{SES}$. First notice that $A(\mathrm{Ext}) \geq \mathfrak{c}^+$ implies the existence of g with $g + F \subset \mathrm{Ext}$. However, an examination of the proof of $A(\mathrm{Ext}) \geq \mathfrak{c}^+$ from [22] shows, that we can choose g such that for some $\mathfrak{c}$-dense subset D of $\mathbb{R}$, any modification $\bar{g}$ of g on D still has the property that $g + F \subset \mathrm{Ext}$. (In [22, lemma 3.2] choose sets $D_\xi \supset h_\xi[M]$ disjoint with some $\mathfrak{c}$-dense $D \subset \mathbb{R}$.) Now, an easy induction shows that there exists an $h\colon D \to \mathbb{R}$ such that $h + f \restriction D$ is SES for every $f \in F$. Therefore, if $\bar{g} \restriction D = h$ and $\bar{g}$ agrees with g outside D, then $\bar{g} + F \subset \mathrm{Ext} \cap \mathrm{SES}$, as needed.

(II): First, notice that it is enough to prove that

$$A(\mathcal{S}) \leq A(\mathrm{AC} \cap \mathrm{PES}). \tag{1}$$

Indeed, the inclusions between the classes and inequality (1) imply that $A(\mathrm{AC} \cap \mathrm{PES}) \leq A(\mathcal{F} \cap \mathcal{G}) \leq A(\mathcal{G}) \leq A(\mathcal{S}) \leq A(\mathrm{AC} \cap \mathrm{PES})$. Therefore, all these quantities are equal and, by Proposition 1.8, we conclude that $\mathfrak{c}^+ \leq A(\mathcal{F} \cap \mathcal{G}) = A(\mathcal{G}) = A(\mathrm{SES}) = A(\mathrm{AC}) = A(\mathrm{Conn}) = A(\mathrm{D}) \leq 2^{\mathfrak{c}}$, as the statement claims.

To see (1), first notice that

$$A(\mathcal{S}) \leq A(\mathrm{ES}).$$

To see this inequality, choose $\kappa < A(\mathcal{S})$ and $F \subset \mathbb{R}^{\mathbb{R}}$ with $|F| = \kappa$. We need to find a $g \in \mathbb{R}^{\mathbb{R}}$ with $g + F \subset \mathrm{ES}$. For this, let $\mathcal{J}$ be the family of all non-empty intervals and let $\{P_I \in [I]^{\mathfrak{c}}\colon I \in \mathcal{J}\}$ be a partition of $\mathbb{R}$. For every $I \in \mathcal{J}$ the family $\{f \restriction P_I\colon f \in F\}$ has cardinality not bigger than $\kappa < A(\mathcal{S})$. Therefore, there exists a $g_I\colon P_I \to \mathbb{R}$ such that $(g_I + f \restriction P_I)[P_I] = \mathbb{R}$ for every $f \in F$. Then, $g = \bigcup_{I \in \mathcal{J}} g_I \in \mathbb{R}^{\mathbb{R}}$ and $g + F \subset \mathrm{ES}$, as required.

Now, let J the class of Jones functions. (See the proof of (III) in Proposition 1.9.) Then $J \subset \mathrm{AC} \cap \mathrm{PES}$. Besides, it is proved in [26] that $A(J) = A(\mathrm{SES})$. Therefore,

$$A(\mathcal{S}) \leq A(\mathrm{ES}) = A(\mathrm{SES}) = A(J) \leq A(\mathrm{AC} \cap \mathrm{PES}),$$

proving the needed inequality (1). $\square$

Next, we turn our attention to the families SZ, $F_{<\mathfrak{c}}$, and $\mathbb{R}^{\mathbb{R}} \setminus \mathrm{SES}$ and their intersections. We start here with noticing that $\mathrm{SZ} \cap (\mathrm{SCIVP} \cup \mathrm{SES}) = \emptyset$. This immediately implies

Proposition 2.2. $A(\mathcal{F}) = \mathcal{L}(\mathcal{F}) = 1$ *for any* $\mathcal{F} \subset \mathrm{SZ} \cap (\mathrm{SCIVP} \cup \mathrm{SES})$.

Therefore, we will drop the classes from Proposition 2.2 from further considerations. The next result can be found in [27].

Proposition 2.3. $\mathcal{L}(\mathrm{SZ})$ *is the smallest cardinality for which there is no almost disjoint family on* $\mathfrak{c}$. *In particular,* $\mathfrak{c}^{++} \le \mathcal{L}(\mathrm{SZ}) \le (2^{\mathfrak{c}})^+$ *and this is all that can be proved in ZFC.*

Recall also the following result from [20].

Proposition 2.4. $A(\mathrm{SZ})$ *is equal to the number*

$$d_{\mathfrak{c}} = \min\{|F| \colon F \subset \mathfrak{c}^{\mathfrak{c}} \ \&\ \forall h \in \mathfrak{c}^{\mathfrak{c}}\ \exists f \in F\ |f \cap h| = \mathfrak{c}\}.$$

In particular, $\mathfrak{c}^+ \le A(\mathrm{SZ}) \le 2^{\mathfrak{c}}$ *and this is all that can be proved in ZFC.*

From this, we immediately conclude

Corollary 2.5. *The equations* $\mathcal{L}(\mathrm{SZ}) = (2^{\mathfrak{c}})^+$ *and* $A(\mathrm{SZ}) = 2^{\mathfrak{c}}$ *are independent of ZFC.*

The following theorem shows that we can still have some ZFC results related to the number $A(\mathrm{SZ})$, in spite of the fact that its exact value is not determined in ZFC.

Theorem 2.6. $A(\mathcal{F} \cap \mathrm{SZ}) = \mathfrak{c}^+$ *for* $\mathcal{F} \in \{\mathrm{CIVP}, \mathrm{PR}\}$ *and* $A(\mathrm{PC} \cap \mathrm{SZ}) = A(\mathrm{SZ})$.

Proof. For the first part notice that $\mathrm{CIVP} \cap \mathrm{SZ} \subset \mathrm{PR} \cap \mathrm{SZ} \subset \mathrm{PR}$ implies $A(\mathrm{CIVP} \cap \mathrm{SZ}) \le A(\mathrm{PR} \cap \mathrm{SZ}) \le A(\mathrm{PR}) = \mathfrak{c}^+$, where $A(\mathrm{PR}) = \mathfrak{c}^+$ follows from Proposition 1.8. Thus, it is enough to show that $A(\mathrm{CIVP} \cap \mathrm{SZ}) \ge \mathfrak{c}^+$.

Let $F \subset \mathbb{R}^{\mathbb{R}}$ with $|F| \le \mathfrak{c}$. We will construct a function g such that $g + F \subset \mathrm{SZ}$ and, besides, for every $f \in F$, every perfect set K, and every open interval (a, b), there exists a perfect set $C \subset (a, b)$ such that $(g + f)[C] \subset K$. The latter claim implies that $g + F \subset \mathrm{CIVP}$.

To this end, let $\mathcal{G} = \{(a, b) \colon a < b\}$, $\mathcal{P} = \{K \subset \mathbb{R} \colon K \text{ is perfect}\}$, and $\mathcal{B}$ be that family of all Borel functions from $\mathbb{R}$ to $\mathbb{R}$. Enumerate $\mathcal{G} \times \mathcal{P} \times F$ as $\{\langle I_\alpha, K_\alpha, f_\alpha\rangle \colon \alpha < \mathfrak{c}\}$, $\mathcal{B}$ as $\{\varphi_\alpha \colon \alpha < \mathfrak{c}\}$, and $\mathbb{R}$ as $\{x_\alpha \colon \alpha < \mathfrak{c}\}$. It is a standard fact that we can choose a sequence $\langle X_\alpha \subset I_\alpha \colon \alpha < \mathfrak{c}\rangle$ forming a partition of $\mathbb{R}$ with each X_α containing a perfect set C_α.

We define $\{g(x_\alpha) \colon \alpha < \mathfrak{c}\}$, by induction on $\alpha < \mathfrak{c}$, as follows. For every $\alpha < \mathfrak{c}$ choose the unique $\beta < \mathfrak{c}$ with $x_\alpha \in X_\beta$ and pick

$$g(x_\alpha) \in (-f_\beta(x_\alpha) + K_\beta) \setminus \{\varphi_\gamma(x_\alpha) - f_\delta(x_\alpha) \colon \gamma, \delta < \alpha\}.$$

It is a simple task to check that the so constructed function g satisfies what we need. Indeed, every $g + f_\delta \in \mathrm{SZ}$ since $|(g + f_\delta) \cap \varphi_\gamma| < \mathfrak{c}$ for every Borel function φ_γ: $(g + f_\delta)(x_\alpha) = \varphi_\gamma(x_\alpha)$ implies that $\alpha \le \max\{\gamma, \delta\}$. Also, to see that $g + f \in \mathrm{CIVP}$ for every $f \in F$, choose a perfect set $K \subset \mathbb{R}$ and a non-trivial interval $I = (a, b)$. Then, there exists a $\beta < \mathfrak{c}$ for which

$\langle I_\beta, K_\beta, f_\beta\rangle = \langle I, K, f\rangle$. So, there is a perfect set C_β contained in X_β and we have $(g+f)(x) = (g+f_\beta)(x) \in K_\beta = K$ for every $x \in C_\beta \subset X_\beta \subset I_\beta = (a,b)$.

For the second part, take $F \subset \mathbb{R}^\mathbb{R}$ with $|F| < A(\mathrm{SZ})$. Then there exists an $h \in \mathbb{R}^\mathbb{R}$ such that $h + F \subset \mathrm{SZ}$. As $|h+F| = |F| < A(\mathrm{SZ}) \le 2^\mathfrak{c} = A(\mathrm{PC})$, there exists $g \in \mathbb{R}^\mathbb{R}$ such that $g + (h+F) \subset \mathrm{PC}$. Actually, according to the proof of [22, Thm. 1.7(3)], g can be chosen to take values only in $\mathbb{Q}$. Then, it is immediate that, for every $h + f \in \mathrm{SZ}$, we have also $g + (h+f) \in \mathrm{SZ}$. Therefore, $(g+h) + F \subset \mathrm{PC} \cap \mathrm{SZ}$. □

The next two theorems show that the classes SZ and $F_{<\mathfrak{c}}$ have the same coefficients A and $\mathcal{L}$. This stands in contrast with what we prove later: that the classes $\mathrm{SZ} \cap \mathrm{ES}$ and $F_{<\mathfrak{c}} \cap \mathrm{ES}$ are actually quite different with respect to the operators A and $\mathcal{L}$.

Theorem 2.7. $\mathcal{L}(F_{<\mathfrak{c}}) = \mathcal{L}(\mathrm{SZ})$.

Proof. The inequality $\mathcal{L}(\mathrm{SZ}) \le \mathcal{L}(F_{<\mathfrak{c}})$ is justified by the inclusion $\mathrm{SZ} \subset F_{<\mathfrak{c}}$. To see the other inequality, notice that if $\kappa < \mathcal{L}(F_{<\mathfrak{c}})$, then the class $F_{<\mathfrak{c}}$ is κ-lineable with some space W witnessing this. Then, there exists an almost disjoint family of subsets of $\mathfrak{c}$ of cardinality κ: the graphs of functions in W are an example. Hence, by Proposition 2.3, the class SZ is κ-lineable, implying that $\kappa < \mathcal{L}(\mathrm{SZ})$. So, indeed $\mathcal{L}(F_{<\mathfrak{c}}) \le \mathcal{L}(\mathrm{SZ})$, as needed. □

Similarly, we have

Theorem 2.8. $A(F_{<\mathfrak{c}}) = A(\mathrm{SZ}) = d_\mathfrak{c}$.

Proof. The equation $A(\mathrm{SZ}) = d_\mathfrak{c}$ follows from Proposition 2.3, while the inequality $A(\mathrm{SZ}) \le A(F_{<\mathfrak{c}})$ is justified by the inclusion $\mathrm{SZ} \subset F_{<\mathfrak{c}}$. Therefore, it is enough to prove that $A(F_{<\mathfrak{c}}) \le d_\mathfrak{c}$. To see this, choose a cardinal $\kappa < A(F_{<\mathfrak{c}})$. It is enough to show that $\kappa < d_\mathfrak{c}$.

Indeed, choose an $F \subset \mathbb{R}^\mathbb{R}$ such that $|F| \le \kappa$. It is enough to show that $|F| < d_\mathfrak{c}$, that is, that there is an $h \in \mathbb{R}^\mathbb{R}$ such that $|f \cap h| < \mathfrak{c}$ for every $f \in F$. But $|F| < A(F_{<\mathfrak{c}})$ implies that there exists a $g \in \mathbb{R}^\mathbb{R}$ for which we have $g + F \subset F_{<\mathfrak{c}}$. Then $h = -g$ has the property that, for every $f \in F$, $|(-h+f)^{-1}(0)| < \mathfrak{c}$, that is, $|f \cap h| < \mathfrak{c}$, as needed. □

Thus, we have the following analog of Corollary 2.5.

Corollary 2.9. *The equations $\mathcal{L}(F_{<\mathfrak{c}}) = (2^\mathfrak{c})^+$ and $A(F_{<\mathfrak{c}}) = 2^\mathfrak{c}$ are independent of ZFC.*

The next theorem shows that $A(\mathrm{ES} \setminus \mathrm{SES})$ is surprisingly small.

Theorem 2.10. *If $\mathfrak{c}$ is regular, then $A(\mathrm{ES} \setminus \mathrm{SES}) \le \mathfrak{c}$ and $A(F_{<\mathfrak{c}} \cap \mathcal{S}) \le \mathfrak{c}$.*

Proof. Let $\{r_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of $\mathbb{R}$ and, for every $\xi < \mathfrak{c}$, define $A_\xi = \{r_\zeta \colon \zeta < \xi\}$. Let $F = \{r\chi_{A_\xi} + y \colon r, y \in \mathbb{R} \ \&\ \xi < \mathfrak{c}\}$, where χ_A is the characteristic function of A. Then $|F| = \mathfrak{c}$. Fix a $g \colon \mathbb{R} \to \mathbb{R}$. We will see that $g + F \not\subset \mathrm{ES} \setminus \mathrm{SES}$.

Indeed, this is clearly the case when $g = g + \chi_{A_0} \in$ SES. So, assume that $g \notin$ SES and let $a, b, y \in \mathbb{R}$ be such that $a < b$ and $A = g^{-1}(y) \cap (a, b)$ has cardinality smaller than $\mathfrak{c}$. Then, $A = (g - y)^{-1}(0) \cap (a, b)$. Let $\xi < \mathfrak{c}$ be such that $A \subset A_\xi$ and choose a non-zero $r \in \mathbb{R} \setminus (g - y)[A_\xi]$. Then $g(x) - y \neq r\chi_{A_\xi}(x)$ for every $x \in (a, b)$: for $x \in A_\xi$ by the choice of r and for $x \in (a, b) \setminus A_\xi$ as $g(x) - y \neq 0 = r\chi_{A_\xi}(x)$. In particular, $(g - y - r\chi_{A_\xi})(x) \neq 0$ for every $x \in (a, b)$, that is, $g + (-r\chi_{A_\xi} - y) \notin$ ES while $f = -r\chi_{A_\xi} - y \in F$, finishing the proof.

The inequality $A(F_{<\mathfrak{c}} \cap \mathcal{S}) \leq \mathfrak{c}$ is justified by the same family F. More precisely, for every $g \in \mathbb{R}^{\mathbb{R}}$, $g + F \not\subset F_{<\mathfrak{c}} \cap \mathcal{S}$ since either $g = g + \chi_{A_0} \notin F_{<\mathfrak{c}}$ or otherwise the above argument works for $A = g^{-1}(y)$ for every $y \in \mathbb{R}$. $\square$

We can now get a quite precise view of how different additivity and lineability coefficients can be for the intersections of SZ with classes of surjective Darboux-like functions. This is shown in the following result.

Theorem 2.11.

(I) *It is consistent with ZFC (it follows from the Covering Property Axiom, CPA, [21]) that* $\mathrm{SZ} \cap (\mathrm{D} \cup \mathcal{S}) = \emptyset$. *In this case,* $A(\mathcal{F}) = \mathcal{L}(\mathcal{F}) = 1$ *for any* $\mathcal{F} \subset \mathrm{SZ} \cap (\mathrm{D} \cup \mathcal{S})$.

(II) *If the union of less than continuum many meager sets does not cover* $\mathbb{R}$ *(i.e., when* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*), then* $\mathcal{L}(\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{ES}) \geq \mathfrak{c}^{++}$.

(III) *If* $\mathfrak{c}$ *is regular, then* $A(\mathrm{SZ} \cap \mathcal{S}) \leq \mathfrak{c}$.

(IV) *It is consistent with ZFC, follows from GCH, that*

$$A(\mathrm{SZ} \cap \mathcal{F} \cap \mathcal{G}) \leq \mathfrak{c} \quad \text{and} \quad \mathcal{L}(\mathrm{SZ} \cap \mathcal{F} \cap \mathcal{G}) = (2^{\mathfrak{c}})^+$$

for every $\mathcal{F} \in \{\mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathbb{R}^{\mathbb{R}}\}$ *and* $\mathcal{G} \in \{\mathrm{ES}, \mathcal{S}\}$.

Proof. (I) The equation $\mathrm{SZ} \cap (\mathrm{D} \cup \mathcal{S}) = \emptyset$ is consistent with ZFC since it holds in the iterated perfect set model, as it was proved by Balcerzak, the first named author, and Natkaniec in [8]. For the proof that this follows from the CPA axiom see [21].

(II) In [32] it is proved that CH implies that $\mathcal{L}(\mathrm{SZ} \cap \mathrm{AC}) \geq \mathfrak{c}^{++}$. A quick examination of the proof reveals that the argument works also under this weaker assumption and that it actually gives $\mathcal{L}(\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{ES}) \geq \mathfrak{c}^{++}$.

(III) It follows from Theorem 2.10, since $\mathrm{SZ} \cap \mathcal{S} \subset F_{<\mathfrak{c}} \cap \mathcal{S}$.

(IV) It follows from (II) and (III). $\square$

Let us recall that, in [25], the authors showed that ES $\setminus$ SES is $\mathfrak{c}$-lineable. However, the sets ES and SES are both $2^{\mathfrak{c}}$-lineable, see [5, 25]. Thus, it is natural to wonder about the maximal lineability of ES $\setminus$ SES. Let us first study the lineability of the class $F_{<\mathfrak{c}} \cap \mathrm{ES}$, which is contained in ES $\setminus$ SES.

Theorem 2.12. *If* $\mathfrak{c}$ *is a regular cardinal, then* $F_{<\mathfrak{c}} \cap \mathrm{ES}$ *is* $\mathfrak{c}^+$*-lineable, that is,* $\mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{ES}) > \mathfrak{c}^+$.

Proof. Let $\mathcal{G}$ be a linear subspace of $(F_{<\mathfrak{c}} \cap \mathrm{ES}) \cup \{0\}$ of cardinality not bigger than $\mathfrak{c}$. It is enough to show that $\mathcal{G}$ is not maximal, since then we can keep extending the linear subspaces of $(F_{<\mathfrak{c}} \cap \mathrm{ES}) \cup \{0\}$ until we get one of cardinality $\mathfrak{c}^+$. To see that $\mathcal{G}$ is not maximal, it is enough to find an $f\colon \mathbb{R} \to \mathbb{R}$ for which $f - \mathcal{G} \subset F_{<\mathfrak{c}} \cap \mathrm{ES}$, since then $\mathbb{R}(f - \mathcal{G}) \cup \mathcal{G} \subset (F_{<\mathfrak{c}} \cap \mathrm{ES}) \cup \{0\}$ is a desired proper extension of $\mathcal{G}$.

So, let $\{\langle g_\xi, r_\xi\rangle \colon \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{G} \times \mathbb{R}$ with no repetitions. Define, by induction on $\xi < \mathfrak{c}$, a sequence $\{X_\xi \in [\mathbb{R}]^\omega \colon \xi < \mathfrak{c}\}$ of pairwise disjoint sets and the values of $f \restriction X_\xi$ such that

- (I_ξ) if $r_\xi \notin \bigcup_{\eta<\xi} X_\eta$, then $r_\xi \in X_\xi$ and $f(r_\xi) \notin \{g_\zeta(r_\xi) + r_\zeta \colon \zeta < \xi\}$;
- (J_ξ) $D_\xi = X_\xi \setminus \{r_\xi\}$ is countable, dense in $\mathbb{R}$, and disjoint with the set $\bigcup_{\zeta<\xi} (g_\zeta - g_\xi)^{-1}(r_\xi - r_\zeta)$; moreover, for every $d \in D_\xi$ we put $f(d) = g_\xi(d) + r_\xi$.

The choice of the set D_ξ in (J_ξ) is possible by the set-theoretical assumption we made, the regularity of $\mathfrak{c}$, since each set $(g_\zeta - g_\xi)^{-1}(r_\xi - r_\zeta)$ is of cardinality smaller than $\mathfrak{c}$ (as either $g_\zeta - g_\xi \in F_{<\mathfrak{c}}$ or $g_\zeta = g_\xi$, in which case $r_\zeta \neq r_\xi$, since our enumeration of $\mathcal{G} \times \mathbb{R}$ is with no repetitions). Notice that (I_ξ) ensures that $\mathbb{R} = \bigcup_{\xi<\mathfrak{c}} X_\xi$.

To see that $f - \mathcal{G} \subset F_{<\mathfrak{c}} \cap \mathrm{ES}$, choose a $g \in \mathcal{G}$ and let $r \in \mathbb{R}$. We need to show that $(f - g)^{-1}(r)$ is a dense subset of $\mathbb{R}$ of cardinality less than $\mathfrak{c}$. To see this, choose a $\xi < \mathfrak{c}$ such that $\langle g, r\rangle = \langle g_\xi, r_\xi\rangle$. Then, by (J_ξ), we have $(f - g)(d) = (f - g_\xi)(d) = r_\xi = r$ for every $d \in D_\xi$. Therefore, $(f - g)^{-1}(r)$ contains the dense set D_ξ.

To see that $(f - g)^{-1}(r)$ has cardinality less than $\mathfrak{c}$ it is enough to show that $(f - g)^{-1}(r) = (f - g_\xi)^{-1}(r_\xi)$ is disjoint with X_α whenever $\xi < \alpha < \mathfrak{c}$. So, choose an $x \in X_\alpha$. We need to show that $(f - g_\xi)(x) \neq r_\xi$, that is, that $f(x) \neq g_\xi(x) + r_\xi$.

Indeed, if $x = r_\alpha$, then $f(x) = f(r_\alpha) \neq g_\xi(r_\alpha) + r_\xi = g_\xi(x) + r_\xi$ is ensured by (I_α), while for $x = d \in D_\xi = X_\xi \setminus \{r_\xi\}$ the condition (J_α) implies that $(g_\xi - g_\alpha)(d) \neq r_\alpha - r_\xi$ so, once again, $f(x) = f(d) = g_\alpha(d) + r_\alpha \neq g_\xi(d) + r_\xi$, finishing the proof. □

Notice also

Theorem 2.13. $\mathcal{L}(\mathrm{SZ} \cap \mathcal{S}) = \mathcal{L}(\mathrm{SZ} \cap \mathrm{ES}) \leq \mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{ES}) = \mathcal{L}(F_{<\mathfrak{c}} \cap \mathcal{S})$ *and this is all that can be proved in ZFC, as GCH implies that* $\mathcal{L}(\mathrm{SZ} \cap \mathrm{ES}) = \mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{ES})$ *while CPA implies that* $\mathcal{L}(\mathrm{SZ} \cap \mathrm{ES}) = 1 < \mathfrak{c}^+ < \mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{ES})$.

Proof. First we prove the equation $\mathcal{L}(F_{<\mathfrak{c}} \cap \mathcal{S}) = \mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{ES})$. Clearly, we have $\mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{ES}) \leq \mathcal{L}(F_{<\mathfrak{c}} \cap \mathcal{S})$ as $F_{<\mathfrak{c}} \cap \mathrm{ES} \subset F_{<\mathfrak{c}} \cap \mathcal{S}$. To see the other inequality, let $\kappa < \mathcal{L}(F_{<\mathfrak{c}} \cap \mathcal{S})$ and let W witness κ-lineability of $F_{<\mathfrak{c}} \cap \mathcal{S}$ (i.e., W is a linear subspace of $(F_{<\mathfrak{c}} \cap \mathcal{S}) \cup \{0\}$ of dimension κ). It is enough to prove that $F_{<\mathfrak{c}} \cap \mathrm{ES}$ is κ-lineable.

Indeed, let $V \subset \mathbb{R}$ be a Vitali set and let $h\colon V \to \mathbb{R}$ be a bijection. For $f \in \mathbb{R}^\mathbb{R}$ define $\hat{f} \in \mathbb{R}^\mathbb{R}$ via the formula $\hat{f}(v + q) = f(h(v))$, where $v \in V$ and $q \in \mathbb{Q}$. It is easy to see that $\hat{W} = \{\hat{f} \colon f \in W\}$ witnesses κ-lineability of $F_{<\mathfrak{c}} \cap \mathrm{ES}$.

Next we prove $\mathcal{L}(\mathrm{SZ}\cap\mathcal{S}) = \mathcal{L}(\mathrm{SZ}\cap\mathrm{ES})$. As $\mathcal{L}(\mathrm{SZ}\cap\mathrm{ES}) \le \mathcal{L}(\mathrm{SZ}\cap\mathcal{S})$ follows from $\mathrm{SZ}\cap\mathrm{ES} \subset \mathrm{SZ}\cap\mathcal{S}$, it is enough to prove the other inequality. So, let $\kappa < \mathcal{L}(\mathrm{SZ}\cap\mathcal{S})$ and let W witness κ-lineability of $\mathrm{SZ}\cap\mathcal{S}$. It is enough to prove that $\mathrm{SZ}\cap\mathrm{ES}$ is κ-lineable.

For this, let $\{P_n \colon n < \omega\}$ be a family of pairwise disjoint compact perfect sets such that each non-empty open interval contains one of the P_n's. For every $n < \omega$ let h_n be a bijection from $\mathbb{R}$ onto $S_n \subset P_n$ such that $h_n \restriction (\mathbb{R}\setminus\mathbb{Q})$ is a homeomorphic embedding. (It exists, since every perfect set in $\mathbb{R}$ is a universal space for zero-dimensional separable metric spaces.) Let $T = \mathbb{R}\setminus\bigcup_{n<\omega} S_n$ and for every $f \in \mathbb{R}^{\mathbb{R}}$ let $\hat{f} = (f \restriction T) \cup \bigcup_{n<\omega}(f\circ h_n^{-1})$. It is easy to see that $\hat{W} = \{\hat{f} \colon f \in W\}$ witnesses κ-lineability of $\mathrm{SZ}\cap\mathrm{ES}$. (For every $f \in \mathrm{SZ}$ the map $\hat{f}$ is also SZ, since it is a countable union of SZ maps: $f\restriction T$ and $f\circ h_n^{-1}$, where $f\circ h_n^{-1} \in \mathrm{SZ}$ since it is a union of a countable set and of $f\circ(h_n^{-1}\restriction h_n[\mathbb{R}\setminus\mathbb{Q}]) \in \mathrm{SZ}$.)

Finally, the inequality $\mathcal{L}(\mathrm{SZ}\cap\mathrm{ES}) \le \mathcal{L}(F_{<\mathfrak{c}}\cap\mathrm{ES})$ follows from the inclusion $\mathrm{SZ}\cap\mathrm{ES} \subset F_{<\mathfrak{c}}\cap\mathrm{ES}$. GCH implies equality, and then, by Theorem 2.11(ii), $\mathfrak{c}^{++} \le \mathcal{L}(\mathrm{SZ}\cap\mathrm{AC}\cap\mathrm{ES}) \le \mathcal{L}(\mathrm{SZ}\cap\mathrm{ES}) \le \mathcal{L}(F_{<\mathfrak{c}}\cap\mathrm{ES}) \le (2^{\mathfrak{c}})^+ = \mathfrak{c}^{++}$. On the other hand, CPA implies that $\mathrm{SZ}\cap\mathcal{S} = \emptyset$, giving $\mathcal{L}(\mathrm{SZ}\cap\mathrm{ES}) = 1$, and that $\mathfrak{c} = \omega_2$ is regular, hence, by Theorem 2.12, $\mathcal{L}(F_{<\mathfrak{c}}\cap\mathrm{ES}) > \mathfrak{c}^+$. □

Theorem 2.14. $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^{\kappa}$ *for every* $\kappa < \mathfrak{c}$.

Proof. Let $\omega \le \kappa < \mathfrak{c}$. We need to show that $\mathrm{ES}\setminus\mathrm{SES}$ is $\mathfrak{c}^{\kappa}$-lineable. Let $\{X_\xi \colon \xi < \kappa\}$ be a partition of $\mathbb{R}$ into $\mathfrak{c}$-dense sets. For every $\xi < \kappa$ choose an $f_\xi \in \mathbb{R}^{\mathbb{R}}$ such that $f_\xi \restriction (\mathbb{R}\setminus X_\xi) \equiv 0$ and, for every $y \in \mathbb{R}$, $X_\xi \cap f_\xi^{-1}(y)$ is a countable dense subset of $\mathbb{R}$. Notice that the family

$$\mathcal{F} = \left\{ \sum_{\xi<\kappa} h(\xi) f_\xi \colon h \in \mathbb{R}^{\kappa} \right\}$$

is as needed. □

As a consequence of the previous results, we have:

Corollary 2.15. $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^+$.

Proof. If $\mathfrak{c}$ is regular, this follows from Theorem 2.12 and the fact that

$$\mathrm{ES}\setminus\mathrm{SES} \supset \mathrm{ES}\cap F_{<\mathfrak{c}}.$$

If $\mathfrak{c}$ is singular, this follows from Theorem 2.14 used with $\kappa = \mathrm{cof}(\mathfrak{c})$. □

3. Open problems

We have elucidated many of the values of lineability and additivity coefficients for the considered families of functions. However, the exact values of these operators for some of these classes are still unknown as we indicate below.

A consequence of Corollary 2.15 is that under the assumption of GCH (or just that $2^{\mathfrak{c}} = \mathfrak{c}^+$) we have $\mathcal{L}(\mathrm{ES} \setminus \mathrm{SES}) = (2^{\mathfrak{c}})^+$. However, the answer to the following question is still unknown.

Problem 3.1. Can equation $\mathcal{L}(\mathrm{ES} \setminus \mathrm{SES}) = (2^{\mathfrak{c}})^+$ be proved in ZFC?

Concerning the additivity operator, Theorem 2.10 assures that, assuming that $\mathfrak{c}$ is a regular cardinal, the values of $A(\mathrm{ES} \setminus \mathrm{SES})$ and $A(\mathrm{ES} \cap F_{<\mathfrak{c}})$ do not exceed $\mathfrak{c}$. But, what can be said about these coefficients in ZFC?

Problem 3.2. Can we prove $A(\mathrm{ES} \setminus \mathrm{SES}) \leq \mathfrak{c}$ in ZFC? What about $A(\mathrm{ES} \cap F_{<\mathfrak{c}}) \leq \mathfrak{c}$? What else can be said about $A(ES \setminus \mathrm{ES})$ or $A(\mathrm{ES} \cap F_{<\mathfrak{c}})$?

According to Theorem 2.11, the lineability numbers for $\mathcal{S} \cap \mathrm{SZ}$ and $\mathrm{D} \cap \mathrm{SZ}$ can be as small as 1 and as large as $(2^{\mathfrak{c}})^+$. Nevertheless, the exact relations between these values remains unclear.

Problem 3.3. Are any of the coefficients $A(\mathrm{D} \cap \mathrm{SZ})$, $A(\mathrm{ES} \cap \mathrm{SZ})$, and $A(\mathcal{S} \cap SZ)$ provably equal (in ZFC)? What about $\mathcal{L}(\mathrm{D} \cap \mathrm{SZ})$ and $\mathcal{L}(\mathcal{S} \cap SZ)$?

Related to this last question is also the following

Problem 3.4. Does the assumption $\mathrm{SZ} \cap \mathcal{S} \neq \emptyset$ imply that $\mathrm{SZ} \cap \mathcal{S}$ is $\mathfrak{c}^+$-lineable? Does it imply that $\mathrm{SZ} \cap \mathcal{S}$ is κ-lineable, where $\kappa = \mathcal{L}(\mathrm{SZ})$?

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