AFFINITY FUNCTIONS IN FUZZY CONNECTEDNESS BASED IMAGE SEGMENTATION

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Affinity functions in fuzzy connectedness based image segmentation

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***Abstract*— Fuzzy connectedness (FC) constitutes an important class of image segmentation schemas. Although affinity functions represent the fundamental aspect (main variability parameter) of FC algorithms, they have not been studied systematically in the literature. In this paper, we present a through study to fill this gap. Our analysis is based on the notion of equivalent affinities: if any two equivalent affinities are used in the same FC schema to produce two versions of the algorithm, then these algorithms are strongly equivalent in the sense that they lead to identical segmentations. We give a complete characterization of the affinity equivalence and show that many natural parameters used in the definitions of affinity functions are redundant in the sense that different values of such parameters lead to equivalent affinities. We also show that two main affinity types, homogeneity based and object feature based, are equivalent, respectively, to the difference quotient of the intensity function and Rosenfeld's degree of connectivity. In addition, we note that any segmentation obtained via relative fuzzy connectedness (RFC) algorithm can be viewed as segmentation obtained via absolute fuzzy connectedness (AFC) algorithm with an automatic threshold detection. We finish with theoretical and experimental analysis of possible ways of combining different affinities.**

I. INTRODUCTION

Image segmentation — the process of partitioning the image domain into meaningful object regions — is perhaps the most challenging and critical problem in image processing and analysis. Research in this area will probably continue indefinitely long because the solution space is infinite dimensional, and since any single solution framework is unlikely to produce an optimal solution (in the sense of the best possible precision, accuracy, and efficiency) for all possible application domains. It is important to distinguish between two types of activities in segmentation research — the first relating to the development of application domain-independent general solution frameworks, and the second pertaining to the construction of domain-specific solutions starting from a known general solution framework. The latter is not a trivial task most of the time. Both these activities are crucial, the former for advancing the theoretical aspects of, and shedding new light on, segmentation research, and the latter for bringing the theoretical advances to actual practice. The topic of this paper touches both of these activities.

General segmentation frameworks [1]–[12] may be broadly classified into three groups: boundary-based [1]–[5], region-based [6]–[10], and hybrid [11], [12]. As the nomenclature indicates, in the first two groups the focus is on recognizing and delineating the boundary or the region occupied by the object in the image. In the third group, the focus is on exploiting the complementary strengths of each of boundary-based and region-based strategies to overcome their individual shortcomings. The segmentation framework discussed in the present paper belongs to the region-based group and constitutes an extension of the fuzzy connectedness (abbreviated from now on as FC) methodology [8].

In the FC framework [8], a fuzzy topological construct, called fuzzy connectedness, characterizes how the spatial elements (abbreviated as *spels*) of an image hang together to form an object. This construct is arrived at roughly as follows. A function called *affinity* is defined on the image domain; the strength of affinity between any two spels depends on how close the spels are spatially and how similar their intensity-based properties are in the image. Affinity is intended to be a local relation. A global fuzzy relation called *fuzzy connectedness* is induced on the image domain by affinity as follows. For any two spels c and d in the image domain, all possible paths connecting c and d are considered. Each path is assigned a strength of fuzzy connectedness which is simply the minimum of the affinities of consecutive spels along the path. The level of fuzzy connectedness between c and d is considered to be the maximum of the strengths of all paths between c and d. For segmentation purposes, FC is utilized in several ways as described below. (Compare also Section II-C.) See [13] for a review of the different FC definitions and how they are employed in segmentation and applications.

In *absolute FC* (abbreviated AFC) [8], the support of a segmented object is considered to be the maximal set of spels, containing one or more seed spels, within which the level of FC is at or above a specific threshold. To obviate the need for a threshold, *relative FC* (or RFC) [14] was developed by letting all objects in the image to compete simultaneously via FC to claim membership of spels in their sets. Each co-object is identified by one or more seed spels. Any spel c in the image domain is claimed by that co-object with respect to whose seed spels c has the largest level of FC compared to the level of FC with the seed sets of all other objects. To avoid treating the core aspects of an object (that are very strongly connected to its seeds) and the peripheral subtle aspects (that may be less strongly connected to the seeds) in the same footing, an iterative refinement strategy is devised in *iterative RFC* (or IRFC) [15]–[18]. This has been shown to lead to better object definition than RFC with a theoretical construct similar to that of RFC. The proper design of affinity is crucial to the effectiveness of the segmentations that ensue, no matter what type of FC is used. In *scale-based* [13] and *vectorial FC* [19],

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which are applicable to all of AFC, RFC, and IRFC, affinity definition is not based just on the scalar properties of the two spels under question but also on the vectorial properties of all spels in the local scale region around the two spels. The FC family of methods developed to date [13]–[24] consists of various combinations of absolute, relative, and iterative FC with scale-based and vectorial versions.

The fundamental construct in any FC method is the affinity function. Its choice determines the effectiveness of the particular FC method. In the published literature on FC, affinity functions have not been studied in depth, leaving open several basic questions relating to their form, parameters, and effectiveness. In the present paper, we address these issues in a fundamental manner and make two sets of fundamental contributions. (1) We define the notion of equivalent affinities (Section II-B), and prove that if any two equivalent affinities are used in the same FC schema to produce two versions of the algorithm, then these algorithms are strongly equivalent in the sense that they lead to identical segmentations when applied to any digital image initialized with the same seeds (Section II-C). The notion of strong equivalence of algorithms is at the foundation of our more general study of the equivalences among segmentation algorithms, the theory of which we initiated in [25]. (2) We present a detailed discussion on how the two most commonly used affinities, homogeneity based (Section III-A) and object feature based (Section III-B), should be properly and intuitively defined. By using these results we discuss (in Section IV) the possible ways of combining two or more different affinities into a single affinity and examine which parameters in the definitions of the combined affinities are redundant. In the process, we present a host of new affinity functions and demonstrate that they may lead to better segmentations. We also show (in Section II-D) that the RFC segmentation can be viewed to some extent as an AFC segmentation with an automatic threshold selection.

II. AFFINITIES EQUIVALENT IN THE FC SENSE

The main purpose of this section is to uncover the essence of the relationship between the local measure of connectedness of pairs of spels, the affinity function, and the resulting segmentations obtained via FC algorithms. In particular, we will introduce the notion of the *equivalence (in the sense of FC) of the affinities* and show that equivalent affinities are indistinguishable from the point of view of FC segmentations.

To make this work complete and useful, our definition of the affinity function will be more general than the one commonly used in the literature. However, we will show that each class of equivalent affinities contains at least one standard (meaning commonly used) affinity.

A. Preliminary definitions

We will use the following interpretation of the notions of (hard) functions and relations, which is standard in set theory (see e.g. [26], [27]) and is used in many calculus books. A binary relation R from a set X to a set Y is identified with its graph; that is, R *equals* $\{\langle x,y\rangle \in X\times Y\colon xRy \text{ holds}\}$. Since a function $f\colon X\to Y$ is a (special) binary relation from X to Y, in particular we have $f=\{\langle x,f(x)\rangle\colon x\in X\}$. With this interpretation, fuzzy sets and fuzzy relations have the following representations. Let $\mathcal{Z}$ be a *fuzzy subset* of a hard set X with a membership function $\mu_{\mathcal{Z}}\colon X\to[0,1]$. For each $x\in X$ we interpret $\mu_{\mathcal{Z}}(x)$ as the degree to which this x belongs to $\mathcal{Z}$. Usually such a fuzzy set $\mathcal{Z}$ is defined as $\{\langle x,\mu_{\mathcal{Z}}(x)\rangle\colon x\in X\}$, which is the graph of $\mu_{\mathcal{Z}}$. Thus, according to our interpretation, $\mathcal{Z}$ actually equals $\mu_{\mathcal{Z}}$. Note that this interpretation agrees quite well with the situation when $\mathcal{Z}$ is a hard subset Z of X, as then $\mathcal{Z}=\mu_{\mathcal{Z}}$ is equal to the *characteristic function* χ_Z of Z (defined as $\chi_Z(x)=1$ for $x\in Z$ and $\chi_Z(x)=0$ for $x\in X\setminus Z$), and the identification of Z with χ_Z is quite common in analysis and set theory. Notice also that a fuzzy binary relation ρ from X to Y is just a fuzzy subset of $X\times Y$, so it is equal to its membership function $\mu_\rho\colon X\times Y\to[0,1]$.

Let $\mathfrak{n}\ge 2$ and let $\mathbb{Z}^{\mathfrak{n}}$ stand for the set of all $\mathfrak{n}$-tuples of integer numbers. A binary fuzzy relation α on $\mathbb{Z}^{\mathfrak{n}}$ is said to be a *fuzzy adjacency* binary relation if α is *symmetric* (i.e., $\alpha(c,d)=\alpha(d,c)$) and *reflexive* (i.e., $\alpha(c,c)=1$). The value of $\alpha(c,d)$ depends only on the relative spatial position of c and d. Usually $\alpha(c,d)$ is decreasing with respect to the distance function $||c-d||$. In most applications, α is just a hard case relation like 4-adjacency relation for $\mathfrak{n}=2$ or 6-adjacency in the three-dimensional case. By an $\mathfrak{n}$*-dimensional fuzzy digital space* we will understand a pair $\langle\mathbb{Z}^{\mathfrak{n}},\alpha\rangle$. The elements of the digital space are called *spels*. (For $\mathfrak{n}=2$ also called *pixels*, while for $\mathfrak{n}=3$ – *voxels*.)

Let $k\ge 1$. A *scene* over a fuzzy digital space $\langle\mathbb{Z}^{\mathfrak{n}},\alpha\rangle$ is a pair $\mathcal{C}=\langle C,f\rangle$, where $C=\prod_{j=1}^{\mathfrak{n}}[-b_j,b_j]\subset\mathbb{Z}^{\mathfrak{n}}$, each $b_j>0$ being an integer, and $f\colon C\to\mathbb{R}^k$ is a *scene intensity function*. The value of f represents either the original acquired image intensity or an estimate of certain image properties (such as gradients and texture measures) obtained from the given image. The notion most important for this paper is that of an affinity function. The affinity function, defined in its general form in the next subsection, is usually denoted by κ and it associates to any pair $\langle c,d\rangle\in C^2$ of spels the strength $\kappa(c,d)$ of their local hanging togetherness in $\mathcal{C}$. Within this class, a special role is played by *standard affinities*, that is, mappings $\kappa\colon C\times C\to[0,1]$ which, treated as fuzzy binary relations, are symmetric and reflexive. In all practical applications, the value of $\kappa(c,d)$ depends on the adjacency strength $\alpha(c,d)$ of c and d (i.e., on the spatial relative position of c and d) as well as on the intensity function f. So far, only standard affinities have been used in applications in the literature. The exception is [17] wherein an asymmetric affinity was employed. It has been demonstrated [14] that to fulfill certain desirable properties of FC segmentations (such as robustness with respect to seed points), affinities must be symmetric. In this paper, therefore, we will restrict ourselves to symmetric affinities. However, we will go quite afar from previous publications otherwise in considering affinity in its very general form.

B. Equivalent affinities

In this subsection we define the notion of the affinity function in its general form and introduce the concept of equivalent affinities. The motivation for developing equivalent affinities comes from our desire to recognize those differences among affinities that are inessential and therefore lead to the same FC segmentations from those that are essential and may give rise to different segmentations.

Let $\preceq$ be a linear order relation on a set L and let C be an arbitrary finite non-empty set. We say that a function $\kappa\colon C \times C \to L$ is an *affinity function* (from C into $\langle L, \preceq\rangle$) provided κ is symmetric (i.e., $\kappa(a,b) = \kappa(b,a)$ for every $a, b \in C$) and $\kappa(a,b) \preceq \kappa(c,c)$ for every $a,b,c \in C$. Clearly, any standard affinity, as defined above, is an affinity function with $\langle L, \preceq\rangle = \langle [0,1], \leq\rangle$. Note that $\kappa(d,d) \preceq \kappa(c,c)$ for every $c, d \in C$. So, there exists an element in L, which we will denote by a symbol $\mathbf{1}_\kappa$, such that $\kappa(c,c) = \mathbf{1}_\kappa$ for every $c \in C$. Notice that $\mathbf{1}_\kappa$ is the largest element of $L_\kappa = \{\kappa(a,b)\colon a,b \in C\}$, although it does not need to be the largest element of L. In what follows, the strict inequality related to $\preceq$ will be denoted by $\prec$, that is, $a \prec b$ if and only if $a \preceq b$ and $a \neq b$.

Certainly, in image processing, C will be always the domain of the scene intensity function. In all specific cases used in this paper, we will take $\langle L, \preceq\rangle$ as either the standard range $\langle [0,1], \leq\rangle$ or, more often, $\langle [0,\infty], \geq\rangle$. Note that, in this second case, the order relation $\preceq$ is the *reversed* standard order relation $\geq$. We say that the affinities $\kappa_1\colon C \times C \to \langle L_1, \preceq_1\rangle$ and $\kappa_2\colon C \times C \to \langle L_2, \preceq_2\rangle$ *equivalent (in the FC sense)* provided, for every $a,b,c,d \in C$

$$\kappa_1(a,b) \preceq_1 \kappa_1(c,d) \quad \text{if and only if} \quad \kappa_2(a,b) \preceq_2 \kappa_2(c,d)$$

or, equivalently,

$$\kappa_1(a,b) \prec_1 \kappa_1(c,d) \quad \text{if and only if} \quad \kappa_2(a,b) \prec_2 \kappa_2(c,d).$$

Equivalent affinities can be characterized as follows, where $\circ$ stands for the composition of functions, that is, $(g \circ \kappa_1)(a,b) = g(\kappa_1(a,b))$.

Proposition 1: Affinities $\kappa_1\colon C \times C \to \langle L_1, \preceq_1\rangle$ *and* $\kappa_2\colon C \times C \to \langle L_2, \preceq_2\rangle$ *are equivalent if and only if there exists a strictly increasing function* g *from* $\langle L_{\kappa_1}, \preceq_1\rangle$ *onto* $\langle L_{\kappa_2}, \preceq_2\rangle$ *such that* $\kappa_2 = g \circ \kappa_1$.

PROOF. If κ_1 and κ_2 are equivalent, define g by putting $g(\kappa_1(a,b)) = \kappa_2(a,b)$ for every $a,b \in C$. Note that g is well defined, since $\kappa_1(a,b) = \kappa_1(c,d)$ implies that $\kappa_2(a,b) = \kappa_2(c,d)$. Also, inequality $\kappa_1(a,b) \preceq_1 \kappa_1(c,d)$ implies that $\kappa_2(a,b) \preceq_2 \kappa_2(c,d)$, so g is a strictly increasing map from L_{κ_1} onto L_{κ_2}. Conversely, if $\kappa_2 = g \circ \kappa_1$, where g is strictly increasing, then κ_1 is equivalent to κ_2 since for every $a,b,c,d \in C$ we have: $\kappa_2(a,b) \preceq_2 \kappa_2(c,d) \Leftrightarrow g(\kappa_1(a,b)) \preceq_2 g(\kappa_1(c,d)) \Leftrightarrow \kappa_1(a,b) \preceq_1 \kappa_1(c,d)$. ■

One of the specific conclusions from Proposition 1 is the following fact.

Corollary 2: If $\kappa\colon C \times C \to \langle [0,\infty], \geq\rangle$ *is an affinity, then, for every strictly decreasing function* g *from* $[0,\infty]$ *onto* $[0,1]$, *a map* $g \circ \kappa\colon C \times C \to \langle [0,1], \leq\rangle$ *is an affinity equivalent to* κ.

Our interest in equivalent affinities comes from the fact (see Theorem 4) that any FC segmentation of a scene $\mathcal{C}$ remains unchanged if an affinity on C used to get the segmentation is replaced by an equivalent affinity. Keeping this in mind, it makes sense to find for each affinity function an equivalent affinity in a nice form:

Theorem 3: Every affinity function is equivalent (in the FC sense) to a standard affinity.

PROOF. Let $\kappa\colon C \times C \to \langle L, \preceq\rangle$ be an arbitrary affinity. Note that there is a strictly increasing function $g\colon L_\kappa \to [0,1]$ with $g(\mathbf{1}_\kappa) = 1$. (If $L_\kappa = \{l_1, \ldots, l_m\}$ with $l_1 = \mathbf{1}_\kappa$, then such a g can be constructed by an easy induction on m.) Let $\kappa_2(c,d) = g(\kappa(c,d))$ for every $c,d \in C$. Then, by Proposition 1, κ is equivalent to the standard affinity $\kappa_2\colon C \times C \to \langle [0,1], \leq\rangle$. ■

Once we agree that equivalent affinities lead to the same segmentations, Theorem 3 says that we can restrict our attention to standard affinities without losing any generality of our method. Thus, one may wonder why study other affinities at all. The answer to this question is simple — in most cases, it is more natural to define an affinity function with more abstract range, and any translation of such affinity to the standard one is a redundant step adding only unnecessary computational burden. Moreover, in some of these cases there is no simple (i.e., continuous) translation of the natural affinity to the standard one. (See Example 10.) On the other hand, Theorems 3 and 4 tell us that all the theoretical results that are true for the standard affinities hold also for the affinities as we defined them. Thus, there is no particular reason to restrict our attention to the affinities in the standard form.

We will discuss several examples of equivalent affinities in Section III. For now, as an example, consider the so called homogeneity based affinity [30]. Its natural form for a scene $\mathcal{C} = \langle C, f\rangle$ is a function $\psi\colon C \times C \to \langle [0,\infty, \geq]$ defined in (3), given by $\psi(c,d) = |f(c) - f(d)|$ for adjacent spels $c, d \in C$ and $\psi(c,d) = \infty$ otherwise. The more commonly used version of the homogeneity based affinity [30] is the standard affinity $\psi_\sigma(c,d) = e^{-\psi(c,d)^2/\sigma^2}$, which is the composition of ψ with the Gaussian function $g_\sigma(x) = e^{-x^2/\sigma^2}$. Note that, by Corollary 2, ψ and ψ_σ are equivalent, independently of the value of the parameter σ, since g_σ is strictly decreasing from $[0,\infty]$ onto $[0,1]$. In particular, the parameter σ in the definition of ψ_σ is non-essential from the FC segmentation point of view.

C. FC segmentations for equivalent affinities

Fix an affinity $\kappa\colon C \times C \to \langle L, \preceq\rangle$. To define fuzzy connectedness segmentation of $\mathcal{C}$, we need first to translate the local measure of connectedness given by κ into the global strength of connectedness. For this, we will need the notions of a path and its strength. A *path in* $A \subseteq C$ is any sequence[1] $p = \langle c_1, \ldots, c_l\rangle$, where $l > 1$ and $c_i \in A$ for every $i = 1, \ldots, l$. (Notice that there is no assumption on any adjacency of the consecutive spels in a path.) The family of all paths in A is denoted by $\mathcal{P}^A$. If $c, d \in A$, then the family of all paths $\langle c_1, \ldots, c_l\rangle$ in A from c to d (i.e., such that $c_1 = c$ and $c_l = d$) is denoted by $\mathcal{P}^A_{cd}$. The *strength* $\mu_\kappa(p)$ *of a path* $p = \langle c_1, \ldots, c_l\rangle \in \mathcal{P}^C$ is defined as the strength of its κ-weakest link; that is, $\mu_\kappa(p) \stackrel{\text{def}}{=} \min\{\kappa(c_{i-1}, c_i)\colon 1 < i \leq l\}$.

[1]Notice that the paths must have length greater than 1. We make this requirement to ease some technical difficulties, while it creates no real restriction as, in whatever we do, a "path" $\langle c\rangle$ can be always replaced by a path $\langle c, c\rangle$.

(Note that, if one follows the common practice of defining $\kappa(c,d)$ to be the minimal element of L_κ for any non-adjacent c and d, then only paths with adjacent consecutive spels can have non-minimal strength.) For $c,d \in A \subseteq C$, the (global) *κ-connectedness strength in A between c and d* is defined as the strength of a strongest path in A between c and d; that is,

$$\mu_\kappa^A(c,d) \stackrel{\text{def}}{=} \max\left\{\mu_\kappa(p)\colon p \in \mathcal{P}_{cd}^A\right\}. \tag{1}$$

Notice that $\mu_\kappa^A(c,c) = \mu_\kappa(\langle c,c\rangle) = \mathbf{1}_\kappa$. We will often refer to function $\mu_\kappa^A\colon C \times C \to L$ as a *connectivity measure* (on A) induced by κ. For $c \in A \subset C$ and a non-empty $D \subset A$, we also define $\mu_\kappa^A(c,D) \stackrel{\text{def}}{=} \max_{d\in D}\mu_\kappa^A(c,d)$. We will write μ for μ_κ and μ^A for μ_κ^A when κ is clear from the context. The issue of why μ_κ^A should be defined from κ by the procedure described above is discussed in detail in [28]. Note that if κ is a hard binary relation (i.e., when $L = \{0,1\}$), then μ_κ^C is a relation (or, more precisely, its characteristic function) known as a *transitive closure of* κ, which is defined as the set of all pairs $\langle c,d\rangle \in C \times C$ for which there exists a sequence $c = c_0, c_1, \ldots, c_m = d$ such that $\kappa(c_i, c_{i+1}) = 1$ for every $i < m$.

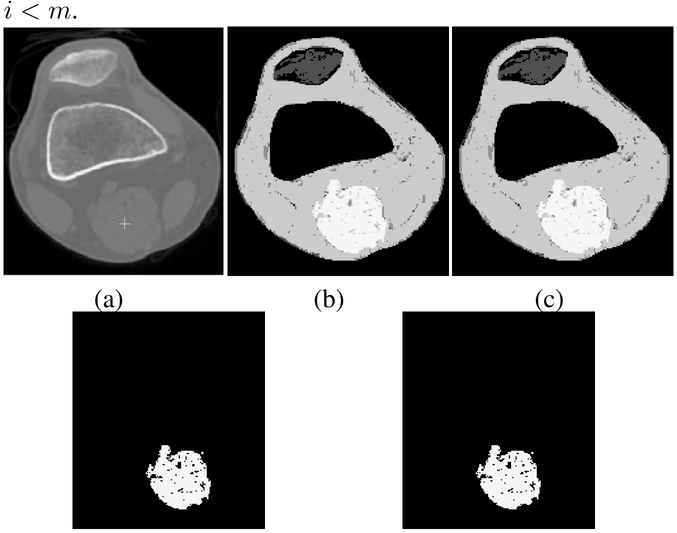

Fig. 1. Illustration of equivalent affinities. (a) A 2D scene — a CT slice of a human knee. (b), (c) Connectivity scenes corresponding to affinities ψ and ψ_σ ($\sigma = 1$), respectively, and the same seed spel (indicated by + in (a)) specified in a soft tissue region of the scene in (a). (d), (e) Identical AFC objects obtained from the scenes in (b) and (c), respectively.

To define fuzzy objects delineated by FC segmentations, we start with a family $\mathcal{S}$ of non-empty pairwise disjoint subsets of C, where each $S \in \mathcal{S}$ represents a set of spels, known as *seeds*, which will belong to the object generated by it. Also, fix a threshold $\theta \in L$, $\theta \le \mathbf{1}_\kappa$. For every $S \in \mathcal{S}$, put $W = \bigcup(\mathcal{S} \setminus \{S\})$ and, similarly as in [18] (see also [29]), define

- $P_{S\theta}^\kappa = \left\{c \in C\colon \theta \preceq \mu_\kappa^C(c,S)\right\}$;
- $P_{S\mathcal{S}}^\kappa = \left\{c \in C\colon \mu_\kappa^C(c,W) \prec \mu_\kappa^C(c,S)\right\}$;
- $P_{S\mathcal{S}}^{I\kappa} = \bigcup_{i=0}^\infty P_{S\mathcal{S}}^{i,\kappa}$, where sets $P_{S\mathcal{S}}^{i,\kappa}$ are defined inductively by the formulas $P_{S\mathcal{S}}^{0,\kappa} = \emptyset$ and $P_{S\mathcal{S}}^{i+1,\kappa} = P_{S\mathcal{S}}^{i,\kappa} \cup \{c \in C \setminus P_{S\mathcal{S}}^{i,\kappa}\colon \mu_\kappa^{C\setminus P_{S\mathcal{S}}^{i,\kappa}}(c,W) \prec \mu_\kappa^C(c,S)\}$.

Then AFC, RFC, and IRFC segmentations of C are defined, respectively, as $\mathbb{P}_\kappa^\theta(\mathcal{S}) = \{P_{S\theta}^\kappa\colon S \in \mathcal{S}\}$, $\mathbb{P}_\kappa(\mathcal{S}) = \{P_{S\mathcal{S}}^\kappa\colon S \in \mathcal{S}\}$, and $\mathbb{P}_\kappa^I(\mathcal{S}) = \left\{P_{S\mathcal{S}}^{I\kappa}\colon S \in \mathcal{S}\right\}$. Now we can formalize our previous claim that the fuzzy connectedness segmentations (i.e., those obtained via AFC, RFC, and IRFC algorithms) are unchanged if an affinity function is replaced by an equivalent one.

Theorem 4: *Let $\kappa_1\colon C \times C \to \langle L_1, \preceq_1\rangle$ and $\kappa_2\colon C \times C \to \langle L_2, \preceq_2\rangle$ be equivalent affinity functions and let $\mathcal{S}$ be a family of non-empty pairwise disjoint subsets of C. Then for every $\theta_1 \preceq_1 \mathbf{1}_{\kappa_1}$ in L_1, there exists a $\theta_2 \preceq_2 \mathbf{1}_{\kappa_2}$ in L_2 such that, for every $S \in \mathcal{S}$ and $i \in \{0,1,2,\ldots\}$, we have $P_{S\theta_1}^{\kappa_1} = P_{S\theta_2}^{\kappa_2}$, $P_{S\mathcal{S}}^{\kappa_1} = P_{S\mathcal{S}}^{\kappa_2}$, and $P_{S\mathcal{S}}^{i,\kappa_1} = P_{S\mathcal{S}}^{i,\kappa_2}$. In particular, $\mathbb{P}_{\kappa_1}^{\theta_1}(\mathcal{S}) = \mathbb{P}_{\kappa_2}^{\theta_2}(\mathcal{S})$, $\mathbb{P}_{\kappa_1}(\mathcal{S}) = \mathbb{P}_{\kappa_2}(\mathcal{S})$, and $\mathbb{P}_{\kappa_1}^I(\mathcal{S}) = \mathbb{P}_{\kappa_2}^I(\mathcal{S})$.*

PROOF. First note that, for any paths $p = \langle c_1, \ldots, c_l\rangle$ and $q = \langle d_1, \ldots, d_m\rangle$ from $\mathcal{P}^C$, we have

$$\begin{aligned} &\mu_{\kappa_1}(p) \preceq_1 \mu_{\kappa_1}(q)\\ &\quad\Leftrightarrow\quad (\forall j)\ (\exists i)\ \kappa_1(c_{i-1},c_i) \preceq_1 \kappa_1(d_{j-1},d_j)\\ &\quad\Leftrightarrow\quad (\forall j)\ (\exists i)\ \kappa_2(c_{i-1},c_i) \preceq_2 \kappa_2(d_{j-1},d_j)\\ &\quad\Leftrightarrow\quad \mu_{\kappa_2}(p) \preceq_2 \mu_{\kappa_2}(q). \end{aligned}$$

Similarly, for every $a,c \in A \subseteq C$ and $b,d \in B \subseteq C$, we have

$$\begin{aligned} &\mu_{\kappa_1}^A(a,c) \preceq_1 \mu_{\kappa_1}^B(b,d)\\ &\quad\Leftrightarrow\quad \left(\forall p \in \mathcal{P}_{ac}^A\right)\ \left(\exists q \in \mathcal{P}_{bd}^B\right)\ \mu_{\kappa_1}(p) \preceq_1 \mu_{\kappa_1}(q)\\ &\quad\Leftrightarrow\quad \left(\forall p \in \mathcal{P}_{ac}^A\right)\ \left(\exists q \in \mathcal{P}_{bd}^B\right)\ \mu_{\kappa_2}(p) \preceq_2 \mu_{\kappa_2}(q)\\ &\quad\Leftrightarrow\quad \mu_{\kappa_2}^A(a,c) \preceq_2 \mu_{\kappa_2}^B(b,d). \end{aligned}$$

If, in addition, $\emptyset \ne W \subseteq A$ and $S \subseteq B$, then also

$$\begin{aligned} &\mu_{\kappa_1}^A(a,W) \preceq_1 \mu_{\kappa_1}^B(b,S)\\ &\quad\Leftrightarrow\quad (\forall c \in W)\ (\exists d \in S)\ \mu_{\kappa_1}^A(a,c) \preceq_1 \mu_{\kappa_1}^B(b,d)\\ &\quad\Leftrightarrow\quad (\forall c \in W)\ (\exists d \in S)\ \mu_{\kappa_2}^A(a,c) \preceq_2 \mu_{\kappa_2}^B(b,d)\\ &\quad\Leftrightarrow\quad \mu_{\kappa_2}^A(a,W) \preceq_2 \mu_{\kappa_2}^B(b,S). \end{aligned} \tag{2}$$

Let $a,b \in C$ be such that $\kappa_1(a,b) = \min_{\theta_1 \preceq_1 \kappa_1(x,y)} \kappa_1(x,y)$ and put $\theta_2 = \kappa_2(a,b)$. Then $P_{S\theta_1}^{\kappa_1} = \left\{c\colon \theta_1 \preceq_1 \mu_{\kappa_1}^C(c,S)\right\} = \left\{c\colon \kappa_1(a,b) \preceq_1 \mu_{\kappa_1}^C(c,S)\right\} = \left\{c\colon \kappa_2(a,b) \preceq_2 \mu_{\kappa_2}^C(c,S)\right\} = P_{S\theta_2}^{\kappa_2}$. Similarly, $P_{S\mathcal{S}}^{\kappa_1} = \left\{c \in C\colon \mu_{\kappa_1}^C(c,W) \prec_1 \mu_{\kappa_1}^C(c,S)\right\} = \left\{c \in C\colon \mu_{\kappa_2}^C(c,W) \prec_2 \mu_{\kappa_2}^C(c,S)\right\} = P_{S\mathcal{S}}^{\kappa_2}$. The final equation we need to prove is $P_{S\mathcal{S}}^{i,\kappa_1} = P_{S\mathcal{S}}^{i,\kappa_2}$. This will be proved by induction on $i \ge 0$. For $i = 0$ this is true, since by definition both sets are empty. So assume that for some i we have $P_{S\mathcal{S}}^{i,\kappa_1} = P_{S\mathcal{S}}^{i,\kappa_2}$. Then $P_{S\mathcal{S}}^{i+1,\kappa_1}$ equals

$$\begin{aligned} &P_{S\mathcal{S}}^{i,\kappa_1} \cup \{c \in C \setminus P_{S\mathcal{S}}^{i,\kappa_1}\colon \mu_{\kappa_1}^{C\setminus P_{S\mathcal{S}}^{i,\kappa_1}}(c,W) \prec_1 \mu_{\kappa_1}^C(c,S)\}\\ =\ &P_{S\mathcal{S}}^{i,\kappa_2} \cup \{c \in C \setminus P_{S\mathcal{S}}^{i,\kappa_2}\colon \mu_{\kappa_1}^{C\setminus P_{S\mathcal{S}}^{i,\kappa_2}}(c,W) \prec_1 \mu_{\kappa_1}^C(c,S)\}\\ =\ &P_{S\mathcal{S}}^{i,\kappa_2} \cup \{c \in C \setminus P_{S\mathcal{S}}^{i,\kappa_2}\colon \mu_{\kappa_2}^{C\setminus P_{S\mathcal{S}}^{i,\kappa_2}}(c,W) \prec_2 \mu_{\kappa_2}^C(c,S)\\ =\ &P_{S\mathcal{S}}^{i+1,\kappa_2}, \end{aligned}$$

where the first equation follows from the inductive assumptions and the second from (2). ■

In summary, Theorem 3 says that for every affinity function there is a standard affinity equivalent to it, while Theorem 4 says that for any two equivalent affinities we get the same FC segmentations in each of AFC, RFC, and IRFC. To further illustrate this, we present an example in Fig. 1 for AFC by using two affinities ψ and ψ_σ defined at the end of Section II-B. Figures 1(b) and (c) display the connectivity scenes $\mathcal{C}_\kappa =$

$\langle C, f_k \rangle$ for the 2D scene of Fig. 1(a), where for any $c \in C$ and the same fixed spel $s \in C$, $f_\kappa(c) = \mu_\kappa^C(c,s)$, where κ is either ψ or ψ_σ. The resulting identical AFC objects are displayed in (d) and (e) as binary scenes. Of course, different thresholds were used in producing scenes (d) and (e) from those in (b) and (c), respectively, which precisely makes our point that segmented object information in Figures 1(b) and (c) is identical.

Theorems 3 and 4 also imply that any result proved for the FC segmentations in the context of standard affinities remains valid for the affinities in our general setting. For example, the following is a translation of some of the results from [18]. In what follows, if affinity κ is clear from the context, we will drop the symbol κ from the object symbols $P^\kappa_{S\theta}$, $P^\kappa_{S\mathcal{S}}$, and $P^{I\kappa}_{S\mathcal{S}}$.

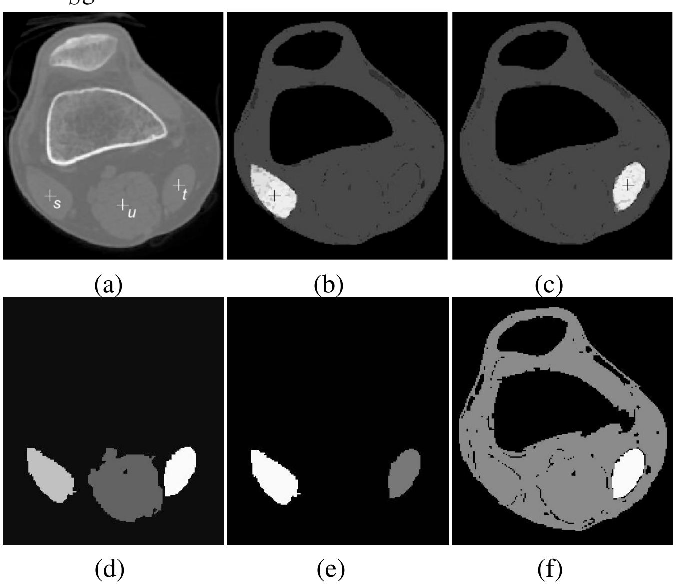

Fig. 2. (a) A 2D scene, same as in Fig. 1(a), with three indicated seeds. (b), (c) Connectivity scenes corresponding to the two AFC objects indicated by s and t. (d) The RFC segmentation for the three indicated objects. (e) The AFC objects initiated with seeds s and t obtained with the threshold $\theta_{\{s\}} < \theta_{\{t\}}$ determined automatically by RFC. Although the result is a binary image, the two objects are shown at two gray levels. The object indicated by seed s agrees with its counterpart in (d). The smaller threshold caused the t-indicated object to be slightly smaller than in 2d. (f) Same as (e) but with threshold $\theta_{\{t\}}$. The object indicated by seed t agrees with its counterpart in (d). However, the larger threshold caused the s-indicated object (grey) to leak to a big part of the scene.

Corollary 5: *Let $\kappa\colon C \times C \to \langle L, \preceq \rangle$ be an arbitrary affinity function.*

(a) *For any $a, b, c \in A \subseteq C$, if $\mu^A(b,c) \prec \mu^A(a,b)$, then $\mu^A(a,c) = \mu^A(b,c)$.*

(b) *(Robustness) Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a family of singletons, and for every $i \in \{1, \ldots, m\}$, let $T_i \subset P_{S_i\mathcal{S}}$ be a singleton. If $\mathcal{T} = \{T_1, \ldots, T_m\}$, then $P_{T_i\mathcal{T}} = P_{S_i\mathcal{S}}$ for every $i \in \{1, \ldots, m\}$.*

(c) *For any family $\mathcal{S}$ of pairwise disjoint non-empty subsets of C, we have $P^I_{S\mathcal{S}} \cap P^I_{U\mathcal{S}} = \emptyset$ for every distinct $S, U \in \mathcal{S}$.*

PROOF. This follows directly from our remark above and, respectively, from [18, Proposition 2.1], [18, Proposition 2.2], and [18, Theorem 2.4]. ∎

D. Relative fuzzy connectedness segmentation as absolute fuzzy connectedness segmentation

In AFC, to obtain the FC object $P^\kappa_{S\theta}$, a threshold θ for the strength of connectedness must be specified. This threshold is obviated in defining RFC objects $P^\kappa_{S\mathcal{S}}$ (see definition above) simply by determining the membership of a spel c in an object by its largest strength of connectedness with respect to the seed sets assigned to the different objects. In this subsection, we will show that the RFC segmentation can be viewed to some extent as an AFC segmentation wherein the required threshold is determined automatically.

Theorem 6: *Let $\kappa\colon C \times C \to \langle L, \preceq \rangle$ be an arbitrary affinity function and $\mathcal{S}$ be a non-empty family of pairwise disjoint, non-empty sets of seeds in C. Fix an $S \in \mathcal{S}$ and let $W = \bigcup(\mathcal{S} \setminus \{S\})$. For every $s \in S$ let $\theta_s = \mu^C_\kappa(s, W)$. Then $P_{S\mathcal{S}} = \bigcup_{s\in S} \bigcup_{\theta_s \prec \theta} P_{\{s\}\theta}$.*

PROOF. Note that, by Corollary 5(a), for every $c, s, w \in C$,

$$\mu^C_\kappa(c,w) \prec \mu^C_\kappa(c,s) \ \text{ if and only if } \ \mu^C_\kappa(s,w) \prec \mu^C_\kappa(c,s).$$

Thus $P_{S\mathcal{S}} = \{c \in C\colon \mu^C_\kappa(c,W) \prec \mu^C_\kappa(c,S)\}$ equals

$$\begin{aligned}
& \{c \in C\colon (\exists s \in S)\ (\forall w \in W)\ \mu^C_\kappa(c,w) \prec \mu^C_\kappa(c,s)\} \\
= \ & \{c \in C\colon (\exists s \in S)\ (\forall w \in W)\ \mu^C_\kappa(s,w) \prec \mu^C_\kappa(c,s)\} \\
= \ & \{c \in C\colon (\exists s \in S)\ \mu^C_\kappa(s,W) \prec \mu^C_\kappa(c,s)\} \\
= \ & \textstyle\bigcup_{s\in S} \{c \in C\colon \theta_s \prec \mu^C_\kappa(c,s)\} \\
= \ & \bigcup_{s\in S} \bigcup_{\theta_s \prec \theta} \{c \in C\colon \theta \preceq \mu^C_\kappa(c,s)\} = \bigcup_{s\in S} \bigcup_{\theta_s \prec \theta} P_{\{s\}\theta}. \quad \blacksquare
\end{aligned}$$

For an affinity $\kappa\colon C \times C \to \langle L, \preceq \rangle$ and $\theta < \mathbf{1}_\kappa$, let θ^+ be the smallest element of $L_\kappa = \{\kappa(a,b)\colon a, b \in C\}$ greater than θ; that is, $\theta^+ \stackrel{\text{def}}{=} \min\{\rho \in L_\kappa\colon \theta \prec \rho\}$.

Theorem 6 has the nicest form when each object is initiated by just one single seed spel.

Corollary 7: *Let $\langle C, \kappa, \preceq \rangle$ be an arbitrary affinity structure and $\mathcal{S}$ be a non-empty family of singletons in C such that $\mu^C_\kappa(s,t) \neq \mathbf{1}_\kappa$ for every distinct $S = \{s\}$ and $T = \{t\}$ from $\mathcal{S}$. For $S = \{s\} \in \mathcal{S}$, let $\theta_S = \mu^C_\kappa(s, \bigcup(\mathcal{S} \setminus \{S\}))$. Then $P_{S\mathcal{S}} = P_{S\theta_S^+}$ for every $S \in \mathcal{S}$. In particular, $\mathbb{P}_\kappa(\mathcal{S}) = \{P_{S\theta_S^+}\colon S \in \mathcal{S}\}$.*

PROOF. Let $S = \{s\} \in \mathcal{S}$. Then $\theta_S = \theta_s$ and, by Theorem 6, we have $P_{S\mathcal{S}} = \bigcup_{\theta_S \prec \theta} P_{S\theta} = \left\{c \in C\colon \theta_S \prec \mu^C_\kappa(c,S)\right\} = \left\{c \in C\colon \theta_S^+ \preceq \mu^C_\kappa(c,S)\right\} = P_{S\theta_S^+}$. ∎

Notice that if for a family $\mathcal{S}$ containing only singletons there exist distinct $S, T \in \mathcal{S}$ such that $\mu^C_\kappa(S,T) \stackrel{\text{def}}{=} \max_{s\in S} \mu^C_\kappa(s,T) = \mathbf{1}_\kappa$, then $P_{S\mathcal{S}} = P_{T\mathcal{S}} = \emptyset$. That is, in this case, S and T are in the same object, and therefore, the sets that contain S and T and that separate them in the FC sense are obviously empty. Thus, in all practical cases we are interested only in the families $\mathcal{S}$ of seeds for which $\mu^C_\kappa(S,T) \neq \mathbf{1}_\kappa$ for any distinct $S, T \in \mathcal{S}$. Thus, this assumption in Corollary 7 does not really restrict its usefulness.

If $\mathcal{S}$ from Corollary 7 has just two elements, say $\mathcal{S} = \{\{s\},\{t\}\}$, then $\theta_{\{s\}} = \theta_{\{t\}}$ and for $\theta = \theta^+_{\{s\}}$ we have $\mathbb{P}_\kappa(\mathcal{S}) = \{P_{S\theta}\colon S \in \mathcal{S}\} = \mathbb{P}^\theta_\kappa(\mathcal{S})$. Thus, in this case, the RFC segmentation $\mathbb{P}_\kappa(\mathcal{S})$ is just an AFC segmentation $\mathbb{P}^\theta_\kappa(\mathcal{S})$, where θ was automatically set by the RFC procedure. However, when there are more than two object involved in RFC and $\mathcal{S}$ contains three or more singletons, the thresholds θ^+_S, $S \in \mathcal{S}$, need not be equal. In this case each P_{SS} from $\mathbb{P}_\kappa(\mathcal{S})$ is an AFC object $P_{S\theta_S^+}$, where the different thresholds are automatically tailored for the different objects under consideration. That is the beauty of RFC compared to AFC.

To illustrate this property of RFC vis-a-vis AFC, in Fig. 2, we consider three objects. The first two objects, denoted by seeds s and t, correspond to soft tissue regions and are really our objects of interest in this example. The third object is the rest of the background and is denoted by seed u. The 2D scene is the one employed in Fig. 1. Identical seed spels denoted by $+$'s in Fig. 2(a) were specified for AFC and RFC. The two connectivity scenes corresponding to the two AFC objects are displayed in Fig. 2(b) and (c), and the resulting AFC objects obtained with two different thresholds θ_S^+ from the scenes in (b) and (c) are shown in Fig. 2(e) and (f). The RFC objects obtained appear in Fig. 2(d), which are identical to the AFC objects in (e) and (f).

Note also that the main reason we could represent RFC objects in terms of AFC objects was that two appearances of c in the inequality $\mu_\kappa^C(c,w) \prec \mu_\kappa^C(c,s)$ could be reduced to one: $\mu_\kappa^C(s,w) \prec \mu_\kappa^C(c,s)$, as both these inequalities are equivalent. In the case of IRFC, the defining inequality is $\mu_\kappa^A(c,w) \prec \mu_\kappa^C(c,s)$ for an appropriate $A \subset C$, and there is no equivalent form of this inequality with just one appearance of c. Thus, no natural AFC representation of IRFC object seems possible. Although increasing sophistication from AFC to RFC to IRFC has been previously demonstrated qualitatively via segmentation experiments [14], [16], [18], in this section we have now given a mathematical justification of that behavior.

III. Two commonly used affinities and their natural definitions

In this section, we will discuss the definitions of two main classes of affinities that have been employed in the FC literature, namely, homogeneity based and object feature based, and examine the connectivity measures they induce.

From now on, we will work with a fixed digital space $\langle \mathbb{Z}^n, \alpha \rangle$ and a scene $\mathcal{C} = \langle C, f \rangle$. Unless otherwise specified, we will assume that the adjacency relation α is a hard relation defined as $\alpha(c,d) = \chi_{[0,1]}(||c-d||)$; that is, $\alpha(c,d) = 1$ when $||c-d|| \le 1$ and $\alpha(c,d) = 0$ for $||c-d|| > 1$, where $||c-d||$ represents a distance between c and d. If we use the Euclidean distance, then α represents 4-adjacency in two-dimensional space and 6-adjacency in three-dimensional space. Note that fixing adjacency has influence only on our discussion of the definition of affinity function and has no direct influence on any FC segmentation outcome, as the definition of the global connectivity measure, μ^C, depends only on the affinity function. From this point on, we will drop the superscript from μ^C, so that symbol $\mu_\kappa(c,d)$ will stand for $\mu_\kappa^C(c,d)$.

In what follows, we will assume that the intensity function has scalar values only, $f\colon C \to \mathbb{R}$. Also, to make our presentation more transparent, we will assume that f represents not necessarily the original scene intensity function, but rather a result of any filtering that could have been done on such acquired scene. In particular, we will not use any scale based approach to the affinity definitions (see [30]), since any scale-based affinity is essentially equal to a non-scale-based affinity applied to an appropriately filtered version of the intensity function. (This is precisely true for the object feature based affinities used in the literature. In the case of homogeneity based affinities, the affinity obtained by what we suggest above is slightly different from that defined in [30]; however, these two versions are very close to each other.)

The range of the two kinds of the affinity functions defined in the following sections will be the space $\langle L, \preceq \rangle = \langle [0,\infty], \ge \rangle$. In other words, the order relation $\preceq$ will be the reversed standard order relation $\ge$. In such a setting, "$\preceq$-stronger" means "less than" in terms of the standard order $\le$. Also, the meanings of the terms $\min$ and $\max$ are switched: "$\min$ in terms of $\preceq$" means "$\max$ in terms of $\le$," and "$\max$ in terms of $\preceq$" becomes "$\min$ in terms of $\le$."

A. Homogeneity based affinity

Intuitively, this function, denoted $\psi(c,d)$, is defined as the maximum of $|f'(x)|$, with x between c and d (where f' is the derivative of f): the higher the magnitude of the slope of f between c and d is, the weaker is the affinity (connectivity) between c and d. Of course, there is more than one way to interpret the symbol $|f'(x)|$. In this section we will interpret this as a magnitude of the directional derivative $D_{\overrightarrow{cd}} f(x)$ in the direction of the vector $\overrightarrow{cd}$. This agrees with the standard FC approach used in the research conducted so far. (See e.g. [8], [12], [17], [20], [21].) Alternatively, it is possible to treat $|f'(x)|$ as a gradient magnitude. True gradient induced homogeneity based affinity will be incorporated in our future work. (See e.g. [25].)

The value $|f'(x)| = |D_{\overrightarrow{cd}} f(x)|$ is best approximated by a difference quotient $\psi_0(c,d) = \left|\frac{f(c)-f(d)}{||c-d||}\right|$. Although this expression has no sense for $c=d$, it should be clear that we should define $\psi_0(c,c)$ as equal to 0, the "highest" possible connectivity in this setting. (Recall that "highest" in terms of $\preceq$ defined as $\ge$ translates into "least" in terms of the standard order $\le$. That is, the greater ψ_0 is, the weaker is the affinity between c and d.) Is the definition $\psi_0(c,d) = \left|\frac{f(c)-f(d)}{||c-d||}\right|$ what we are looking for?

Certainly this is not a local measurement of connectedness when $||c-d||$ is large. In this case, the difference quotient is a poor approximation of the definition of the derivative. We also have a better way of estimating the highest slope on the road from c to d: crawl from c to d along a path with steps of length 1, estimating the slope of each step separately. Because of this, it makes sense to consider the number $\psi_0(c,d)$ as a good value for $\psi(c,d)$ only when $||c-d|| \le 1$, in all other cases we should assign to it it the worst possible value; that is, ∞. This leads to the definition $\psi(c,d) = \psi_0(c,d)/\alpha(c,d)$; that is,

$$\psi(c,d) = \begin{cases} |f(c)-f(d)| & \text{for } ||c-d|| \le 1 \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

It is easy to see that ψ satisfies our definition of affinity function. It should be stressed here that such a function approximates only the magnitude of the directional derivative of f in the direction $\overrightarrow{cd}$, and gives no information on the slope of f in a direction perpendicular to $\overrightarrow{cd}$.

If one likes to express this affinity by an equivalent standard affinity, our definition of ψ can be replaced by $g_1(\psi(c,d))$, where g_σ is a Gaussian function $g_\sigma(x) = e^{-x^2/\sigma^2}$. Notice that if $\alpha(c,d) = \chi_{[0,1]}(||c-d||)$, as we defined earlier, then $g_1(\psi(c,d)) = \alpha(c,d) \cdot g_1(|f(c) - f(d)|)$, the formula defining purely homogeneity based affinity in [30, pp. 149–150]. (We use the weights $w_1 = 0$ and $w_2 = 1$.) However, if α is an arbitrary fuzzy adjacency relation, then the formula $\alpha(c,d) \cdot g_1(|f(c) - f(d)|)$ disagrees with the derivative intuition. For example, if $\alpha(c,d) = g_1(||c-d||)$, then $\alpha(c,d) \cdot g_1(|f(c) - f(d)|) = e^{-(|f(c)-f(d)|^2 + ||c-d||^2)} = g_1(\sqrt{|f(c) - f(d)|^2 + ||c-d||^2})$, rather than the more appropriate $g_1\left(\frac{|f(c)-f(d)|}{||c-d||}\right)$ (possibly multiplied by number $\alpha(c,d)$). In what follows, we will use the homogeneity based affinity $\psi(c,d)$ as defined in (3), rather than $g_1(\psi(c,d))$, as it is more intuitive, and, by Corollary 2, these two affinities are equivalent. We refer the reader to Fig. 1 for an illustration demonstrating the equivalence of $\psi(c,d)$ and $g_\sigma(\psi(c,d))$. Thus, the parameter σ in the homogeneity based affinity $\psi_\sigma = g_\sigma \circ \psi$ is of no consequence for the FC algorithms, although in all FC literature, this σ has been considered as a parameter of the method in the description of the methods and their evaluation.

The homogeneity based connectivity measure, $\mu_\psi = \mu_\psi^C$, can be elegantly interpreted if our scene $\mathcal{C} = \langle C, f\rangle$ is considered as a topographical map in which $f(c)$ represents an elevation at the location $c \in C$. Then, $\mu_\psi(c,d)$ is the highest possible step (a slope of f) that one must make in order to get from c to d with each step on a location (spel) from C and of unit length. In particular, the object $P^\psi_{s\theta} = \{c \in C \colon \theta \geq \mu_\psi(s,c)\}$ represents those spels $c \in C$ which can be reached from s without ever making a step higher than θ. Note that all we measure in this setting is the actual change of the altitude while making the step. Thus, this value can be small, even if the step is made on a very steep slope, as long as the motion is approximately perpendicular to the hill-side gradient. On the other hand, the measure of the same step would be large, if measured with some form of gradient induced homogeneity based affinity!

B. *Object feature based affinity*

There are two principal differences between the object feature based and the homogeneity based affinities. (1) The definition of the object feature based affinity requires some prior knowledge on the objects we like to uncover, while the definition of the homogeneity based affinity is completely independent of such knowledge. (2) The homogeneity based affinity is represented in terms of (the approximation of) the derivative f' of the intensity function f, while the object feature based affinity is defined directly from the intensity function f. In the rest of this subsection, we will consider object feature based affinity for the cases of single and multiple objects separately.

1) Object feature based affinity for one object: We will start with the definition of the object feature based affinity, denoted $\phi(c,d)$, in terms of only a single object O. To define ϕ, we need to start with an approximate expected (average) intensity value m for the spels in the object. We will also assume that we have a standard deviation $\sigma > 0$ of the distribution of intensity for this object. Then, the intuitive definition of affinity ϕ is just $\bar{\varphi}_0(c) = |f(c) - m|$. In other words, the smaller the value of $\bar{\varphi}_0(c)$ is, the closer is c's intensity to the object intensity, and the better c is connected to object O. (Since the range of ϕ is $\langle L, \preceq\rangle = \langle [0,\infty], \geq\rangle$, the notion of "$\preceq$-stronger" translates into "smaller in the $\leq$ sense.") It is also convenient, for facilitating a definition of the object feature based affinity for multiple objects, to rescale this formula to $\bar{\varphi}(c) = |f(c) - m|/\sigma$. (This is related to the Mahalanobis distance [35].) Now, one may attempt to define the strength of a path $p = \langle c_1, \ldots, c_l\rangle$ as

$$\mu_{\bar{\varphi}}(p) = \max_{i=1,\ldots,l} \bar{\varphi}(c_i) \tag{4}$$

and the connectivity measure as $\mu_{\bar{\varphi}}(c,d) = \min_{p \in \mathcal{P}^C_{cd}} \mu_{\bar{\varphi}}(p)$. (Once again, the use of inverse inequality $\geq$ as $\preceq$ makes the $\leq$-largest value to be the $\preceq$-smallest value.) However, since in this definition we do not assume that the consecutive spels in a path are adjacent, there is nothing local in this definition. In particular, if $f(c) = f(d) = m$, then $\mu_{\bar{\varphi}}(\langle c, d\rangle) = 0$ is not a good connectivity measure: the best possible connectivity in $\mu_{\bar{\varphi}}$-sense, $\mu_{\bar{\varphi}}(\langle c,d\rangle) = 0$, means only that the intensities at both spels equal m, and it may still happen that such spels are spatially separated by spels with very different intensities; on the other hand, if distinct c and d are adjacent (next to each other), then the fact that $f(c) = f(d) = m$ is very informative — such spels are indeed perfectly connected. The situation can be rescued if one considers only the paths from the family $\bar{\mathcal{P}}_{cd}$ of all paths from c to d in which the consecutive spels are <u>distinct</u> and adjacent. Then, for $c \neq d$, the formula

$$\mu_{\bar{\varphi}}(c,d) = \min_{p \in \bar{\mathcal{P}}_{cd}} \mu_{\bar{\varphi}}(p) \tag{5}$$

agrees with our intuition and with the formula for μ_ϕ defined below. (See (7).) So, why can we not we use formula (5) as a definition of μ_ϕ? Although we could, there are two inconveniences connected with this approach: first we would need to replace $\mathcal{P}^C_{cd}$ with $\bar{\mathcal{P}}_{cd}$; second, the value of $\mu_{\bar{\varphi}}(p)$ is not defined by using any affinity function ($\bar{\varphi}(c_i)$ cannot be treated as affinity, since it is a function of one variable), so the general results on the FC theory could not be applied to a connectivity measure so defined. Moreover, affinity formula (5) carries some other dangers, which we will mention below.

Thus, we will define ϕ properly, as a function on the pairs $\langle c, d\rangle$ of spels. We like to define ϕ in such a way that, for every $p \in \bar{\mathcal{P}}_{cd}$, the strength $\mu_\phi(p)$ of p is equal $\mu_{\bar{\varphi}}(p)$. To ensure this, for distinct adjacent c and d, $\phi(c,d)$ must be defined as $\max\{\bar{\varphi}(c), \bar{\varphi}(d)\} = \max\{|f(c) - m|, |f(d) - m|\}/\sigma$. Thus, in general, we define $\phi(c,d) = \max\{\bar{\varphi}(c), \bar{\varphi}(d)\}/\alpha(c,d)$; that is, $\phi(c,d) = 0$ for $c = d$, $\phi(c,d) = |f(c) - m|, |f(d) - m|\}/\sigma$, when $||c - d|| = 1$, and $\phi(c,d) = \infty$ otherwise. Clearly function ϕ is an affinity function in our sense. Moreover,

$$\mu_\phi(p) = \max_{i=1,\ldots,l} \bar{\varphi}(c_i) \text{ for every } p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathcal{P}}_{cd}, \tag{6}$$

since $\mu_\phi(p) = \max_i \max\{\bar{\varphi}(c_i), \bar{\varphi}(c_{i+1})\} = \max_i \bar{\varphi}(c_i)$. In particular, by (4), $\mu_\phi(p) = \mu_{\bar{\varphi}}(p)$ for every $p \in \bar{\mathcal{P}}_{cd}$. Notice also that for every $c \neq d$ function μ_ϕ agrees with $\mu_{\bar{\varphi}}$:

$$\mu_\phi(c,d) = \mu_{\bar{\varphi}}(c,d), \tag{7}$$

since $\mu_\phi(c,d) = \min_{p \in \mathcal{P}^C_{cd}} \mu_\phi(p) = \min_{p \in \bar{\mathcal{P}}_{cd}} \mu_\phi(p) = \min_{p \in \bar{\mathcal{P}}_{cd}} \mu_{\bar{\varphi}}(p) = \mu_{\bar{\varphi}}(c,d)$. Here the first and the last equations come from (1) and (5), respectively. The third equation follows from the above argument, while the second one is justified by the fact that for every $q \in \mathcal{P}_{cd}$ either $\mu_\phi(q) = \infty$ (when q contains non-adjacent consecutive spels) or $\mu_\phi(q) = \mu_\phi(p)$ for $p \in \bar{\mathcal{P}}_{cd}$ obtained from q by collapsing all constant consecutive subsequences of q to a single occurrence of the repeated value.

Note that, in reference [30], for distinct adjacent spels c and d the authors define $\phi(c,d)$ as $\bar{\varphi}\left(\frac{f(c)+f(d)}{2}\right) = \left|\frac{f(c)+f(d)}{2} - m\right|$ in place of $\max\{\bar{\varphi}(c), \bar{\varphi}(d)\}$. Although this caries similar intuitions, the averaging of the values of $f(c)$ and $f(d)$ looses information on how far the intensity of each spel is from m. For example, if $f(c) = m+r$ and $f(d) = m-r$ for some $r > 0$, then $\bar{\varphi}\left(\frac{f(c)+f(d)}{2}\right) = 0$ and $\mu_\phi(\langle c, d\rangle)$ associated with such affinity equals 0, which does not satisfy (6) and is counterintuitive for large values of r.

Once again, we can replace $\phi(c,d)$ with $g_1(\phi(c,d))$ for some Gaussian-like function to get an equivalent affinity in the standard form. In particular, for $g_1(x) = e^{-x^2}$ this leads to $\bar{\varphi}(c) = e^{-\frac{(f(c)-m)^2}{\sigma^2}}$, one of the formulas used in [30]. (See also [8], [14], [16], [19].)

The object feature based connectivity measure of one object has also a nice topographical map interpretation. For understanding this, consider a modified scene $\bar{\mathcal{C}} = \langle C, |f(\cdot) - m|\rangle$ as a topographical map. Then the number $\mu_\phi(c,d)$ represents the lowest possible elevation (in $\bar{\mathcal{C}}$) which one must reach (a mountain pass) in order to get from c to d, where each step is on a location from C and is of unit length. Notice that $\mu_\phi(c,d)$ is precisely the *degree of connectivity* as defined by Rosenfeld [31]–[33]. (Compare also [34], where it is used under the name *pass value*.)

2) Object feature based affinity for multiple objects: The single object connectivity measure μ_ϕ can be useful in object definition only if we define it by using absolute connectedness definition, AFC. To find an object via RFC or IRFC methods, we need to have μ_ϕ defined for at least two objects. So, suppose that the scene consists of $n > 1$ objects with expected average intensities $m_1, \ldots, m_n$ and standard deviations $\sigma_1, \ldots, \sigma_n$, respectively. Then we have n different object feature based affinities $\hat{\phi}_i(c,d)$, defined for $c \neq d$ as $\max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\alpha(c,d)$, where $\bar{\varphi}_i(c) = \frac{|f(c)-m_i|}{\sigma_i}$, and their respective connectivity measures $\mu_{\hat{\phi}_i}$. We like to combine affinities $\hat{\phi}_i$ to get the cumulative object feature based affinity ϕ. (Obtaining a single affinity at the end becomes essential in order to fulfill the theoretical requirements that lead to dynamic programing as the efficient computational tool in all of AFC, RFC, and IRFC.) But how to define such a ϕ? We will build our intuition for such a ϕ by assuming that each object O_i is generated by a single seed s_i with $f(s_i) = m_i$. Although this situation is not general, any discussion of this subject must include this important case. Therefore, we will decide on the form of a definition of ϕ in this situation first, and then argue that the notion we come up with has the desired properties without requiring any extra assumptions.

First note that σ_i's help us to compare different $\hat{\phi}_i$'s. Specifically, each number $\bar{\varphi}_i(c)$ measures the distance $|f(c) - m_i|$ of the image intensity $f(c)$ from the average intensity m_i of the i-th object. However, if we like to compare the numbers $\bar{\varphi}_i(c)$ for different i's, we need to fix a reasonable measuring unit. The most natural measuring unit for $\bar{\varphi}_i$ is the associated standard deviation σ_i: with our definition $\bar{\varphi}_i(c) = \frac{|f(c)-m_i|}{\sigma_i}$, the equation $\bar{\varphi}_i(c) = K$ means that the intensity $f(c)$ at c is K standard deviations apart from m_i (like the Mahalanobis distance [35]). Then, equation $\bar{\varphi}_1(c) = \bar{\varphi}_2(c)$ caries the correct intuition: $f(c)$ is the same number of σ_i's apart from m_i for $i = 1$ and $i = 2$.

Now, by equation (6), if $p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathcal{P}}_{s_i c}$ and $s_i \neq c$, then the strength of the i-th object connectivity between s_i and c on this path is given by $\mu_{\hat{\phi}_i}(p) = \max_{t=1,\ldots,l} \bar{\varphi}_i(c_t)$. Similarly, the strength of the j-th object connectivity between s_j and $c \neq s_j$ on a path $q = \langle d_1, \ldots, d_\ell\rangle \in \bar{\mathcal{P}}_{s_j c}$ is equal to $\mu_{\hat{\phi}_j}(q) = \max_{t=1,\ldots,\ell} \bar{\varphi}_j(d_t)$. Therefore, by the analysis given in the above paragraph, the i-th object connectivity strength $\mu_{\hat{\phi}_i}(p)$ of p exceeds (in the $\preceq$ sense) the j-th object connectivity strength $\mu_{\hat{\phi}_j}(q)$ of q provided $\mu_{\hat{\phi}_i}(p) = \max_{t=1,\ldots,l} \bar{\varphi}_i(c_t) < \max_{t=1,\ldots,\ell} \bar{\varphi}_j(d_t) = \mu_{\hat{\phi}_j}(q)$. So, by (5), c is better $\hat{\phi}_i$-connected to s_i than it is $\hat{\phi}_j$-connected to s_j precisely when $\mu_{\hat{\phi}_i}(s_i,c) < \mu_{\hat{\phi}_j}(s_j,c)$.

For some of the key results of FC theory, which eventually determine the good properties of the FC objects, we need to arrive at one affinity defined over the whole scene. We shall examine this issue at the higher level in Section IV. In this section, our goal is to focus on a lower level, that is, to study how to combine the affinities $\hat{\phi}_i$ into a single object feature based affinity ϕ so that it preserves the information given by all affinities $\hat{\phi}_i$ to the fullest possible extent. (The reason why we cannot use two different affinities and define an object via inequality $\mu_{\hat{\phi}_i}(s_i,c) < \mu_{\hat{\phi}_j}(s_j,c)$ is explained below.) In particular, since for every i, the value of $\mu_\phi(s_i,c)$ should approximate, as much as possible, the i-th object connectivity strength between s_i and c, it would be most desirable if we could have insured that $\mu_\phi(s_i,c) = \mu_{\hat{\phi}_i}(s_i,c)$. In particular, we would like to insure that $\mu_\phi(s_i,c) < \mu_\phi(s_j,c)$ if and only if $\mu_{\hat{\phi}_i}(s_i,c) < \mu_{\hat{\phi}_j}(s_j,c)$. Unfortunately, we will see below that there is no way to have such a strong property, since in the process of combining $\hat{\phi}_i$'s we always lose some information. Nevertheless, at the very least, we should insure that inequality $\mu_\phi(s_i,c) < \mu_\phi(s_j,c)$ never happens when $\mu_{\hat{\phi}_i}(s_i,c) \geq \mu_{\hat{\phi}_j}(s_j,c)$. This can be expressed as

$$\mu_\phi(s_i,c) < \mu_\phi(s_j,c) \text{ implies } \mu_{\hat{\phi}_i}(s_i,c) < \mu_{\hat{\phi}_j}(s_j,c). \tag{8}$$

This implication represents the most fundamental property that we will impose on the definition of ϕ. In particular, in what follows we will define the object based affinity ϕ which satisfies (8) under some simple assumptions connecting each s_k with m_k. We will also argue (see Example 12 in Appendix) that other seemingly natural definitions of ϕ, like the one used in [14] (compare also [30]), do not satisfy this property.

Another way to look at property (8) is that, when $n = 2$, it insures that the RFC object $P^{\phi}_{s_i\{s_j\}}$ is contained in a set $O_{ij} = \{c \in C\colon \mu_{\hat{\phi}_i}(s_i, c) < \mu_{\hat{\phi}_j}(s_j, c)\}$. One may wonder whether we should consider sets O_{ij} (or their intersections $O_i = \bigcap_{j \neq i} O_{ij}$, if $n > 2$) as our objects. The argument against this consideration can be given at two levels. The simple one is that there is a very nice theory for the objects defined with a single connectivity measure and this theory does not extend, in general, to sets defined as in O_{ij}. (Of course, IRFC sets are also defined in this form, but the different connectivity measures used there have a very specific form.) A slightly deeper argument is that the sets O_{ij} do not have nice properties. For example, it was proved in [14] that, unlike $P^{\phi}_{s_i\{s_j\}}$, the object O_{ij} does not have the *robustness* property (which says that, if in the definition of an object, the generating seed s_i is replaced by any other spel from the object, then the new object obtained that way is equal to the original one) and *path connectedness* property (which says that for any two spels in the object, the strongest path connecting these spels is contained in the object). In fact, the failure of path connectedness for O_{ij} can also be seen in the situation from Example 12 ($c, s_2 \in O_{21}$, but if $f(b) = 33$, then neither b nor any other spel belongs to O_{21}; so, O_{21} is not connected).

The idea behind the formula for ϕ is to define $\phi(c, d)$ as the best among all numbers $\hat{\phi}_i(c, d)$. One possible choice for $\phi(c, d)$ is $\min_{i=1,\ldots,n} \hat{\phi}_i(c, d)$. The problem with this choice is that we never know which value of $\hat{\phi}_i(c, d)$ was used to determine $\phi(c, d)$. Since the values of $\hat{\phi}_i(c, d) = \max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\sigma_i$ are the most valuable when this number is small and because difficulties occur when $\hat{\phi}_i(c, d) = \hat{\phi}_j(c, d)$ for $i \neq j$, we will eliminate the information in $\bar{\varphi}_i(c)$ when this value exceeds $\bar{\varphi}_j(c)$ for some j. This is made formal below.

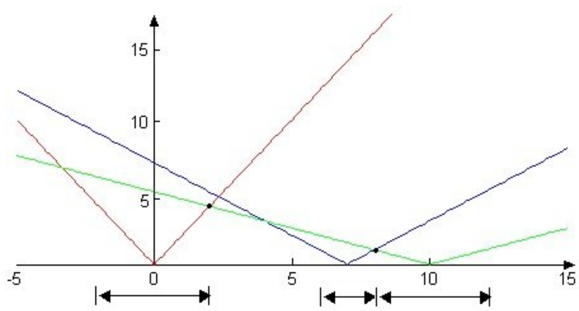

Fig. 3. The graphs of three functions $\bar{\varphi}_i$ with $m_1 = 0$, $m_2 = 7$, $m_3 = 10$, $\sigma_1 = 0.5$, $\sigma_2 = 1$, and $\sigma_3 = 2$. We have $\delta_1^3 = 2 < \delta_1^2$, leading to $I_1 = (-2, 2)$. Also, $\varepsilon_2 = \delta_2^3 = 1$, so $I_2 = (6, 8)$ and $\varepsilon_3 = \delta_3^2 = 2$ leading to $I_3 = (8, 12)$.

For distinct $i, j \in \{1, \ldots, n\}$, let $\delta_i^j \geq 0$ be the largest number with the property that $\frac{|x-m_i|}{\sigma_i} < \frac{|x-m_j|}{\sigma_j}$ for every $x \in (m_i - \delta_i^j, m_i + \delta_i^j)$. (If $\sigma_i = \sigma_j$, then δ_i^j is just half of the distance between m_i and m_j.) Thus, if $x_i^j \in \{m_i - \delta_i^j, m_i + \delta_i^j\}$ is between m_i and m_j, then for each $c \in C$

$$\bar{\varphi}_i(c) < \frac{|x_i^j - m_i|}{\sigma_i} = \frac{\delta_i^j}{\sigma_i} = \frac{|x_i^j - m_j|}{\sigma_j} < \bar{\varphi}_j(c) \tag{9}$$

provided $|f(c) - m_i| < \delta_i^j$. Let $\varepsilon_i = \min_{j \neq i} \delta_i^j$ and $I_i = (m_i - \varepsilon_i, m_i + \varepsilon_i)$. Then intervals I_i, $i \in \{1, \ldots, n\}$, are pairwise disjoint. Function φ_i is defined as a truncation of $\bar{\varphi}_i$ to the interval I_i, that is, by a formula

$$\varphi_i(c) = \varphi_i^{I_i}(c) = \begin{cases} \bar{\varphi}_i(c) & \text{for } f(c) \in I_i \\ \infty & \text{otherwise.} \end{cases}$$

Then $\varphi_i(c) < \infty$ implies $f(c) \in I_i = (m_i - \varepsilon_i, m_i + \varepsilon_i)$. Fig. 3 gives an example of the graphical representation for numbers δ_i^j and intervals I_i. For $c \neq d$ put $\phi_i(c, d) = \max\{\varphi_i(c), \varphi_i(d)\}/\alpha(c, d)$; that is, $\phi_i(c, d) = 0$ when $c = d$, $\phi_i(c, d) = \max\{\varphi_i(c), \varphi_i(d)\}$ for $||c - d|| = 1$, and $\phi_i(c, d) = \infty$ otherwise, and let

$$\phi(c, d) = \min_{i=1,\ldots,n} \phi_i(c, d). \tag{10}$$

We define $\mu_\phi(p)$ for a path p and a connectivity measure μ_ϕ^A according to our general method. The following theorem shows that ϕ so defined satisfies (8) we promised. The proof of this theorem and the associated machinery are provided in Appendix Section VII.

Theorem 8: Fix $i \in \{1, \ldots, n\}$ *and* $c, s_i, s_j \in C$ *such that* $f(s_i) \notin \bigcup_{k \neq i} I_k$. *If* $\mu_\phi(s_i, c) < \mu_\phi(s_j, c)$, *then* $\mu_{\hat{\phi}_i}(s_i, c) < \mu_{\hat{\phi}_k}(s_j, c)$ *for every* $k \in \{1, \ldots, n\}$.

The role of a seed s_i is not only to indicate an approximate position of an object but also to indicate its approximate average intensity m_i. This is the only way to insure that s_i indicates the correct object. Thus, if one allows the situation in which $s_i \in I_j$ for some $j \neq i$, then s_i would really represent j's object and $\mu_\phi(s_i, c)$ would be represented by $\mu_{\hat{\phi}_j}(s_i, c)$ rather than by $\mu_{\hat{\phi}_i}(s_i, c)$. Not surprisingly, in such a situation, the conclusion of the theorem cannot be expected. This explains our assumption $f(s_i) \notin \bigcup_{k \neq i} I_k$. In fact, we could as well assume $f(s_i) \in I_i$, as otherwise (i.e., when $f(s_i)$ belongs to no I_k) $\mu_\phi(s_i, c) = \infty$ for every c, so $\mu_\phi(s_i, c)$ caries no valuable information, and the conclusion of the theorem is satisfied in void.

Clearly, truncating each $\bar{\varphi}_i$ to $\varphi_i = \varphi_i^{I_i}$ is causing the loss of some information. In fact, the most common definition of ϕ used in the literature till now, see e.g. [14], coincides with ours if one drops the matter of truncation: define $\bar{\phi}(c, d) = \min_{i=1,\ldots,n} \bar{\phi}_i(c, d)$, where $\bar{\phi}_i(c, d) = \max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\alpha(c, d)$ for $c \neq d$. Then $\mu_{\bar{\phi}}$ is defined as usual. Clearly, at the first glance it seems that affinity $\bar{\phi}$ is superior to its truncated version ϕ defined above and that the information truncation makes the ability to distinguish among objects weaker. Although, to some extent, this is a legitimate concern, it should be noted that the objects obtained with the use of $\bar{\phi}$ may be bigger than those obtained with the use of ϕ. However, since $\bar{\phi}$ is not required to satisfy (8), it is possible that a spel c is assigned to object $P^{\bar{\phi}}_{s_i\theta}$ while it truly belongs to another object. (See Example 12.) Thus, under the circumstances, we believe that it is better to leave c unassigned to any object, rather than to run into the risk of assigning it to an incorrect object.

Another possible way for defining object feature based connectivity, $\mu_{\bar{\varphi}}$, is to put $\bar{\varphi}(c) = \min_{i=1,\ldots,n} \bar{\varphi}_i(c)$ and define it as in (4) and (5). Although $\mu_{\bar{\varphi}}$ is equal to $\mu_{\bar{\phi}}$ when $n = 1$, in general this is not the case. This is best seen in Example 11 in Appendix, which fully discredits $\mu_{\bar{\varphi}}$ as a valid definition of an object feature based connectivity measure. Example 12 shows that the motivational implication (8) fails for $\mu_{\bar{\phi}}$. The algorithm for computing object feature based affinity ϕ as defined in (10) is summarized below.

Algorithm $\phi AFFINITY$

Input: Scene $\mathcal{C} = \langle C, f\rangle$, number $n \geq 1$ of objects' to be delineated, and objects expected average intensities $m_1, \ldots, m_n$ together with their respective standard deviations $\sigma_1, \ldots, \sigma_n$.

Output: The scene $\langle C \times C, \phi\rangle$ representing object feature based affinity ϕ as defined in (10).

Auxiliary Data Structures: The sequence $\delta_1, \ldots, \delta_n$ of radii of the truncation intervals. Auxiliary variables x, y, and a sequence $\phi_1, \ldots, \phi_n$.

begin

1. initialize all δ_i's to ∞;
2. *for* $i = 1$ *to* n *do*
3.   *for* $j = 1$ *to* n *do*
4.   *if* $j \neq i$ *then*
5.   set $x = \frac{m_j\sigma_i + m_i\sigma_j}{\sigma_i + \sigma_j}$;
6.   set $\delta_i = \min\{\delta_i, |x - m_i|\}$;
7.   *endif*;
8.   *endfor*;
9. *endfor*;
10. initialize all $\phi(c, d)$ with $c \neq d \in C$ as ∞, and as 0 for $c = d \in C$;
11. *for* all distinct adjacent $c, d \in C$ *do*
12.   *for* $i = 1$ *to* n *do*
13.   *if* $|f(c) - m_i| < \delta_i$ *then* set $x = \frac{|f(c)-m_i|}{\sigma_i}$ *else* set $x = \infty$;
14.   *if* $|f(d) - m_i| < \delta_i$ *then* set $y = \frac{|f(d)-m_i|}{\sigma_i}$ *else* set $y = \infty$;
15.   set $\phi_i = \max\{x, y\}$;
16.   *endfor*;
17.   set $\phi(c, d) = \min_{i=1,\ldots,n} \phi_i$;
18. *endfor*;
19. output $\langle C \times C, \phi\rangle$;

end

In steps 1–9 the algorithm calculates the radii δ_i's. Number $x = \frac{m_j\sigma_i + m_i\sigma_j}{\sigma_i + \sigma_j}$ constitutes the solution of the equation $\frac{|x-m_i|}{\sigma_i} = \frac{|x-m_j|}{\sigma_j}$ when x is between m_i and m_j. If $\sigma_i \neq \sigma_j$, this equation has also another solution x' on the other side of m_i. However, we always have $|x - m_i| < |x' - m_i|$, so δ_i^j from (9) is always equal $|x - m_i|$. Note also that for $n = 1$, the resulted radius δ_1 is correctly calculated as ∞.

In steps 12–17, the algorithm calculates the value of $\phi(c, d)$ for distinct adjacent c and d. Number x calculated in step 13 represents $\varphi_i(c)$, while y from step 14 is equal $\varphi_i(d)$. Number ϕ_i from step 15 is equal $\phi_i(c, d)$. Note that, for non-adjacent c and d, the algorithm returns $\phi(c, d) = \infty$.

C. Homogeneity versus object feature based affinity

First note that the homogeneity based connectivity measure μ_ψ and the object feature based connectivity measure μ_ϕ, although related (as function f is related to its derivative f'), behave quite differently. For example, μ_ψ, unlike μ_ϕ, is not very sensitive to the slow background intensity variation often found in medical images as an artifact. To see this, imagine that the image consists of a long straight tube (say an artery) with the intensity of each spel in a tube around 10, and the intensity of each spel outside the tube around 20. Now, assume that a slow (spatially) changing artifact is applied to the image. This artifact is often multiplicative in nature. For simplicity, assume that it is additive and that it changes along the length of the tube from 0 to 20. Then, the beginning of the tube will have intensity around 10, while its end will have a value around 30. Now, the artifact we added changes little the value of μ_ψ, so the entire tube can still be easily obtained as $O_{s\theta}$ or O_{st} if one uses μ_ψ as a connectivity measure. On the other hand, if s is a seed located at the beginning of the tube and $O_{s\theta} = \{c \in C \colon \mu_{\phi_{f(s)}}(s, c) \leq \theta\}$ contains a spel t from the end of the tube, then $\theta \geq \mu_{\phi_{f(s)}}(s, t) \approx 20$. Therefore, $O_{s\theta}$ must contain also many spels outside the tube, since for any spel c outside the tube close to the beginning, we have $f(c) \approx 20$, so $\mu_{\phi_{f(s)}}(s, c) \approx 10 < \theta$.

On the other hand, if a scene $\mathcal{C}$ contains seeds s and t with $|f(s) - f(t)|$ being large, it still may happen that there is a long path p from s to t along which the intensity changes very slowly. Then $\mu_\psi(s, t) \leq \mu_\psi(p)$ is very small, which makes it nearly impossible to distinguish s and t by means of homogeneity based connectivity measure alone. However, since $\mu_\phi(s, t)$ is large, we can easily distinguish s and t with the help of object feature based connectivity measure.

As pointed out in [30], these two concepts — one related to homogeneity (a derivative $f'(c)$ concept) and another related to the intensity $f(c)$ — are fundamental to any segmentation methods that are based purely on information derived from the given image.

IV. How to combine different affinities?

In this section, we will discuss the issue of how to combine two or more different affinities of the sort described in the previous section into one affinity. We will also examine which parameters in the definitions of the combined affinity are redundant, in the sense that their change leads to an equivalent affinity.

A. Affinity combining methods

Assume that for some $k \geq 2$ we have affinity functions $\kappa_i \colon C \times C \to \langle L_i, \preceq_i\rangle$ for $i = 1, \ldots, k$. For example, we can have $k = 2$, $\kappa_1 = \psi$, and $\kappa_2 = \phi$. The most flexible way of combining all these affinities into a single affinity κ is to put $\kappa(c, d) = \langle \kappa_1(c, d), \ldots, \kappa_k(c, d)\rangle$ and define an appropriate linear order $\preceq$ on $L = L_1 \times \cdots \times L_k$. To understand this formalism better, we will start with the following examples, which also constitute our practical approach to the affinity combining problem.

Example 9: **(Weighted Averages)** Assume that all linear orderings L_i are equal to the same ordering $\langle L_0, \preceq_0\rangle$ which is either $\langle [0, \infty], \geq\rangle$ or $\langle [0, 1], \leq\rangle$ and fix a vector $\mathbf{w} = \langle w_1, \ldots, w_k\rangle$ of numbers from $[0, 1]$ (weights) such that $w_1 + \cdots + w_k = 1$; we allow a weight w_i to be equal to 0 (meaning "ignore influence of κ_i") assuming that $0 \cdot \infty = 0$ and $0^0 = \infty^0 = 1$.

Additive Average: Let $h^{add}_{\mathbf{w}}(\mathbf{a}) = w_1 a_1 + \cdots + w_k a_k$ for $\mathbf{a} = \langle a_1, \ldots, a_k\rangle \in (L_0)^k$. If $\mathbf{a} \leq^{add}_{\mathbf{w}} \mathbf{b} \;\Leftrightarrow\; h^{add}_{\mathbf{w}}(\mathbf{a}) \preceq_0 h^{add}_{\mathbf{w}}(\mathbf{b})$, then $\kappa\colon C \times C \to \langle L, \leq^{add}_{\mathbf{w}}\rangle$ is equivalent to $\kappa^a_{\mathbf{w}}\colon C \times C \to \langle L_0, \preceq_0\rangle$ defined as $\kappa^a_{\mathbf{w}}(c,d) = h^{add}_{\mathbf{w}}(\kappa_1(c,d), \ldots, \kappa_k(c,d))$.
Multiplicative Average: Let $h^{mul}_{\mathbf{w}}(\mathbf{a}) = a_1^{w_1} \cdots a_k^{w_k}$ for $\mathbf{a} = \langle a_1, \ldots, a_k\rangle \in (L_0)^k$. If $\mathbf{a} \leq^{mul}_{\mathbf{w}} \mathbf{b} \;\Leftrightarrow\; h^{mul}_{\mathbf{w}}(\mathbf{a}) \preceq_0 h^{mul}_{\mathbf{w}}(\mathbf{b})$, then $\kappa\colon C \times C \to \langle L, \leq^{mul}_{\mathbf{w}}\rangle$ is equivalent to $\kappa^m_{\mathbf{w}}\colon C \times C \to \langle L_0, \preceq_0\rangle$ defined as $\kappa^m_{\mathbf{w}}(c,d) = h^{mul}_{\mathbf{w}}(\kappa_1(c,d), \ldots, \kappa_k(c,d))$.

Notice that, for $k = 2$, the affinities $\kappa^a_{\mathbf{w}} = w_1\kappa_1 + w_2\kappa_2$ and $\kappa^m_{\mathbf{w}} = \kappa_1^{w_1}\kappa_2^{w_2}$ have been already considered in [30]. Recall that the *lexicographical order* $\leq_{lex}$ on $L = L_1 \times \cdots \times L_k$ is defined for distinct $\mathbf{a} = \langle a_1, \ldots, a_k\rangle, \mathbf{b} = \langle b_1, \ldots, b_k\rangle \in L$ as

$$\mathbf{a} <_{lex} \mathbf{b} \quad\Leftrightarrow\quad a_i \prec_i b_i, \text{ where } i = \min\{j\colon a_j \neq b_j\}.$$

Example 10: **(Lexicographical Order)** Affinity function $\kappa_{lex}\colon C \times C \to \langle L, \leq_{lex}\rangle$ establishes the strongest possible hierarchy between the coordinate affinities κ_i: in establishing whether $\kappa_{lex}(a,b) \leq_{lex} \kappa_{lex}(c,d)$, the values $\kappa_i(a,b)$ and $\kappa_i(c,d)$ are completely irrelevant, unless $\kappa_j(a,b) = \kappa_j(c,d)$ for all $j < i$, in which case $\kappa_i(a,b) \prec_i \kappa_i(c,d)$ implies $\kappa_{lex}(a,b) <_{lex} \kappa_{lex}(c,d)$.

Notice that κ_{lex} cannot be expressed in the form of $h(\kappa_1, \ldots, \kappa_k)$ for any continuous function on $[0,1]^k$ or on $[0,\infty]^k$. In what follows, we will restrict our attention to the situation when $k = 2$. In this case the lexicographical order is defined as $\langle a_1, a_2\rangle <_{lex} \langle b_1, b_2\rangle$ provided either $a_1 \prec_1 b_1$ or $a_1 = b_1$ and $a_2 \prec_2 b_2$. Notice, that the lexicographical order approach is quite appealing in case when $\kappa_1 = \psi$ and $\kappa_2 = \phi$ as the decision whether $\mu_\kappa(c,s) \leq_{lex} \mu_\kappa(c,t)$ is hierarchical in nature: if $\mu_\psi(c,s) < \mu_\psi(c,t)$, then $\mu_\kappa(c,s) \leq_{lex} \mu_\kappa(c,t)$ independent of the values of $\mu_\phi(c,s)$ and $\mu_\phi(c,t)$; only when the homogeneity based connectivity measure cannot decide the matter, that is, when $\mu_\psi(c,s) = \mu_\psi(c,t)$, we decide on the direction of $\leq_{lex}$ between $\mu_\kappa(c,s)$ and $\mu_\kappa(c,t)$ based on the direction of $\preceq_2$ between $\mu_\phi(c,s)$ and $\mu_\phi(c,t)$. Thus, we treat the homogeneity based connectivity measure as dominant over object feature based connectivity measure. However, there is more in it. If $\mu_\psi(c,s) = \mu_\psi(c,t)$, then we decide about $\mu_\kappa(c,s) \leq_{lex} \mu_\kappa(c,t)$ only along the paths $p \in \mathcal{P}_{cs}$ and $q \in \mathcal{P}_{ct}$ with $\mu_\psi(p) = \mu_\psi(q) = \mu_\psi(c,s)$. Only to these paths we apply μ_ϕ measure. Thus, we use the object based feature measure in this schema in considerably a more subtle way than what is suggested by the threshold-like interpretation described in Section III. It should be also clear that, if we agree that we should give priority to homogeneity based connectivity measure in the RFC approach, this is precisely the way we should proceed.

Next, consider the coordinate order preserving property of the combined affinity $\kappa(c,d) = \langle \kappa_0(c,d), \kappa_1(c,d)\rangle$:

(C) for every $i = 0,1$ and c, d, c', d', if $\kappa_i(c,d) = \kappa_i(c',d')$, then $\kappa(c,d) \prec \kappa(c',d') \Leftrightarrow \kappa_{1-i}(c,d) \prec_{1-i} \kappa_{1-i}(c',d')$.

Property (C) says that if one of the coordinate affinities does not distinguish between two pairs of spels, then the combined affinity decides on this pair according to the other coordinate affinity. This seems to be a very natural and desirable property. It is easy to see that the κ_{lex} affinity has this property. However, in general, (C) is not satisfied for the multiplicative average $\kappa^m_{\mathbf{w}}$: if $\kappa_i(c,d) = \kappa_i(c',d') = 0$, then $\kappa^m_{\mathbf{w}}(c,d) = \kappa^m_{\mathbf{w}}(c',d') = 0$ independently of the value of κ_{1-i} on these pairs. A similar problem appears for $\kappa_i(c,d) = \kappa_i(c',d') = \infty$, though for $\kappa_i(c,d) = \kappa_i(c',d') \in (0,\infty)$ the equivalence from (C) is satisfied. This creates problem especially with the truncated version of the object-feature based affinity, since in this case affinity is equal to ∞ for many adjacent pairs of spels. Condition (C) also fails for $\kappa^{add}_{\mathbf{w}}$ when $\kappa^{add}_{\mathbf{w}}(c,d) = \kappa^{add}_{\mathbf{w}}(c',d') = \infty$, though for $\kappa^{add}_{\mathbf{w}}(c,d) = \kappa^{add}_{\mathbf{w}}(c',d') < \infty$ the equivalence is satisfied. In particular, (C) holds for $\kappa^{add}_{\mathbf{w}}$ formed with the coordinate affinities with range $\langle [0,1], \leq\rangle$.

B. Counting essential parameters

Next, let us turn our attention to the determination of the number of parameters essential in defining the affinities presented in the previous section. We will consider here only the parameters explicitly mentioned there, since any implicit parameters (like the parameters for getting intensity function from the actual acquisition data) could not be handled by the methods we will employ. This exercise is useful in tuning the FC segmentation methods to different applications. It is also useful in comparing these methods with others. Recall that for a $\sigma \in (0,\infty)$ we defined $g_\sigma\colon [0,\infty] \to [0,1]$ by $g_\sigma(x) = e^{-x^2/\sigma^2}$.

Homogeneity based affinity, ψ, is defined as $\psi(c,d) = |f(c) - f(d)|$ for $||c-d|| \leq 1$ and $\psi(c,d) = \infty$ otherwise. As such, there are no parameters in this definition. In its standard form, $g_\sigma \circ \psi$, the parameter σ is redundant, since, by Corollary 2, $g_\sigma \circ \psi$ is equivalent to ψ.

Object feature based affinity for one object, ϕ, is defined by a formula $\phi(c,d) = \max\{|f(c) - m_1|, |f(d) - m_1|\}/\sigma_1$ for $||c-d|| = 1$, $\phi(c,d) = 0$ for $c = d$, and $\phi(c,d) = \infty$ otherwise. From the two parameters, m_1 and σ_1, present in this definition, only m_1 is essential. Parameter σ_1 is redundant, since function $\sigma_1 \cdot \phi$ is independent of its value and $\sigma_1 \cdot \phi$ is equivalent to ϕ, as $\sigma_1 \cdot \phi = h \circ \phi$ for an increasing function $h(x) = \sigma_1 x$. As before, the standard form $g_\sigma \circ \phi$ of ϕ is equivalent to it, so the only essential parameter in the definition of $g_\sigma \circ \phi$ is the number m_1.

Object feature based affinity for multiple objects. Suppose that the affinity is defined for $n > 1$ different objects for which $\bar{m} = \langle m_1, \ldots, m_n\rangle$ and $\bar{\sigma} = \langle \sigma_1, \ldots, \sigma_n\rangle$ represent their average intensities and standard deviations, respectively. Let $\phi_{\bar{m},\bar{\sigma}}$ represent the object feature affinity in its main truncated form and let $\bar{\phi}_{\bar{m},\bar{\sigma}}$ stand for its untruncated version. (See Section III-B.2.) Then $\sigma_1 \cdot \phi_{\bar{m},\bar{\sigma}} = \phi_{\bar{m},\bar{\delta}}$ and $\sigma_1 \cdot \bar{\phi}_{\bar{m},\bar{\sigma}} = \bar{\phi}_{\bar{m},\bar{\delta}}$, where $\bar{\delta} = \langle 1, \delta_2, \ldots, \delta_n\rangle$ and $\delta_i = \sigma_i/\sigma_1$. Since $\sigma_1 \cdot \phi_{\bar{m},\bar{\sigma}}$ is equivalent to $\phi_{\bar{m},\bar{\sigma}}$, affinity $\phi_{\bar{m},\bar{\sigma}}$ depends essentially only on $2n-1$ parameters $m_1, \ldots, m_n, \delta_2, \ldots, \delta_n$. The same is true for its standard form $g_\sigma \circ \phi_{\bar{m},\bar{\sigma}}$ as well as for their untruncated counterparts $\bar{\phi}_{\bar{m},\bar{\sigma}}$ and $g_\sigma \circ \bar{\phi}_{\bar{m},\bar{\sigma}}$.

In what follows we will assume that $w, \sigma, \tau \in (0,1)$ and that ϕ is equal to either $\phi_{\bar{m},\bar{\delta}}$ or to $\bar{\phi}_{\bar{m},\bar{\delta}}$, so it has $2n-1$ essential parameters. Then we have the following methods of combining, denoted m1–m5, for homogeneity and object feature based affinities.

m1 The additive average $\kappa = (1-w)\psi + w\phi$ of ψ and ϕ has $2n$ parameters. It is equivalent to $\psi + x\phi$, where $x = \frac{w}{1-w} \in (0,\infty)$. Notice that if ϕ is replaced by an equivalent affinity $\sigma_1\phi$, then the resulting average affinity $(1-w)\psi + w\sigma_1\phi$ is also equivalent to $\psi + x\phi$ with $x \in (0,\infty)$. Note that κ does not satisfy property (C).

m2 The additive average $\kappa = (1-w)g_\sigma \circ \psi + w g_\tau \circ \phi$ of $g_\sigma \circ \psi$ and $g_\tau \circ \phi$ has $2n+2$ essential parameters. Since $\kappa = e^{\ln(1-w)-\psi^2/\sigma^2} + e^{\ln w - \phi^2/\tau^2}$, this operation strangely mixes additive and multiplicative modifications of ψ and ϕ. The additional two parameters, σ and τ, are of importance in this mix. This affinity does satisfy property (C).

m3 The multiplicative average $\kappa = \psi^{(1-w)}\phi^w$ of ψ and ϕ has $2n$ parameters and it is equivalent to $\psi\phi^x$, where $x = \frac{w}{1-w} \in (0,\infty)$, as $\kappa = (\psi\phi^x)^{1-w}$. If ϕ is replaced by an equivalent affinity $\sigma_1\phi$, then the resulting average $(\psi\sigma_1^x\phi^x)^{1-w}$ is also equivalent to $\psi\phi^x$ with $x \in (0,\infty)$, since function $h(t) = (\sigma_1^x t)^{1-w}$ is increasing as a composition of two increasing functions. This κ does not satisfy property (C).

m4 The multiplicative average $\kappa = (g_\sigma \circ \psi)^{(1-w)}(g_\tau \circ \phi)^w$ of $g_\sigma \circ \psi$ and $g_\tau \circ \phi$ has $2n+2$ parameters, but only $2n$ of them are essential. This is so since $\kappa = \left(e^{-\psi^2/\tau^2}\right)^{1-w}\left(e^{-\phi^2/\sigma^2}\right)^w = \left(e^{-\psi^2 - x\phi^2}\right)^{(1-w)/\tau^2}$, where $x = \frac{\tau^2}{\sigma^2}\frac{w}{1-w} \in (0,\infty)$, is equivalent to $\psi^2 + x\phi^2$. The same is true if ϕ is replaced by $\sigma_1\phi$. This κ does not satisfy property (C).

m5 There are only two essential possibilities for lexicographical order of ψ and ϕ: $\langle\psi,\phi\rangle$ and $\langle\phi,\psi\rangle$, even if we allow replacement of each of the coordinate affinities by any of their equivalent forms, including but not restricted to $g_\sigma \circ \psi$ and $\sigma_1\phi$, $g_\tau \circ \phi$, or $g_\tau \circ (\sigma_1\phi)$. This follows from Proposition 1, since for any pair $\langle\psi^*,\phi^*\rangle$ such that ψ^* is equivalent to ψ and ϕ^* is equivalent to ϕ, there are strictly monotone functions g and h such that $\psi^* = g\circ\psi$ and $\phi^* = h \circ \phi$, and then $\langle\psi^*,\phi^*\rangle = \langle g,h\rangle \circ \langle\psi,\phi\rangle$, so $\langle g,h\rangle$ establishes the equivalence of $\langle\psi,\phi\rangle$ and $\langle\psi^*,\phi^*\rangle$.

V. Experimental results

In our discussion up to the previous section, we presented a rigorous theory behind the design of affinity functions for FC. This theory led us to a host of forms for affinity which where listed under m1–m5 in the previous section. We note that, by considering both truncated ϕ and untruncated $\bar{\phi}$ versions of object feature based affinities, there are 10 different forms of affinity functions we arrived at. All truncated versions under m1–m5, as well as the untruncated versions under m1, m3, and m5, are novel and not considered in the literature so far. m2 and m4 with untruncated ϕ have been previously proposed [30]. The most interesting among these are the truncated and untruncated versions of m5.

It is desirable to restrict the combination methods of to those that satisfy property (C). From the above list, only methods m2 and m5 fall under this category, and so, we will restrict our experiments only to these two cases. It should be stressed that the lack of property (C) in the combining methods m1, m2, and m4 is most visible when the truncated version ϕ of object-feature affinity is used, since then $\phi(c,d) = \infty$ for many adjacent c and d.

VI. Concluding remarks

The analysis presented in Section II shows that, from the perspective of FC methodology, the only essential attribute of an affinity function is its order. In particular, many transformations (like gaussian) of the natural affinity definitions (like derivative-driven homogeneity based affinity) are of esthetic value only and do not influence the FC segmentation outcomes. Nevertheless, such transformations may play a role in combining different affinities, as can be seen in methods m1 and m2, since only one of them has the property (C).

The analysis from Section II forms also the foundation of the investigation, presented in Section IV, of which parameters in the definitions of homogeneity and object-feature based affinities, as well as their combinations, are of importance. In particular, we uncovered that many of the parameters in these definitions are of no consequence. Thus, for the tasks of application-driven optimization of the parameters, the number of parameters to be optimized is reduced. This aspect of setting values of parameters for segmentation methods is ridden with confusion. There are no scientific and systematic solutions for this problem. We indicated a solution in [37] which consisted of simultaneously minimizing false positive and false negative regions as a function of the parameter values. It make sense to first identify what the essential parameters of a segmentation method are, since such an attempt does not seem to have been made in the literature. This especially is relevant if we choose optimal parameter settings as mentioned by an optimization process.

In Section III, we discussed two commonly used affinities, homogeneity and object-feature based, and interpreted them, respectively, as approximations of the directional derivatives and the distance from the object's average intensities. We also pointed out some theoretical deficiencies with the standard format of the object-feature based affinity in the case of multiple objects and proposed a truncated version of such affinity, which avoids theoretical difficulties, but loses some information along the way.

Other possible ways of defining affinities. Note that in the definition of the "object feature based affinity," described in Section III, the only prior knowledge of the object we used was the image intensity distribution of the object. More elaborate object feature affinity can use some other prior knowledge on the object(s) to be delineated. For example, the general shape of the object(s) can constitute such prior knowledge. If shape for the family of the object under consideration is modeled in a statistical manner [2], then we can consider a model based component of affinity $\beta(c,d)$ between c and d to be higher only if c and d are inside or close to the boundary of the mean shape, and smaller otherwise. A simple strategy based on the distance from mean shape boundary has been employed in [36] in an attempt to bring in prior shape information into FC. This discussion of how to properly define β and how to combine this with ψ and ϕ, however, requires fundamental investigation along the lines of this paper.

Also, as mentioned in Section III-A, in the definition of the homogeneity based affinity it makes sense to use the notion of the gradient as a base for its definition, instead of the notion of the directional derivative. The discussion of the gradient induced homogeneity based affinity is a part of our forthcoming paper.

VII. Appendix

The following example fully discredits $\mu_{\bar{\varphi}}$ as a valid definition of an object feature based connectivity measure, while Example 12 shows that the motivational implication (8) fails for $\mu_{\bar{\phi}}$.

Example 11: Let $\mathcal{C}$ be a binary scene with two intensities $m_2 > m_1 = 0$ and $\sigma_1 = \sigma_2 = 1$. We will consider $\mathcal{C}$ as a two object scene: for $i = 1,2$ object O_i consists of all spels with the intensity m_i. Then for every $c, d \in \mathcal{C}$ we have $\mu_{\bar{\varphi}}(c,d) = 0$, while $\mu_{\bar{\phi}}(c,d) = m_2 > 0$ provided $f(c) \neq f(d)$.

Example 12: Let $p = \langle s_1, a, c, b, s_2\rangle$ be a sequence of spels in scene $\mathcal{C}$ in which only consecutive spels are adjacent and assume that $\langle 0, 8, 14, 20 \pm 13, 20\rangle$ represents their intensities, respectively. We also assume that any other spel in $\mathcal{C}$ adjacent to one listed in p has the intensity at least 80. Consider s_1 and s_2 as the seeds of objects O_1 and O_2 with averages $m_1 = f(s_1) = 0$ and $m_2 = f(s_2) = 20$ and standard deviations $\sigma_1 = \sigma_2 = 1$, respectively. Then $\mu_{\bar{\phi}}(s_1, c) = 12 < 13 = \mu_{\bar{\phi}}(s_2, c)$. However, $\mu_{\hat{\phi}_1}(s_1, c) = 14 > 13 = \mu_{\hat{\phi}_2}(s_2, c)$.

Proof. For any adjacent spels s and t we have $\bar{\phi}(s,t) = \min\{\max\{|f(s)|, |f(t)|\}, \max\{|20 - f(s)|, |20 - f(t)|\}\}$. So, $\bar{\phi}(s_1, a) = \min\{8, 20\} = 8$, $\bar{\phi}(a,c) = \min\{14, 12\} = 12$, and $\mu_{\bar{\phi}}(s_1, c) = \mu_{\bar{\phi}}(\langle s_1, a, c\rangle) = \max\{8, 12\} = 12$. Similarly $\bar{\phi}(s_2, b) = \min\{\max\{20, 20 \pm 13\}, \max\{0, |\pm 13|\}\} = 13$ and $\bar{\phi}(b,c) = \min\{\max\{20 \pm 13, 14\}, \max\{|\pm 13|, 6\}\} = 13$, which leads to $\mu_{\bar{\phi}}(s_2, c) = \mu_{\bar{\phi}}(\langle s_2, b, c\rangle) = \max\{13, 13\} = 13$. On the other hand, by property (6), we have $\mu_{\hat{\phi}_1}(s_1, c) = \mu_{\hat{\phi}_1}(\langle s_1, a, c\rangle) = \max\{0, 8, 14\} = 14$, while $\mu_{\hat{\phi}_2}(s_2, c) = \mu_{\hat{\phi}_2}(\langle s_2, b, c\rangle) = \max\{0, 13, 6\} = 13$. ■

To understand this example better, let x_1^2 be as in (9); that is, such that $\frac{|x_1^2 - m_1|}{\sigma_1} = \frac{|x_1^2 - m_2|}{\sigma_2}$. So, in the setting of Example 12, we have $\delta_1 = \delta_2 = x_1^2 = 10$. The key characteristics of this example, that allows us to negate property (8), is that the intensities present in the path $q = \langle s_2, b, c\rangle$ (i.e., $\{f(s_2), f(b), f(c)\}$) are not in $I_2 = (m_2 - \varepsilon_2, m_2 + \varepsilon_2)$, despite the fact that $f(s_2), f(c) \in I_2$. Indeed, if the equation $\mu_{\hat{\phi}_2}(s_2, c) = \mu_{\hat{\phi}_2}(q)$ was satisfied with the intensities of all spels in q belonging to J_2, then (by Lemmas 13 and 14) we would have had $\mu_\phi(s_2, c) = \mu_{\hat{\phi}_2}(s_2, c) < \mu_\phi(s_1, c)$ and $\mu_\phi(s_2, c) = \mu_{\hat{\phi}_2}(s_2, c) < \mu_{\hat{\phi}_1}(s_1, c)$, which is in agreement with (8).

In case when $f(b) = 20 - 13 = 7$, all the intensities under question are between m_1 and m_2. Moreover, $f(b)$ is just barely below the threshold $m_2 - \delta_2$. (A slight modification of the example can make it arbitrarily close to $m_2 - \delta_2$.) The case when $f(b) = 20 + 13 = 33$ shows that it is not enough to stay within the interval $I = (m_2 - \delta_2, \infty)$, for which we have $\frac{|x - m_1|}{\sigma_1} > \frac{|x - m_2|}{\sigma_2}$ for every $x \in I$. Thus, the symmetry of I_i's around m_i's is essential in proving (8). In other words, the above discussion shows that, if ϕ is defined via the truncation technique, then the intervals I_i are the smallest with which we can still prove property (8).

For the rest of the discussion, we will assume that $f(s_i) \in I_i$ for every i. What is the format of the objects generated with μ_ϕ under such assumption? First notice that in the case of the absolute connectedness definition we get

$$P^{\phi}_{s_i\theta} = \begin{cases} \{c \in C\colon \theta \geq \mu_{\phi_i}(s_i, c)\} & \text{for } \theta < \frac{\varepsilon_i}{\sigma_i}, \\ \left\{c \in C\colon \frac{\varepsilon_i}{\sigma_i} > \mu_{\phi_i}(s_i, c)\right\} & \text{for } \frac{\varepsilon_i}{\sigma_i} \leq \theta. \end{cases}$$

In other words, $P^{\phi}_{s_i\theta}$ can be expressed in terms of the objects defined via AFC with respect to the affinity ϕ_i: $P^{\phi_i}_{s_i\theta} = \{c \in C\colon \theta \geq \mu_{\phi_i}(s_i, c)\}$. It is also easy to see that the i-th object defined via RFC is the largest among the above objects: $\bigcap_{j \neq i} P^{\phi_i}_{s_i\{s_j\}} = \left\{c \in C\colon \frac{\varepsilon_i}{\sigma_i} > \mu_{\phi_i}(s_i, c)\right\} = \bigcup_{\theta < \frac{\varepsilon_i}{\sigma_i}} P^{\phi_i}_{s_i\theta}$. The same remains true for the IRFC case.

Since the above reduces RFC and IRFC objects defined with respect to ϕ to the unions of absolute connectedness objects $P^{\phi_i}_{s_i\theta}$ with respect to ϕ_i, one might wonder whether there is any sense at all in defining object feature based affinity ϕ. However, the full definition of ϕ is necessary in order to amalgamate ϕ with any other affinity, as discussed in Section IV.

The remainder of this paper is devoted to the proof of Theorem 8.

Lemma 13: Let $p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathcal{P}}_{cs}$ and $i \in \{1, \ldots, n\}$. If $f(c_k) \in I_i$ for every $k \in \{1, \ldots, l\}$, then $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\varepsilon_i}{\sigma_i}$.

Proof. Notice that for every distinct $i, j \in \{1, \ldots, n\}$ and for every index $k \in \{1, \ldots, l-1\}$ we have $\phi_j(c_k, c_{k+1}) = \max\{\varphi_j(c_k), \varphi_j(c_{k+1})\} \geq \varphi_j(c_k) = \infty$, since $f(c_k) \notin I_j$. So, $\phi(c_k, c_{k+1}) = \min_{j=1,\ldots,n} \phi_j(c_k, c_{k+1}) = \phi_i(c_k, c_{k+1}) = \max\{\varphi_i(c_k), \varphi_i(c_{k+1})\} = \max\{\bar{\varphi}_i(c_k), \bar{\varphi}_i(c_{k+1})\} = \hat{\phi}_i(c_k, c_{k+1})$ and thus $\mu_\phi(p) = \max_{k=1,\ldots,l-1} \phi(c_k, c_{k+1}) = \max_{k=1,\ldots,l-1} \hat{\phi}_i(c_k, c_{k+1}) = \mu_{\hat{\phi}_i}(p)$. In addition, by (6), we have $\mu_{\hat{\phi}_i}(p) = \max_{k=1,\ldots,l} \bar{\varphi}_i(c_k)$. So, there is a $k \in \{1, \ldots, l\}$ for which $\mu_{\hat{\phi}_i}(p) = \bar{\varphi}_i(c_k) = \frac{|f(c_k) - m_i|}{\sigma_i} < \frac{\varepsilon_i}{\sigma_i}$, since $f(c_k) \in I_i = (m_i - \varepsilon_i, m_i + \varepsilon_i)$. ■

Lemma 14: Let $p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathcal{P}}_{cs}$ be such that $\mu_\phi(p) < \infty$. Then, for every $i \in \{1, \ldots, n\}$, the following conditions are equivalent.

(a) $f(c) \in I_i$.
(b) $f(c_k) \in I_i$ *for every* $k \in \{1, \ldots, l\}$.
(c) $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\varepsilon_i}{\sigma_i} < \bar{\varphi}_j(c)$ *for every* $j \neq i$.
(d) $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p)$.

Moreover, there is an $i \in \{1, \ldots, n\}$ for which each of these conditions holds.

Proof. Note that $\alpha(c_k, c_{k+1}) = 1$ for every $k = 1, \ldots, l-1$, since $p \in \bar{\mathcal{P}}_{cs}$. To see that $\mu_\phi(p) < \infty$ implies that $f(c) \in I_i$ for some i, note that $\infty > \mu_\phi(p) = \max_{k=1,\ldots,l-1} \phi(c_k, c_{k+1}) \geq \phi(c_1, c_2) = \min_{i=1,\ldots,n} \phi_i(c_1, c_2)$. So, there exists an $i \in \{1, \ldots, n\}$ with

$\infty > \phi_i(c_1, c_2) = \frac{\max\{\varphi_i(c_1), \varphi_i(c_2)\}}{\alpha(c_1, c_2)}$. Thus, $\infty > \varphi_i(c_1)$ and $f(c) = f(c_1) \in I_i$.

"(a)$\Longrightarrow$(b)" Let $Z = \{k \in \{1, \dots, l\} \colon f(c_k) \in I_i\}$. Then (a) says that $1 \in Z$. We need to prove that $Z = \{1, \dots, l\}$. By way of contradiction, assume that this is not the case and let $m \in \{1, \dots, l\}$ be the smallest such that $m \notin Z$. Then $m > 1$, as $1 \in Z$, so $k = m - 1 \in Z$. In particular, $f(c_k) \in I_i$, so, for $j \in \{1, \dots, n\}$,

$$\varphi_j(c_k) < \infty \quad \Leftrightarrow \quad j = i.$$

Since $\infty > \mu_\phi(p) \geq \phi(c_k, c_{k+1}) = \min_{j=1,\dots,n} \phi_j(c_k, c_{k+1})$, there exists a j with $\infty > \phi(c_k, c_{k+1}) = \phi_j(c_k, c_{k+1}) = \max\{\varphi_j(c_k), \varphi_j(c_{k+1})\}/\alpha(c_k, c_{k+1})$. In particular, $\infty > \varphi_j(c_k)$ and $\infty > \varphi_j(c_{k+1})$. Hence, from the first of these inequalities, we get $j = i$. Therefore, the second inequality becomes $\infty > \varphi_i(c_{k+1}) = \varphi_i(c_m)$, implying that $m \in Z$, contrary to our assumption. Thus, $Z = \{1, \dots, l\}$ and (b) holds.

"(b)$\Longrightarrow$(c)" By Lemma 13, we have $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\varepsilon_i}{\sigma_i}$. Also, since $f(c) \in I_i = (m_i - \varepsilon_i, m_i + \varepsilon_i) \subseteq (m_i - \delta_i^j, m_i + \delta_i^j)$, condition (9) implies that $\frac{\varepsilon_i}{\sigma_i} \leq \frac{\delta_i^j}{\sigma_i} < \bar{\varphi}_j(c)$.

Implication "(c)$\Longrightarrow$(d)" is obvious.

"(d)$\Longrightarrow$(a)" Condition (d) implies that $\mu_\phi(p) < \infty$. So, by the first remark, $f(c) \in I_j$ for some j. If $j = i$, we are done. So, by way of contradiction, assume that $j \neq i$. Then, using the implication "(a)$\Longrightarrow$(c), " we have $\mu_\phi(p) = \mu_{\hat{\phi}_j}(p) < \frac{|x_j^k - m_j|}{\sigma_j} < \bar{\varphi}_k(c)$ for every $k \neq j$. In particular, for $k = i$ we get $\mu_\phi(p) < \bar{\varphi}_i(c) \leq \max_{k=1,\dots,l} \bar{\varphi}_i(c_k) = \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$, a contradiction. ■

Lemma 15: Let $p, q \in \bar{\mathcal{P}}_{cs}$ *and* $i \in \{1, \dots, n\}$. *If* $\mu_{\hat{\phi}_i}(q) \leq \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$, *then* $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q)$. *In particular, if* $\mu_{\hat{\phi}_i}(p) = \mu_\phi(p) = \mu_\phi(s, c)$, *then also* $\mu_{\hat{\phi}_i}(s, c) = \mu_{\hat{\phi}_i}(p) = \mu_\phi(s, c)$.

PROOF. Let us choose two paths, $p = \langle c_1, \dots, c_l \rangle$ and $q = \langle d_1, \dots, d_m \rangle$. Since we have $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \infty$ (remember that $\hat{\phi}_i$ is a non-truncated version of the object feature base affinity for the i-th object) we can apply Lemma 14. Since Lemma 14(d) holds, so must also Lemma 14(c). Hence, by (6), for every index $k \in \{1, \dots, m\}$ we have $\frac{|f(d_k) - m_i|}{\sigma_i} = \bar{\varphi}_i(d_k) \leq \max_{j=1,\dots,l} \bar{\varphi}_i(d_j) = \mu_{\hat{\phi}_i}(q) \leq \mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\varepsilon_i}{\sigma_i}$. Thus, $f(d_k) \in I_i$ for every $k \in \{1, \dots, m\}$. So, again by Lemma 14, we have $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q)$.

The additional part is obvious when $s = c$, since then $\mu_{\hat{\phi}_i}(p) = \mu_\phi(s, c) = 0 = \mu_{\hat{\phi}_i}(s, c)$. So, assume that $s \neq c$ and that $\mu_{\hat{\phi}_i}(p) = \mu_\phi(p) = \mu_\phi(s, c)$. Then, by (7), there exists a path $q \in \bar{\mathcal{P}}_{cs}$ with $\mu_{\hat{\phi}_i}(s, c) = \mu_{\hat{\phi}_i}(q)$. Then $\mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_i}(s, c) \leq \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$. So, by the first part, $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_i}(s, c) \leq \mu_{\hat{\phi}_i}(p) = \mu_\phi(s, c) \leq \mu_\phi(q)$, proving that $\mu_{\hat{\phi}_i}(s, c) = \mu_{\hat{\phi}_i}(p)$. ■

To see the additional part, assume that $\mu_{\hat{\phi}_i}(p) = \mu_\phi(p) = \mu_\phi(s, c)$. Take $q \in \bar{\mathcal{P}}_{cs}$ with $\mu_{\hat{\phi}_i}(s, c) = \mu_{\hat{\phi}_i}(q)$. Then $\mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_i}(s, c) \leq \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$. So, by the first part, $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_i}(s, c) \leq \mu_{\hat{\phi}_i}(p) = \mu_\phi(s, c) \leq \mu_\phi(q)$, proving that $\mu_{\hat{\phi}_i}(s, c) = \mu_{\hat{\phi}_i}(p)$. ■

PROOF OF THEOREM 8. Assume that $c, s_i, s_j \in C$ are as in the theorem, that is, such that $f(s_i) \notin \bigcup_{k \neq i} I_k$ and $\mu_\phi(s_i, c) < \mu_\phi(s_j, c)$. Fix a $k \in \{1, \dots, n\}$. We need to show that $\mu_{\hat{\phi}_i}(s_i, c) < \mu_{\hat{\phi}_k}(s_j, c)$.

Note that $s_j \neq c$, since otherwise $\mu_\phi(s_i, c) < \mu_\phi(s_j, c) = 0$, which is impossible. Thus, by (7), there exists a $q = \langle d_1, \dots, d_m \rangle \in \bar{\mathcal{P}}_{cs_j}$ such that $\mu_{\hat{\phi}_k}(s_j, c) = \mu_{\hat{\phi}_k}(q)$. Also, if $s_i = c$ then, by the definition (10) of μ_ϕ, we have $\mu_{\hat{\phi}_i}(s_i, c) = 0 = \mu_\phi(s_i, c) < \mu_\phi(s_j, c) \leq \mu_{\hat{\phi}_k}(s_j, c)$. Thus, we can assume that $s_i \neq c$. In particular, using the argument utilized in the proof of (7), we can show that there exists a $p = \langle c_1, \dots, c_l \rangle \in \bar{\mathcal{P}}_{cs_i}$ such that $\mu_\phi(p) = \mu_\phi(c, s_i)$.

We have $\mu_\phi(p) = \mu_\phi(s_i, c) < \mu_\phi(s_j, c)$, so $\mu_\phi(p) < \infty$. Thus, by Lemma 14, there exists an i' for which $f(s_i) = f(c_l) \in I_{i'}$. Therefore $i' = i$, since $f(s_i) \notin \bigcup_{k \neq i} I_k$. So, by Lemma 14(c), $\mu_\phi(s_i, c) = \mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\varepsilon_i}{\delta_i} < \bar{\varphi}_j(c)$ for every $j \neq i$. Also, by Lemma 15, we have $\mu_{\hat{\phi}_i}(s_i, c) = \mu_\phi(s_i, c)$.

Now, if $k \neq i$, then, by (9) and the above, $\mu_{\hat{\phi}_i}(s_i, c) \leq \mu_{\hat{\phi}_i}(p) < \bar{\varphi}_k(c) \leq \max_{r=1,\dots,m} \bar{\varphi}_k(d_r) = \mu_{\hat{\phi}_k}(q) = \mu_{\hat{\phi}_k}(s_j, c)$. So, assume that $k = i$. If there is an $r \in \{1, \dots, m\}$ for which $f(d_r) \notin I_i$, then, by Lemma 14(c), $\mu_{\hat{\phi}_i}(s_i, c) \leq \mu_{\hat{\phi}_i}(p) < \frac{\varepsilon_i}{\sigma_i} < \frac{|f(d_r) - m_i|}{\sigma_i} = \bar{\varphi}_i(d_r)$, so $\mu_{\hat{\phi}_i}(s_i, c) < \bar{\varphi}_i(d_r) = \bar{\varphi}_k(d_r) \leq \max_{r=1,\dots,m} \bar{\varphi}_k(d_r) = \mu_{\hat{\phi}_k}(q) = \mu_{\hat{\phi}_k}(s_j, c)$. So, assume that $f(d_r) \in I_i$ for every $r \in \{1, \dots, m\}$. Then, by Lemma 13, $\mu_{\hat{\phi}_i}(q) = \mu_\phi(q)$. So, $\mu_{\hat{\phi}_i}(s_i, c) = \mu_\phi(s_i, c) < \mu_\phi(s_j, c) \leq \mu_\phi(q) = \mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_k}(q) = \mu_{\hat{\phi}_k}(s_j, c)$ finishing the proof. ■

REFERENCES

[1] M. Kass, A. Witkin, and D. Terzopoulos, Snakes: Active contour models. *Int. J. Comput. Vision* **1** (1987), 321–331.

[2] T. Cootes, C. Taylor, and D. Cooper, Active shape models-their training and application. *Computer Vision and Image Understanding* **61** (1995), 38–59.

[3] T. McInerney and D. Terzopoulos, Deformable models in medical image analysis: A survey. *Medical Image Analysis* **1**(2) (1996), 91–108.

[4] A.X. Falcão, J.K. Udupa, S. Samarasekera, S. Sharma, B.E. Hirsch, and R. Lotufo, User-steered image segmentation paradigms: live wire and live lane. *Graph. Models Image Process* **60**(4) (1998), 233–260.

[5] T. Cootes, G. Edwards, and C. Taylor, Active appearance models. *IEEE Trans. Pattern Anal. Machine Intell.* **23**(6) (2001), 681–685.

[6] M. Trivedi and J. Bezdek, Low-level segmentation of aerial images with fuzzy clustering. *IEEE Trans. Systems, Man, and Cybernetics* **16**(4) (1986), 589–598.

[7] S. Beucher, The watershed transformation applied to image segmentation. In: *10th Pfefferkorn Conf. Signal and Image Processing in Microscopy and Microanalysis* (1992), 299–314.

[8] J.K. Udupa and S. Samarasekera, Fuzzy connectedness and object definition: theory, algorithms, and applications in image segmentation. *Graphical Models and Image Processing* **58**(3) (1996), 246–261.

[9] J.A. Sethian, *Fast Marching Methods and Level Sets Methods. Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science*, Cambridge Univ. Press, 1999.

[10] Y. Boykov, O. Veksler, and R. Zabih, Fast approximate energy minimization via graph cuts. *IEEE Trans. Pattern Anal. Machine Intell.* **23**(11) (2001), 1222–1239.

[11] A. Chakraborty, L. Staib, and J. Duncan, Deformable boundary finding in medical images by integrating gradient and region information. *IEEE Trans. Med. Imag.* **15**(6) (1996), 859–870.

[12] C. Imielinska, D. Metaxas, J.K. Udupa, Y. Jin, and T. Chen, Hybrid segmentation of anatomical data. In: *Proceedings of MICCAI* (2001), 1048–1057.

[13] J.K. Udupa and P.K. Saha, Fuzzy connectedness in image segmentation. *Proceedings of the IEEE*, **91**(10) (2003), 1649–1669.

[14] P.K. Saha and J.K. Udupa, Relative fuzzy connectedness among multiple objects: Theory, algorithms, and applications in image segmentation. *Computer Vision and Image Understanding* **82(1)** (2001), 42–56.

[15] P.K. Saha and J.K. Udupa, Iterative relative fuzzy connectedness and object definition: theory, algorithms, and applications in image segmentation. In *Proceedings of IEEE Workshop on Mathematical Methods in Biomedical Image Analysis*, Hilton Head, South Carolina 2002, 28–35.

[16] J.K. Udupa, P.K. Saha, and R.A. Lotufo, Relative fuzzy connectedness and object definition: Theory, algorithms, and applications in image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **24** (2002), 1485–1500.

[17] G.T. Herman and B.M. Carvalho, Multiseeded segmentation using fuzzy connectedness. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **23** (2001), 460–474.

[18] K.C. Ciesielski, J.K. Udupa, P.K. Saha, and Y. Zhuge, Iterative Relative Fuzzy Connectedness for Multiple Objects, Allowing Multiple Seeds. *Computer Vision and Image Understanding*, to appear; Technical Report MIPG-327, Medical Image Processing Group, Department of Radiology, University of Pennsylvania, Philadelphia, 2005; for preprint see http://jacobi.math.wvu.edu/~kcies/publications.html

[19] Y. Zhuge, J.K. Udupa, and P.K. Saha, Vectorial scale-based fuzzy connected image segmentation. *Computer Vision and Image Understanding* **101** (2006), 177–193.

[20] A. Pednekar, I.A. Kakadiaris, Image segmentation based on fuzzy connectedness using dynamic weights. *IEEE Trans Image Process.* **15**(6) (2006), 1555–1562.

[21] X. Fan, J. Yang, and L. Cheng, A novel segmentation method for MR brain images based on fuzzy connectedness and FCM. *Lecture Notes in Computer Science* **3613** (2005), 505–513.

[22] B.M. Carvalho, C.J. Gau, G.T. Herman, and Y.T. Kong, Algorithms for fuzzy segmentation. *Pattern Analysis and Applications* **2** (1999), 73–81.

[23] B.M. Carvalho, G.T. Herman, and Y.T. Kong, Simultaneous fuzzy segmentation of multiple objects. *Discrete Applied Mathematics* **151** (2005), 65–77.

[24] J. Betancur, F. Prieto, Fuzzy Connectedness applied to Coffee Fruit Image Segmentation, preprint.

[25] K.C. Ciesielski and J.K. Udupa, A general theory of image segmentation and its practical applications. Technical Report MIPG-335, Medical Image Processing Group, Department of Radiology, University of Pennsylvania, Philadelphia, 2006.

[26] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Stud. Texts **39**, Cambridge Univ. Press, Cambridge, 1997.

[27] K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1983.

[28] P.K. Saha and J.K. Udupa, Fuzzy connected Object Delineation: Axiomatic Path Strength Definition and the Case of Multiple Seeds. *Computer Vision and Image Understanding* **83** (2001), 275–295.

[29] J.K. Udupa and P.K. Saha, Fuzzy Connectedness. In: *Insight into Images: Principles and Practice for Segmentation, Registration, and Image Analysis*, ed. Terry Yoo, A K Peters Ltd (2004).

[30] P.K. Saha, J.K. Udupa, and D. Odhner, Scale-Based Fuzzy Connectedness Image Segmentation: Theory, Algorithms, and Validation. *Computer Vision and Image Understanding* **77** (2000), 145–174.

[31] A. Rosenfeld, Fuzzy digital topology. *Information and Control* **40** (1979), 76–87.

[32] A. Rosenfeld, On connectivity properties of grayscale pictures. *Pattern Recognition* **16** (1983), 47–50.

[33] A. Rosenfeld, The fuzzy geometry of image subsets. *Pattern Recognition Letters* **2** (1984), 311–317.

[34] L. Najman, M. Couprie, and G. Bertrand, Watersheds, mosaics and the emergence paradigm. *Discrete Applied Math.* **147** (2005), 301–324.

[35] R. O. Duda, P. E. Hart, D. G. Stork, *Pattern Classification*, (2nd Edition), John Wiley and Sons, 2001.

[36] Seong-Jee Lim, *Hybrid Method of Fuzzy Connectedness and Active Shape Model Algorithms for Three-dimensional Medical Image Segmentation*. PhD Thesis, Kwangju Institute of Science and Technology (South Korea), Department of Information & Communications, March 30, 2006.

[37] J.K. Udupa and Y. Zhuge, Delineation operating characteristic (DOC) curve for assessing the accuracy behavior of image segmentation algorithms, *SPIE Proceedings* **5370**(1) (2004), 640-647.