A unifying graph-cut image segmentation framework: algorithms it encompasses and equivalences among them

Krzysztof Chris Ciesielski,[a,b] Jayaram K. Udupa,[b] A.X. Falcão,[c] and P.A.V. Miranda[d]

[a]Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310

[b]Dept. of Radiology, MIPG, Univ. of Pennsylvania, Blockley Hall – 4th Floor, 423 Guardian Dr., Philadelphia, PA 19104-6021

[c]Institute of Computing, University of Campinas, Campinas, SP, Brazil

[d]Department of Computer Science, IME, University of São Paulo (USP), São Paulo, SP, Brazil

ABSTRACT

We present a general graph-cut segmentation framework GGC, in which the delineated objects returned by the algorithms optimize the energy functions associated with the ℓ_p norm, $1 \leq p \leq \infty$. Two classes of well known algorithms belong to GGC: the standard graph cut GC (such as the min-cut/max-flow algorithm) and the relative fuzzy connectedness algorithms RFC (including iterative RFC, IRFC). The norm-based description of GGC provides more elegant and mathematically better recognized framework of our earlier results from [18, 19]. Moreover, it allows precise theoretical comparison of GGC representable algorithms with the algorithms discussed in a recent paper [22] (min-cut/max-flow graph cut, random walker, shortest path/geodesic, Voronoi diagram, power watershed/shortest path forest), which optimize, via ℓ_p norms, the intermediate segmentation step, the labeling of scene voxels, but for which the final object need not optimize the used ℓ_p energy function. Actually, the comparison of the GGC representable algorithms with that encompassed in the framework described in [22] constitutes the main contribution of this work.

1. INTRODUCTION

The image segmentation field has a rich literature dating back to the 60's. For the consideration of this paper, it is useful to categorize the segmentation algorithms into three groups: purely image-based (pI), appearance model-based (AM), and hybrid. pI methods focus on delineating objects based entirely on the information about the object that can be harnessed from the given image. AM approaches bring in information about the object family in terms of its appearance variation in the form of statistical/fuzzy texture and/or shape models to bear on the segmentation problem. Hybrid approaches are recent; they combine synergistically the pI and AM approaches in an attempt to overcome the weaknesses of the individual approaches. The major frameworks existing under the pI approaches include level sets (LS), active boundaries, fuzzy connectedness (FC), graph cut (GC), watershed (WS), clustering, and Markov Random Field.

In this paper we study the group of purely image-based (pI) segmentation algorithms. Since the top-rated pI algorithms harness the information with equal effectiveness, there must exist similarity or even equivalence among such algorithms. This observation prompted researchers to study the possibility of explaining such algorithms in a common framework [17, 1, 31]. In the same spirit, the popular graph cut (GC) framework has been generalized recently to, what we refer to as, *Generalized GC* (GGC). This framework was proposed by the authors in [18, 19, 20], and studied in a slightly different form in [22], to describe GC, fuzzy connectedness (FC) and watershed (WS) algorithms in a unified manner. A byproduct of such a unification effort is a deeper understanding of the strengths and weaknesses of the individual algorithms, which can lead to new methods with improved performance, a subject of our current research (not described here).

The GGC framework is described in detail in the next section. Briefly, in GGC, the image information is represented in the form of a weighted graph $G = \langle V, E, w \rangle$ and the delineated objects P minimize the energy functions $||F_P||_q$ for different $q \in [1, \infty]$, where F_P is a map that assigns to every element e from the boundary of object P its weight $w(e)$. In this formulation, our approach is similar to that from papers [42, 22]. We notice that all minimization problems associated with the energies $||F_P||_q$ can be solved by only two types of algorithms: $\mathrm{GC}_{\mathrm{sum}}$ and $\mathrm{GC}_{\mathrm{max}}$, that solve, respectively, the minimization problems for the energies $||F_P||_1$ and $||F_P||_\infty$.

The *graph cut* $\mathrm{GC}_{\mathrm{sum}}$ algorithms, minimizing the energy $\varepsilon^{\mathrm{sum}}(P) = ||F_P||_1$, have a rich literature [12, 7, 8, 9, 6, 10, 11]. (See also [41, 31, 29].) The energy $\varepsilon^{\max}(P) = ||F_P||_\infty$ used as an optimizer is a relatively new phenomenon — it seems to appear so far only in the papers [42, 22] and in a slightly different setting from the one

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we use in this paper. (But see also [30, 2, 3, 24].) However, as shown in [18, 19, 20], the energy $\varepsilon^{\max}$ is actually minimized by most of the algorithms from the *fuzzy connectedness, FC*, framework, which was extensively studied since 1996 [43, 37, 44, 38, 21, 45]. (See also [13, 28, 14], where a slightly different approach to this methodology is used. For other extensions of FC, compare e.g., [36, 27], and the references listed in [15, 16].) Recall also that the *watershed, WS,* framework [5, 40, 35, 23, 24, 4, 34] can be encompassed in the FC framework [2, 24, 31], so it too minimizes the energy $\varepsilon^{\max}$.

2. BACKGROUND: GENERALIZED GRAPH CUT FRAMEWORK

In every algorithm within GGC, a digital image $I = \langle C, f\rangle$ (where C is its *domain* and $f\colon C \to \mathbb{R}^\ell$ its *intensity function*) is identified with a weighted directed graph $G = \langle C, E, w\rangle$ such that:

- C is the set of vertices of the graph, which coincides with the image domain.
- E is the image scene adjacency relation. In particular, $E \subset C \times C$ is a binary relation representing the set of all directed edges of G, that is, $\langle c, d\rangle$ is an edge if, and only if, $\langle c, d\rangle \in E$. It is assumed that E is symmetric, that is, $\langle d, c\rangle$ is an edge provided so is $\langle c, d\rangle$.
- $w\colon E \to [0, 1]$ is a weight function associating with any edge $e \in E$ its weight $w(e)$. It is assumed that w is symmetric: $w(c, d) = w(d, c)$ for every edge $\langle c, d\rangle$. An example of one of the most standard weight assignments, measuring the level of homogeneity between a pair of spels, is given by the following formula, where $\sigma > 0$ is a fixed constant: $w(c, d) = e^{-||f(c)-f(d)||^2/\sigma^2}$ for every $\langle c, d\rangle \in E$.

For every weighted graph $G = \langle C, E, w\rangle$, consider the space $\tilde{\mathcal{X}}$ of all functions $x\colon C \to [0, 1]$, referred to as *fuzzy subsets* of C, with the value $x(c)$ indicating a degree of membership with which c belongs to the set. The family $\mathcal{X}$ of all functions $x \in \tilde{\mathcal{X}}$ with the only allowed values of 0 and 1 (i.e., $x\colon C \to \{0, 1\}$) will be referred to as the family of all *hard subsets* of C. Each $x \in \mathcal{X}$ is identified with the true subset $P = \{c \in C \colon x(c) = 1\}$ of C. Notice that, in such a case, x is the *characteristic function* χ_P of $P \subset C$. The GGC framework is expressed exclusively in terms of $\mathcal{X}$, the hard subsets of C. The fuzzy subsets $\tilde{\mathcal{X}}$ are used in the framework considered in [22] and discussed in detail in the next section.

The goal of the segmentation algorithms we consider here is to indicate, in the input image $I = \langle C, f\rangle$, a "desired" object $P \subset C$, which is identified with its characteristic function $\chi_P \in \mathcal{X}$. We usually restrict the collection $\mathcal{X}$ of all allowable "desirable" objects by indicating two disjoint sets, referred to as *seeds*: $S \subset C$ indicating the object and $T \subset C$ indicating the background. This restricts the collection of allowable outputs of the algorithm to the family $\mathcal{X}(S, T)$ of all $x \in \mathcal{X}$ with $x(s) = 1$ and $x(t) = 0$ for all $s \in S$ and $t \in T$. Note that $\mathcal{X}(S, T) = \{\chi_P \colon S \subset P \subset C \setminus T\}$. We also use the notation $\mathcal{P}(S, T) = \{P \subset C \colon \chi_P \in \mathcal{X}(S, T)\}$, indicating standard set representation of $\mathcal{X}(S, T)$.

For $q \in [1, \infty]$ consider the energy functional $\varepsilon_q \colon \tilde{\mathcal{X}} \to [0, \infty)$, where, for every $x \in \tilde{\mathcal{X}}$, $\varepsilon_q(x)$ is defined as the q-norm of the functional $F_x\colon E \to \mathbb{R}$, given by a formula $F_x(c, d) = w(c, d)|x(c) - x(d)|$ for $\langle c, d\rangle \in E$. That is,

$$\varepsilon_\infty(x) = ||F_x||_\infty = \max\nolimits_{\langle c,d\rangle \in E} w(c, d)|x(c) - x(d)| \quad \text{and}$$

$$\varepsilon_q(x) = ||F_x||_q = \sqrt[q]{\textstyle\sum_{\langle c,d\rangle\in E} \left(w(c, d)|x(c) - x(d)|\right)^q} \quad \text{for } q < \infty.$$

Notice that $\lim_{q\to\infty} \varepsilon_q(x) = \varepsilon_\infty(x)$, since q-norms converge, as $q \to \infty$, to the ∞-norm. In the GGC framework these functionals are used only for $x = \chi_P \in \mathcal{X}$. In this case, if $\mathrm{bd}(P)$ is defined as the set of all edges $e = \langle c, d\rangle$ with $x(c) \neq x(d)$, then $\varepsilon_q(\chi_P) = \sqrt[q]{\sum_{\langle c,d\rangle\in \mathrm{bd}(P)} \left(w(c, d)\right)^q}$ and $\varepsilon_\infty(\chi_P) = \max_{\langle c,d\rangle\in\mathrm{bd}(P)} w(c, d)$.

For $1 \le q \le \infty$, graph $G = \langle C, E, w\rangle$ (associated with $I = \langle C, f\rangle$), and seed sets S and T, let $\varepsilon^q_{\min}$ be the minimum of the energy $\varepsilon_q(x)$ over all ST-allowable objects $x \in \mathcal{X}(S, T)$, that is, $\varepsilon^q_{\min} = \min\{\varepsilon_q(x) \colon x \in \mathcal{X}(S, T)\}$. Any element of $\mathcal{X}_q(S, T) = \{x \in \mathcal{X}(S, T) \colon \varepsilon_q(x) = \varepsilon^q_{\min}\}$ will be referred to as an energy ε_q *minimizer of* $\mathcal{X}(S, T)$. Any algorithm A that, given an image I and seed sets S and T, returns an object $A(I, S, T)$ from $\mathcal{X}_q(S, T)$ will be referred to as an ε_q*-minimizing algorithm.* Notice that any such algorithm has also a hidden aspect: a subroutine, denote it $I \mapsto w$, that translates the input image I into its associated graph $G = \langle C, E, w\rangle$. We will write $A_{I\mapsto w}$ in place of A if we like to stress this parameter.

The standard min-cut/max-flow algorithm is, obviously, an ε_1-minimizing algorithm. We will use a symbol $\mathrm{GC}_{\mathrm{sum}}$ to denote this (or any other ε_1-minimizing) algorithm. The ε_∞-minimizing algorithms have also been in use for a long time, although they were not recognized as such until very recently. (See [18]. Compare also [19] and [20].) More precisely, the *Relative Fuzzy Connectedness, RFC,* and *Iterative, Relative Fuzzy Connectedness, IRFC*, algorithms are the ε_∞-minimizing algorithms, as discussed in more detail in the remainder of this section.

The ε_q-minimizing algorithms for $1 < q < \infty$ do not bring anything new to the GGC framework, since any ε_q-minimizing algorithm $A_{I\mapsto w}$ is also an ε_1-minimizing algorithm $A_{I\mapsto w^q}$. This is so, since for both these algorithms the associated sets $\mathcal{X}_q$ and $\mathcal{X}_1$ are identical. However, the situation becomes more complicated in the fuzzy optimization case, discussed in the next section.

The ε_1- and ε_∞-minimization problems (and so, the associated algorithms) are truly distinct, as discussed in [18, 19, 20] and below. Nevertheless, there is an interesting connection between them, as proved in [20]: for every image I there exists a $q < \infty$ such that the family $\mathcal{X}_1(S,T)$ associated with any ε_1-minimizing algorithm $A_{I\mapsto w^q}$ (e.g., for $A = \mathrm{GC}_{\mathrm{sum}}$) is contained in the family $\mathcal{X}_\infty(S,T)$. In particular, the output of $A_{I\mapsto w^q}$ minimizes ε_∞ in $\mathcal{X}(S,T)$ and, in the case when $\mathcal{X}_\infty(S,T)$ has only one element, $A_{I\mapsto w^q}(I,S,T) = \mathrm{GC}_{\max}(I,S,T)$.

2.1. RFC and IRFC as ε_∞-optimizers; fast $\mathrm{GC}_{\max}$ algorithm returning IRFC objects

The FC objects are, usually, defined in terms of paths in the weighted graph $G = \langle C, E, w\rangle$ associated with an image $I = \langle C, f\rangle$. A *path* p in G is any finite sequence $\langle c_1, \ldots, c_k\rangle$ of vertices such that any consecutive vertices c_i, c_{i+1} in p are adjacent (i.e., $\langle c_i, c_{i+1}\rangle \in E$). The *strength* $\mu(p)$ of a path $p = \langle c_1, \ldots, c_k\rangle$, $k > 1$, is defined as the strength of the w-weakest link of p, that is, as $\mu(p) = \min\{w(c_{i-1}, c_i)\colon 1 < i \leq k\}$. For $k = 1$ (i.e., when p has length 1) we associate with p the strongest possible value, $\mu(p) = 1$. For $c, d \in C$, the *connectedness strength* $\mu^C(c,d)$ *between* c *and* d is defined as the strength of a strongest path in G from c to d (i.e., of a path $\langle c_1, \ldots, c_k\rangle$ in G with $c_1 = c$ and $c_k = d$). Also, for non-empty $S, T \subset C$ we define $\mu^C(c,T) = \max_{t\in T}\mu^C(c,t)$ and $\mu^C(S,T) = \max_{s\in S}\mu^C(s,T)$. A path $p = \langle c_1, \ldots, c_k\rangle$ from c to T (i.e., with $c_1 = c$ and $c_k \in T$) is *optimal* provided $\mu(p) = \mu^C(c,T)$. The *RFC object*, indicated by S and T, is defined via competition of sets S and T for attracting a given $c \in C$ to their realms (see [37]):

$$P_{S,T} \stackrel{\text{def}}{=} \{c \in C\colon \mu^C(c,S) > \mu^C(c,T)\}.$$

Notice that $P_{S,T} = \{c \in C\colon (\exists s \in S)\mu^C(c,s) > \mu^C(c,T)\} = \bigcup_{s\in S} P_{\{s\},T}$, as $\mu^C(c,S) = \max_{s\in S}\mu^C(c,s)$. The following theorem (see [18], [19], or [20]) shows that the RFC object minimizes energy $\varepsilon^{\max} = \varepsilon_\infty$ and that, under simple assumptions, it is the smallest among all minimizers.

THEOREM 2.1. *Assume that* $\mu^C(S,T) < 1$. *Then* $P_{S,T}$ *minimizes the energy* $\varepsilon^{\max}$ *on* $\mathcal{X}_\infty(S,T)$. *Moreover,*

(i) *The number* $\mu^C(S,T)$ *is the minimum of* $\varepsilon^{\max}$ *on* $\mathcal{X}(S,T)$ *(i.e.,* $\mu^C(S,T) = \varepsilon^\infty_{\min}$*).*

(ii) *If* S *is a singleton, then* $P_{S,T}$ *is contained in any object* P *with* $\varepsilon^{\max}(\chi_P) = \mu^C(S,T)$.

The original definition of the IRFC object $P^\infty_{S,T}$ was in terms of expanding the RFC object $P_{S,T}$ iteratively, with $P_{S,T}$ being the first iteration of $P^\infty_{S,T}$. However, for the discussion presented in this papers, it is more appropriate to use another equivalent description of $P^\infty_{S,T}$, given below.

Let $W = S \cup T \subset C$. A *forest* for a graph $G = \langle C, E, w\rangle$ is any of its subgraph $\mathbb{F} = \langle C, E'\rangle$ free of cycles; a forest $\mathbb{F}$ is *spanning with respect to* W provided any connected component of $\mathbb{F}$ contains precisely one element of W. In particular, for any such forest $\mathbb{F}$ and any $c \in C$, there is a unique path p_c in $\mathbb{F}$ from c to W. We associate with any such forest $\mathbb{F}$ a set $P(S,\mathbb{F}) \in \mathcal{P}(S,T)$ of all vertices connected to S by a path in $\mathbb{F}$. Also, we say that a spanning forest $\mathbb{F} = \langle C, E'\rangle$ with respect to W is: an *optimum path forest, OPF*, provided every path p_c in $\mathbb{F}$ is optimal (i.e., when $\mu(p_c) = \mu^C(c,W)$ for every $c \in C$); it is a *maximal spanning forest, MSF*, provided the number $\sum_{e\in E'} w(e)$ is maximal among all numbers $\sum_{e\in E''} w(e)$, with $\langle C, E''\rangle$ being a spanning forest with respect to W. Let

$$\mathcal{P}^{OPF}(S,T) = \{P(S,\mathbb{F})\colon \mathbb{F} \text{ is an OPF with respect to } S \cup T\}.$$

According to the following theorem, which is a compilation of the results from [18, 19] (see also [20]), the IRFC object $P^\infty_{S,T}$ can be defined as the smallest (with respect to set inclusion) of the sets belonging to $\mathcal{P}^{OPF}(S,T)$.

Note also that, by our new Theorem 4.1, $P^{\infty}_{S,T}$ can be defined as well as the smallest object belonging to the family $\mathcal{P}^{MSF}(S,T) = \{P(S,\mathbb{F})\colon \mathbb{F} \text{ is an MSF with respect to } S \cup T\}$.

THEOREM 2.2. *Let $G = \langle C, E, w\rangle$ be a weighted graph associated with an image $I = \langle C, f\rangle$ and let S and T be non-empty subsets of C indicating, respectively, the foreground and the background seeds. If $\mu^C(S,T) < 1$, then*

(i) The family $\mathcal{P}^{OPF}(S,T)$ has the smallest element, which coincides with the IRFC object $P^{\infty}_{S,T}$.

(ii) $P^{\infty}_{S,T}$ minimizes the energy $\varepsilon^{\max} = \varepsilon_{\infty}$ on $\mathcal{P}(S,T)$.

(iii) $P^{\infty}_{S,T}$ is returned by the algorithm $GC_{\max}$ (indicated below), which runs (provably, worst case scenario) in a linear time with respect to the scene size $|C|$.

Algorithm $\mathrm{GC}_{\max}$	
Input:	A weighted graph $G = \langle C, E, w\rangle$ associated with an image $I = \langle C, f\rangle$; sets $S, T \subset C$, with $W = S \cup T$ being non-empty.
Output:	A connectedness strength function $\mu^C(\cdot, W)$; an OPF $\mathbb{F}$ with respect to W such that, if $\mu^C(S,T) < 1$, then $P^{\infty}_{S,T} = P(S,\mathbb{F})$.

Notice that by running $\mathrm{GC}_{\max}$ twice, with $W = S$ and $W = T$, we can find functions $\mu^C(\cdot, S)$ and $\mu^C(\cdot, T)$. In particular, $\mathrm{GC}_{\max}$ can be used also to find, in linear time, the RFC object $P_{S,T}$.

3. HARD VERSUS FUZZY GRAPH CUT MINIMIZATION PROBLEMS

This section contains new results. It can be treated as a comparison of GGC with the results presented in the papers [42] and [22]. In these papers the authors discuss image delineation algorithms that use a very similar approach to that described above: the same weighted graphs are associated with the images and the same energy functions ε_p are used to find their minimizers which, in turn, are transformed to final image delineations. However, for most cases, the actual outputs of these algorithms need to minimize the energy functionals ε_p which they employ, see Theorem 3.2. As such, they actually do not fit the GGC framework described in the previous section. Nevertheless, there are interesting relationships between the two approaches, as we describe in more detail below.

A *fuzzy subset* of a set C (i.e., an element of $\tilde{\mathcal{X}}$, which in [42, 22] is referred to as a *labeling*) is any function $x\colon C \to [0,1]$, with the value $x(c)$ indicating a degree of membership with which c belongs to the set. Many delineation algorithms considered in the literature, as those surveyed in [22], deal with the fuzzy minimization problems, the notion obtained from that of hard minimization problem upon replacing in its definition the "hard" subsets of C by the "fuzzy" subsets of C. More precisely, for disjoint sets $S, T \subset C$, we define $\mathcal{P}^F(S,T)$ as the family of all fuzzy sets $x \in \tilde{\mathcal{X}}$ with $x(c) = 1$ for all $c \in S$ and $x(c) = 0$ for all $c \in T$. For a threshold θ and an energy map $\hat{\varepsilon}$ from $\tilde{\mathcal{X}}$ into $[0,\infty)$, we define $\mathcal{P}^F_{\theta}(S,T)$ as the family of all $x \in \mathcal{P}^F(S,T)$ such that $\hat{\varepsilon}(x) \le \theta$. Then, a *fuzzy minimization problem*, $\mathrm{MP}^F(\hat{\varepsilon}_f)$, is a map $\langle f, S, T\rangle \mapsto \mathcal{P}^F_{\hat{\theta}_{\min}}(S,T)$, where $\hat{\theta}_{\min}$ is the smallest number θ for which the family $\mathcal{P}^F_{\theta}(S,T)$ is non-empty. Finally, a delineation algorithm for $\mathrm{MP}^F(\hat{\varepsilon}_f)$ is any specific numerical recipe that, given f and $\langle S, T\rangle$, returns an $x_{\min}$ from $\mathcal{P}^F_{\hat{\theta}_{\min}}(S,T)$.

Recall that any *hard set* $P \subset C$ can be treated as a fuzzy set, by identifying it with its *characteristic function* $\chi_P\colon C \to \{0,1\}$. This allows us to identify the family $\mathcal{P}(S,T)$ with $\mathcal{P}^H(S,T) = \mathcal{X}(S,T)$ and recognize $\chi_{P_{\min}}$ as a minimizer of the energy ε^H defined as $\varepsilon^H(\chi_P) = \varepsilon(P)$. In particular, if ε^H is equal to the restriction $\hat{\varepsilon} \restriction \mathcal{P}^H$ of $\hat{\varepsilon}$ to $\mathcal{P}^H \overset{\text{def}}{=} \mathcal{P}^H(\emptyset,\emptyset)$, then the three minimization problems $\mathrm{MP}(\varepsilon)$, $\mathrm{MP}^F(\varepsilon^H)$, and $\mathrm{MP}^F(\hat{\varepsilon} \restriction \mathcal{P}^H)$ coincide. In what follows, we will often write ε to denote ε^H.

If we are happy to accept a fuzzy minimizer that returns an x from $\mathcal{P}^F(S,T)$, not necessarily from $\mathcal{P}^H$, as a "desired object," this is a viable approach. However, often, we are after the hard delineated objects. In particular, although the delineation algorithms presented in [42, 22] minimize the "fuzzy" energy functions $\hat{\varepsilon}\colon \mathcal{P}^F \to [0,\infty)$, they actually return a characteristic function $\bar{x}\colon C \to \{0,1\}$ of a hard object, rather than a fuzzy minimizing object (labeling) $x_{\min} \in \mathcal{P}^F_{\hat{\theta}_{\min}}(S,T)$ indicated by the fuzzy minimization problem, where

$$\bar{x}(c) = 1 \text{ when } x_{\min}(c) \ge 0.5 \text{ and } \bar{x}(c) = 0 \text{ for } x_{\min}(c) < 0.5. \tag{1}$$

In [22], the authors consider the energy functions on $\mathcal{P}^F$ defined for every $p \in [0,\infty)$ and $q \in [1,\infty)$ via formula*

$$E_{p,q}(x) = \sum_{\langle c,d\rangle \in E} [w(c,d)]^p |x(c)-x(d)|^q. \tag{2}$$

Actually, for $p=0$, the formula (2) is undefined whenever $w(c,d)=0$. We interpret $E_{0,q}$ as $\lim_{p\to 0^+} E_{p,q}$, which leads to treating $[w(c,d)]^0$ as $\operatorname{sgn}[w(c,d)]$, where the value of the sign function $\operatorname{sgn}(a)$ is defined as 0 for $a=0$ and 1 for $a>0$. In [42, 22], the authors also allow $p=\infty$ and $q=\infty$, through different limiting processes, which we will discuss below.

Notice that, for every q, $E_{p,q} \restriction \mathcal{P}^H = E_{p,1} \restriction \mathcal{P}^H$ (i.e., $E_{p,q}$ agrees with $E_{p,1}$ for the hard delineations), rendering the parameter q in $E_{p,q}$ redundant for the hard optimization problem set-up:

Remark 3.1. If, for any energy functions $\bar{\varepsilon}$ and $\hat{\varepsilon}$ defined on $\mathcal{P}^F$, their restrictions $\bar{\varepsilon} \restriction \mathcal{P}^H$ and $\hat{\varepsilon} \restriction \mathcal{P}^H$ are equal, then the hard minimization problems $\mathrm{MP}(\bar{\varepsilon} \restriction \mathcal{P}^H)$ and $\mathrm{MP}(\hat{\varepsilon} \restriction \mathcal{P}^H)$ associated with them coincide.

3.1. Cases $p,q \in \mathbb{R}$ and $p=q\to\infty$

Paper [42] discusses the following variants of $E_{p,q}$:

$$\varepsilon_q(x) \stackrel{\text{def}}{=} (E_{q,q}(x))^{1/q} = \sqrt[q]{\sum_{\langle c,d\rangle\in E} \left(w(c,d)|x(c)-x(d)|\right)^q} = ||F_x||_q, \tag{3}$$

where $F_x\colon E\to\mathbb{R}$, $F_x(c,d)=w(c,d)|x(c)-x(d)|$ for $\langle c,d\rangle\in E$, and $||\cdot||_q$ is the standard ℓ_q-norm. In particular, for $q=\infty$, the formula (3) is interpreted as $\varepsilon_\infty(x) \stackrel{\text{def}}{=} \lim_{q\to\infty}(E_{q,q}(x))^{1/q} = \lim_{q\to\infty}||F_x||_q$, leading to

$$\varepsilon_\infty(x) = ||F_x||_\infty = \max_{\langle c,d\rangle\in E} w(c,d)|x(c)-x(d)|. \tag{4}$$

This energy function is the only form of the energy $E_{p,q}$, with $p,q\to\infty$, considered in [22] and in this paper.

The following theorem summarizes the relationships between these minimization problems.

Theorem 3.2. *Let $1\le q<\infty$ and $0\le p<\infty$.*

- *(a) The hard delineation optimization problem associated with ε_∞ coincides with $\mathrm{MP}(\varepsilon^{\max})$.*
- *(b) The hard delineation optimization problems associated with $E_{p,q}$ and with $(E_{p,q})^{1/q}$ (so, also with $\varepsilon_q = (E_{q,q})^{1/q}$) coincide with $\mathrm{MP}(\varepsilon_p^{\text{sum}})$, where $\varepsilon_p^{\text{sum}}$ is the energy ε^{sum} associated with the graph $G=\langle C,E,w^p\rangle$.*
- *(c) Moreover, if $q\ne 1$, then the hard object $\bar{x}$ associated, as in (1), with a fuzzy minimizer $x_{\min}$ for the fuzzy energy function $E_{p,q}$ need not minimize the associated hard delineation energy function; that is, $\bar{x}$ need not belong to the appropriate family $\mathcal{P}^H_{\theta_{\min}}(S,T)$.*

Proof. (a) Clearly $\varepsilon_\infty \restriction \mathcal{P}^H = \varepsilon^{\max}$, so also $\mathrm{MP}(\varepsilon_\infty \restriction \mathcal{P}^H) = \mathrm{MP}(\varepsilon^{\max})$.

(b) The map $y\mapsto y^{1/q}$ is strictly increasing for every $q\in[1,\infty)$, so the optimization problem (fuzzy or hard) associated with $E_{p,q}$ is clearly equivalent to that for $(E_{p,q})^{1/q}$ (since the associated families $\mathcal{P}_{\theta_{\min}}(S,T)$ are identical). Since $E_{p,q}\restriction\mathcal{P}^H = \varepsilon_p^{\text{sum}}$, (b) follows.

(c) This part is justified by the following example. ∎

Example 3.3. For the energies ε_q and $E_{q,q}$ with $q\in(1,\infty]$ it is possible that $\mathcal{P}^F_{\hat{\theta}_{\min}}(S,T)$ and $\mathcal{P}^H_{\theta_{\min}}(S,T)$ are disjoint and that $\bar{x}\in\mathcal{P}^H(S,T)$ associated with $x_{\min}\in\mathcal{P}^F_{\hat{\theta}_{\min}}(S,T)$ does not belong to $\mathcal{P}^H_{\theta_{\min}}(S,T)$.

Proof. Take $C=\{s,c,d,t\}$, where s is a foreground seed and t is a background seed, that is, $S=\{s\}$ and $T=\{t\}$. Consider a graph on C with just three symmetric edges, $\{s,c\}$, $\{c,d\}$, and $\{d,t\}$ (so, with six directed

*Actually, the most general energy formula defined in [22] is of the form $\hat{E}_{p,q}(x) = E_{p,q}(x) + \sum_{c\in C}(w_c)^p|x(c)-y(c)|^q$ for a $y\in\mathcal{P}^F$. However, in all theoretical investigations there, the unary constants w_c are taken as 0, in which case $\hat{E}_{p,q}=E_{p,q}$. Our analysis here applies only to this simplified case.

edges) with the respective weights 1, v, and v, for $v > 1$ to be determined. Then, $\mathcal{P}^F(S,T)$ consists of all fuzzy sets $x_{y,z}\colon C \to [0,1]$ with $y, z \in [0,1]$, where $x_{y,z}(s) = 1$, $x_{y,z}(c) = y$, $x_{y,z}(d) = z$, and $x_{y,z}(t) = 0$.

First fix a $q \in (1,\infty)$. Then, $E_{q,q}(x_{y,z}) = 2[(1-y)^q + v^q|y-z|^q + v^q z^q]$ is a function of two variables, y and z. It has precisely one minimum† at $z_q = (v^{q/(q-1)}+2)^{-1}$ and $y_q = 2(v^{q/(q-1)}+2)^{-1}$. Thus, $\mathcal{P}^F_{\hat{\theta}_{\min}}(S,T) = \{x_{y_q,z_q}\}$, leading to $x_{\min} = x_{y_q,z_q}$. Now, if $v \in (1, 2^{(q-1)/q})$, then $1 < v^{q/(q-1)} < 2$ and we have $0 < z_q < 0.5 < y_q < 1$, leading to $\bar{x}$ with $\bar{x}(s) = \bar{x}(c) = 1$ and $\bar{x}(d) = \bar{x}(t) = 0$. But this implies that $E_{q,q}(\bar{x}) = v^q > 1 = E_{q,q}(\chi_{\{s\}})$, so indeed $\bar{x} \notin \mathcal{P}^H_{\theta_{\min}}(S,T)$.

To see that the same example works for $q = \infty$, fix a $v \in (1, 2^{1/2})$. Then, for every $q > 2$ and $y, z \in \mathbb{R}$, we have $||F(x_{y,z})||_q \geq ||F(x_{y_q,z_q})||_q$. Taking the limit, as $q \to \infty$, gives $||F(x_{y,z})||_\infty \geq ||F(x_{y_\infty,z_\infty})||_\infty$, where $z_\infty = \lim_{q\to\infty} z_q = (v+2)^{-1}$ and $y_\infty = 2z_\infty$. Then, similarly as above, $\mathcal{P}^F_{\theta_{\min}}(S,T) = \{x_{y_\infty,z_\infty}\}$, leading to $x_{\min} = x_{y_q,z_q}$ and $\bar{x}$ with $\bar{x}(s) = \bar{x}(c) = 1$ and $\bar{x}(d) = \bar{x}(t) = 0$. But this implies that $\varepsilon_\infty(\bar{x}) = v > 1 = \varepsilon_\infty(\chi_{\{s\}})$, so once again $\bar{x} \notin \mathcal{P}^H_{\theta_{\min}}(S,T)$. ∎

It was noticed in [42] that, for the energy $\varepsilon_1 = E_{1,1}$ (i.e., for $q = 1$), we have $\mathcal{P}^F_{\theta_{\min}}(S,T) = \mathcal{P}^H_{\theta_{\min}}(S,T)$, so, in this case, the fuzzy MP($E_{1,1}$) and the hard MP($E_{1,1} \restriction \mathcal{P}^H$) minimization problems coincide with the "classic" min-cut/max-flow problem MP($\varepsilon_p^{\text{sum}}$). For all other energy functions considered in Theorem 3.2 (including the cases of random walk $\varepsilon_2 = (E_{2,2})^{1/2}$ and of the ℓ_∞ energy ε_∞ studied in [42, 22]), the algorithmic output $\bar{x}$ (derived from $x_{\min}$) does not constitute (an exact) solution to the related hard optimization problem. This, for example, explains why the experimental results from [42] for the ℓ_∞ algorithm are not robust (i.e., the delineations lack stability with changing seeds), in spite of a theorem (see e.g. [18]) according to which the related hard optimization problem is provably robust.

3.2. The case of $q \to \infty$ and $p \in \mathbb{R}$

The most natural understanding of this case would be to define the energy as $E_{p,\infty}(x) \stackrel{\text{def}}{=} \lim_{q\to\infty} E_{p,q}(x) = \sum_{\langle c,d\rangle\in E}[w(c,d)]^p \lim_{q\to\infty} |x(c) - x(d)|^q = \sum_{\langle c,d\rangle\in E}[w(c,d)]^p \lfloor x(c) - x(d)\rfloor$, where the value of the floor function $\lfloor a\rfloor$ is defined as the largest integer less than or equal to a. (This is the case since, for $a \in [0,1]$, we have $\lim_{q\to\infty} a^q = \lfloor a\rfloor$.) However, for such function, $\mathcal{P}^F_{\hat{\theta}_{\min}}(S,T)$ contains all $x \in \mathcal{P}^F(S,T)$ with $0 < x(c) < 1$ for all $c \in C$ not in $S \cup T$. In particular, any element of $\mathcal{P}^H(S,T)$ could end up as $\bar{x}$, rendering such $E_{p,\infty}$ useless. Instead, in [22, sec. 3.3] the authors use the function $\varepsilon_{p,\infty}(x) \stackrel{\text{def}}{=} \lim_{q\to\infty}(E_{p,q}(x))^{1/q} = \lim_{q\to\infty} \sqrt[q]{\sum_{\langle c,d\rangle\in E}\left([w(c,d)]^{p/q}|x(c) - x(d)|\right)^q}$, that is, $\varepsilon_{p,\infty}(x) = \max_{\langle c,d\rangle\in E} \operatorname{sgn}(w(c,d))|x(c) - x(d)|$, and relate it to the Voronoi diagram delineation. It is clear, that $\varepsilon_{p,\infty}$ is equal to ε_∞ associated with the graph $G = \langle C, E, \operatorname{sgn}(w)\rangle$, so Theorem 3.2(a) is applicable in this case.

3.3. The case of $p \to \infty$ and $q \in \mathbb{R}$

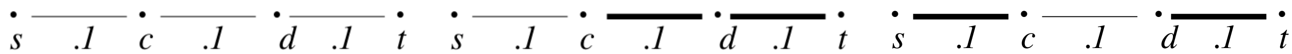
The delineation algorithm in [22] associated with $q \in [1,\infty)$ and $p \to \infty$ is referred to as *Power Watershed, PW, algorithm.* Although, as in the previous cases, its hard set output $\bar{x}$ is obtained from a fuzzy object (labeling) x, it is proved in [22, property 2] that such an $\bar{x}$ belongs to the family $\mathcal{P}^{MSF}(S,T)$, that is, $\bar{x}$ is generated by an MSF with respect to $S \cup T$. At the same time, every object from $\mathcal{P}^{MSF}(S,T)$ maximizes the energy $\varepsilon^{\max}$ on $\mathcal{P}(S,T)$, as we prove in Theorem 4.1. (This last result is closely related to the subject of papers [3, 24].)

The above shows that PW returns an optimizer for the energy $\varepsilon^{\max}$. Since the same is true about IRFC returned objects (see Theorem 2.2), it can be argued that PW is nothing more than a version of Fuzzy Connectedness algorithm. This impression is even deepened by the fact (Theorem 4.1) that the output of the GC$_{\max}$

†This can be found by simple multivariable calculus. First notice, that both second partial derivatives, $\frac{\partial^2}{\partial z^2}E_{q,q}(x_{y,z}) = 2q(q-1)v^q[|y-z|^{q-2} + z^{q-2}]$ and $\frac{\partial^2}{\partial y^2}E_{q,q}(x_{y,z}) = 2q(q-1)[(1-y)^{q-2} + v^q|y-z|^{q-2}]$ are positive, so the function $E_{q,q}$ is convex and it can have only one global minimum. For $y \geq z$, $\frac{\partial}{\partial z}E_{q,q}(x_{y,z}) = 2[-qv^q(y-z)^{q-1} + qv^q z^{q-1}]$ equals 0 when $(y-z)^{q-1} = z^{q-1}$, that is, when $y = 2z$. Similarly, the other derivative $\frac{\partial}{\partial y}E_{q,q}(x_{y,z}) = 2[-q(1-y)^{q-1} + qv^q(y-z)^{q-1}]$ equals 0 when $(1-y)^{q-1} = v^q(y-z)^{q-1}$, which, with $y = 2z$, leads to $\left(\frac{1-2z}{2z-z}\right)^{q-1} = v^q$ and $\frac{1}{z} - 2 = v^{q/(q-1)}$. So, $z = (v^{q/(q-1)}+2)^{-1}$ and $y = 2(v^{q/(q-1)}+2)^{-1}$ minimize $E_{q,q}$ on $[0,1]\times[0,1]$, since for $z > y$, the derivative $\frac{\partial}{\partial z}E_{q,q}(x_{y,z}) = 2[qv^q(z-y)^{q-1} + qv^q z^{q-1}]$ never equals 0.

algorithm also belongs to the family $\mathcal{P}^{MSF}(S,T)$ of objects indicated by MSF. In particular, if $\mathcal{P}^{MSF}(S,T)$ has only one element (no tie-zones), then the outputs of PW and $\mathrm{GC}_{\max}$ are identical.

Nevertheless, the algorithms PW and $\mathrm{GC}_{\max}$ use different paradigms to choose their outputs from $\mathcal{P}^{MSF}(S,T)$: $\mathrm{GC}_{\max}$ always chooses its smallest element, while, within each plateau of the graph, PW chooses the object that minimizes the energy $E_{p,q}$ for a current value of q (which, for $q>1$, is unique). In particular, Figure 1 provides an example of a graph, in which outputs of $\mathrm{GC}_{\max}$ and PW are different.



(a) Weighted graph G with $S=\{s\}$ and $T=\{t\}$ (b) MSF $\mathbb{F}$, indicated by thicker edges, returned by $\mathrm{GC}_{\max}$; $P(s,\mathbb{F})=\{s\}$ (c) MSF $\hat{\mathbb{F}}$, indicated by thicker edges, returned by PW; $P(s,\hat{\mathbb{F}})=\{s,c\}$

Figure 1. Example of different outputs of $\mathrm{GC}_{\max}$ and PW used with $q>1$; the intermediate labeling x for PW is given by $x(c)=1/3$ and $x(d)=2/3$ (these numbers can be verified by multivariable calculus technique)

The association of PW with the limiting process $p\to\infty$ comes from the following (fuzzy) limiting property, that holds for $q>1$ and essentially no restriction on the seed choice (see [22, theorem 3]):

(LP^{F}) the fuzzy labeling x returned by PW is equal to a limit, as $p\to\infty$, of the fuzzy sets (labelings) x_p minimizing the energy $E_{p,q}$.

Since, in this paper, we are predominantly interested in the hard set objects, of considerably more interest to us is the following (hard) limiting property, which is analogous to the property of the $\mathrm{GC}_{\max}$ algorithm mentioned earlier:

(LP^{H}) the output $\bar{x}$ of PW is equal to a limit, as $p\to\infty$, of $\bar{x}_p$, where each is a fuzzy set (labeling) x_p minimizing the energy $E_{p,q}$.

However, unlike (LP^{F}), the property (LP^{H}) is proved in [22, theorem 1] only under an additional strong assumption‡ on seeds S and T. Moreover, as noted in [22, figure 2], (LP^{H}) may be false without the assumption on seeds.

In summary,

- The algorithms $\mathrm{GC}_{\max}$ and PW return outputs with very similar properties: they both minimize the same energy $\varepsilon^{\max}$ and, in both cases, can be generated by an MSF. Nevertheless, their outputs can be different (Figure 1).

- The algorithm $\mathrm{GC}_{\max}$ is very fast: it provably runs in a (quasi-) linear time with respect to the image size (Theorem 2.2). There is no similar theoretical result for PW. (The experimental results presented in [22] suggest that PW runs in quasi-linear time, at least for a simple case of $q=2$. It is also true, that components of PW algorithm, Kruskal's algorithm and plateau optimizations for $E_{p,2}$, run, provably, in a linear time with respect to the image size. However, their complicated amalgamation, formation of merged graphs, puts under question, whether a provable quasi-linear time implementation of PW can be found.)

Finally, note that, at first glance, the most natural candidate for $E_{\infty,q}(x)$ is the limit $L(x)\stackrel{\text{def}}{=}\lim_{p\to\infty}E_{p,q}(x)=\lim_{p\to\infty}\sum_{\langle c,d\rangle\in E}[w(c,d)]^p|x(c)-x(d)|^q$, rather than $\varepsilon_{\infty,q}$. However, $L(x)$ does not exist, unless $w(c,d)\le 1$ for all $\langle c,d\rangle\in E$. Moreover, even if the limit exists (i.e., when $w(c,d)\le 1$ for all $\langle c,d\rangle\in E$), the energy $E_{\infty,q}(x)\stackrel{\text{def}}{=}L(x)=\sum_{\langle c,d\rangle\in E}\lfloor w(c,d)\rfloor|x(c)-x(d)|^q$ does not lead to a new optimization problem, as such $E_{\infty,q}$ is equal to $E_{1,q}$ for the graph $G=\langle C,E,\lfloor w\rfloor\rangle$.

‡The assumption is that for every threshold t, the set $S\cup T$ intersects every connected component of the graph $\langle C,E_t\rangle$, where $E_t=\{e\in E\colon w(e)\ge t\}$.

4. IRFC OBJECT AS OPTIMUM PATH AND MAXIMUM SPANNING FORESTS

According to Theorem 2.2, the output $P^{\infty}_{S,T}$ of $\mathrm{GC}_{\max}$, the IRFC object, is given by an OPF. The next theorem shows, in particular, that it is also given by a maximal spanning forest, MSF. It also relates the family $\mathcal{P}_\theta(S,T)$ of all $\varepsilon^{\max}$ optimizing objects with the families $\mathcal{P}^{MSF}(S,T)$ and $\mathcal{P}^{OPF}(S,T)$ of all objects $P(S,\mathbb{F})$ associated with MSF and OPF, respectively.

THEOREM 4.1. *Let $G = \langle C, E, w\rangle$ be a weighted graph and $S,T \subset C$ be non-empty disjoint sets of seeds. If $\mu(S,T) < 1$, then*

$$P^{\infty}_{S,T} \in \mathcal{P}^{MSF}(S,T) \subset \mathcal{P}^{OPF}(S,T) \cap \mathcal{P}_\theta(S,T). \tag{5}$$

In particular, the families $\mathcal{P}^{MSF}(S,T)$ and $\mathcal{P}^{OPF}(S,T)$ share the same minimal element, $P^{\infty}_{S,T}$.

Notice, that the OPF $\mathbb{F}$ returned by $\mathrm{GC}_{\max}$ need not be MSF. (See Figure 2.) However, by Theorem 4.1, there is always an MSF $\hat{\mathbb{F}}$ for which $P^{\infty}_{S,T} = P(S,\mathbb{F}) = P(S,\hat{\mathbb{F}})$. Moreover, if one is after MSF $\hat{\mathbb{F}}$ for which $P(S,\hat{\mathbb{F}}) = P^{\infty}_{S,T}$, such an $\hat{\mathbb{F}}$ can still be found (in linear time) as follows: (1) Run $\mathrm{GC}_{\max}$ (which returns $P^{\infty}_{S,T}$ as $P(S,\mathbb{F})$ for some OPF $\mathbb{F}$, which need not be an MSF). (2) Find an MSF $\hat{\mathbb{F}}$ with $P(S,\hat{\mathbb{F}}) = P(S,\mathbb{F})$ using Kruskal's algorithm, as indicated in the proof of the theorem.

PROOF OF THEOREM 4.1. It was proved in [3, proposition 8] that every MSF is also an OPF. (The same result, proved independently, is also included in [24, theorem 21]. In both papers optimum path spanning forests are referred to as shortest path forests.) This justifies inclusion $\mathcal{P}^{MSF}(S,T) \subset \mathcal{P}^{OPF}(S,T)$.

Next, we prove that $P^{\infty}_{S,T} \in \mathcal{P}^{MSF}(S,T)$. Let $\mathbb{F}$ be the OPF returned by $\mathrm{GC}_{\max}$, so that we have $P^{\infty}_{S,T} = P(S,\mathbb{F})$. We will find an MSF $\hat{\mathbb{F}}$ relative to $W = S \cup T$ which returns the same object, that is, such that $P(S,\hat{\mathbb{F}}) = P(S,\mathbb{F})$.

Recall, that the Kruskal's algorithm creates MSF $\hat{\mathbb{F}} = \langle C, \hat{E}\rangle$ as follows:

- it lists all edges of the graph in a queue Q such that their weights form a decreasing sequence;
- it removes consecutively the edges from Q, adding to $\hat{E}$ those, whose addition creates in the expanded $\hat{\mathbb{F}} = \langle C, \hat{E}\rangle$ neither a cycle nor a path between different vertices from W; other edges are discarded.

This schema has a leeway in choosing the order of the edges in Q: those that have the same weight can be ordered arbitrarily.

Let B be the boundary of $P(S,\mathbb{F})$, $B = \mathrm{bd}(P(S,\mathbb{F}))$. Assume, that we create the list Q in such a way that, among the edges with the same weight, all those that do not belong to B precede all those that belong to B. We will show that Kruskal's algorithm with Q so chosen, indeed returns MSF $\hat{\mathbb{F}}$ with $P(S,\hat{\mathbb{F}}) = P(S,\mathbb{F})$.

Clearly, by the power of Kruskal's algorithm, the returned $\hat{\mathbb{F}} = \langle C, \hat{E}\rangle$ will be MSF relative to W. We will show that $\hat{E}$ is disjoint with B. This easily implies the equation $P(S,\hat{\mathbb{F}}) = P(S,\mathbb{F})$.

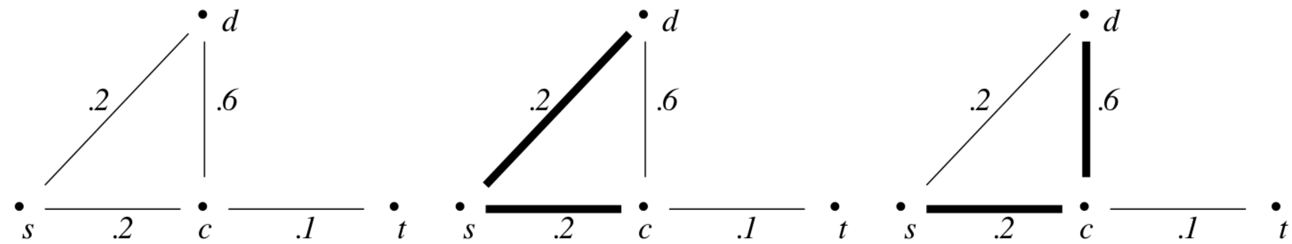
To prove that $\hat{E}$ is disjoint with B, choose an edge $e = \{c,d\} \in B$. Consider the step in Kruskal's algorithm when we remove e from Q. We will argue, that adding e to the already existing part of $\hat{E}$ would add a path from S to T, which implies that e would not be added to $\hat{E}$.

Let p_c and p_d be the paths in $\mathbb{F}$ from W to c and d, respectively. By symmetry, we can assume that $c \in C \setminus P(S,\mathbb{F}) = P(T,\mathbb{F})$ and $d \in P(S,\mathbb{F})$. We will first show that

$$\mu(p_c) \geq w_e \text{ and } \mu(p_d) \geq w_e. \tag{6}$$

Indeed, if $\mu(p_c) > \mu(p_d)$, then $w_e \leq \mu(p_d)$, since otherwise $\mu(d,S) = \mu(p_d) < \min\{\mu(p_c), w_e\} \leq \mu(d,T)$, implying that d belongs to the RFC object $P_{T,S} \subset P(T,\mathbb{F})$, which is disjoint with $P(S,\mathbb{F})$. Similarly, if $\mu(p_c) < \mu(p_d)$, then $w_e \leq \mu(p_c)$, since otherwise $\mu(c,T) = \mu(p_c) < \min\{\mu(p_d), w_e\} \leq \mu(c,S)$, implying that c belongs to the RFC object $P_{S,T} \subset P(S,\mathbb{F})$. Finally, assume that $\mu(p_c) = \mu(p_d)$. Then $w_e < \mu(p_c) = \mu(p_d)$, since otherwise $\mathrm{GC}_{\max}$ would reassign d to $P(T,\mathbb{F})$, which is disjoint with $P(S,\mathbb{F})$. So, (6) is proved.

Next, let $E' = \{e' \in E\colon w_{e'} \geq w_e\} \setminus B$. Then, every edge in E' is already considered by the Kruskal's algorithm by the time we remove e from Q. In particular, $\hat{E} \cap E'$ is already constructed. We claim, that there is a path $\hat{p}_d$ in $\hat{G} = \langle C, \hat{E} \cap E'\rangle$ from S to d.



(a) Weighted graph G, with s indicating the object and t the background (b) OPF returned by $\mathrm{GC}_{\max}$, which is not MSF (c) MSF for G, which indicates the same object $\{s, c, d\}$ as OPF from (b)

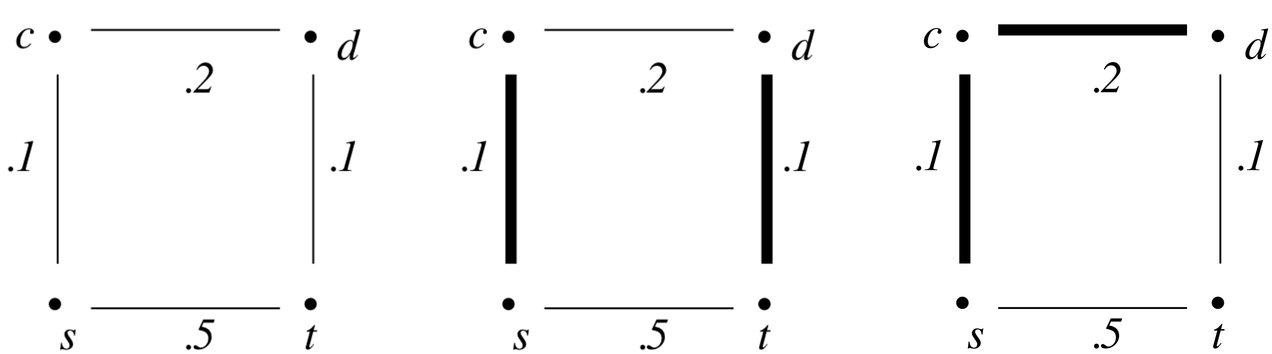
Figure 2. Weighted graph G, with $S = \{s, t\}$

Indeed, the component of d in $\hat{G}$ must intersect S, since otherwise there is an edge $\hat{e}$ in p_d (so, in E') only one vertex of which intersects this component. But this means that $\hat{e} \in E'$ would have been added to $\hat{E}$, which was not the case. So, indeed, there is a path $\hat{p}_d$ in $\hat{G}$ from S to d. Similarly, there is a path $\hat{p}_c$ in $\hat{G}$ from T to c. But this means that adding e to $\hat{E}$ would create a path from S to T, which is a forbidden situation. Therefore, indeed, Kruskal's algorithm discards e, what we had to prove. This completes the argument for $P^{\infty}_{S,T} \in \mathcal{P}^{MSF}(S,T)$.

To finish the proof, we need to show that $\mathcal{P}^{MSF}(S,T) \subset \mathcal{P}_\theta(S,T)$. So, fix a $P \in \mathcal{P}^{MSF}(S,T)$. Then, there is an MSF $\mathbb{F} = \langle C, E' \rangle$ with respect to W for which $P = P(S, \mathbb{F})$. Clearly, $P = P(S, \mathbb{F}) \in \mathcal{P}(S,T)$. So, it is enough to show that $\varepsilon^{\max}(P) \le \theta_{\min} = \mu(S,T)$.

By way of contradiction, assume that this is not the case. Then, there exists an edge $e = \{c, d\} \in E$ with $c \in P = P(S, \mathbb{F})$ and $d \in C \setminus P = P(T, \mathbb{F})$ for which $w_e > \theta_{\min} = \mu(S,T)$. Let p_c and p_d be the paths in $\mathbb{F}$ from W to c and d, respectively. Then either $\mu(p_c) < w_e$ or $\mu(p_d) < w_e$, since otherwise the path p starting with p_c, followed by e, and then by p_d is a path from S to T with $\mu(p) = w_e > \mu(S,T)$, a contradiction.

Assume that $\mu(p_c) < w_e$. Then $p_c = \langle c_1, \ldots, c_k \rangle$ with $k > 1$ and the edge $e' = \{c_{k-1}, c_k\}$ has weight $\le \mu(p_c) < w_e$. But then $\hat{\mathbb{F}} = \langle C, \hat{E} \rangle$ with $\hat{E} = E' \cup \{e\} \setminus \{e'\}$ is a spanning forest rooted at W with $\sum_{e \in \hat{E}} w(e) = \sum_{e \in E'} w(e) + w_e - w_{e'} > \sum_{e \in E'} w(e)$, what contradicts maximality of $\mathbb{F}$. This completes the proof of the theorem. ■



(a) Weighted graph G, with $S = \{s\}$ and $T = \{t\}$ (b) OPF $\mathbb{F}$ indicating object $P(S, \mathbb{F}) = \{s, c\}$ (c) The OPF $\hat{\mathbb{F}}$ with $w(\hat{\mathbb{F}}) = .3 > .2 = w(\mathbb{F})$

Figure 3. The OPF $\mathbb{F}$ in (b) indicates object $P(S, \mathbb{F}) \in \mathcal{P}_\theta(S,T) \setminus \mathcal{P}^{MSF}(S,T)$

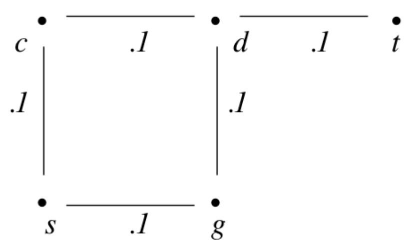
Finally, we provide several examples, indicating that little can be improved in the statement of Theorem 4.1. In all figures forest edges are indicated by thicker lines. Figure 2(b) shows that the OPF $\mathbb{F}$ returned by $\mathrm{GC}_{\max}$ (i.e., with $P^{\infty}_{S,T} = P(S, \mathbb{F})$) need to be MSF. Thus, the additional work for finding MSF $\hat{\mathbb{F}}$ (indicated on Figure 2(c)) with $P(S,\hat{\mathbb{F}}) = P(S,\mathbb{F}) = P^{\infty}_{S,T}$ is essential.

An example of an object in $\mathcal{P}^{OPF}(S,T) \cap \mathcal{P}_\theta(S,T)$ but not in $\mathcal{P}^{MSF}(S,T)$ is given in Figure 3. So, the inclusion in Theorem 4.1 cannot be reversed.

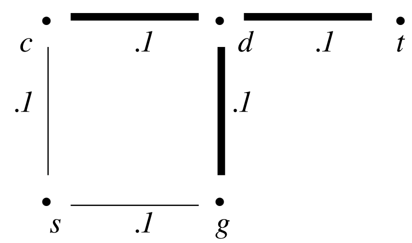
Also, there is no inclusion between $\mathcal{P}^{OPF}(S,T)$ and $\mathcal{P}_\theta(S,T)$. An object in $\mathcal{P}_\theta(S,T) \setminus \mathcal{P}^{OPF}(S,T)$ can be chosen as $\{s,d\}$ for the graph from Figure 1. The object indicated in Figure 1(b) belongs to $\mathcal{P}^{OPF}(S,T) \setminus \mathcal{P}_\theta(S,T)$, if the weight of the middle edge is changed to .5.

We will finish this section, by relating the above results to the minimizers of the energy $\varepsilon^{\text{sum}}(P) = \sum_{e \in \text{bd}(P)} w_e$, which are usually calculated via graph cut algorithm GC_{sum}. It is well known that the graph cut algorithms have the so called shrinking problem: if the object is indicated only by a small set of seeds, it is likely that the object minimized by ε^{sum} will have a short boundary composed of edges with high weights, even if there is another object with a long boundary of edges with very small weight. In such a case, the families of minimizers of ε^{sum} and $\varepsilon^{\max}$ are disjoint, indicating no relation between such minimizers.

Still an interesting question is: what happens if we know that the objects minimizing ε^{sum} also minimize $\varepsilon^{\max}$? Is it true, that an object $P^{\infty}_{S,T}$ returned by $\text{GC}_{\max}$ (so, minimizing $\varepsilon^{\max}$) minimizes also ε^{sum}? A negative answer to this question is provided in Figure 4. Actually, the results presented in Figure 4 remain the same, if all weights in the graph are raised to some finite power p.



(a) Weighted graph G, with $S = \{s\}$ and $T = \{t\}$



(b) An example of OPF $\mathbb{F}$ returned by $\text{GC}_{\max}$

Figure 4. The object $P^{\infty}_{S,T} = P(S,\mathbb{F})$ has ε^{sum}-energy .2, while minimum ε^{sum}-energy on $\mathcal{P}(S,T)$ is .1, for the object $\{s,c,d,g\}$

5. CONCLUDING REMARKS

- Two classes of distinct algorithms, GC_{sum} and $\text{GC}_{\max}$, are enough to find minimizers for all GGC energies: ε_q with $1 \le q \le \infty$.
- PW algorithm minimizes the ε_∞ energy, but does not run in linear time, while $\text{GC}_{\max}$ algorithm, returning IRFC object, has both properties.
- The IRFC (returned by $\text{GC}_{\max}$) and PW objects can be different although they are usually close (since, in most cases, the tie zones are small).
- The output of $\text{GC}_{\max}$, the IRFC object, is provably robust to seed choice. Neither PW not GC_{sum} algorithm has this property.
- GC_{sum} usually produces smoother boundaries than $\text{GC}_{\max}$.
- Any MSF is also an OPF but not vice versa.

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