An auto-homeomorphism of a Cantor set with derivative zero everywhere

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Abstract

We construct a closed bounded subset $\mathfrak{X}$ of $\mathbb{R}$ with no isolated points which admits a differentiable bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}'(x) = 0$ for all $x \in \mathfrak{X}$. We also show that any such function admits a restriction $\mathfrak{f}\restriction P$ to an uncountable closed $P \subseteq \mathfrak{X}$ forming a minimal dynamical system. The existence of such a map $\mathfrak{f}$ *seems* to contradict several well know results. The map $\mathfrak{f}$ marks a limit beyond which Banach Fixed-Point Theorem cannot be generalized.

1 Introduction

Recall, that a subset $X \subseteq \mathbb{R}$ is *perfect*, if it is closed and has no isolated points. A map $f\colon X \to X$ (or, more formally, a pair $\langle X, f\rangle$) is a *minimal dynamical system*, provided X is non-empty, f is surjective, and $f[P] \neq P$ for any non-empty closed proper subset $P \subsetneq X$.

The main contribution of this article is the construction and discussion of a perfect set $\mathfrak{X}$ and a seemingly paradoxical (see Fact 2) map $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$, a bijection with $\mathfrak{f}' \equiv 0$. More importantly, $\mathfrak{f}$ satisfies certain local contraction properties but does not have a fixed point. Hence it indicates the boundaries beyond which local versions of Banach fixed-point theorem cannot be generalized.

*Key words: differentiable minimal dynamical systems; fixed point theorem;
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Theorem 1 *There exists a non-empty compact perfect set $\mathfrak{X} \subset \mathbb{R}$ and a differentiable bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}'(x) = 0$ for every $x \in \mathfrak{X}$. Moreover,*

(i) $\mathfrak{f}$ is a minimal dynamical system;

(ii) $\mathfrak{f}$ can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.

The identity $\mathfrak{f}' \equiv 0$ readily implies that $\mathfrak{f}$ is *locally radially shrinking* in a sense that

(LRS) for every $x \in \mathfrak{X}$ there exists an $\varepsilon_x > 0$ such that $|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for any $y \in \mathfrak{X}$ with $0 < |x - y| < \varepsilon_x$

and it seems impossible for a function with such property to map an infinite compact set $\mathfrak{X}$ onto itself.

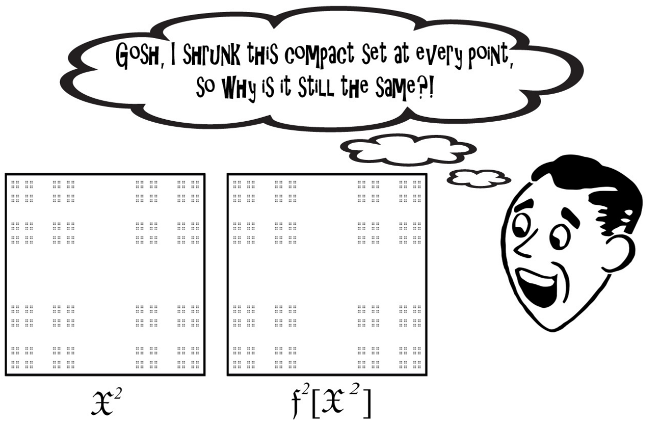

Figure 1: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

The (incorrect) intuition against the existence of the function $\mathfrak{f}$ from Theorem 1 is also supported by the following three facts.

Fact 2 *Assume that $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$.*

(i) $X \not\subseteq f[X]$ when X is a bounded closed interval and $|f'| \le \lambda < 1$ on X since then, by the Mean Value Theorem, $|f(y) - f(z)| \le \lambda |y - z|$ for every $y, z \in X$, so that the diameter of $f[X]$ is strictly smaller than the diameter of X. If $\mathfrak{f}' \equiv 0$, then f is constant.

(ii) $X \not\subseteq f[X]$ when X has a positive finite Lebesgue measure $m(X)$ and $|f'| \le \lambda < 1$ on X, since then $m(f[X]) \le \lambda m(X)$, see e.g. [9].

(iii) $X \not\subseteq f[X]$ when $|f'| < 1$ on X and f can be extended to a **continuously** *differentiable function $F\colon \mathbb{R} \to \mathbb{R}$. This has been proved by the authors in [5, lemma 3.3].*

The nonexistence of an example such as one from Theorem 1 must have been suspected by Edrei, when in his 1952 paper [8] he made the following conjecture.

> *If $\langle X, d\rangle$ is a compact metric space and $f\colon X \to X$ is surjection such that for every $x \in X$ there exists an $\varepsilon_x > 0$ such that $d(f(x), f(y)) \leq d(x, y)$ for every $y \in X$ with $d(x, y) < \varepsilon_x$, then every point of X is a point of isometry of f (i.e., for every $x \in X$ there exists an $\delta_x > 0$ such that $d(f(x), f(y)) = d(x, y)$ for every $y \in X$ with $d(x, y) < \delta_x$).*

Clearly, Theorem 1 contradicts this conjecture.

In Section 2 we discuss the relation of the dynamical system $\langle \mathfrak{X}, \mathfrak{f}\rangle$ from Theorem 1 to the fixed-point theory of locally contractive functions. Section 3 contains the details of a rather delicate construction of $\langle \mathfrak{X}, \mathfrak{f}\rangle$. In Section 4 we prove that any infinite dynamical system $\langle X, f\rangle$ on a compact space X and with surjective (LRS) map f must contain an uncountable minimal dynamical system. This illuminates the role of property (i) in Theorem 1.

2 The example, minimal dynamics, and Banach Fixed-Point Theorem

Let $\langle X, d\rangle$ be a metric space. A map $f\colon X \to X$ is *contractive* with a contraction constant $\lambda \in [0, 1)$ if $d(f(y), f(z)) \leq \lambda d(y, z)$ for every $y, z \in X$. An $x \in X$ is a *fixed point* of f whenever $f(x) = x$.

A famous 1922 theorem of Banach [1], known as Banach Fixed-Point Theorem or the Contractive Mapping Principle, states that

Theorem 3 *If X is a complete metric space and $f\colon X \to X$ is contractive, then f has a fixed point.*

Let us recall some notation we need to discuss the dynamics of a continuous function $f\colon X \to X$. For a number $n \in \omega = \{0, 1, 2, \ldots\}$, the n-th iteration $f^{(n)}$ of f is defined as $f \circ \cdots \circ f$, the composition of n instances of f. In particular, $f^{(1)} = f$ and $f^{(0)}$ is the identity function. The *orbit* of $x \in X$ with respect to f is the set $O(x) = \{f^{(n)}(x)\colon n \in \omega\}$. It is easy to see that f is a minimal dynamical system if, and only if, the orbit $O(x)$ of every $x \in X$ is dense in X (i.e., for every $c \in X$ and $\varepsilon > 0$, the open ball $B(c, \varepsilon) = \{y \in X\colon d(c, y) < \varepsilon\}$ intersects $O(x)$).

Recall, that a simple application of Zorn's Lemma[1] gives the following 1912 theorem of Birkhoff [2].

Theorem 4 *For every compact X and continuous $f\colon X \to X$ there exists a non-empty compact $Z \subseteq X$ such that $f \restriction Z$ is a minimal dynamical system.*

Of course, the set Z from Birkhoff's theorem 4 can be a singleton. Actually, it must be a singleton whenever f is a contraction, since otherwise, the diameter of $f[Z]$ would be smaller than the diameter of Z.

[1]Applied to the family $\mathcal{Z}$ of all closed non-empty $Z \subseteq X$ such that $f[Z] \subseteq Z$.

> Does it mean, that *the only compact minimal dynamical systems to which Banach Fixed-Point Theorem is applicable are the systems with singleton spaces*?

For the original Banach Fixed-Point Theorem, the answer is affirmative. However, in this note, we discuss its generalizations in which the assumption that f is contractive is relaxed to a "local contracting" condition, see Theorems 6 and 7 below. In particular, under such relaxed assumptions, the interplay between the generalized Banach fixed-point theorems and the minimal dynamical systems is considerably more intricate.

In the rest of this section, we will discuss two notions of locally contractive maps: one defined via standard topological localization technique, the other motivated by a calculus interpretation of contractive maps.

Locally contractive maps via standard localization technique: We say that a map $f\colon X \to X$ is *locally contractive*, (LC), provided for every $x \in X$ there exists an $\varepsilon_x > 0$ such that $f \restriction B(x, \varepsilon_x)$ is contractive with some constant $\lambda_x \in [0, 1)$. For a compact space X, (LC) is equivalent to the following *uniform local contraction* property[2]

Fact 5 *If X is compact, then $f\colon X \to X$ is locally contractive if, and only if,*

(ULC) there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that $d(f(y), f(z)) \leq \lambda d(y, z)$ for every $x \in X$ and $y, z \in B(x, \varepsilon)$.

Recall that an $x \in X$ is a periodic point of a function $f\colon X \to X$ provided $f^{(n)}(x) = x$ for some $n > 0$. In particular, $x \in X$ is a fixed point of f if, and only if, it is a periodic point of f with period 1, that is, $f^{(1)}(x) = x$. For (LC) functions, using Fact 5, Edelstein's generalizations of Banach Fixed-Point Theorem [7, Remark 5.1], and [6, Theorem 5.2], we obtain the following:

Theorem 6 *Assume that $f\colon X \to X$ is locally contractive and that X is compact. Then*

(i) f has a periodic point;

(ii) f has a fixed point provided X is connected.

Notice, that the assumption of connectedness in (ii) is essential, as justified by the function $f\colon X \to X$, with $X = [-2, -1] \cup [1, 2]$, defined as $f(x) = -\mathrm{sgn}(x) = -\frac{x}{|x|}$ for all $x \in X$. Clearly, it satisfies (LC) with $\lambda = 0$ and it has no fixed point, though points 1 and -1 are periodic.

[2]Let $\{B(x, \varepsilon_x)\colon x \in X_0\} \subseteq \{B(x, \varepsilon_x)\colon x \in X\}$ be a finite subcover of X. Then the number $\lambda = \max_{x \in X_0} \lambda_x \in [0, 1)$ satisfies (LC), though with possibly smaller numbers ε_x.

Locally contractive maps via calculus interpretation: Differentiable contractive maps on $\mathbb{R}$ have a very nice characterization. Namely, if $X \subseteq \mathbb{R}$ is a closed interval and $f\colon X \to X$ is differentiable, then, by the Mean Value Theorem, f is contractive if, and only if,

(D) there exists a $\lambda \in [0,1)$ such that $|f'(x)| \leq \lambda$ for every $x \in X$.

More generally, notice that if $X \subseteq \mathbb{R}$ has no isolated points, then the standard definition of the derivative makes sense for $f\colon X \to X$ and, if f is differentiable, then (D) is equivalent to the following property, which uses no notion of the derivative

(LRC) there is a $\lambda \in [0,1)$ such that for every $x \in X$ there exists an $\varepsilon_x > 0$ with a property that $d(f(x), f(z)) \leq \lambda d(x,z)$ for every $z \in B(x, \varepsilon_x)$.

(LRC) was studied, for arbitrary metric spaces X, by several authors [12, 13, 15] and was referred to as the *local radial contraction* property of f.

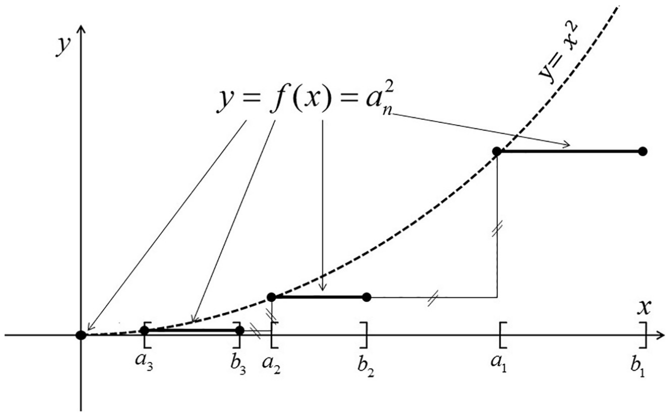

Figure 2: $f(0) = 0$ and $f(x) = (a_n)^2$ for any $x \in [a_n, b_n]$ and $n = 1, 2, 3, \ldots$

Clearly (ULC)⇒(LRC). The fact that this implication cannot be reversed is justified by a function $f\colon X \to X$ depicted in Figure 2, where $X = \{0\} \cup \bigcup_{n=1}^{\infty}[a_n, b_n]$, $1 = b_1 > a_1 > b_2 > a_2 > \cdots > \lim_n a_n = 0$, and $f(a_n) - f(b_{n+1}) = a_n - b_{n+1}$ for all $n = 1, 2, 3, \ldots$. This f is (LRC) since $f'(x) = 0$ for every $x \in X$. At the same time (LC) fails for f at $x = 0$, since any open $U \ni 0$ contains distinct a and b with $f(a) - f(b) = a - b$.

Now, returning to Banach Fixed-Point Theorem, the following generalization to (LRC) functions first appeared in a 1978 paper [13] of Hu and Kirk. However, its proof contained a gap, as it relied on a false proposition from [12]. The first complete proof of this theorem appeared in the 1982 paper [15] of Jungck.

Theorem 7 *Assume that X is a complete metric space and that every two points of X can be connected by a path in X of finite length.*[3] *If $f\colon X \to X$ satisfies (LRC), then f has a fixed point.*

But what happens if, in Theorem 7, we replace all the assumptions on the space X with a simple requirement that X is compact? In other words,

is Theorem 6(i) true for (LRC) maps?

The negative answer is provided by the function $\mathfrak{f}$ from Theorem 1; it shows the limits to the localized generalizations of Banach Fixed-Point Theorem. As $\mathfrak{f}$ forms a minimal dynamical system, it is fair to say that $\mathfrak{f}$ marks the spot where *the minimal dynamical systems "meet" Banach Fixed-Point Theorem*. See also Theorem 9.

The results discussed in this section are summarized in Table 1.

Convexity of X assumed?	$f\colon X \to X$ has periodic/fixed point when f is		
	contractive	locally contractive (LC)	locally radially contractive (LRC)
Yes	fixed point Banach, Thm 3	fixed point Edelstein, Thm 6(ii)	fixed point Hu & Kirk, Thm 7
No	fixed point Banach, Thm 3	periodic point Edelstein, Thm 6(i)	NEITHER KC & JJ, Thm 1

Table 1: Fixed/periodic point properties implied by various contractive properties of the function $f\colon X \to X$, where X is compact and either arbitrary, or a convex subspace of a Banach space

Remark 8 It is interesting to notice that, according to the property (6) proven below, function $\mathfrak{f}$ from Theorem 1 is (LC) at all points but one. Of course, this single exception is of paramount importance, since, by Theorem 6(i), any everywhere (LC) function has periodic points.

3 Construction of the example from Theorem 1

The Adding machine: On the set 2^ω of infinite 0-1 sequences define the following "add one and carry" operation $\sigma\colon 2^\omega \to 2^\omega$, often referred to as *adding machine* (see e.g. [18] or [4]) and representing odometer-like action: for $s = \langle s_0, s_1, s_2, \ldots\rangle \in 2^\omega$, $\sigma(s) = s + \langle 1, 0, 0, ...\rangle$ or, more precisely,

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle & \text{if } s_k = 0 \text{ and } s_i = 1 \text{ for all } i < k. \end{cases}$$

[3] A length of a path $p\colon [0,1] \to X$ is defined as a supremum over all numbers $\sum_{i=1}^{n} d(p(t_i), p(t_{i-1}))$, where $0 = t_0 < t_1 < \cdots < t_n = 1$. In particular, every convex subset X of a Banach space is path connected in the sense of Theorem 7.

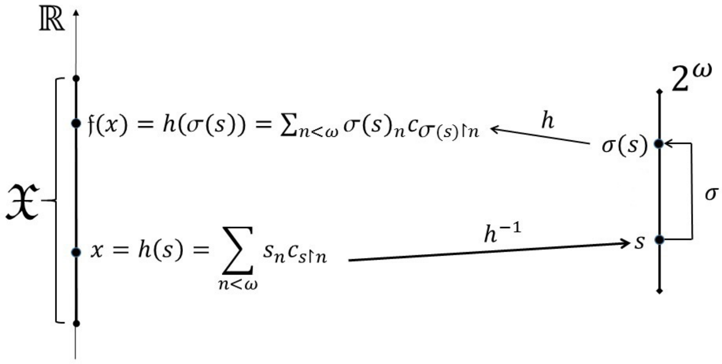

Figure 3: $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

In other words, if for $k < \omega$ we let $w_k \in 2^{k+1}$ to be $w_k = \langle 1, \ldots, 1, 0\rangle$ (a sequence of k-many 1s followed by a single 0) and $z_k \in 2^{k+1}$ to be $z_k = \langle 0, \ldots, 0, 1\rangle$ (a sequence of k-many 0s followed by a single 1), then

$$\begin{aligned} \sigma(1,1,1,\ldots) &= \langle 0,0,0,\ldots\rangle \\ \sigma(w_k, s_{k+1}, s_{k+2}, \ldots) &= \langle z_k, s_{k+1}, s_{k+2}, \ldots\rangle. \end{aligned}$$

It is well known and easy to see that σ is a continuous bijection and that

$$\text{the orbit of every } s \in 2^\omega \text{ is dense in } 2^\omega.^4 \tag{1}$$

In particular, σ is a minimal dynamical system, see e.g. [17].

For $s \in 2^\omega$ and $\nu < \omega$ let $N_\nu(s) = \sum_{i<\nu} s_i 2^i$, with $N_0(s)$ understood as 0. An important property of σ is that for every $s \in 2^\omega$ and $k < \omega$

$$\text{if } s \restriction (k+1) = w_k, \text{ then } N_\nu(\sigma(s)) = N_\nu(s) + 1 \text{ for every } \nu > k. \tag{2}$$

Let $\bar{1} = \langle 1, 1, 1, \ldots\rangle$. Then, in particular,

$$N_\nu(s) < N_\nu(\sigma(s)) \text{ for every } s \in 2^\omega \text{ with } s \neq \bar{1} \text{ and any large enough } \nu < \omega.$$

However, the inequality $N_\nu(s) < N_\nu(\sigma(s))$ is false for any $\nu < \omega$, when $s = \bar{1}$.

Format of the example: We will find a continuous injection $h \colon 2^\omega \to \mathbb{R}$ such that $\mathfrak{X} = h[2^\omega]$ and $\mathfrak{f} = h \circ \sigma \circ h^{-1}$ forms the example from Theorem 1, see Figure 3. (Note that h^{-1} is a homeomorphism between 2^ω and X.) Since

[4] For $\tau \in 2^n$ let $[\tau] = \{t \in 2^\omega : t \restriction n = \tau\}$. By induction on $n < \omega$, we can easily see that $O(s) \cap [\tau] \neq \emptyset$ for any $s \in 2^\omega$.

$\mathfrak{f}^{(n)} = h \circ \sigma^{(n)} \circ h^{-1}$ whenever $n < \omega$, (1) implies that for any $x \in \mathfrak{X}$ the orbit $O(x)$ of $\mathfrak{f}$ is dense in $\mathfrak{X}$.

Note that $\mathfrak{f} = h \circ \sigma \circ h^{-1}$ is, what is usually called, a *topological conjugate* of (or *isomorphic* to) the adding machine σ. In particular, the mapping h can be considered as a generator of a metric ρ on 2^ω defined as $\rho(s,t) = |h(s) - h(t)|$.

Format of the function h: The map $h\colon 2^\omega \to \mathbb{R}$ will be defined via formula

$$h(s) = \sum_{n<\omega} s_n c_{s\restriction n} \text{ for every } s \in 2^\omega \tag{3}$$

for appropriately chosen numbers $c_\tau \in \mathbb{R}$ for $\tau \in 2^{<\omega}$. To ensure that $\mathfrak{f}'(x) = 0$ for $x = h(s)$ with $s \in 2^\omega$, it needs to be shown that for every $y = h(t)$ with $t \in 2^\omega$ and $t \neq s$, the numbers

$$\Delta_{st} = \frac{|\mathfrak{f}(x) - \mathfrak{f}(y)|}{|x-y|} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|}$$

converge to 0 when $\ell = \min\{i < \omega\colon s_i \neq t_i\}$ diverges to infinity.

For $s \neq \bar{1}$, that is, of the form $\langle w_k, s_{k+1}, s_{k+2}, \ldots\rangle$, the choice of c_τ's will guarantee this convergence by ensuring, for large enough ℓ, and the $u \in \{s,t\}$ with $u_\ell = 1$,

$$\begin{aligned} |h(\sigma(s)) - h(\sigma(t))| &\leq \tfrac{3}{2} \sum_{n\geq\ell} u_n |c_{\sigma(u)\restriction n}| \\ |h(s) - h(t)| &\geq \tfrac{1}{2} \sum_{n\geq\ell} u_n |c_{u\restriction n}| > 0 \end{aligned} \tag{4}$$

as well as the existence of a constant $E_k > 0$ depending only on k, and a sequence $\langle \beta_n : n < \omega\rangle$ with $\beta_n^{-1} \searrow 0$ for which

$$\frac{|c_{\sigma(u)\restriction n}|}{|c_{u\restriction n}|} = E_k \beta_n^{-1} \leq E_k \beta_\ell^{-1} \text{ for every } n \geq \ell. \tag{5}$$

This guarantees the desired convergence, as then

$$\Delta_{st} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \leq \frac{\frac{3}{2}\sum_{n\geq\ell} u_n |c_{\sigma(u)\restriction n}|}{\frac{1}{2}\sum_{n\geq\ell} u_n |c_{u\restriction n}|} \leq 3E_k\beta_\ell^{-1} \to_{\ell\to\infty} 0. \tag{6}$$

The case $s = \bar{1}$ requires essentially different argument, based on the following two properties, satisfied for $\ell > 0$:

$$|h(\sigma(s)) - h(\sigma(t))| \leq \frac{1}{\ell+1}\frac{1}{\ell} \tag{7}$$

and

$$|h(s) - h(t)| \geq \sum_{n\geq\ell} |c_{s\restriction n}| \geq \sum_{n\geq\ell} \frac{1}{(n+2)^{1/2}} \frac{1}{n+2} \frac{1}{n+1}. \tag{8}$$

Since $\sum_{n\geq\ell}\frac{1}{(n+2)^{1/2}}\frac{1}{n+2}\frac{1}{n+1} \geq \sum_{n\geq\ell}\frac{1}{(n+2)^{2.5}} \geq \int_{\ell+2}^{\infty} x^{-2.5}\,dx = \frac{1}{1.5}\frac{1}{(\ell+2)^{1.5}}$, (7) and (8) imply the required convergence:

$$\Delta_{st} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s)-h(t)|} \leq \frac{\frac{1}{\ell(\ell+1)}}{\frac{1}{1.5}\frac{1}{(\ell+2)^{1.5}}} = 1.5\frac{(\ell+2)^{1.5}}{\ell(\ell+1)} \to_{\ell\to\infty} 0.$$

Definition of the coefficients $c_{s\restriction n}$ from (3): We can see by now that a lot is expected of the coefficients c_τ. So, their definition is quite delicate and it will not be fully completed until we reach equation (14).

To ensure satisfaction of the properties (4)-(8), for every $s \in 2^\omega$ and $n < \omega$ we let $\beta_n = \ln(n+3) > 1$, and define

$$c_{s\restriction n} = a_{s\restriction n}\beta_n^{-b_{s\restriction n}} d_{s\restriction n}, \tag{9}$$

where $d_{s\restriction n} > 0$ is defined below in (14), $a_{s\restriction 0} = -1$, $b_{s\restriction 0} = 0$, and, for $n > 0$,

$$a_{s\restriction n} = \begin{cases} -1 & \text{when } s\restriction n = \langle 1,1,\ldots,1\rangle, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad b_{s\restriction n} = N_{\nu_n}(s) = \sum_{i<\nu_n} s_i 2^i,$$

where $\nu_n = \max\left\{m < \omega\colon (\beta_n)^{2^m-1} < \sqrt{n+2}\right\}$. Notice that the definition of ν_n gives $(\beta_n)^{b_{s\restriction n}} \leq (\beta_n)^{2^{\nu_n}-1} < \sqrt{n+2}$, that is, that

$$\beta_n^{-b_{s\restriction n}} > \frac{1}{(n+2)^{1/2}} \quad \text{for every } s \in 2^\omega \text{ and } n<\omega. \tag{10}$$

Reduction of property (8): The sole purpose of the coefficients $a_{s\restriction n}$ is to facilitate the following argument for the first inequality from (8), in case $s = \bar{1}$, where the equations hold since $s\restriction n = t\restriction n$ for all $n < \ell$, while $a_{s\restriction n} = -1$ and $a_{t\restriction n} = 1$ for all $n \geq \ell$

$$|h(s)-h(t)| = \left|\sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n\geq\ell} t_n c_{t\restriction n}\right| = \left|-\sum_{n\geq\ell}|c_{s\restriction n}| - \sum_{n\geq\ell} t_n|c_{t\restriction n}|\right| \geq \sum_{n\geq\ell}|c_{s\restriction n}|.$$

Also, by (10), for every $n > 0$ we have $|c_{s\restriction n}| = \beta_n^{-b_{s\restriction n}} d_{s\restriction n} \geq \frac{1}{(n+2)^{1/2}} d_{s\restriction n}$. Thus, the second inequality from (8) is ensured by the following requirement:

$$d_{s\restriction n} = \frac{1}{n+2}\frac{1}{n+1} \text{ for every } n<\omega \text{ and } s = \bar{1}. \tag{11}$$

Reduction of property (5): For $s = \langle w_k, s_{k+1}, s_{k+2}, \ldots\rangle$ and large enough ℓ, the property (5) holds, as long as we ensure that

$$d_{\sigma(s)\restriction n} = E_k d_{s\restriction n} \text{ for every } s = \langle w_k, s_{k+1}, s_{k+2},\ldots\rangle \text{ and } n > k. \tag{12}$$

Indeed, since $\frac{(\beta_n)^{2^{k+1}-1}}{\sqrt{n+2}} = \frac{(\ln(n+3))^{2^{k+1}-1}}{\sqrt{n+2}} \to_{n\to\infty} 0$, there exists an $\ell > k$ such that $(\beta_n)^{2^{k+1}-1} \leq \sqrt{n+2}$ for any $n \geq \ell$. This choice of ℓ ensures (5) as then,

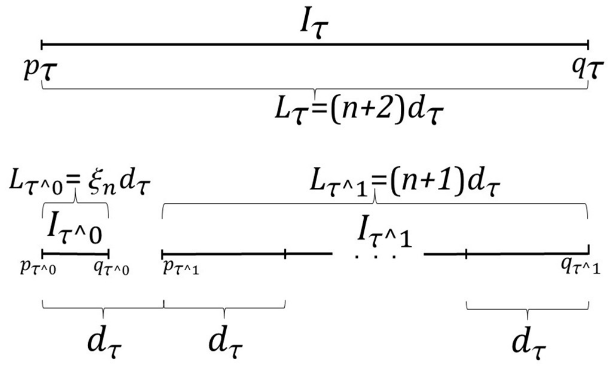

Figure 4: I_τ, $I_{\tau\hat{}0}$, and $I_{\tau\hat{}1}$ for $\tau \in 2^n$

by the definition of numbers ν_n, for every $n \geq \ell$ we have $k+1 \leq \nu_n$. So, by (2), $N_{\nu_n}(\sigma(u)) = N_{\nu_n}(u) + 1$ and

$$\frac{|c_{\sigma(u)\restriction n}|}{|c_{u\restriction n}|} = \frac{\beta_n^{-N_{\nu_n}(\sigma(u))} d_{\sigma(u)\restriction n}}{\beta_n^{-N_{\nu_n}(u)} d_{u\restriction n}} = \beta_n^{-1} \frac{d_{\sigma(u)\restriction n}}{d_{u\restriction n}} = E_k \beta_n^{-1}.$$

To finish the construction, it is enough to define the coefficients $d_{t\restriction n}$ that ensure: the properties (11) and (12), the fact that h is a continuous injection, and the estimates (4) and (7).

Definition of the coefficients $d_{s\restriction n}$: For every $n < \omega$ let

$$\xi_n = \frac{1}{2} \frac{1}{(n+4)^{1/2}}.$$

Then, by (10), for every $s \in 2^\omega$, $\ell < \omega$, and $0 < m < \omega$,

$$\xi_\ell < \frac{1}{2} \beta_\ell^{-b_{s\restriction \ell}} \quad \text{and} \quad \xi_m < \beta_{m-1}^{-b_{s\restriction(m-1)}}. \tag{13}$$

Mimicking the classical construction of Cantor's ternary set, we define, for $\tau \in 2^{<\omega}$, the intervals $I_\tau = [p_\tau, q_\tau]$ in the following way, see Figure 4. For τ of length 0 (i.e., $\tau = \langle\rangle$), we put $I_\tau = [p_\tau, q_\tau] = [0,1]$. If, for some $\tau \in 2^n$, the interval I_τ is already defined and $\tau\hat{}i \in 2^{n+1}$ is an extension of τ by a term $i \in \{0,1\}$, then $I_{\tau\hat{}1}$ is the terminal $\frac{n+1}{n+2}$-th part of I_τ, while $I_{\tau\hat{}0}$ the initial $\frac{\xi_n}{n+2}$-th part of I_τ. More specifically, if $L_\tau = q_\tau - p_\tau$ is the length of I_τ, then $I_{\tau\hat{}0} = [p_{\tau\hat{}0}, q_{\tau\hat{}0}] = [p_\tau, p_\tau + \frac{\xi_n}{n+2} L_\tau]$, $I_{\tau\hat{}1} = [p_{\tau\hat{}1}, q_{\tau\hat{}1}] = [p_\tau + \frac{1}{n+2} L_\tau, q_\tau]$, $L_{\tau\hat{}0} = \frac{\xi_n}{n+2} L_\tau$, and $L_{\tau\hat{}1} = \frac{n+1}{n+2} L_\tau$. We define

$$d_{s\restriction n} = \frac{1}{n+2} L_{s\restriction n}. \tag{14}$$

Observe that, for any $\tau \in 2^n$ and $i \in \{0,1\}$, we have $L_{\tau\hat{}0} = \frac{\xi_n}{n+2} L_\tau < \frac{n+1}{n+2} L_\tau = L_{\tau\hat{}1}$. So, $L_{\tau\hat{}i} \le L_{\tau\hat{}1} = \frac{n+1}{n+2} L_\tau$ and, by induction on $n < \omega$,

$$L_{s\restriction n} \le L_{\bar{1}\restriction n} = \frac{1}{n+1} \quad \text{for every } s \in 2^\omega \text{ and } n < \omega. \tag{15}$$

Also, an easy inductive argument shows that

$$\sum_{n<\ell} s_n d_{s\restriction n} = p_{s\restriction \ell} \in I_{s\restriction \ell} \quad \text{for every } s \in 2^\omega \text{ and } \ell < \omega.$$

In particular, $\bigcap_{n<\omega} I_{s\restriction n} = \left\{\sum_{n<\omega} s_n d_{s\restriction n}\right\}$ for every $s \in 2^\omega$. Moreover

$$\sum_{n\ge \ell} s_n d_{s\restriction n} \le L_{s\restriction \ell} \quad \text{for every } s \in 2^\omega \text{ and } \ell < \omega \tag{16}$$

as $p_{s\restriction \ell} + \sum_{n\ge\ell} s_n d_{s\restriction n} = \sum_{n<\omega} s_n d_{s\restriction n} \in I_{s\restriction \ell} = [p_{s\restriction \ell}, p_{s\restriction \ell} + L_{s\restriction \ell}]$. This will be of special importance in the case when $s_\ell = 0$, since then we have $\sum_{n\ge\ell} s_n d_{s\restriction n} = \sum_{n\ge \ell+1} s_n d_{s\restriction n} \le L_{s\restriction(\ell+1)} = L_{(s\restriction \ell)\hat{}0} = \xi_\ell d_{s\restriction \ell}$, that is,

$$\sum_{n>\ell} s_n d_{s\restriction n} = \sum_{n\ge\ell} s_n d_{s\restriction n} \le \xi_\ell d_{s\restriction \ell} \text{ for every } s \in 2^\omega \text{ and } \ell < \omega \text{ with } s_\ell = 0. \tag{17}$$

Proof of (11) and (12): The property (11) follows immediately from (14) and (15).

To see (12) notice that for every $\tau, \eta \in 2^m$ and $i \in \{0,1\}$ we have $\frac{L_{\tau\hat{}i}}{L_{\eta\hat{}i}} = \frac{L_\tau}{L_\eta}$. So, an easy induction shows that for every $k < n < \omega$ and $\tau, \eta \in 2^n$ we have

$$\frac{L_{\tau\restriction(k+1)}}{L_{\eta\restriction(k+1)}} = \frac{L_\tau}{L_\eta} \text{ provided } \tau_i = \eta_i \text{ for all } i \text{ with } k < i < n.$$

Since, in (12), $s_i = \sigma(s)_i$ for all i with $k < i < n$, by (14) and the above equation we have $\frac{d_{\sigma(s)\restriction n}}{d_{s\restriction n}} = \frac{L_{\sigma(s)\restriction n}}{L_{s\restriction n}} = \frac{L_{\sigma(s)\restriction(k+1)}}{L_{s\restriction(k+1)}} = \frac{L_{z_k}}{L_{w_k}}$. Thus, (12) holds with $E_k = \frac{L_{z_k}}{L_{w_k}}$.

Proof of the estimate (7): Here $s = \bar{1}$. Then, the use of (17), with $\ell - 1$ in place of ℓ and $\sigma(t)$ in place of s, and (15) gives us the required estimate:

$$\begin{aligned}
|h(\sigma(s)) - h(\sigma(t))| &= \sum_{n\ge\ell} \sigma(t)_n c_{\sigma(t)\restriction n} = \sum_{n\ge \ell-1} \sigma(t)_n \beta_n^{-b_{\sigma(t)\restriction n}} d_{\sigma(t)\restriction n} \\
&\le \sum_{n\ge\ell-1} \sigma(t)_n d_{\sigma(t)\restriction n} \le d_{\sigma(t)\restriction(\ell-1)} \xi_{\ell-1} \\
&\le d_{\sigma(t)\restriction(\ell-1)} = \frac{1}{\ell+1} L_{\sigma(t)\restriction(\ell-1)} \le \frac{1}{\ell+1}\frac{1}{\ell}.
\end{aligned}$$

Proof of the estimates (4): Here $s = \langle w_k, s_{k+1}, s_{k+2}, \ldots\rangle$ and $\sigma(s) = \langle z_k, s_{k+1}, s_{k+2}, \ldots\rangle$ for some $k < \omega$. Also $t \in 2^\omega$ does not equal s and $\ell = \min\{i < \omega \colon s_i \neq t_i\} > 0$. By symmetry of expressions $|h(s) - h(t)|$ and $|h(\sigma(s)) - h(\sigma(t))|$ we can assume, without loss of generality, that $s_\ell = 1$ and $t_\ell = 0$. So, the estimates will be proved for $u = s$.

Now, as $t_\ell = 0$, by (17) and (13), we obtain

$$\sum_{n>\ell} t_n \beta_n^{-b_{t\restriction n}} d_{t\restriction n} \le \sum_{n>\ell} t_n d_{t\restriction n} \le \xi_\ell d_{t\restriction \ell} = \xi_\ell d_{s\restriction \ell} \le \frac{1}{2}\beta_\ell^{-b_{s\restriction \ell}} d_{s\restriction \ell}. \tag{18}$$

Hence, we get the second estimate of (4):

$$\begin{aligned} h(s) - h(t) &= \sum_{n\ge\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>\ell} t_n \beta_n^{-b_{t\restriction n}} d_{t\restriction n} \\ &\ge \sum_{n\ge\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \frac{1}{2}\beta_\ell^{-b_{s\restriction \ell}} d_{s\restriction \ell} \\ &\ge \sum_{n\ge\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \frac{1}{2}\sum_{n\ge\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} \\ &= \frac{1}{2}\sum_{n\ge\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} = \frac{1}{2}\sum_{n\ge\ell} s_n |c_{s\restriction n}| > 0. \end{aligned}$$

The first estimate of (4) is obtained as follows:

$$\begin{aligned} |h(\sigma(s)) - h(\sigma(t))| &= \left| \sum_{n\ge\ell} s_n c_{\sigma(s)\restriction n} - \sum_{n>\ell} t_n c_{\sigma(t)\restriction n} \right| \quad (19) \\ &\le \sum_{n\ge\ell} s_n |c_{\sigma(s)\restriction n}| + \sum_{n>\ell} t_n |c_{\sigma(t)\restriction n}| \\ &= \sum_{n\ge\ell} s_n \beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} + \sum_{n>\ell} t_n \beta_n^{-b_{\sigma(t)\restriction n}} d_{\sigma(t)\restriction n} \\ &\le \sum_{n\ge\ell} s_n \beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} + \frac{1}{2}\beta_\ell^{-b_{\sigma(s)\restriction \ell}} d_{\sigma(s)\restriction \ell} \quad (20) \\ &\le \sum_{n\ge\ell} s_n \beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} + \frac{1}{2}\sum_{n\ge\ell} s_n \beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} \\ &= \frac{3}{2}\sum_{n\ge\ell} s_n |c_{\sigma(s)\restriction n}|, \end{aligned}$$

where (19) is ensured by the fact that $\sigma(s)_n = s_n$ and $\sigma(t)_n = t_n$ for every $n \ge \ell$ and by the equation $\sigma(s) \restriction \ell = \sigma(t) \restriction \ell$, while (20) follows from (18) applied to the pair $\sigma(s)_\ell$ and $\sigma(t)_\ell$.

Proof of continuity of h: By (9), (16), and (15), for any $s \in 2^\omega$ and $\ell < \omega$ we have $\left|\sum_{n\ge\ell} s_n c_{s\restriction n}\right| \le \sum_{n\ge\ell} s_n |c_{s\restriction n}| \le \sum_{n\ge\ell} s_n d_{s\restriction n} \le L_{s\restriction \ell} \le \frac{1}{\ell+1}$. Therefore, for distinct $s, t \in 2^\omega$ and $\ell = \min\{i < \omega \colon s_i \neq t_i\}$, $|h(s) - h(t)| =$

$\left|\sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n\geq\ell} t_n c_{t\restriction n}\right| \leq \left|\sum_{n\geq\ell} s_n c_{s\restriction n}\right| + \left|\sum_{n\geq\ell} t_n c_{t\restriction n}\right| \leq \frac{2}{\ell+1}$, that is, h is continuous.

Proof of injectivity of h: To see that the function h is one-to-one, fix distinct $s,t \in 2^\omega$ and let $\ell = \min\{i<\omega \colon s_i \neq t_i\}$. By symmetry, we can assume that $s_\ell = 1$ and $t_\ell = 0$. Then, we have

$$h(s)-h(t) = \sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n\geq\ell} t_n c_{t\restriction n} = \sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n>\ell} t_n c_{t\restriction n}.$$

We need to show that $h(s)-h(t)\neq 0$. For this we will consider the following cases.

Case 1: *s equals to $\bar{1} = \langle 1,1,1,\ldots\rangle$.* Then $a_{s\restriction n} = -1$ for all $n<\omega$ and $a_{t\restriction n} = 1$ for all $n>\ell$. Hence

$$h(s)-h(t) = -\sum_{n\geq\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>\ell} t_n \beta_n^{-b_{t\restriction n}} d_{t\restriction n} < 0.$$

Case 2: *there exists an $i<\ell$ such that $t_i = s_i = 0$.* Then, $a_{s\restriction n} = a_{t\restriction n} = 1$ for all $n\geq\ell$. So, using the fact that $\beta_n^{-b_{t\restriction n}} \leq 1$ for all $n<\omega$ and the equations $s_\ell = 1$ and $s\restriction\ell = t\restriction\ell$, and, afterwards, applying (17) to t, followed by (13), we get

$$\begin{aligned} h(s)-h(t) &= \sum_{n\geq\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>\ell} t_n \beta_n^{-b_{t\restriction n}} d_{t\restriction n} \\ &\geq s_\ell \beta_\ell^{-b_{s\restriction\ell}} d_{s\restriction\ell} - \sum_{n>\ell} t_n d_{t\restriction n} \\ &\geq \beta^{-b_{t\restriction\ell}} d_{t\restriction\ell} - \xi_\ell d_{t\restriction\ell} = d_{t\restriction\ell}(\beta_\ell^{-b_{t\restriction\ell}} - \xi_\ell) > 0. \end{aligned}$$

Case 3: *neither Case 1 nor Case 2 hold.* Let $m = \min\{i<\omega\colon s_i = 0\}$. Then $m>\ell$, $s_{m-1} = 1$, and $s_m = 0$. Hence, as $a_{s\restriction n} = -1$ for $n\leq m$ and $a_{s\restriction n} = 1$ for $n>m$, using (17) we get

$$\begin{aligned} -\sum_{n\geq\ell} s_n c_{s\restriction n} &= \sum_{\ell\leq n\leq m} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>m} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} \\ &\geq s_{m-1}\beta_{m-1}^{-b_{s\restriction(m-1)}} d_{s\restriction(m-1)} - \sum_{n>m} s_n d_{s\restriction n} \\ &= \beta_{m-1}^{-b_{s\restriction(m-1)}} d_{s\restriction(m-1)} - \sum_{n\geq m} s_n d_{s\restriction n} \\ &\geq \beta_{m-1}^{-b_{s\restriction(m-1)}} d_{s\restriction(m-1)} - \xi_m d_{s\restriction m}. \end{aligned}$$

Now, $d_{s\restriction m} = \frac{1}{m+2} L_{s\restriction m} = \frac{1}{m+2} L_{(s\restriction(m-1))^\frown 1} = \frac{1}{m+2}\frac{m}{m+1} L_{s\restriction(m-1)} = \frac{m}{m+2} d_{s\restriction(m-1)}$ so that $d_{s\restriction(m-1)} = \frac{m+2}{m} d_{s\restriction m} \geq d_{s\restriction m}$. Thus, by (13),

$$\begin{aligned} -\sum_{n\geq\ell} s_n c_{s\restriction n} &\geq \beta_{m-1}^{-b_{s\restriction(m-1)}} d_{s\restriction(m-1)} - \xi_m d_{s\restriction m} \\ &\geq \beta_{m-1}^{-b_{s\restriction(m-1)}} d_{s\restriction m} - \xi_m d_{s\restriction m} = d_{s\restriction m}\left(\beta_{m-1}^{-b_{s\restriction(m-1)}} - \xi_m\right) > 0. \end{aligned}$$

So, $h(t) - h(s) = \sum_{n>\ell} t_n c_{t\restriction n} - \sum_{n\geq\ell} s_n c_{s\restriction n} \geq -\sum_{n\geq\ell} s_n c_{s\restriction n} > 0$.

Proof of (i) and (ii) of Theorem 1: Item (i) was addressed earlier, see (1) and the discussion in Section 4 below.

Item (ii) follows from a theorem of Jarník [14] that *every differentiable function f from a compact perfect subset of $\mathbb{R}$ into $\mathbb{R}$ can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.* (More on Jarník's theorem can be found in [16]. The theorem has also been independently proved in [19, theorem 4.5].)

This concludes the proof of Theorem 1.

4 Must the example be based on a minimal dynamics?

Recall that for a metric space X, a function $f\colon X \to X$ is *locally radially shrinking* if

(LRS) for every $x \in X$ there exists an $\varepsilon_x > 0$ such that $d(f(x), f(y)) < d(x, y)$ for any $y \in B(x, \varepsilon_x)$, $y \neq x$.

The function $\mathfrak{f}$ from Theorem 1(i), constructed in Section 3, is (LRS) and forms a minimal dynamical system. Our goal here is to prove, that this is not a coincidence, since any surjective (LRS) self map of an infinite compact space X contains a minimal dynamics of an uncountable $Y \subset X$:

Theorem 9 *Let X be an infinite compact metric space and assume that a map $f\colon X \to X$ is an (LRS) surjection. Then there exists a perfect subset $Y \subseteq X$ such that $f \restriction Y$ is a minimal dynamical system.*

The proof of this theorem is based on several lemmas. We will also use the following standard notation: for $\delta > 0$ and non-empty $A \subseteq X$ we define $B(A, \delta) = \bigcup_{a\in A} B(a, \delta)$.

Lemma 10 *If $X_0 \subseteq X$, $f\colon X_0 \to X$ satisfies* (LRS)*, and finite $A \subseteq X_0$ is such that $f[A] \subseteq A$, then exists a $\delta > 0$ such that $f[X_0 \cap B(A, \varepsilon)] \subseteq B(A, \varepsilon)$ for every $\varepsilon \in (0, \delta]$.*

Proof. For every $a \in A$ let $\delta_a > 0$ be such that $d(f(x), f(a)) \leq d(x, a)$ whenever $x \in X_0 \cap B(a, \delta_a)$. Then $\delta = \min_{a\in A} \delta_a > 0$ is as needed.

Indeed, fix an $\varepsilon \in (0, \delta]$ and choose an $x \in X_0 \cap B(A, \varepsilon)$. To see that $f(x) \in B(A, \varepsilon)$ pick an $a \in A$ with $x \in B(a, \varepsilon)$. Then, since $f(a) \in A$, we have $d(f(x), A) \leq d(f(x), f(a)) \leq d(x, a) < \varepsilon$ so that $f(x) \in B(A, \varepsilon)$, as needed. ■

Our next lemma states that the existence of a surjection with (LRS) property implies that the space X must be uncountable. In the proof, we use the notion of Cantor-Bendixon rank, defined as follows. For a metric space X we let $(X)'$ to be the set of all accumulation points of X. For the ordinal numbers $\alpha, \lambda < \omega_1$, where λ is a limit ordinal, we define

$$X^{(0)} = X,\ X^{(\alpha+1)} = (X^{(\alpha)})', \text{ and } X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}.$$

An easy inductive argument shows that for every $\alpha < \omega_1$, if $A \subseteq B \subseteq X$, then $A^{(\alpha)} \subseteq B^{(\alpha)}$.

We define the *Cantor-Bendixon rank* of X, denoted $|X|_{CB}$, to be the least ordinal number $\alpha < \omega_1$ such that $X^{(\alpha+1)} = X^{(\alpha)}$. Recall, that if X is compact, then $\alpha = |X|_{CB}$ is either zero or a successor ordinal, that is, of the form $\alpha = \beta + 1$. Moreover, if X is also countable, then $\alpha > 0$ and $X^{(\alpha)} = \emptyset$.

Lemma 11 *If $X_0 \subseteq X$ is infinite compact and $f\colon X_0 \to X$ is a surjection with* (LRS) *property, then X_0 is uncountable.*

Proof. Assume, towards a contradiction, that there exists a function f as in the lemma with a countable infinite X_0. Let X_0 be such an example with the smallest possible Cantor-Bendixon rank $\alpha = |X_0|_{CB}$. Then, since X_0 is compact and infinite, $\alpha = \beta + 1$ for some ordinal $\beta \geq 1$. Clearly $X \subseteq f[X_0]$ implies that $X^{(\beta)} \subseteq f[X_0]^{(\beta)}$. Also, an easy inductive argument shows that $f[X_0]^{(\beta)} \subseteq f[(X_0)^{(\beta)}]$. (See e.g. (I_β) in [5, lemma 4.3].) It follows that $X^{(\beta)} \subseteq f[(X_0)^{(\beta)}]$. Since, $(X_0)^{(\beta)} \subseteq X^{(\beta)}$ and, by compactness of X_0, $(X_0)^{(\beta)}$ is finite, the inclusions $(X_0)^{(\beta)} \subseteq X^{(\beta)} \subseteq f[(X_0)^{(\beta)}]$ imply the equality $(X_0)^{(\beta)} = f[(X_0)^{(\beta)}]$. The set $A = (X_0)^{(\beta)}$ satisfies the assumptions of Lemma 10. So, let $\delta > 0$ be as in this lemma.

If $\beta = 1$, put $B = B(A, \delta)$. Then $f[X_0 \cap B] \subseteq B$. We need to show that the inclusion is proper. Indeed, $X_0 \cap B$ is closed, since it contains $A = (B)'$. So, there exists an $x \in X_0 \cap B$ of maximal distance $\eta = d(x, A)$ to A. Notice, that $\eta > 0$, as $X_0 \cap B \not\subseteq A$. We claim that $x \neq f(z)$ for every $z \in B$. Indeed, it is obvious when $z \in A$, since then $d(f(z), A) = 0 < \eta = d(x, A)$; otherwise, there is an $a \in A$ with $0 < d(z, a) = d(z, A) \leq \eta < \delta$ and so, by (LRS), $d(f(z), A) \leq d(f(z), f(a)) < d(z, a) \leq \eta = d(x, A)$, once again giving $x \neq f(z)$. So, we proved that $f[B] \subsetneq B$.

The contradiction is obtained by noticing that, f being surjective, $f[X_0 \setminus B]$ must contain $X \setminus f[X_0 \cap B]$, which is impossible, since finite a set $X_0 \setminus B$ cannot be mapped onto its proper superset $X \setminus f[X_0 \cap B]$.

If $\beta > 1$, then, for some $\varepsilon \in (0, \delta]$, the set $X_1 = X_0 \setminus B(A, \varepsilon)$ is infinite and contains a limit point. Moreover, $X_1 \subseteq f[X_1]$ and X_1 has the Cantor-Bendixon rank less than α, contradicting the choice of α. ■

Lemma 12 *If X is compact and $f: X \to X$ satisfies* (LRS)*, then, for every positive $m < \omega$, the set $F_m = \{x \in P: f^{(m)}(x) = x\}$ is finite.*

PROOF. An easy induction shows that $f^{(m)}$ also satisfies (LRS). If F_m was infinite, then, being compact, it would contain an accumulation point, say $x \in F_m$. But then, $f^{(m)}$ would not be shrinking in any neighborhood of x, a contradiction. ■

In what follows, we will use notation F_m to the sets from Lemma 12.

Lemma 13 *If X is compact uncountable and $f: X \to X$ is a surjective map satisfying* (LRS)*, then there exists an open $U \subseteq X$ containing $\bigcup_{m=1}^{\infty} F_m$ such that $X \setminus U$ is uncountable and $X \setminus U \subseteq f[X \setminus U]$.*

PROOF. Let μ be a Borel probability measure on X vanishing on points (e.g., defined as an appropriate product measure on a copy of a Cantor set in X). Clearly the orbit $O(x)$ of each $x \in F_m$ is finite. So, by Lemma 12, the set $A_m = \bigcup_{x \in F_m} O(x)$ is finite and, clearly, $f[A_m] \subseteq A_m$.

By Lemma 10 applied to $A = A_m$, for every positive $m < \omega$ there is a $\delta_m > 0$ such that $f[B(A_m, \varepsilon)] \subseteq B(A_m, \varepsilon)$ for every $\varepsilon \in (0, \delta_m]$. Choose $\varepsilon_m \in (0, \delta_m]$ small enough so that $\mu(B(A_m, \varepsilon_m)) \leq 2^{-(m+2)}$. Then $U = \bigcup_{m=1}^{\infty} B(A_m, \varepsilon_m)$ is as desired, since $\mu(U) \leq 1/2 < \mu(X)$ and $f[U] \subseteq U$. ■

Proof of Theorem 9. Let f and X be as in Theorem 9. Then, by Lemma 11, X is uncountable. Hence, we can use Lemma 13. Let U be as in Lemma 13 and put $T = X \setminus U$. Then,

(∗) $T \subseteq f[T]$.

A simple application of Zorn's Lemma, following an idea from Birkhoff [2], implies that there exists a minimal non-empty compact $Y \subseteq T$ satisfying (∗). Notice, that this minimality of Y implies $Y = f[Y]$, as otherwise $Y \cap f^{-1}(Y)$ would be a proper closed subset of Y satisfying (∗).

To finish the argument, notice that Y is infinite, since otherwise it would be contained in $\bigcup_{m=1}^{\infty} F_m \subseteq U$, which is disjoint with Y. Thus, by Lemma 11, Y is uncountable and, being minimal, it must be perfect. (It cannot have isolated points, since the orbit of any point of Y must be dense in Y.) ■

Finally, notice that the careful choice of a metric on a copy $\mathfrak{X}$ of the Cantor set 2^ω is essential to the example from Theorem 1.

Remark 14 If d is the standard metric on 2^ω defined, for distinct $s, t \in 2^\omega$ as $d(s,t) = 2^{-\min\{n<\omega\colon s_n \neq t_n\}}$, then $2^\omega \not\subseteq f[2^\omega]$ for every (LRS) map f on $\langle 2^\omega, d\rangle$. Indeed, $\langle 2^\omega, d\rangle$ is ultrametric (i.e., satisfies $d(s,u) \leq \max\{d(s,t), d(t,u)\}$ for every $s,t,u \in 2^\omega$) while F. George has recently proved [11] that $X \not\subseteq f[X]$ for any (LC) map f on a compact ultrametric space $\langle X, d\rangle$. However, George's proof works for the (LRS) functions as well, because for any $Y \subseteq X$, and $a \in Y$ the diameter of Y equals $\sup\{d(a,y)\colon y \in Y\}$, see [20, p. 49].

Notice that the perfect subsets X of $\mathbb{R}$ that admit a function f as in Theorem 1 are rare, in a sense that they are of first category in the space $\mathcal{K}$ of non-empty compact subsets of $\mathbb{R}$ furnished with the Hausdorff metric. This has been proved by Bruckner and Steele in [3].

The fact that the set $\mathfrak{X}$ is compact is crucial. The examples of this kind for non-compact complete metric spaces are considerably easier to come by. In particular, Hu and Kirk [13] give an example of a complete metric ρ on $\mathbb{R}$, inducing the standard topology, such that the map $f(x) = x + 1$ has derivative zero everywhere in a sense that $\lim_{y\to x} \frac{\rho(f(y),f(x))}{\rho(y,x)} = 0$ for all $x \in \mathbb{R}$.

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