A general theory of image segmentation: level set segmentation in the fuzzy connectedness framework

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ABSTRACT

In the current vast image segmentation literature, there is a serious lack of methods that would allow theoretical comparison of the algorithms introduced by using different mathematical methodologies. The main goal of this article is to introduce a general theoretical framework for image segmentation that would allow such comparison. The framework is based on the formal definitions designed to answer the following fundamental questions: What is the relation between an idealized image and its digital representation? What properties a segmentation algorithm must satisfy to be acknowledged as acceptable? What does it mean that a digital image segmentation algorithm truly approximates an idealized segmentation model? We use the formulated framework to analyze the front propagation (FP) level set algorithm of Malladi, Sethian, and Vemuri and compare it with the fuzzy connectedness family of algorithms. In particular, we prove that the FP algorithm is weakly model-equivalent with the absolute fuzzy connectedness algorithm of Udupa and Samarasekera used with gradient based affinity. Experimental evidence of this equivalence is also provided. The presented theoretical framework can be used to analyze any arbitrary segmentation algorithm. This line of investigation is a subject of our forthcoming work.

1. INTRODUCTION

Image segmentation—the process of partitioning the image domain into meaningful object regions—is perhaps the most challenging and critical problem in image processing and analysis. Its central position in image processing comes from the fact that the delineation of objects is usually the first step in other higher level processing tasks, like image interpretation, diagnosis, analysis, visualization, virtual object manipulation, and often even registration. Image segmentation may be thought of as consisting of two related processes: recognition and delineation. *Recognition* is the high-level process of determining roughly the whereabouts of an object of interest in the image. *Delineation* is the low-level process of determining the precise spatial extent and point-by-point composition (material membership percentage) of the object in the image. The topic of this paper concerns image delineation.

General segmentation frameworks may be classified into three groups: boundary-based [4, 8, 9, 11, 13, 14, 16], region-based [2, 21, 24–26], and hybrid [12]. As the nomenclature indicates, in the first two groups the focus is on recognizing and delineating the boundary or the region occupied by the object in the image. In the third group, the focus is on exploiting the complementary strengths of each of boundary-based and region-based strategies to overcome their individual shortcomings.

The rationale for the development of a general theoretical study of image segmentation methodologies is to address several serious gaps that currently exist in our knowledge in this subject, which are denoted (G1)–(G3) in the following: (G1) Are all different families of segmentation methods/models (e.g., functional optimization, usually implemented via level sets [14, 16, 20, 21], graph-cut [2], active contour [13], live wire [11], active shape [8], active appearance [9], fuzzy connectedness [6, 7, 19, 24, 26], and watershed [22]) really fundamentally independent or are there similarities, or even theoretical equivalences, among them? Although there are some rare attempts here and there to compare the methods at a theoretical level, this is largely an open question. (G2) Segmentation research has two clearly distinct components: the practical, focused on describing efficient segmentation algorithms that can be practically implemented; and theoretical, concerning development and use of sophisticated tools of infinite (i.e., not discrete) mathematics for the purpose of describing segmentation models of idealized images. One of the peculiarities of the current state of segmentation research is that these two tracks are not

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connected in any formal way. True, the papers that start with a description of a segmentation model of idealized images usually transcribe such a model into a digital image segmentation procedure. However, all of these translations are done only at the intuitive level, without a formal, mathematical argument. In fact, there is even no evidence of the use of any definition formally connecting idealized images (infinite objects) with their digital representations (which are finite). (G3) Another element clearly missing from current segmentation research is a set of certain properties that any digital segmentation algorithm must or should have. For example, it seems that an output of any reasonable segmentation algorithm should be reasonably stable if it is fed with the digital approximations of the same idealized image with better and better resolution. It would be also desirable for the segmentation output to remain reasonably unchanged when applied to the same resolution digital representations of the same idealized image that was rotated and/or shifted. (This latter aspect becomes important when we keep in mind that, in many areas such as medical imaging, there is no guarantee that the same object with subtle and fine features will be digitized in the same manner in repeated scans/digitizations.) So far, there is very little research done along these lines, especially for the algorithms that were not motivated by idealized image segmentation models. This paper is a first attempt to fill some of these gaps via the general theory proposed in Sec. 2.

In the FC framework [26], a fuzzy topological construct, called fuzzy connectedness, characterizes how the spatial elements (abbreviated as *spels*) of an image hang together to form an object. This construct is arrived at roughly as follows. A function called *affinity* is defined on the image domain; the strength of affinity between any two spels depends on how close the spels are spatially and how similar their intensity-based properties are in the image. Affinity is intended to be a local relation. A global fuzzy relation called *fuzzy connectedness* is induced on the image domain by affinity as follows. For any two spels c and d in the image domain, all possible paths connecting c and d are considered. Each path is assigned a strength of fuzzy connectedness which is simply the minimum of the affinities of consecutive spels along the path. The level of fuzzy connectedness between c and d is considered to be the maximum of the strengths of all paths between c and d. For segmentation purposes, FC is utilized in several ways as described below. See [24] for a review of the different FC definitions and how they are employed in segmentation and applications.

In *absolute FC* (abbreviated AFC) [26], the support of a segmented object is considered to be the maximal set of spels, containing one or more seed spels, within which the level of FC is at or above a specific threshold. To obviate the need for a threshold, *relative FC* (or RFC) [19] was developed by letting all objects in the image to compete simultaneously via FC to claim membership of spels in their sets. To avoid treating the core aspects of an object (that are very strongly connected to its seeds) and the peripheral subtle aspects (that may be less strongly connected to the seeds) in the same footing, an iterative refinement strategy is devised in *iterative RFC* (or IRFC) [6, 25]. The FC family of methods developed to date consists of various combinations of absolute, relative, and iterative FC. In this paper we will study (in Sec. 3.1) only the AFC algorithm considered with a gradient based affinity. Note that gradient based affinity is a generalized affinity notion, in a format introduced and examined in [7]. The other forms of FC algorithms will be examined within the general framework of Sec. 2 in our future work.

The level set method refers to the specific model of an evolving front (surface or curve) in a time dependent manner and to the numerical algorithm tracking such propagating fronts. The model and the associated narrow band propagation algorithm were introduced in 1985 by Sethian [20] which made their way into image segmentation research in 1995 with paper [14]. The popularity of the level set method in segmentation tasks led to a multitude of research papers, as exemplified by the books [17, 18, 21]. Although the level set method in image segmentation is nowadays more often used indirectly to solve the PDE optimizing the segmentation cost functions (see e.g., [5, 15, 27]), the original segmentation algorithms are still studied [23]. Therefore, in this paper, for the purpose of using the theoretical framework for comparing methods, we have chosen the level set method with front propagation, because of its popularity, and FC, because of our familiarity with it.

Our general theoretical framework is described in Sec. 2. Its application to the analysis of a particular model of FC [26] and to a comparison of its algorithms with the level set delineation algorithm of [14] is presented in Sec. 3. (An attempt of expressing this level set algorithm [14] without PDE can also be found in [23].) Although in this paper we focus only on the algorithms of [26] and [14] for a theoretical comparison, the general framework can be utilized to compare any methods in the literature. In Sec. 4, we present a practical segmentation example to illustrate the equivalence proved in Sec. 3.

2. THE GENERAL IMAGE SEGMENTATION THEORY

Despite the enormous volume of image segmentation research, so far no formal, general, and coherent theoretical framework of a segmentation theory has been published that would allow a theoretical comparison of segmentation algorithms independently of the mathematical framework in which they were introduced. In this section, we describe a theoretical structure designed for facilitating such comparisons. Moreover, the framework will help us to identify the basic properties that any reasonable segmentation algorithm should satisfy. The specific application of the theory developed here will be presented in the next section.

2.1. Stage set up: What is an image?

We will start off by formalizing the notions of an "idealized image" and the "physical image" associated with it. This formalization is rather standard in imaging literature. However, most of the imaging papers concentrate only on one of these two kinds of images, leaving unanswered or hazy the fundamental question at to what the relation between them is. One of the most important goals of this paper is to clearly describe such relation in a general setting and to apply it to each of the segmentation models that is considered here.

There are many different kinds of physical phenomena that lead to images. These include: biological visual (human or animal) perception leading to internal imagery, different analogue pictures (like those obtained by standard analog cameras or x-ray films), and digital images (including digital camera pictures, synthesized computer images, and computed images created from image acquisition devices like ultrasound, CT, PET, and MRI scanners). Every such image can be identified with an intensity function, say f, associating to each sensor c from some finite set C of sensors the image intensity value $f(c)$ at c. Note that set C is finite even in the case of chemical sensors used by analog images, although in this case their number is usually large. We often ignore the physical dimensions of the sensors once the image is formed, but keep track of their relative position. This allows us to treat set C as a subset of the Euclidean space $\mathbb{R}^n$ of appropriate dimension. Note that we do not restrict our attention to two-dimensional images, since we are particularly interested in medical imaging, where three-dimensional images have become a predominant entity of interest lately.

The intensity value $f(c)$ does not need to be expressible as a single number; however, it can usually be treated as a vector in $\mathbb{R}^\ell$. For example, $f(c)$ can be a vector of the intensities of different color components recorded at c. This leads us to the following definition of a "physical" image, to which we refer as a *digital image*, where $n, \ell \geq 1$ are arbitrary natural numbers. We will always assume that $n \geq 2$, although we will alow it to be larger than 3, as a time sequence of 3-dimensional images, for example, can be interpreted as a 4-dimensional image.

DEFINITION 2.1. An *(n-dimensional) digital image* is any function f from a finite subset C of $\mathbb{R}^n$ into $\mathbb{R}^\ell$.

In this definition, we slightly depart from the standard assumption that the coordinates of C are the integer numbers, that is, that $C \subset \mathbb{Z}^n$. Our generality will help us to lay our theory, while it creates no real implementation difficulty, especially in the most important case when C is a subset of a rectangular grid $\{hk\colon k \in \mathbb{Z}\}^n$, where $h > 0$ is a fixed number.

Although Definition 2.1 captures practical, computer driven, aspects of image representation, it misses the fact that essentially all images we are interested in (with the exception of synthesized computer images) are representations of some "true" images of some real objects. A digital image of a real object is simply an approximation of the "true" image, and (in the ideal setting) it usually improves with an improvement of image acquisition resolution, that is, with increasing number of sensors in C. (We realize that this is somewhat idealistic. Often, just improving resolution alone may not improve the image; the time duration of acquisition and the energy used in the imaging process may need to be increased. We assume that such provisions are made concomitantly with increasing resolution to obtain truly better images.) So, what is the "true" image of a real object? The simplest and the most common way to represent this idealized notion is to treat it as an intensity function F defined on some infinite subset (usually open bounded region) Ω of $\mathbb{R}^n$ and with the values in $\mathbb{R}^\ell$. Here Ω is the geometrical (distance preserving) representation of the depicted object, independent of any attributed intensity. We will refer to such a "true" image as an *idealized image*.

DEFINITION 2.2. An *(n-dimensional) idealized image* is any function F from a bounded connected subset Ω of the n-dimensional Euclidean space $\mathbb{R}^n$ into $\mathbb{R}^\ell$. In what follows, we will always assume that Ω is an open subset of $\mathbb{R}^n$, and often it will be just an n-dimensional cube $\Omega = (a, b)^n$.

In general, we do not assume any nice properties for function F. However, we will find that for the algorithms to have desired properties, it will be often necessary to assume that F is continuous or that it has continuous

derivatives. The relation between these two types of images can be expressed as follows.

DEFINITION 2.3. A digital image $f\colon C \to \mathbb{R}^\ell$ is a *digitization of an idealized image* $F\colon \Omega \to \mathbb{R}^\ell$ provided f is the restriction $F \restriction C$ of F to C, that is, $C \subset \Omega$ and $f(c) = F(c)$ for every $c \in C$.

2.2. The segmentation algorithms

The following definition treats a delineation algorithm as a "black box:" Given an input (a digital image map and the parameters, which may include also some prior knowledge on the segmented object), the only outcome that is considered is the actual output of the algorithm which is a segmented image. Thus, this definition is general enough to encompass essentially all possible delineation algorithms. In particular, this definition includes the fuzzy connectedness, FC, Udupa-Samarasekera algorithms [26] (see Sec. 3.1) as well as level set Malladi-Sethian-Vemuri algorithm [14] (see Sec. 3.2). Also, we do not restrict the algorithms, as we define them, to any particular application domain, as it is not important for our considerations. Nevertheless, for practical purposes, many algorithms are often considered only for some specific applications. Thus, the same (general) algorithm may be "good" in one application domain, while it may give unacceptable results in another domain. This discrepant behavior, however, will have no effect on the theory presented below.

DEFINITION 2.4. A *(digital) delineation algorithm* $\mathcal{A}$ is any effectively defined mapping $\langle f, \vec{\theta}\rangle \overset{\mathcal{A}}{\mapsto} S$ which to any digital image $f\colon C \to \mathbb{R}^\ell$ (possibly restricted to some subclass) and a parameter vector $\vec{\theta}$ associates a subset S of C interpreted as a segment of the image f indicated by the parameters. We will write $\mathcal{A}(f, \vec{\theta})$ for the output S of $\mathcal{A}$ applied to $\langle f, \vec{\theta}\rangle$.

The parameters may include a threshold number $\theta \in \mathbb{R}$ and some subsets of $\mathbb{R}^n$ (like a simple closed curve, as in the case of some level set algorithms) approximating respective subsets of the domain C of f which carry information on the objects we seek. Often, a seed point $s \in C$ is used as a parameter which indicates the segment S, that is, with the goal that $s \in S$. Some algorithms use also another seed point $t \in C$ indicating the background, that is, with the goal that $t \notin S$. We will often treat the parameter set as a pair $\langle \vec{\theta}, \vec{p}\rangle$ of parameter sequences (i.e., replacing $\mathcal{A}(f, \vec{\theta})$ with $\mathcal{A}(f, \vec{\theta}, \vec{p})$), as this is often more suitable for our theory.

We will often refer to a delineation algorithm as a *segmentation algorithm*. The segmentation algorithm, in general, can return as an output a finite sequence $\langle S_1, \ldots, S_k\rangle$ of (usually pairwise disjoint) subsets of C, while a delineation algorithm returns only one set $S \subset C$. The theory presented below is considerably more easily expressed for the delineation algorithms, while with some effort it can be applied also to any general segmentation procedure. (Any segmentation algorithm $\mathcal{A}$ that returns a k-element sequence $\mathcal{A}(f, \vec{\theta}) = \langle \mathcal{A}_1(f, \vec{\theta}), \ldots, \mathcal{A}_k(f, \vec{\theta})\rangle$ can be treated as k separate segmentation algorithms $\mathcal{A}_1, \ldots, \mathcal{A}_k$.)

Next, we will formalize what we believe to be the most fundamental property that any reasonable delineation algorithm should possess: *The better the resolution of the digital approximation of the idealized image, the closer the algorithm's output is to the "real object" in the idealized image.* To express this intuition formally, we will use the following definitions.

For a subset A of an underlying space X, a *characteristic (*or *indicator) function* χ_A of A is defined as $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in X \setminus A$. Recall that for a sequence $\langle A_i\rangle_{i=1}^\infty$ of subsets of X we define $\limsup_i A_i = \bigcap_{j=1}^\infty \bigcup_{i\geq j} A_i$. Note that $A = \limsup_i A_i$ holds precisely when $\chi_A = \limsup_i \chi_{A_i}$. Similarly, we define $\liminf_i A_i = \bigcup_{j=1}^\infty \bigcap_{i\geq j} A_i$ and we have $A = \liminf_i A_i$ precisely when $\chi_A = \liminf_i \chi_{A_i}$. The limit $\lim_i A_i$ exists and is equal to $\limsup_i A_i$ provided $\limsup_i A_i = \liminf_i A_i$.

For $h > 0$, let $(h\mathbb{Z})^n = \{hk\colon k \in \mathbb{Z}\}^n$ be the rectangular grid of points in $\mathbb{R}^n$ with the basic distance h. The most natural (and commonly used) discretization of an idealized image $F\colon \Omega \to \mathbb{R}^\ell$ is of the form $f_h = F \restriction \Omega_h$, where $\Omega_h = \Omega \cap (h\mathbb{Z})^n$. For $i = 0, 1, 2, \ldots$ let $C_i = \Omega_{h/2^i}$. Thus, we are doubling the resolution when passing from C_i to C_{i+1}. In this notation, our intuitive definition requires that any "good" delineation algorithm should have the property that the sequence $A_i = \mathcal{A}(F \restriction C_i, \vec{\theta})$ converges to some single set $A \subset \Omega$. Moreover, if the algorithm approximates an idealized segmentation model $\mathcal{M}$ which associates with F and $\vec{\theta}$ an object $\mathcal{M}(F, \vec{\theta}) \subset \Omega$, then A should reasonably approximate $\mathcal{M}(F, \vec{\theta})$.

The most natural means to express the above intuition in a formal way seems to be to require that the limit $L = \lim_i \mathcal{A}(F \restriction C_i, \vec{\theta})$, as defined above, exists and, if the idealized model $\mathcal{M}$ is provided, the limit is a dense subset of $\mathcal{M}(F, \vec{\theta})$ (i.e., the topological closure $\mathrm{cl}(L)$ of L contains $\mathcal{M}(F, \vec{\theta})$). Unfortunately, even for some of

the most common algorithms the limit $\lim_i \mathcal{A}(F \restriction C_i, \vec{\theta})$ may not exist. Nevertheless, $\limsup_i \mathcal{A}(F \restriction C_i, \vec{\theta})$ can still reasonably approximate a set $\mathcal{M}(F, \vec{\theta})$ if we choose an appropriate notion of approximation. A related way to solve this difficulty (and the one which we favor in this paper) is to change the notion of the limit so that the result of the new limiting process applied to the sequence $\mathcal{A}(F \restriction C_i, \vec{\theta})$ exists and is dense in $\mathcal{M}(F, \vec{\theta})$. In general, we will denote such a limit as $\lim^*_{i,\vec{\theta}} \mathcal{A}(F \restriction C_i, \vec{\theta})$, where calculation of $\lim^*$ may require calculation of several limits in a hierarchical manner, as in the example below. The limit notion that we will use for this purpose in Subsection 3.1 is defined for the families of sets $\{A_i(\theta)\colon \theta \in \mathbb{R} \ \&\ i = 1, 2, 3, \ldots\}$ by a formula

$$\lim_{i,\theta}{}^* A_i(\theta) \stackrel{\text{def}}{=} \lim_{\eta \to 0^+} (\limsup_{i \to \infty} A_i(\theta - \eta)) = \lim_{\eta \to 0^+} \left(\bigcap_{j=1}^{\infty} \bigcup_{i \geq j} A_i(\theta - \eta) \right), \tag{1}$$

where we define $B = \lim_{\eta \to 0^+} B(\eta)$ if and only if $\chi_B = \lim_{\eta \to 0^+} \chi_{B(\eta)}$. However, in Subsection 3.2, the family of sets A_i will have two real parameters, that is, we will deal with sets $\{A_i(\eta, \varepsilon)\colon \eta, \varepsilon \in \mathbb{R} \ \&\ i = 1, 2, 3, \ldots\}$. This will require a modification of $\lim^*$ to $\lim^\dagger$ defined as: $\lim_{i,\eta,\varepsilon}{}^\dagger A_i(\eta, \varepsilon) = \lim_{\varepsilon \to 0^+} (\lim_{i,\eta}{}^* A_i(\eta, \varepsilon))$. Notice that if $A_i(\eta)$ does not depend on ε, then the two limit notions coincide, that is, $\lim_{i,\eta,\varepsilon}{}^\dagger A_i(\eta) = \lim_{i,\eta}{}^* A_i(\eta)$. In general, different algorithms may require slightly different limiting notions.

The limit notion from (1) will be applied to the segmentation algorithms as follows. We will assume that the algorithm uses one real parameter θ and some other parameters $\vec{p}$. Thus, the output of the algorithm can be expressed as $\mathcal{A}(f, \theta, \vec{p})$. Then, for a fixed parameter $\vec{p}$ and fixed sequence $\langle F \restriction C_i \rangle_i$ of the digital approximations of an idealized image $F\colon \Omega \to \mathbb{R}^\ell$, we will define $A_i(\theta) = \mathcal{A}(F \restriction C_i, \theta, \vec{p})$ and require that the limit $\lim_{i,\theta}{}^* A_i(\theta)$ exists and is dense in $\mathcal{M}(F, \vec{\theta})$. For the algorithms which we consider in this paper, these sets $A_i(\theta)$ will satisfy the assumptions of the following fact. (The proofs of all results presented here will be provided in a full version of the paper.)

PROPOSITION 2.5. *Let $\{A_i(\theta)\colon \theta \in \mathbb{R} \ \&\ i = 1, 2, 3, \ldots\}$ be a family of sets such that $A_i(\theta') \subseteq A_i(\theta)$ for every i and $\theta' < \theta$. Then $\lim_{i,\theta}{}^* A_i(\theta)$ exists and equals $\bigcup_{\eta > 0} \left(\bigcap_{j=1}^{\infty} \bigcup_{i \geq j} A_i(\theta - \eta) \right)$.*

Now, we are ready for our fundamental definitions. In what follows, ε and η denote the real numbers.

DEFINITION 2.6. Let $\mathcal{A}^{\varepsilon,\eta}(f, \vec{\theta})$ be a delineation algorithm, where $\varepsilon, \eta \in \mathbb{R}$ are its parameters, and assume that $\mathcal{A}^{\varepsilon,0}(f, \vec{\theta}) = \lim_{\eta \to 0^+} \mathcal{A}^{\varepsilon,\eta}(f, \vec{\theta})$. If $\mathcal{A}^{\varepsilon}(f, \vec{\theta}) = \mathcal{A}^{\varepsilon,0}(f, \vec{\theta})$, then we say that the algorithm $\mathcal{A}^{\varepsilon}(f, \vec{\theta})$ is *weakly acceptable* for an idealized image $F\colon \Omega \to \mathbb{R}^\ell$ and a parameter $\vec{\theta}$ provided the limit $\lim^\dagger_{i,\eta,\varepsilon} \mathcal{A}^{\varepsilon,\eta}(F \restriction \Omega_{h/2^i}, \vec{\theta})$ exists for every $h > 0$.

Note that, if the algorithms $\mathcal{A}^{\varepsilon,\eta}(f, \vec{\theta}) = \mathcal{A}^{\eta}(f, \vec{\theta})$ are independent of the parameter ε, then $\mathcal{A}(f, \vec{\theta}) = \mathcal{A}^0(f, \vec{\theta}) = \lim_{\eta \to 0^+} \mathcal{A}^{\eta}(f, \vec{\theta})$ is also independent of ε, and $\mathcal{A}(f, \vec{\theta})$ is weakly acceptable (for F and $\vec{\theta}$) when the limit $\lim^*_{i,\eta} \mathcal{A}^{\eta}(F \restriction \Omega_{h/2^i}, \vec{\theta})$ exists for every $h > 0$. Of course, we would prefer convergence to hold for a wider variety of sequences $\langle C_i\colon i \in \mathbb{N} \rangle$ approximating Ω. The next definition makes this idea more precise.

DEFINITION 2.7. For a non-empty subset C of a bounded set $\Omega \subset \mathbb{R}^n$ define an *Ω-resolution number of C* as $r_\Omega(C) = \inf\{\varepsilon > 0\colon \Omega \subset \bigcup_{c \in C} B(c, \varepsilon)\}$, where $B(c, \varepsilon) = \{x \in \mathbb{R}^n\colon ||x - c|| < \varepsilon\}$ is an open ball in $\mathbb{R}^n$ centered at c and with radius ε. A delineation algorithm $\mathcal{A}$ as in Definition 2.6 is *acceptable* for an image $F\colon \Omega \to \mathbb{R}^\ell$ and parameter $\vec{\theta}$ provided the limit $\lim^\dagger_{i,\eta,\varepsilon} \mathcal{A}^{\varepsilon,\eta}(F \restriction C_i, \vec{\theta})$ exists for every increasing sequence $\langle C_i\colon i \in \mathbb{N} \rangle$ of finite subsets of Ω for which $\lim_i r_\Omega(C_i) = 0$.

Although this last definition is satisfied by many segmentation algorithms and is more desirable than just weak acceptability, the proofs of the acceptability property are more technically involved. Since a vast majority of practical algorithms are concerned only with scenes of the form Ω_h, we will often start the analysis by considering the property of weak acceptability.

2.3. Segmentation models and related algorithms

Now, we will define the notion of a segmentation model for an idealized image.

DEFINITION 2.8. A *delineation model* $\mathcal{M}$ for a class $\mathcal{F}$ of idealized images is any mapping $\langle F, \vec{p} \rangle \stackrel{\mathcal{M}}{\mapsto} O$ which for any image $F\colon \Omega \to \mathbb{R}^\ell$ from $\mathcal{F}$ and any parameters $\vec{p}$ associates a subset O of Ω interpreted as a segment of the image F indicated by the parameters. We will write $\mathcal{M}(F, \vec{p})$ for the output O of $\mathcal{M}$ applied to $\langle F, \vec{p} \rangle$. A *segmentation model* $\mathcal{M}$ that returns k objects is defined as a k-element sequence $\mathcal{M}(f, \vec{p}) = \langle \mathcal{M}_1(f, \vec{p}), \ldots, \mathcal{M}_k(f, \vec{p}) \rangle$ of delineation models $\mathcal{M}_i$.

Notice that in our definition of a delineation model $\mathcal{M}$ there is no assumption of the effectiveness of finding the object $\mathcal{M}(F,\vec{p})$ despite the fact that this might be considered as a departure from a terminology used in many mathematical modeling papers. To justify our choice of this terminology, we note that the modeling papers (including segmentation modeling) frequently start with a non-effectively defined map $\mathcal{M}$, often as an optimizer (minimizer or maximizer) of some functional, and then proceed to find an effectively defined procedure $\hat{\mathcal{M}}$, often via a solution of a differential equation and/or using variational methods, which, as a function, is equal to $\mathcal{M}$. We will refer to such $\hat{\mathcal{M}}$ as a *solution to a model* $\mathcal{M}$. (In the literature, it is often the procedure $\hat{\mathcal{M}}$ itself that is designated as the "model.") Since, treated as functions, $\mathcal{M}=\hat{\mathcal{M}}$, our definition can be applied to $\hat{\mathcal{M}}$ as well as to $\mathcal{M}$. If one plans to use the effective version $\hat{\mathcal{M}}$ of the model to find an algorithm $\mathcal{A}$ that approximates $\mathcal{M}=\hat{\mathcal{M}}$, then $\hat{\mathcal{M}}$ is at the center of the investigation and it makes sense to designate $\hat{\mathcal{M}}$ as "the model" of the process. We think of this modeling schema as a two stage process: $\mathcal{M}\longrightarrow\hat{\mathcal{M}}\longrightarrow\mathcal{A}$. In a large class of delineation methods (including the model discussed in Sec. 3.2, as well as many optimization models), the value of $\hat{\mathcal{M}}$ is found via time dependent front propagation, usually approximated numerically with fast marching level set algorithms. (See e.g. [17, 18].) In many modeling tasks, like modeling of the wave propagation or flame burning, the time sequence of the consecutive approximations is at least as important as the final position of the front, making the front propagation approach the most desirable. However, in segmentation tasks, we are usually not interested in the intermediate stages of object approximation, and we treat the final position of the front as the only output of the model and the algorithm. Thus, in the investigation of the essential aspects of a delineation task, it is more productive to follow directly from $\mathcal{M}$ to $\mathcal{A}$, that is, forgoing the effective version $\hat{\mathcal{M}}$ of the model and follow the schema $\mathcal{M}\longrightarrow\mathcal{A}$. This is the central idea behind our investigation.

The following definition is probably the most fundamental in this paper. It formally relates the segmentation model of an idealized image with the associated segmentation algorithm.

DEFINITION 2.9. A delineation algorithm $\mathcal{A}$ *represents (weakly represents) a delineation model* $\mathcal{M}$ *for a class* $\mathcal{F}$ of idealized images provided, for every $F\in\mathcal{F}$ and parameter $\vec{p}$ appropriate for F, algorithm $\mathcal{A}$ is acceptable (weakly acceptable, respectively) and for every sequence $\mathcal{C}=\langle C_i\colon i\in\mathbb{N}\rangle$ from the definition of (weak) acceptability the limit $\lim^{\dagger}_{i,\vec{\eta},e}\mathcal{A}^{\varepsilon,\eta}(F\restriction C_i,\vec{p})$ is a dense subset of $\mathcal{M}(F,\vec{p})$. A *segmentation algorithm* $\mathcal{A}=\langle\mathcal{A}_1,\ldots,\mathcal{A}_k\rangle$ *represents a segmentation model* $\mathcal{M}=\langle\mathcal{M}_1,\ldots,\mathcal{M}_k\rangle$ provided each $\mathcal{A}_i$ appropriately represents $\mathcal{M}_i$.

Notice that for any sequence $\mathcal{C}=\langle C_i\colon i\in\mathbb{N}\rangle$ considered in the above definition, the set $C=\bigcup_i C_i$ is dense in Ω. Since $D=\lim^{\dagger}_{i,\eta,\varepsilon}\mathcal{A}^{\varepsilon,\eta}(F\restriction C_i,\vec{p})$ is a subset of C, for D to be dense in $\mathcal{M}(F,\vec{p})$, it is necessary that $C\cap\mathcal{M}(F,\vec{p})$ is dense in $\mathcal{M}(F,\vec{p})$. The easiest way to insure this is to guarantee that $\mathcal{M}(F,\vec{p})$ is open in Ω. Thus, in what follows we will consider only the models which will ensure this property. (More generally, the same effect is achieved if we guarantee that $\mathcal{M}(F,\vec{p})$ is contained in the closure of its interior, that is, when $\mathcal{M}(F,\vec{p})\subset\mathrm{cl}_\Omega(\mathrm{int}_\Omega(\mathcal{M}(F,\vec{p})))$.) Then, the above definition is satisfied precisely when D is a dense subset of $C\cap\mathcal{M}(F,\vec{p})$. In most of the cases of good models and their associated algorithms, it can be shown that these two sets are equal. The above definition leads us also to a way of comparing segmentation algorithms on a theoretical level, the key goal of the paper.

DEFINITION 2.10. The segmentation algorithms $\mathcal{A}$ and $\mathcal{A}'$ are *model-equivalent* (*weakly model-equivalent*, respectively) in a class $\mathcal{F}$ of idealized images provided there exists a segmentation model $\mathcal{M}$ for $\mathcal{F}$ such that both $\mathcal{A}$ and $\mathcal{A}'$ represent (weakly represent, respectively) $\mathcal{M}$.

It should be stressed here that the model-equivalent algorithms behave identically only in the limit at the infinitely best resolution. So, their outputs may still be slightly different for given digital images. This may be contrasted with strongly-equivalent algorithms (defined and studied in [7]), which have identical outputs. Note also that equivalent algorithms (in the sense of any of these definitions) may still have very different computational times and/or memory requirements. Thus, equivalent algorithms should still be compared at some more subtle level: by analyzing their computational requirements, by estimating computational errors, and by running comparative simulations. Nevertheless, the equivalence of two segmentation algorithms is a strong *theoretical* evidence that they perform quite similarly.

3. GRADIENT BASED EDGE-THRESHOLD SEGMENTATION MODEL $\mathcal{M}_\triangledown$

In this section, we will show that the front propagation level set algorithm of Malladi, Sethian, and Vemuri from [14] is model-equivalent to the absolute fuzzy connectedness algorithm of Udupa and Samarasekera [26]

used with a gradient based affinity. We will start with the description of an idealized segmentation model $\mathcal{M}_\nabla$ represented by each of these algorithms. The description of the algorithms and the proofs that they indeed represent $\mathcal{M}_\nabla$ are presented in the following subsections.

In the model $\mathcal{M}_\nabla$, the edge (i.e., boundary) of the object $P \subset \Omega$ of interest is identified as the set of points x at which the image intensity, given by F, changes rapidly. Mathematically, this means that at the edge points the gradient magnitude $|\nabla F(x)|$ of F is large. Of course, this has a meaning only when the function $|\nabla F(x)|$ is well defined, that is, when F is differentiable. (A possible meaning of $|\nabla F|$ for non-differentiable F, and its implication to the presented discussion, is outlined in Sec. 4.) Thus, for this model, we will assume that F is of the class $\mathcal{C}^1$, that is, that F has continuous first order partial derivatives. Also, "large gradient" will be interpreted here "as greater than or equal to some threshold number θ." Thus, the object of interest will be a connected component of the set $\Omega(\theta) = \{x \in \Omega\colon |\nabla F(x)| < \theta\}$. The component will be indicated by some connected set $S \subset \Omega(\theta)$ of seeds, usually a single point or a simple closed curve. This component is the result of applying the model $\mathcal{M}_\nabla$ and parameters $\langle \theta, S\rangle$ to F, that is, it is equal to $\mathcal{M}_\nabla(F, \theta, S)$. Note that the continuity of the gradient implies that this set is open. We will usually denote $\mathcal{M}_\nabla(F, \theta, S)$ as $P^F_{S\theta}$ or just $P_{S\theta}$ when F is clear from the context.

Next, we describe a characterization of $P_{S\theta}$ that can be naturally translated into a numerical algorithm of its approximation. For this, we need the following definitions. A *path* p in Ω is any continuous injection from an interval $[a, b]$ into Ω. We say that a path p is from $S \subseteq \Omega$ to $x \in \Omega$ provided $p(a) \in S$ and $p(b) = x$. In this model, a strength $\mu(p)$ of a path $p\colon [a, b] \to \Omega$, which depends on F, is defined as $\mu(p) = \sup_{t\in[a,b]} |\nabla F(p(t))|$. Notice that the compactness of $[a, b]$ and the continuity of $\nabla F(x)$ implies that $\mu(p) = |\nabla F(p(t_0))|$ for some $t_0 \in [a, b]$.

THEOREM 3.1. *For every $\mathcal{C}^1$ image $F\colon \Omega \to \mathbb{R}^\ell$, $\theta \in \mathbb{R}$, and a connected set $S \subset \Omega(\theta)$, the object $P_{S\theta}$ is equal to the set of all $x \in \Omega$ for which there exists a path p from S to x with $\mu(p) < \theta$.*

The next, robustness, theorem tells us that the form of the object $P_{S\theta}$ essentially does not depend on the choice of the seed set S.

THEOREM 3.2. *For every $\mathcal{C}^1$ image $F\colon \Omega \to \mathbb{R}^\ell$, $\theta \in \mathbb{R}$, and connected sets $S, T \subset \Omega(\theta)$: $P_{S\theta} = P_{T\theta} \neq \emptyset$ if and only if there is a path p from S to T with $\mu(p) < \theta$. In particular, if $T \subset P_{S\theta}$, then $P_{S\theta} = P_{T\theta}$.*

3.1. First algorithm representing model $\mathcal{M}_\nabla$: gradient based Udupa-Samarasekera AFC algorithm $\mathcal{A}_\nabla$

We start here by describing a general form of the absolute fuzzy connectedness, AFC, algorithm of [26]. It is used to delineate the images identified with the intensity functions f from the finite subset C of $\mathbb{R}^n$ into $\mathbb{R}^\ell$. The elements of C are referred to as *spels*. We will think of f as a restriction of some idealized image $F\colon \Omega \to \mathbb{R}^\ell$ to a subset C of Ω. In most practical applications, C is a subset of a rectangular grid $(h\mathbb{Z})^n = \{hk\colon k \in \mathbb{Z}\}^n$ which, in terms of the idealized image, can be defined as $C = \Omega_h = \Omega \cap (h\mathbb{Z})^n$. (In fact, for algorithmic implementation, it is usually assumed that $h = 1$, that is, that $C \subset \mathbb{Z}^n$. This does not change the essence of the algorithm, since $(h\mathbb{Z})^n$ and $\mathbb{Z}^n$ can be naturally identified. Nevertheless, to describe the relation of the algorithm with the model, we need to adhere to the assumption that $C \subset \Omega$.) The special case $C = \Omega_h$ is also easier to handle in the analysis that follows, so we will give it special attention. We should also stress that, in the algorithm that follows, we will never use the fact that f is a restriction of an F. This fact will be used only to help our intuition and to express the convergence theorem. However, we will use Ω as a parameter of the algorithm, unless $C = \Omega_h$, in which case this parameter will be dropped.

Adjacency relation: The domain C of the digital image f, for which $\Omega \supset C$ is fixed, comes with an *adjacency relation* telling us which pairs $c, d \in C$ of spels are adjacent, that is, close enough to be considered spatially connected to each other. In this paper, we will assume that the adjacency relation is expressed in terms of the Euclidean distance in $\mathbb{R}^n$ as follows: for some constant $\alpha > 0$, the spels c and d are said to be adjacent provided $||c - d|| \leq \alpha$. We will assume that, for every $h \in (0, 1]$, $\alpha \in (2r_\Omega(C), n^2 r_\Omega(C)]$ and $\alpha \in [h, n^2 r_\Omega(C)]$ when $C = \Omega_h$. Thus, in general, the choice of α depends on the domain C of f and the set Ω. However, the dependency on Ω can be removed in case when $C = \Omega_h$, since then we can assume that $\alpha = h$ or, more generally, that $\alpha \in [h, n^2 h)$. This is important, since the algorithm we will construct depends on f and α. Thus, in the case when $C = \Omega_h$, we can assume that $\alpha = h$ and the algorithm depends only on f. In the general case, however, Ω will be also a parameter of the algorithm.

If $C = \Omega_h$ and $\alpha = h$, then we deal with 4-adjacency for $n = 2$, and with 6-adjacency for $n = 3$. If $C = \Omega_h$

and $\alpha = \sqrt{3}h$, then we deal with 8-adjacency for $n = 2$, and with 26-adjacency for $n = 3$. The idea behind the adjacency relation is to capture the blurring effect of the "point spread function" of imaging devices; that is, that the neighborhood size α should relate to the width of the point spread function.

A path in a digital scene: The choice of α as above ensures the following important property, where $B[T, \varepsilon] = \{x \in \mathbb{R}^n \colon \mathrm{dist}(T, x) \le \varepsilon\}$ is a generalized closed ball in $\mathbb{R}^n$ centered at $T \subset \mathbb{R}^n$ and with radius $\varepsilon > 0$. Recall that a *path* p *in* C is any sequence $\langle c_1, \ldots, c_k \rangle$ of spels in C, where consecutive c_i and c_{i+1} are adjacent; p is from $c \in C$ to $d \in C$ if $c_1 = c$ and $c_k = d$; it is from $S \subset C$ to $T \subset C$ if $c_1 \in S$ and $c_k \in T$.

LEMMA 3.3. *For every path* $\hat{p} \colon [a, b] \to \Omega$ *from* $s \in C$ *to* $d \in C$ *and* $\varepsilon \ge 2n\alpha$, *if* $B[\mathrm{range}(\hat{p}), \varepsilon] \subset \Omega$, *then there exists a path* $p = \langle c_1, \ldots, c_k \rangle$ *in* C *from* s *to* d *which is contained in* $B[\mathrm{range}(\hat{p}), \varepsilon]$.

Affinity function: Recall that any FC algorithm starts with an *affinity function*—a symmetric function κ defined on $C \times C$ for which the value $\kappa(c, d)$ represents a strength of local connectedness of the spels $c, d \in C$. We will use here an approach similar to that from the paper [7] and consider for affinity any symmetric function κ from $C \times C$ into any linearly ordered set $\langle L, \preceq \rangle$; however, in general, we will not assume that κ is *reflexive* (which, in [7], is expressed as a property that $\kappa(a, b) \preceq \kappa(c, c)$ for every $a, b, c \in C$). We drop the assumption of reflexivity of κ since only in this setting we can find an FC-type of algorithm representing $\mathcal{A}_\nabla$. Although this change will restrict our ability to cite any prior results concerning the FC theory results, this will be of no consequence to us, since we will not use any such result. In this particular subsection we will assume that $\langle L, \preceq \rangle = \langle [0, \infty], \ge \rangle$. Thus, the strongest connectedness (in the sense of $\preceq$) will be given by the value 0, and the weakest connectedness by ∞. Note that in the literature usually only standard affinities are considered, that is, those with the range $\langle L, \preceq \rangle = \langle [0, 1], \le \rangle$ and such that $\kappa(c, c) = 1$ for every $c \in C$. However, any reflexive affinity κ as above can be translated into a standard affinity by a formula $\kappa_\sigma(c, d) = (g_\sigma \circ \kappa)(c, d) = g_\sigma(\kappa(c, d))$, where $g_\sigma(x) = e^{-x^2/\sigma^2}$ is a Gaussian function for some $\sigma > 0$. In this situation affinities κ and κ_σ are naturally equivalent (lead to strongly equivalent algorithms) in a sense defined precisely in [7].

Digital path strength and AFC object: The affinity function κ represents the main parameter of the FC algorithms and can be defined differently for different applications. In the algorithm $\mathcal{A}_\nabla$, the definition of κ will be based on the gradient approximation of f. In general, any AFC algorithm, including $\mathcal{A}_\nabla$, depends on the definition of κ as follows. The strength of a path $p = \langle c_1, \ldots, c_k \rangle$ in C is defined as the $\preceq$-weakest link in p:

$$\mu(p) = \max_{i=1,\ldots,k-1} \kappa(c_i, c_{i+1}). \tag{2}$$

We define the AFC object as $P_{s\theta} = \{c \in C \colon$ there is a path p in C from s to c with $\mu(p) < \theta\}$, where $\theta \in \mathbb{R}$ is a threshold and $s \in C$ is a seed. In other words, if we denote our algorithm by a symbol $\mathcal{A}_\nabla$, then $\mathcal{A}_\nabla(f, \theta, s) = P_{s\theta}$. Our goal is to show that, for an appropriately defined function κ, this algorithm represents a segmentation model $\mathcal{M}_\nabla$. Note that if $\hat{\mu}(p)$ equals $\min_{i=1,\ldots,k-1} \kappa_\sigma(c_i, c_{i+1})$ and we put $\hat{\theta} = g_\sigma(\theta)$, then $P_{s\theta} = \{c \in C \colon$ there is a path p from s to c with $\hat{\mu}(p) > \hat{\theta}\}$. This is essentially the usual definition of an AFC object defined with the use of the standard affinity κ_σ, except that we use here the strict inequality $>$ rather than the more common $\ge$. This change is essential for the proof of our convergence theorem.

Gradient based path strength: Our definition of κ will be based on the formula $|\nabla f(c)|$ for the approximation of the magnitude of the gradient of F at c. It will have a property that, under appropriate assumptions on F, the limit $\lim_{r_\Omega(C) \to 0} |\nabla(F \restriction C)(c)|$ converges uniformly to $|\nabla F(c)|$ in a sense that: for every $\varepsilon > 0$ and compact set $B \subset \Omega$, there is a $\delta > 0$ such that for every finite $C \subset \Omega$ with $r_\Omega(C) < \delta$ and every $c \in C$

$$||\nabla F(c)| - |\nabla(F \restriction C)(c)|| < \varepsilon \text{ when } |\nabla(F \restriction C)(c)| \in \mathbb{R}, \text{ and } \nabla(F \restriction C)(c)| \in \mathbb{R} \text{ when } c \in B \cap C. \tag{3}$$

It is relatively easy to find such a formula for functions f defined on the sets $C = \Omega_h$. However, the general case is a bit technical and we will postpone the actual definition of $|\nabla f(c)|$ to the full version of the paper. Thus, in what follows, we assume that $|\nabla f(c)|$ is already defined and that it satisfies (3). From this, we define gradient based affinity as $\kappa(c, d) = \max\{|\nabla f(c)|, |\nabla f(d)|\}$ for adjacent c and d, and $\kappa(c, d) = \infty$ otherwise. In particular, for such affinity, formula (2) for the strength of a path $p = \langle c_1, \ldots, c_k \rangle$ reduces to $\mu(p) = \max_{i=1,\ldots,k} |\nabla f(c_i)|$.

The following theorem shows that the algorithm $\mathcal{A}_\nabla$ indeed represents the segmentation model $\mathcal{M}_\nabla$. Note that the assumption of uniform continuity of $|\nabla F|$ is satisfied if F is a restriction of a $\mathcal{C}^1$ function defined on the closure $\mathrm{cl}(\Omega)$ of Ω.

THEOREM 3.4. *Let $F\colon \Omega \to \mathbb{R}^\ell$ be an idealized $\mathcal{C}^1$ image, where Ω is a convex bounded open subset of $\mathbb{R}^n$. Assume that $|\nabla F|$ is uniformly continuous on Ω. Then for every $\theta > \theta' > 0$, finite set $C \subset \Omega$, and $s \in C$, there exists a $\delta > 0$ such that for every finite set $D \subset \Omega$ containing C for which $r_\Omega(D) < \delta$, we have*

$$C \cap P^F_{s\theta'} \subseteq \mathcal{A}_\nabla(F \restriction D, \theta', s) \subseteq P^F_{s\theta}.$$

In particular, if $\langle C_i \subset \Omega\colon i \in \mathbb{N}\rangle$ is an increasing sequence of finite subsets of Ω with $\lim_i r_\Omega(C_i) = 0$, then $\lim^*_{i,\theta} \mathcal{A}_\nabla(F \restriction C_i, \theta, s) = P^F_{s\theta} \cap \bigcup_i C_i$ for every $s \in C_1$.

COROLLARY 3.5. *The gradient based AFC algorithm $\mathcal{A}_\nabla$ represents the segmentation model $\mathcal{M}_\nabla$ for the class of all functions F from convex bounded open subsets of Ω of $\mathbb{R}^n$ into $\mathbb{R}^\ell$ which can be extended to a $\mathcal{C}^1$ function defined on an open set Ω containing* $\mathrm{cl}(\Omega)$.

3.2. Second algorithm representing model $\mathcal{M}_\nabla$: Malladi-Sethian-Vemuri level set algorithm $\mathcal{A}_{LS}$

In this subsection, we will *argue* that the level set algorithm $\mathcal{A}_{LS}$, which is essentially the fast marching algorithm described by Malladi, Sethian, and Vemuri in [14] (compare [21, Chapter 17]), also represents the segmentation model $\mathcal{M}_\nabla$ for the appropriate class of $\mathcal{C}^1$ functions $F\colon \Omega \to \mathbb{R}^\ell$. Thus, both algorithms $\mathcal{A}_\nabla$ and $\mathcal{A}_{LS}$ are model-equivalent. We use in the above a vague term "argue" rather than "proof," since the model and the algorithm presented in [14] (as well as in essentially all other papers describing this method, see e.g. books [4,17,18]) are in several aspects only sketched, leaving considerable leeway for interpretation. We will choose the interpretations that are the most favorable for our formalism, and only briefly discuss the problems with other interpretations. Moreover, the theoretical justification for the level set algorithm seems not to be fully completed yet. Since we are arguing here that a simple algorithm $\mathcal{A}_\nabla$ does the same job as its level set counterpart $\mathcal{A}_{LS}$ even under the assumption that all theoretical gaps for the level set theory can be patched, we feel no need for a completion of the level set theory framework.

The level set delineation model $\mathcal{M}_{LS}$ of the idealized image is described in terms different from the model $\mathcal{M}_\nabla$. Thus, we will start with its description. The model $\mathcal{M}_{LS}$ is applied to an ideal image $F\colon \Omega \to \mathbb{R}$, where Ω is an open convex bounded subset of $\mathbb{R}^n$. Basically, to use $\mathcal{M}_{LS}$ we pick a smooth simple closed surface Γ_0 (diffeomorphic with $(n-1)$-dimensional sphere) inside the region that is to be delineated—it plays the role of a seed—and then we let Γ_0 propagate outward until it reaches the boundary of the region we seek. The propagation is controlled by the speed function v which indicates at every point z on the front the speed $v(z)$ at which this point propagates in the direction normal to the front. The set of points inside the final position of the front represents the output of $\mathcal{M}_{LS}$. The front Γ_t at time $t \geq 0$ is represented as the zero level set $\{z \in \Omega\colon \Psi(z,t) = 0\}$ for some function $\Psi\colon \Omega \times [0,\infty) \to \mathbb{R}$. To make region R_t inside the front easier to identify, it is also assumed that Ψ is negative inside Γ_t and positive outside Γ_t. In other words, $R_t = \{z \in \Omega\colon \Psi(z,t) < 0\}$. In [14], the authors define Ψ at time $t = 0$ as a signed distance from Γ_0, that is, $\psi(z,0) = \mathrm{dist}(z,\Gamma_0)$ for z outside Γ_0, and $\psi(z,0) = -\mathrm{dist}(z,\Gamma_0)$ for z inside Γ_0.

The boundary of the object is defined as the set of points where the image intensity changes rapidly, that is, when the magnitude of the gradient $|\nabla F|$ is "large." To force the front propagation "...to stop in the vicinity of the desired objects' boundaries..." the propagation speed v is defined in such a way that v goes to zero precisely when $|\nabla F|$ approaches the "large" threshold value $\theta \in (0,\infty]$.[†] Neither "large value" θ nor formula for v is uniquely defined in [14]. Formulas (13) and (16) from [14] suggest that the speed should be reduced to zero at the points $z \in \Omega$ when $|\nabla F|(z)$ is equal to the maximum M of $|\nabla F|$ on Ω, which means that $\theta = M$. (The authors of [14] do not explain why such maximum should exist.) Alternatively, formulas (14), (15), and (17) from [14] suggest that the speed should be a product of a positive factor independent of F and a number of the form $(1+|\nabla F|)^{-1}$; that is, the propagation speed should go to zero only as $|\nabla F|$ goes to ∞, meaning that $\theta = \infty$. The first from these options suggests that $\mathcal{M}_{LS}(F,\Gamma_0)$ is equal to $\mathcal{M}_\nabla(F,M,\Gamma_0)$. The second makes $\mathcal{M}_{LS}(F,\Gamma_0)$ equal $\mathcal{M}_\nabla(F,\infty,\Gamma_0)$ which, for $\mathcal{C}^1$ function F, is equal to the entire Ω. To stop the algorithm associated with $\mathcal{M}_{LS}(F,\Gamma_0) = \mathcal{M}_\nabla(F,\infty,\Gamma_0)$, the authors arbitrarily introduce the maximum number of algorithm iterations (see [14, page 164]), which produces an algorithm completely inadequate for a theoretical analysis that we wish to attempt. Both of these approaches can be reconciled making the value of $\mathcal{M}_{LS}$ dependent on θ and reducing

[†]Quote comes from the first paragraph of [14, Section III]. A similar statement can be also found in [21, page 220].

the propagation speed v to 0, when $|\nabla F|$ reaches θ. For example, we may define $v(x) = (|\nabla F|(x) - \theta)^2$. Then, we define $\mathcal{M}_{LS}(F, \theta, \Gamma_0)$ as the set of all points of Ω that are eventually inside the propagating curve, that is, $\mathcal{M}_{LS}(F, \theta, \Gamma_0) = \bigcup_{t \geq 0} R_t$. This general setup allows us to relate models $\mathcal{M}_\nabla$ and $\mathcal{M}_{LS}$ as follows.

LEMMA 3.6. *$\mathcal{M}_{LS}(F, \theta, \Gamma_0) \subset \mathcal{M}_\nabla(F, \theta, \Gamma_0)$ for every $\mathcal{C}^1$ image $F\colon \Omega \to \mathbb{R}^\ell$, $\theta \in \mathbb{R}$, and smooth simple closed surface Γ_0 such that $\Gamma_0 \cup R_0 \subset \mathcal{M}_\nabla(F, \theta, \Gamma_0)$.*

The assumption $\Gamma_0 \cup R_0 \subset \mathcal{M}_\nabla(F, \theta, \Gamma_0)$ ensures that every point z of the initiation set $\Gamma_0 \cup R_0$ satisfies the thresholding condition $|\nabla F(z)| < \theta$. The proof of Lemma 3.6 is topological in nature. The other inclusion is also true, but its proof depends on some missing details concerning the definition of $\mathcal{M}_{LS}$. In particular, we need to clarify the meaning of front propagation, as described in [14]. For every $z \in \Gamma_0$, let $T_z\colon [0, \infty) \to \Omega$ be a trajectory of z propagated according to the rules described above. Then $\Psi(T_z(t), t) = 0$ for every z and t. So, its derivative $\frac{d}{dt}\Psi(T_z(t), t) = 0$ is also equal to 0. By using chain rule, it is easy to transform this last equation (see [14] or [21]) to $\frac{\partial \Psi}{\partial t}(T_z(t), t) + v(T_z(t)) \cdot |\nabla \Psi|\,(T_z(t), t) = 0$, where the gradient $\nabla \Psi$ concerns only spatial variables. In particular, any solution of the PDE

$$\frac{\partial \Psi}{\partial t}(x, t) + v(x) \cdot |\nabla \Psi|\,(x, t) = 0, \quad x \in \Omega, \quad t \geq 0 \tag{4}$$

with the initial condition $\Psi(\cdot, 0) = \Psi_0$ leads to the unique front propagation and the model $\mathcal{M}_{LS}$.

Unfortunately, even in very simple cases, (4) does not need to have a smooth solution. (See e.g. [21].) Thus, the authors in [14] consider its weak solution, which satisfies (4) only at the points of differentiability of Ψ. This is good enough, but there are some difficulties. First, in general, the weak solution of (4) does not need to be unique. However, its *viscosity solution*, introduced by Crandall and Lions [10], is unique and this is the solution chosen in [14]. The viscosity solution of (4) is defined in [14] as a limit $\Psi = \lim_{\varepsilon \to 0^+} \Psi_\varepsilon$, where Ψ_ε is a solution of

$$\frac{\partial \Psi}{\partial t} + (1 - \varepsilon \mathcal{K}) v \cdot |\nabla \Psi| = 0, \quad \Psi(\cdot, 0) = \Psi_0, \tag{5}$$

where $\mathcal{K} = \nabla \cdot \frac{\Psi}{|\nabla \Psi|}$ is the curvature of the level surface. The theoretical value of this approach is based on the following two claims, which were not proved in [14] and which we were unable to locate in the literature in that generality, despite the intensive search and a correspondence with professor Sethian.

(I) PDE (5) has a global smooth solution for smooth Γ_0 and v.

The existence of such a solution is known for some simple speed functions v, but we were not able to find it in that generality required for most imaging tasks. Note that the algorithm from [14] requires only the existence of the solution locally, near the front.

(II) The solutions Ψ_ε of (5) converge to the viscosity solution for (4).

A proof of existence of the viscosity solution for (4) can be found in [1], although the solution is not described as a limit from (II). (Paper [10] contains a proof of uniqueness of the viscosity solution in a general setting, but not the existence.) In what follows we will assume that (I) and (II) are true. With their help we can prove the equality between models $\mathcal{M}_\nabla$ and $\mathcal{M}_{LS}$.

THEOREM 3.7. *$\mathcal{M}_{LS}(F, \theta, \Gamma_0) = \mathcal{M}_\nabla(F, \theta, \Gamma_0)$ for every $\mathcal{C}^1$ image $F\colon \Omega \to \mathbb{R}^\ell$, $\theta \in \mathbb{R}$, and smooth simple closed surface Γ_0 such that $\Gamma_0 \cup R_0 \subset \mathcal{M}_\nabla(F, \theta, \Gamma_0)$.*

The delineation algorithm $\mathcal{A}_{LS}$ described in [14] depends on $\varepsilon > 0$ and finds its value from a numerical approximation for Ψ_ε. In order to prove formally that $\mathcal{A}_{LS}$ weakly represents $\mathcal{M}_{LS}$, we should first show that,

(III) $\mathcal{A}^\varepsilon_{LS}$ weakly represents $\mathcal{M}^\varepsilon_{LS}$ for the class $\mathcal{F}$ of C^1 images with uniformly continuous gradient

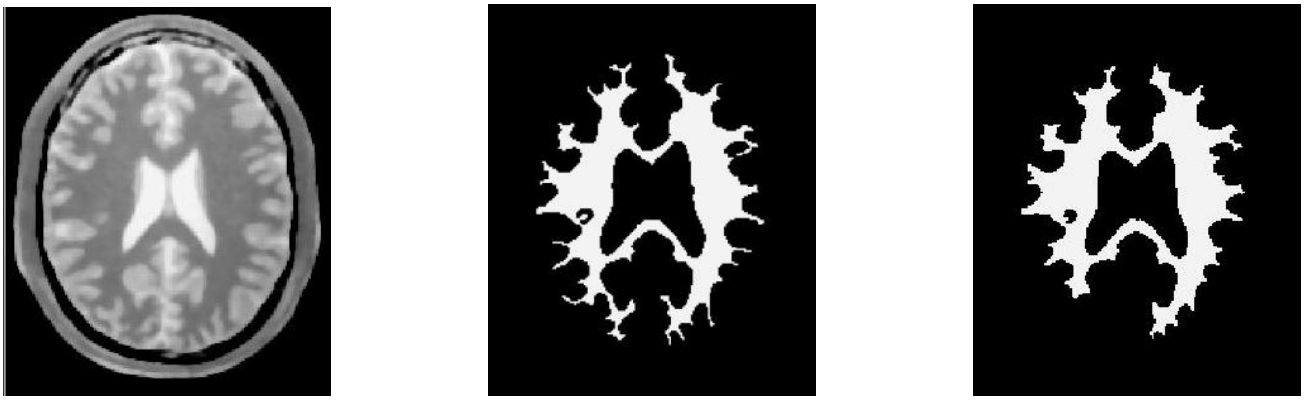
in a sense that for every appropriate $F\colon \Omega \to \mathbb{R}^\ell$, Γ_0, and $\theta, \varepsilon, h > 0$ the limit $\lim^*_{i,\eta} \mathcal{A}^\varepsilon_{LS}(F \restriction \Omega_{h/2^i}, \theta - \eta, \Gamma_0)$ exists and is a dense subset of the idealized model $\mathcal{M}^\varepsilon_{LS}(F, \theta, \Gamma_0)$ outcome $\{z \in \Omega\colon \psi_\varepsilon(z, t) < 0 \text{ for some } t \geq 0\}$. However, once again, we will assume that (III) is true, without proving it. (This fact is the foundation for the entire level set theory. However, we found no proof for it in the published literature.) Notice that the property (II) and Theorem 3.7 imply that $\lim_{\varepsilon \to 0^+} \mathcal{M}^\varepsilon_{LS} = \mathcal{M}_{LS}$. Therefore, if we define $\mathcal{A}^{\varepsilon,\eta}_{LS}(f, \theta, \Gamma_0) = \mathcal{A}^\varepsilon_{LS}(f, \theta - \eta, \Gamma_0)$, then we have $\lim^\dagger_{i,\eta,\varepsilon} \mathcal{A}^{\varepsilon,\eta}_{LS}(F \restriction \Omega_{h/2^i}, \theta, \Gamma_0) = \lim_{\varepsilon \to 0^+} \mathcal{M}^\varepsilon_{LS}(F, \theta, \Gamma_0) = \mathcal{M}_{LS}(F, \theta, \Gamma_0)$ for appropriate F, Γ_0, and θ. In particular, $\mathcal{A}_{LS}$ *weakly represents* $\mathcal{M}_{LS}$. This can be rephrased as follows.

COROLLARY 3.8. *Algorithms $\mathcal{A}_\nabla$ and $\mathcal{A}_{LS}$ are weakly model-equivalent in the class $\mathcal{F}$ of all $\mathcal{C}^1$ images $F\colon \Omega \to \mathbb{R}^\ell$ having uniformly continuous gradient and such that $\Omega \subset \mathbb{R}^n$ is bounded, open, and convex. In this class, both these algorithms represent model $\mathcal{M}_\nabla = \mathcal{M}_{LS}$.*

4. EXPERIMENT

Having proved their model equivalence, we wanted to examine how this equivalence is manifested in actual image segmentation by using $\mathcal{A}_\nabla$ and $\mathcal{A}_{LS}$. So, we compared algorithms $\mathcal{A}_\nabla$ and $\mathcal{A}_{LS}$ at the experimental level. The goal in this paper is not really a formal practical evaluation of the segmentation. Therefore, we provide practical qualitative example illustrating the stronger theoretical results.

We applied the algorithms to a 2D proton density weighted brain MR image, Fig. 1(a), to delineate the white matter object. The image had 20% background non uniformity and 3% noise. The level set results, displayed in Fig. 1(c), were obtained with a version of the algorithm $\mathcal{A}_{LS}$ implemented in the open source software ITK [28]. This algorithm has four steps: (i) it applies a Gaussian filter to the original image; (ii) it calculates gradient magnitude of the filtered image; and (iii) it applies to this image f a non-linear filter $\hat{f}(c) = (Max - Min) \cdot \left(1 + e^{-(f(c)-\beta)/\alpha}\right) + Min$, where Min and Max are the minimum and the maximum of the input image f, respectively, and the default parameter values, that we used, were $\alpha = -0.3$ and $\beta = 2$. To this modified image $\hat{f}$ the curve propagation step is applied. The results of the application of the fuzzy connectedness algorithm $\mathcal{A}_\nabla$ to the same image $\hat{f}$ are presented at Fig. 1(b). To make the comparison fair, we calculated the path connectivity strength from the filtered gradient image $\hat{f}$ from step (iii) described above, rather than from the original gradient magnitude image f. The subtle differences seen in the delineated objects are due to different approximations involved in the otherwise equivalent algorithms. In particular, some object areas accessible from its main part by narrow passages (e.g., lower left) are correctly recognized by the AFC algorithm $\mathcal{A}_\nabla$ but not by the LS algorithm $\mathcal{A}_{LS}$. This is due to the fact that $\mathcal{A}_{LS}$ propagates by using a curvature factor $\varepsilon\mathcal{K}$. The constant ε must approach 0 to get the same result as the algorithm $\mathcal{A}_\nabla$. However, at the resolution of the provided image (pixel size $= 1 \times 1mm^2$), decreasing ε further does not produce the desirable results.



(a) (b) (c)

Fig. 1. The white matter (the region with darker intensities) in a 2D proton density weighted brain MR image (left) delineated with $\mathcal{A}_\nabla$ (center) and $\mathcal{A}_{LS}$ (right) algorithms.

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