Affinity functions: recognizing essential parameters in fuzzy connectedness based image segmentation

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ABSTRACT

Fuzzy connectedness (FC) constitutes an important class of image segmentation schemas. Although affinity functions represent the core aspect (main variability parameter) of FC algorithms, they have not been studied systematically in the literature. In this paper, we present a thorough study to fill this gap. Our analysis is based on the notion of equivalent affinities: if any two equivalent affinities are used in the same FC schema to produce two versions of the algorithm, then these algorithms are equivalent in the sense that they lead to identical segmentations. We give a complete characterization of the affinity equivalence and show that many natural definitions of affinity functions and their parameters used in the literature are redundant in the sense that different definitions and values of such parameters lead to equivalent affinities. We also show that two main affinity types — homogeneity based and object feature based — are equivalent, respectively, to the difference quotient of the intensity function and Rosenfeld's degree of connectivity. In addition, we demonstrate that any segmentation obtained via relative fuzzy connectedness (RFC) algorithm can be viewed as segmentation obtained via absolute fuzzy connectedness (AFC) algorithm with an automatic and adaptive threshold detection. We finish with an analysis of possible ways of combining different component affinities that result in non equivalent affinities.

1. AFFINITIES EQUIVALENT IN THE FC SENSE

Image segmentation — the process of partitioning the image domain into meaningful object regions — is perhaps the most challenging and critical problem in image processing and analysis. The segmentation framework discussed in the present paper belongs to the region-based group of methods and constitutes an extension of the fuzzy connectedness (abbreviated from now on as FC) methodology [9].

In the FC framework, a fuzzy topological construct, called fuzzy connectedness, characterizes how the *spels* (short for *spatial elements*) of an image hang together to form an object. This construct is arrived at roughly as follows. A function called *affinity* is defined on the set $C \times C$ of all pairs of spels from the image domain C; the strength of affinity between any two spels depends on how close the spels are spatially and how similar their intensity-based properties are in the image. Affinity is intended to be a local relation. A global fuzzy relation called *fuzzy connectedness* is induced on the image domain by affinity as follows. For any two spels c and d in the image domain, all possible paths connecting c and d are considered. Each path is assigned a strength of fuzzy connectedness: the minimum of the affinities of consecutive spels along the path. The level of fuzzy connectedness between c and d is considered to be the maximum of the strengths of all paths between c and d.

For segmentation purposes, FC is utilized in three main ways. (See also Section 1.3.) In *absolute FC* (AFC) [9], the support of a segmented object is considered to be the maximal set of spels, containing one or more seed spels, within which the level of FC is at or above a specific threshold. To obviate the need for a threshold, *relative FC* (or RFC) [4] was developed by letting all objects in the image to compete simultaneously via FC to claim membership of spels in their sets. Each co-object is identified by one or more seed spels. Any spel c in the image domain is claimed by that co-object with respect to whose seed spels c has the largest level of FC compared to the level of FC with the seed sets of all other objects. To avoid treating the core aspects of an object and the peripheral subtle aspects in the same footing, an iterative refinement strategy is devised in *iterative RFC* (or IRFC) [2, 4, 5]. See [7] for a review of the different FC definitions and how they are employed in segmentation and applications.

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The main purpose of this section is to uncover the essence of the relationship between the local measure of connectedness of pairs of spels, the affinity function, and the resulting segmentations obtained via FC algorithms. In particular, we will introduce the notion of the *equivalence (in the sense of FC) of the affinities* and show that equivalent affinities are indistinguishable from the point of view of FC segmentations, no matter what the empirical results indicate. Due to the space limitation, the proofs of presented theoretical results will be deferred to a full, journal version of this paper.

To make this work complete and useful, our definition of the affinity function will be more general than the one commonly used in the literature. However, we will show that each class of equivalent affinities contains at least one standard (meaning commonly used) affinity.

1.1. Preliminary definitions

We will use the following interpretation of the notions of (hard) functions and relations, which is standard in set theory (see e.g. [1]) and is used in many calculus books. A binary relation R from a set X to a set Y is identified with its graph; that is, R *equals* $\{\langle x, y\rangle \in X \times Y : xRy \text{ holds}\}$. Since a function $f : X \to Y$ is a (special) binary relation from X to Y, in particular we have $f = \{\langle x, f(x)\rangle : x \in X\}$. With this interpretation, fuzzy sets and fuzzy relations have the following representations. Let $\mathcal{Z}$ be a *fuzzy subset* of a hard set X with a membership function $\mu_{\mathcal{Z}} : X \to [0, 1]$. For each $x \in X$ we interpret $\mu_{\mathcal{Z}}(x)$ as the degree to which x belongs to $\mathcal{Z}$. Usually such a fuzzy set $\mathcal{Z}$ is defined as $\{\langle x, \mu_{\mathcal{Z}}(x)\rangle : x \in X\}$, which is the graph of $\mu_{\mathcal{Z}}$. Thus, according to our interpretation, $\mathcal{Z}$ actually equals $\mu_{\mathcal{Z}}$. Note that this interpretation agrees quite well with the situation when $\mathcal{Z}$ is a hard subset Z of X, as then $\mathcal{Z} = \mu_{\mathcal{Z}}$ is equal to the *characteristic function* χ_Z of Z (defined as $\chi_Z(x) = 1$ for $x \in Z$ and $\chi_Z(x) = 0$ for $x \in X \setminus Z$), and the identification of Z with χ_Z is quite common in analysis and set theory. Notice also that a fuzzy binary relation ρ from X to Y is just a fuzzy subset of $X \times Y$, so it is equal to its membership function $\mu_\rho : X \times Y \to [0, 1]$.

Let $\mathfrak{n} \geq 2$ and let $\mathbb{Z}^{\mathfrak{n}}$ stand for the set of all $\mathfrak{n}$-tuples of integer numbers. A binary fuzzy relation α on $\mathbb{Z}^{\mathfrak{n}}$ is said to be a *fuzzy adjacency* if α is *symmetric* (i.e., $\alpha(c, d) = \alpha(d, c)$) and *reflexive* (i.e., $\alpha(c, c) = 1$). The value of $\alpha(c, d)$ depends only on the relative spatial position of c and d. In most applications, α is just a hard case relation like 4-adjacency relation for $\mathfrak{n} = 2$ or 6-adjacency in the three-dimensional case, defined as $\alpha(c, d) = 1$ for $||c, d|| \leq 1$ and $\alpha(c, d) = 0$ for $||c, d|| > 1$. For $k \geq 1$, a *scene* over a fuzzy digital space $\langle \mathbb{Z}^{\mathfrak{n}}, \alpha\rangle$ is a pair $\mathcal{C} = \langle C, f\rangle$, where $C = \prod_{j=1}^{\mathfrak{n}}[-b_j, b_j] \subset \mathbb{Z}^{\mathfrak{n}}$, each $b_j > 0$ being an integer, and $f : C \to \mathbb{R}^k$ is a *scene intensity function*. The notion most important for this paper is that of an affinity function. The affinity function, defined in its general form in the next subsection, is usually denoted by κ and it associates with any pair $\langle c, d\rangle \in C \times C$ of spels the strength $\kappa(c, d)$ of their local hanging togetherness in $\mathcal{C}$. Within this class, a special role is played by *standard affinities*, that is, mappings $\kappa : C \times C \to [0, 1]$ which, treated as fuzzy binary relations, are symmetric and reflexive. In all practical applications, the value of $\kappa(c, d)$ depends on the adjacency strength $\alpha(c, d)$ of c and d (i.e., on the spatial relative position of c and d) as well as on the intensity function f. So far, only standard affinities have been used in applications in the literature. Of those, the most prominent are (see [6] and Sec. 2): **homogeneity based affinity:** $\psi_\sigma(c, d) = \alpha(c, d)\ e^{-||f(c)-f(d)||^2/\sigma^2}$, where $\sigma > 0$ and $c, d \in C$, with its value being close to 1 when the spels are spatially close and have very similar intensity values (i.e. are well connected); **object feature based affinity** (single object case, with an expected intensity m for the object)**:** $\phi_\sigma(c, d) = \alpha(c, d)\ e^{-\max\{||f(c)-m||, ||f(d)-m||\}^2/\sigma^2}$, where $\sigma > 0$ and $c, d \in C$, with its value being close to 1 when the spels are spatially close and both have intensity values close to m.

It has been demonstrated [4] that, in the standard FC algorithms of AFC and RFC, to fulfill certain desirable properties of segmentations (such as robustness with respect to seed points), affinities must be symmetric. In this paper, therefore, we will restrict ourselves to symmetric affinities. However, we will go quite afar from previous publications otherwise in considering affinity in its very general form.

1.2. Equivalent affinities

In this subsection, we define the notion of the affinity function in its general form, without just confining to the basis of standard affinities (as defined above ψ_σ and ϕ_σ) and introduce the concept of equivalent affinities. The motivation for developing equivalent affinities comes from our desire to recognize those differences among affinities that are inessential, and therefore lead to the same FC segmentations, from those that are essential and may give rise to different segmentations.

Let $\preceq$ be a linear order relation [1] on a set L and let C be an arbitrary finite non-empty set. We say that a function $\kappa\colon C\times C\to L$ is an *affinity function* (from C into $\langle L,\preceq\rangle$) provided κ is symmetric (i.e., $\kappa(a,b)=\kappa(b,a)$ for every $a,b\in C$) and $\kappa(a,b)\preceq\kappa(c,c)$ for every $a,b,c\in C$. Clearly, any standard affinity, as defined above, is an affinity function with $\langle L,\preceq\rangle=\langle[0,1],\leq\rangle$. Note that $\kappa(d,d)\preceq\kappa(c,c)$ for every $c,d\in C$. So, there exists an element in L, which we will denote by a symbol $\mathbf{1}_\kappa$, such that $\kappa(c,c)=\mathbf{1}_\kappa$ for every $c\in C$. Notice that $\mathbf{1}_\kappa$ is the largest element of $L_\kappa=\{\kappa(a,b)\colon a,b\in C\}$, although it does not need to be the largest element of L. In what follows, the strict inequality related to $\preceq$ will be denoted by $\prec$, that is, $a\prec b$ if and only if $a\preceq b$ and $a\neq b$.

Certainly, in image processing, C will be always the domain of the scene intensity function. In all specific cases used in this paper, we will take $\langle L,\preceq\rangle$ as either the standard range $\langle[0,1],\leq\rangle$ or, more often, $\langle[0,\infty],\geq\rangle$. In the second case, the order relation $\preceq$ is the *reversed* standard order relation $\geq$. We say that the affinities $\kappa_1\colon C\times C\to\langle L_1,\preceq_1\rangle$ and $\kappa_2\colon C\times C\to\langle L_2,\preceq_2\rangle$ are *equivalent (in the FC sense)* provided, for every $a,b,c,d\in C$

$$\kappa_1(a,b)\preceq_1\kappa_1(c,d)\quad\text{if and only if}\quad\kappa_2(a,b)\preceq_2\kappa_2(c,d)$$

or, equivalently: $\kappa_1(a,b)\prec_1\kappa_1(c,d)$ if and only if $\kappa_2(a,b)\prec_2\kappa_2(c,d)$. For example, it can be easily seen that for any constants $\sigma,\tau>0$ the homogeneity based affinities ψ_σ and ψ_τ, as defined above, are equivalent, since for any pairs $\langle a,b\rangle$ and $\langle c,d\rangle$ of adjacent spels we have (symbol $\Leftrightarrow$ means "if and only if"):

$$\psi_\sigma(a,b)<\psi_\sigma(c,d)\ \Leftrightarrow\ \ ||f(a)-f(b)||>||f(c)-f(d)||\ \Leftrightarrow\ \ \psi_\tau(a,b)<\psi_\tau(c,d).\tag{1}$$

In the following characterization, $\circ$ stands for the composition of functions, that is, $(g\circ\kappa_1)(a,b)=g(\kappa_1(a,b))$.

Proposition 1.1. *Affinities $\kappa_1\colon C\times C\to\langle L_1,\preceq_1\rangle$ and $\kappa_2\colon C\times C\to\langle L_2,\preceq_2\rangle$ are equivalent if and only if there exists a strictly increasing function g from $\langle L_{\kappa_1},\preceq_1\rangle$ onto $\langle L_{\kappa_2},\preceq_2\rangle$ such that $\kappa_2=g\circ\kappa_1$.*

One of the specific conclusions from Proposition 1.1 is the following fact.

Corollary 1.2. *If $\kappa\colon C\times C\to\langle[0,\infty],\geq\rangle$ is an affinity, then, for every strictly decreasing function g from $[0,\infty]$ onto $[0,1]$, a map $g\circ\kappa\colon C\times C\to\langle[0,1],\leq\rangle$ is an affinity equivalent to κ.*

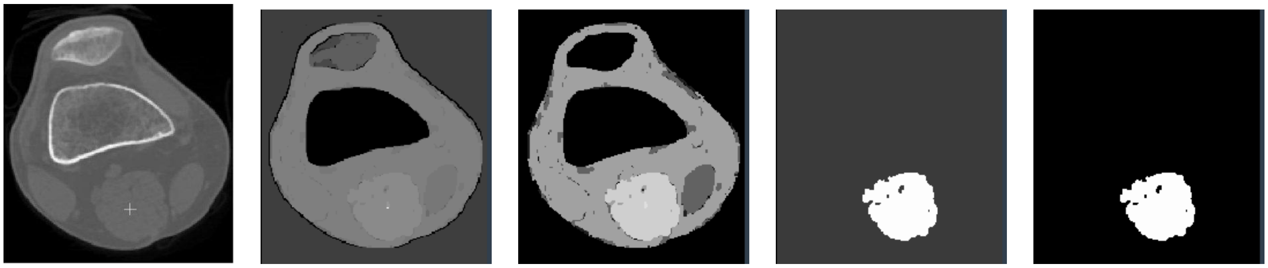
Our interest in equivalent affinities comes from the fact (see Theorem 1.5) that any FC segmentation of a scene $\mathcal{C}$ remains unchanged if an affinity on C used to get the segmentation is replaced by an equivalent affinity. Keeping this in mind, it makes sense to find for each affinity function an equivalent affinity in a nice form:

Theorem 1.3. *Every affinity function is equivalent (in the FC sense) to a standard affinity.*

Once we agree that equivalent affinities lead to the same segmentations, Theorem 1.3 says that we can restrict our attention to standard affinities without losing any generality of our method. Thus, one may wonder why study other affinities at all. The answer to this question is simple — in most cases, it is more natural to define an affinity function with more abstract range, and any translation of such affinity to the standard one is a redundant step adding only unnecessary computational burden, although some researchers may believe, that it helps intuitive understanding. Moreover, in some of these cases there is no simple (i.e., continuous) translation of the natural affinity to the standard one. (See Example 3.2.) On the other hand, Theorems 1.3 and 1.5 tell us that all the theoretical results that are true for the standard affinities hold also for the affinities as we defined them. Thus, there is no particular reason to restrict our attention to the affinities in the standard form.

The next example shows an application of the above described theory. (For more examples, see Sec. 3.)

Example 1.4. For a scene $\mathcal{C}=\langle C,f\rangle$, a natural form of the *homogeneity based affinity* is a function $\psi\colon C\times C\to\langle[0,\infty],\geq\rangle$ given by $\psi(c,d)=||f(c)-f(d)||$ for adjacent spels $c,d\in C$ and $\psi(c,d)=\infty$ otherwise. (See also (2).) The more commonly used version of the homogeneity based affinity is the standard affinity $\psi_\sigma(c,d)=e^{-\psi(c,d)^2/\sigma^2}$ (see above), which is the composition of ψ with the Gaussian function $g_\sigma(x)=e^{-x^2/\sigma^2}$. Note that, by Corollary 1.2, ψ and ψ_σ are equivalent, independently of the value of the parameter σ, since g_σ is strictly decreasing from $[0,\infty]$ onto $[0,1]$. In particular, the parameter σ in the definition of ψ_σ is totally inessential from the FC segmentation point of view, as described below. For now, it is enough to understand that, intuitively, varying σ essentially results in a different scaling of the strength of connectedness. Therefore, for example, the same segmentation of a given image is obtained by using AFC algorithm with (a) affinity ψ and threshold θ; (b) affinity ψ_σ and threshold $g_\sigma(\theta)$, independently of the value of σ. This phenomenon is illustrated in Figure 1 on a 2D scene — a CT slice of a human knee, Fig. 1(a). In Figs. 1 (d) and (e) segmented binary scenes are shown, resulting from the use of ψ_σ with $\sigma=1$ and $\sigma=10.8$, respectively, and the corresponding thresholds $g_\sigma(\theta)$. The results are identical. Figs. 1(b) and (c) show the corresponding connectivity scenes, in which the intensity of each spel c represents the ψ_σ-connectivity strength between the seed and c (i.e., the strength of the strongest path joining the seed and c).



(a) (b) (c) (d) (e)

Fig. 1. Illustration of equivalent affinities. (a) A 2D scene — a CT slice of a human knee. (b), (c) Connectivity scenes corresponding to affinities ψ_σ with $\sigma = 1$ and $\sigma = 10.8$, respectively, and the same seed spel (indicated by + in (a)) specified in a soft tissue region of the scene in (a). (d), (e) Identical AFC objects obtained from the scenes in (b) and (c).

1.3. FC segmentations for equivalent affinities

Fix an affinity $\kappa\colon C \times C \to \langle L, \preceq\rangle$. To define fuzzy connectedness segmentation of $\mathcal{C}$, we need first to translate the local measure of connectedness given by κ into the global strength of connectedness. For this, we will need the notions of a path and its strength. A *path in* $A \subseteq C$ is any sequence $p = \langle c_1, \ldots, c_l\rangle$, where $l > 1$ and $c_i \in A$ for every $i = 1, \ldots, l$. The family of all paths in A is denoted by $\mathcal{P}^A$. If $c, d \in A$, then the family of all paths $\langle c_1, \ldots, c_l\rangle$ in A from c to d (i.e., such that $c_1 = c$ and $c_l = d$) is denoted by $\mathcal{P}^A_{cd}$. The *strength* $\mu_\kappa(p)$ *of a path* $p = \langle c_1, \ldots, c_l\rangle \in \mathcal{P}^C$ is defined as the strength of its κ-weakest link; that is, $\mu_\kappa(p) = \min\{\kappa(c_{i-1}, c_i)\colon 1 < i \le l\}$. For $c, d \in A \subseteq C$, the (global) κ*-connectedness strength in* A *between* c *and* d is defined as the strength of a strongest path in A between c and d; that is, $\mu^A_\kappa(c, d) = \max\left\{\mu_\kappa(p)\colon p \in \mathcal{P}^A_{cd}\right\}$. Notice that $\mu^A_\kappa(c, c) = \mu_\kappa(\langle c, c\rangle) = \mathbf{1}_\kappa$. We will often refer to the function $\mu^A_\kappa\colon C \times C \to L$ as a *connectivity measure* (on A) induced by κ. For $c \in A \subset C$ and a non-empty $D \subset A$, we also define $\mu^A_\kappa(c, D) = \max_{d\in D} \mu^A_\kappa(c, d)$. We will write μ for μ_κ and μ^A for μ^A_κ when κ is clear from the context. Note that if κ is a hard binary relation, then μ^C_κ is a relation (or, more precisely, its characteristic function) known as a *transitive closure of* κ, which is defined as the set of all pairs $\langle c, d\rangle \in C \times C$ for which there exists a sequence $c = c_0, c_1, \ldots, c_m = d$ such that $\kappa(c_i, c_{i+1}) = 1$ for every $i < m$.

To define fuzzy objects delineated by FC segmentations, we start with a family $\mathcal{S}$ of non-empty pairwise disjoint subsets of C, where each $S \in \mathcal{S}$ represents a set of spels, known as *seeds*, which will belong to the object generated by it. Also, fix a threshold $\theta \in L$, $\theta \le \mathbf{1}_\kappa$. For every $S \in \mathcal{S}$, put $W = \bigcup(\mathcal{S}\setminus\{S\})$ and, similarly as in [2], define $P^\kappa_{S\theta} = \left\{c \in C\colon \theta \preceq \mu^C_\kappa(c, S)\right\}$; $P^\kappa_{S\mathcal{S}} = \left\{c \in C\colon \mu^C_\kappa(c, W) \prec \mu^C_\kappa(c, S)\right\}$; and $P^{I\kappa}_{S\mathcal{S}} = \bigcup_{i=0}^\infty P^{i,\kappa}_{S\mathcal{S}}$, where sets $P^{i,\kappa}_{S\mathcal{S}}$ are defined inductively by the formulas $P^{0,\kappa}_{S\mathcal{S}} = \emptyset$ and $P^{i+1,\kappa}_{S\mathcal{S}} = P^{i,\kappa}_{S\mathcal{S}} \cup \{c \in C \setminus P^{i,\kappa}_{S\mathcal{S}}\colon \mu_\kappa^{C\setminus P^{i,\kappa}_{S\mathcal{S}}}(c, W) \prec \mu^C_\kappa(c, S)\}$. Then AFC, RFC, and IRFC segmentations of C are defined, respectively, as $\mathbb{P}^\theta_\kappa(\mathcal{S}) = \{P^\kappa_{S\theta}\colon S \in \mathcal{S}\}$, $\mathbb{P}_\kappa(\mathcal{S}) = \{P^\kappa_{S\mathcal{S}}\colon S \in \mathcal{S}\}$, and $\mathbb{P}^I_\kappa(\mathcal{S}) = \left\{P^{I\kappa}_{S\mathcal{S}}\colon S \in \mathcal{S}\right\}$. Notice that an AFC object $P^\kappa_{S\theta}$ consists of all spels connected with at least one seed s in S with the κ-connectivity strength at least θ. An RFC object is created via competition of seeds for each spel: a spel c belongs to $P^\kappa_{S\mathcal{S}}$ provided there is a seed s in S for which the κ-connectivity between c and s exceeds the κ-connectivity between c and any other seed indicating another object. Finally, IRFC objects are obtained by refining the RFC competition: a spel c is unassigned to any RFC object provided there is a tie between two seeds s and t from different objects, e.g., $\mu^C_\kappa(c, w) \preceq \mu^C_\kappa(c, s) = \mu^C_\kappa(c, t)$ for any seed w. However, such a tie can be resolved if the strongest paths justifying $\mu^C_\kappa(c, s)$ and $\mu^C_\kappa(c, t)$ cannot pass through the spels already assigned to another object. Upon such resolution, the spel under question is assigned to the winning object in the next iteration of IRFC.

Now we can formalize our previous claim that the fuzzy connectedness segmentations (i.e., those obtained via AFC, RFC, and IRFC algorithms) are unchanged if an affinity function is replaced by an equivalent one.

THEOREM 1.5. *Let* $\kappa_1\colon C \times C \to \langle L_1, \preceq_1\rangle$ *and* $\kappa_2\colon C \times C \to \langle L_2, \preceq_2\rangle$ *be equivalent affinity functions and let* $\mathcal{S}$ *be a family of non-empty pairwise disjoint subsets of* C*. Then for every* $\theta_1 \preceq_1 \mathbf{1}_{\kappa_1}$ *in* L_1*, there exists a* $\theta_2 \preceq_2 \mathbf{1}_{\kappa_2}$ *in* L_2 *such that, for every* $S \in \mathcal{S}$ *and* $i \in \{0, 1, 2, \ldots\}$*, we have* $P^{\kappa_1}_{S\theta_1} = P^{\kappa_2}_{S\theta_2}$, $P^{\kappa_1}_{S\mathcal{S}} = P^{\kappa_2}_{S\mathcal{S}}$*, and* $P^{i,\kappa_1}_{S\mathcal{S}} = P^{i,\kappa_2}_{S\mathcal{S}}$*. In particular,* $\mathbb{P}^{\theta_1}_{\kappa_1}(\mathcal{S}) = \mathbb{P}^{\theta_2}_{\kappa_2}(\mathcal{S})$, $\mathbb{P}_{\kappa_1}(\mathcal{S}) = \mathbb{P}_{\kappa_2}(\mathcal{S})$*, and* $\mathbb{P}^I_{\kappa_1}(\mathcal{S}) = \mathbb{P}^I_{\kappa_2}(\mathcal{S})$.

In summary, Theorem 1.3 says that for every affinity function there is a standard affinity equivalent to it, while Theorem 1.5 says that for any two equivalent affinities we get the same FC segmentations in each of AFC, RFC, and IRFC. To further illustrate this, we examine the example in Fig. 1 for AFC by using two affinities ψ_σ, with $\sigma = 1$ and $\sigma = 10.8$. Figures 1(b) and (c) display the connectivity scenes $\mathcal{C}_\kappa = \langle C, f_k\rangle$ for the 2D scene of

Fig. 1(a), where for any $c \in C$ and the same fixed spel $s \in C$, $f_\kappa(c) = \mu_\kappa^C(c,s)$, where κ is either ψ_1 or $\psi_{10.8}$. The resulting identical AFC objects are displayed in (d) and (e) as binary scenes. Of course, different thresholds were used in producing scenes (d) and (e) from those in (b) and (c), respectively, which precisely makes our point that segmented object *information* in Figures 1(b) and (c) is identical.

Practical Considerations The equivalence theorems say that, if a function g is strictly monotone, then the affinities κ and $g \circ \kappa$ are equivalent and they lead to identical segmentations. However, the segmentations are insured to be identical only when there are no rounding errors. In actual implementations, it is possible that for distinct numbers x and y in the range of κ, the actual values $g(x)$ and $g(y)$ are so close that the implemented algorithm identifies $g(x)$ with $g(y)$. In such implementations some information is lost when passing from κ to $g \circ \kappa$, which may lead to different segmentations. This problem must be considered, when performing any experimental comparisons. Note also that, even when there is no rounding error in the algorithm that influences our theoretical results, a human operator may have an impression that some information is lost when passing from κ to $g \circ \kappa$, due to the limited resolution perception of the human eye. This phenomenon can be noticed in Figures 1(b) and (c): it is easier for human eye to identify the object in Fig. 1(c) than it is in Fig. 1(b).

Theorems 1.3 and 1.5 also imply that any result proved for the FC segmentations in the context of standard affinities remains valid for the affinities in our general setting, that is, the FC algorithms used with our general affinities have all nice properties that the FC algorithms have when used with the standard affinities.

1.4. Relative fuzzy connectedness, RFC, segmentation as absolute FC, segmentation

In AFC, to obtain the FC object $P^\kappa_{S\theta}$, a threshold θ for the strength of connectedness must be specified. This threshold is obviated in defining RFC objects $P^\kappa_{S\mathcal{S}}$ (see definition above) simply by determining the membership of a spel c in an object by its largest strength of connectedness with respect to the seed sets assigned to the different objects. In this subsection, we will show that the RFC segmentation can be viewed to some extent as an AFC segmentation wherein the required threshold is determined automatically.

Theorem 1.6. *Let $\kappa\colon C \times C \to \langle L, \preceq\rangle$ be an arbitrary affinity function and $\mathcal{S}$ be a non-empty family of pairwise disjoint, non-empty sets of seeds in C. Fix an $S \in \mathcal{S}$ and let $W = \bigcup(\mathcal{S} \setminus \{S\})$. For every $s \in S$ let $\theta_s = \mu^C_\kappa(s, W)$. Then $P_{S\mathcal{S}} = \bigcup_{s\in S}\bigcup_{\theta_s \prec \theta} P_{\{s\}\theta}$.*

For an affinity $\kappa\colon C \times C \to \langle L, \preceq\rangle$ and $\theta < \mathbf{1}_\kappa$, let θ^+ be the smallest element of $L_\kappa = \{\kappa(a,b)\colon a, b \in C\}$ greater than θ; that is, $\theta^+ \stackrel{\text{def}}{=} \min\{\rho \in L_\kappa\colon \theta \prec \rho\}$.

Theorem 1.6 has the nicest form when each object is initiated by just one single seed spel.

Corollary 1.7. *Let $\langle C, \kappa, \preceq\rangle$ be an arbitrary affinity structure and $\mathcal{S}$ be a non-empty family of singletons in C such that $\mu^C_\kappa(s,t) \neq \mathbf{1}_\kappa$ for every distinct $S = \{s\}$ and $T = \{t\}$ from $\mathcal{S}$. For $S = \{s\} \in \mathcal{S}$, let $\theta_S = \mu^C_\kappa(s, \bigcup(\mathcal{S} \setminus \{S\}))$. Then $P_{S\mathcal{S}} = P_{S\theta_S^+}$ for every $S \in \mathcal{S}$. In particular, $\mathbb{P}_\kappa(\mathcal{S}) = \{P_{S\theta_S^+}\colon S \in \mathcal{S}\}$.*

Notice that if for a family $\mathcal{S}$ containing only singletons there exist distinct $S, T \in \mathcal{S}$ such that $\mu^C_\kappa(S,T) \stackrel{\text{def}}{=} \max_{s\in S}\mu^C_\kappa(s,T) = \mathbf{1}_\kappa$, then $P_{S\mathcal{S}} = P_{T\mathcal{S}} = \emptyset$. That is, in this case, S and T are in the same object, and therefore, the sets that contain S and T and that separate them in the FC sense are obviously empty. Thus, in all practical cases we are interested only in the families $\mathcal{S}$ of seeds for which $\mu^C_\kappa(S,T) \neq \mathbf{1}_\kappa$ for any distinct $S, T \in \mathcal{S}$. Thus, this assumption in Corollary 1.7 does not really restrict its usefulness, but warrants it from practical requirements that the different seeds do not all belong to the same "object."

If $\mathcal{S}$ from Corollary 1.7 has just two elements, say $\mathcal{S} = \{\{s\},\{t\}\}$, then $\theta_{\{s\}} = \theta_{\{t\}}$ and for $\theta = \theta^+_{\{s\}}$ we have $\mathbb{P}_\kappa(\mathcal{S}) = \{P_{S\theta}\colon S \in \mathcal{S}\} = \mathbb{P}^\theta_\kappa(\mathcal{S})$. Thus, in this case, the RFC segmentation $\mathbb{P}_\kappa(\mathcal{S})$ is just an AFC segmentation $\mathbb{P}^\theta_\kappa(\mathcal{S})$, where θ was automatically set by the RFC procedure. However, when there are more than two objects involved in RFC and $\mathcal{S}$ contains three or more singletons, the thresholds θ^+_S, $S \in \mathcal{S}$, need not be equal. In this case each $P_{S\mathcal{S}}$ from $\mathbb{P}_\kappa(\mathcal{S})$ is an AFC object $P_{S\theta_S^+}$, where the different thresholds are automatically tailored to the different objects under consideration. That is the beauty of RFC compared to AFC.

We illustrate this property of RFC vis-a-vis AFC in a schematic (Figure 2), as well as in an actual medical image (Figure 3). In both figures, we consider three objects, indicated by seeds s, t, and u. In Figure 2, region W is more strongly connected to seed u than to either s or t. As such, RFC correctly assigns it to the region $P_{u,\{s,t,u\}}$ indicated by u, as shown in Fig. 2(b). However, there is no single threshold that could lead to an AFC segmentation coinciding with the RFC segmentation: a threshold θ below $(.6)^+$ causes objects $P_{s,\theta}$ and $P_{t,\theta}$ to be equal and too big, as shown in Fig. 2(d), while $\theta \geq (.6)^+$ cuts region W from $P_{u,\theta}$, see Fig. 2(c). Nevertheless, every RFC delineated object is also equal to appropriate AFC object: $P_{s,\{s,t,u\}} = P_{s,(.6)^+}$, $P_{t,\{s,t,u\}} = P_{t,(.6)^+}$, and $P_{u,\{s,t,u\}} = P_{u,(.5)^+}$.

In Fig. 3, we concentrate on the objects indicated by seeds s and t, corresponding to soft tissue regions. The third object is the rest of the background and is denoted by seed u. The 2D scene is the one employed in Fig. 1. Identical seed spels denoted by $+$'s in Fig. 3(a) were specified for AFC and RFC. The two connectivity scenes corresponding to the two AFC objects are displayed in Fig. 3(b) and (c), and the resulting AFC objects obtained with two different thresholds θ_S^+ from the scenes in (b) and (c) are shown in Fig. 3(e) and (f). The RFC objects obtained appear in Fig. 3(d), wherein the two objects of interest are identical to the AFC objects in (e) and (f).

Note also that the main reason we could represent RFC objects in terms of AFC objects was that two appearances of c in the inequality $\mu_\kappa^C(c,w) \prec \mu_\kappa^C(c,s)$ could be reduced to one: $\mu_\kappa^C(s,w) \prec \mu_\kappa^C(c,s)$, as both these inequalities are equivalent. In the case of IRFC, the defining inequality is $\mu_\kappa^A(c,w) \prec \mu_\kappa^C(c,s)$ for an appropriate $A \subset C$, and there is no equivalent form of this inequality with just one appearance of c. Thus, no natural AFC representation of IRFC object seems possible. Although increasing sophistication from AFC to RFC to IRFC has been previously demonstrated via segmentation experiments [2, 4, 8], in this section we have now given a mathematical justification of that behavior.

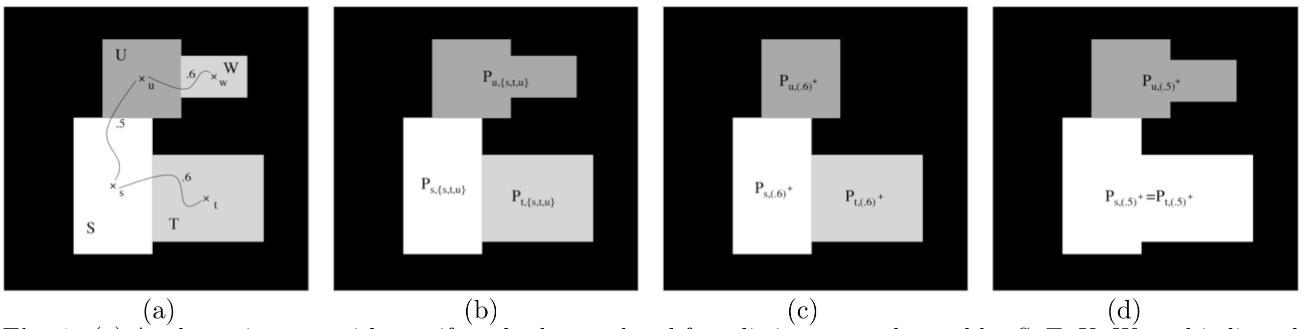


Fig. 2. (a) A schematic scene with a uniform background and four distinct areas denoted by S, T, U, W, and indicated by seeds marked by $\times$. It is assumed that the connectivity strength within each of these areas has the maximal value of 1, the connectivity between the background and any other spel is $\leq .2$, while the connectivity between the adjacent regions is as indicated in the figure: $\mu(s,t) = .6$, $\mu(s,u) = .5$, and $\mu(u,w) = .6$. (b) The RFC segmentation of three objects indicated by seeds s, t, and u, respectively. (c) Three AFC objects indicated by the seeds s, t, u and delineated with threshold $\theta = (.6)^+$. Notice that while $P_{s,\{s,t,u\}} = P_{s,(.6)^+}$ and $P_{t,\{s,t,u\}} = P_{t,(.6)^+}$, object $P_{u,(.6)^+}$ is smaller than RFC indicated $P_{u,\{s,t,u\}}$. (d) Same as in (c) but with $\theta = (.5)^+$. Note that while $P_{u,\{s,t,u\}} = P_{u,(.5)^+}$, objects $P_{s,(.5)^+}$ and $P_{t,(.5)^+}$ coincide and lead to an object bigger than $P_{s,\{s,t,u\}}$ and $P_{t,\{s,t,u\}}$.

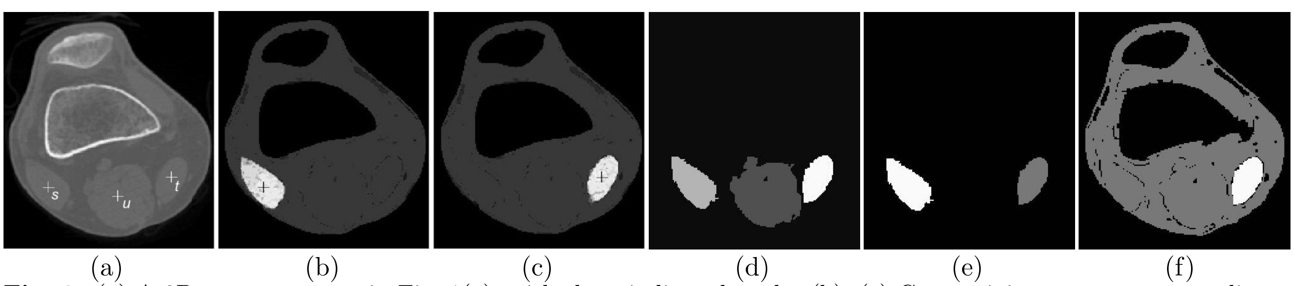


Fig. 3. (a) A 2D scene, same as in Fig. 1(a), with three indicated seeds. (b), (c) Connectivity scenes corresponding to the two AFC objects indicated by s and t. (d) The RFC segmentation for the three indicated objects. (e) The AFC objects initiated with seeds s and t obtained with the threshold $\theta_{\{s\}} < \theta_{\{t\}}$ determined automatically by RFC. Although the result is a binary image, the two objects are shown at two gray levels. The object indicated by seed s agrees with its counterpart in (d). The smaller threshold caused the t-indicated object to be slightly smaller than in (d). (f) Same as (e) but with threshold $\theta_{\{t\}}$. The object indicated by seed t agrees with its counterpart in (d). However, the larger threshold caused the s-indicated object (grey) to leak to a big part of the scene.

2. TWO COMMONLY USED AFFINITIES AND THEIR NATURAL DEFINITIONS

In this section, we briefly describe two main classes of affinities and present a short discussion of their definition. This represents a summary of a detailed discussion to be published in a full, journal version of this paper.

From now on, we will work with a fixed digital space $\langle \mathbb{Z}^n, \alpha \rangle$ and a scene $\mathcal{C} = \langle C, f \rangle$ with a scalar valued intensity function $f : C \to \mathbb{R}$. We assume that the adjacency relation α is a hard relation $\alpha(c,d) = \chi_{[0,1]}(||c-d||)$; that is, $\alpha(c,d) = 1$ when $||c-d|| \le 1$ and $\alpha(c,d) = 0$ for $||c-d|| > 1$, where $||c-d||$ is a distance between c and d. From this point on, we will drop the superscript from μ^C, so that symbol $\mu_\kappa(c,d)$ will stand for $\mu^C_\kappa(c,d)$.

2.1. Homogeneity based affinity

Intuitively, this function, denoted $\psi(c,d)$, is defined as the maximum of $|f'(x)|$, with x on the segment joining c and d (where f' is the derivative of f): the higher the magnitude of the slope of f between c and d is, the weaker is the affinity (connectivity) between c and d. Of course, there is more than one way to interpret the symbol $|f'(x)|$. In this section we will interpret this as a magnitude of the directional derivative $D_{\overrightarrow{cd}} f(x)$ in the direction of the vector $\overrightarrow{cd}$. This agrees with the standard FC approach used in the research conducted so far. (See e.g. [9].) Alternatively, it is possible to treat $|f'(x)|$ as a gradient magnitude. True gradient induced homogeneity based affinity will be incorporated in our future work.

The value $|f'(x)| = |D_{\overrightarrow{cd}} f(x)|$ is best approximated by a difference quotient $\psi_0(c,d) = \left|\frac{f(c)-f(d)}{||c-d||}\right|$. Although this expression has no sense for $c = d$, it should be clear that we should define $\psi_0(c,c)$ as equal to 0, the "highest" possible connectivity in this setting. (Recall that "highest" in terms of $\preceq$ defined as $\ge$ translates into "least" in terms of the standard order $\le$. That is, the greater ψ_0 is, the weaker is the affinity between c and d.)

Is the definition of this affinity as $\psi_0(c,d)$ what we are looking for? Certainly this is not a local measurement of connectedness when $||c-d||$ is large. In this case, the difference quotient is a poor approximation of the definition of the derivative. We also have a better way of estimating the highest slope on the road from c to d: crawl from c to d along a path with steps of length 1, estimating the slope of each step separately. Because of this, it makes sense to consider the number $\psi_0(c,d)$ as a good value for $\psi(c,d)$ only when $||c-d|| \le 1$, in all other cases we should assign to it it the worst possible value; that is, ∞. This leads to the definition $\psi(c,d) = \psi_0(c,d)/\alpha(c,d)$; that is,

$$\psi(c,d) = \begin{cases} |f(c) - f(d)| & \text{for } ||c-d|| \le 1 \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

It is easy to see that ψ satisfies our definition of affinity function. It should be stressed here that such a function approximates only the magnitude of the directional derivative of f in the direction $\overrightarrow{cd}$, and gives no information on the slope of f in a direction perpendicular to $\overrightarrow{cd}$. If one likes to express this affinity by an equivalent standard affinity, our definition of ψ can be replaced by $g_\sigma(\psi(c,d))$, where g_σ is a Gaussian function $g_\sigma(x) = e^{-x^2/\sigma^2}$.

The homogeneity based connectivity measure, $\mu_\psi = \mu^C_\psi$, can be elegantly interpreted if our scene $\mathcal{C} = \langle C, f \rangle$ is considered as a topographical map in which $f(c)$ represents an elevation at the location $c \in C$. Then, $\mu_\psi(c,d)$ is the highest possible step (a slope of f) that one must make in order to get from c to d with each step on a location (spel) from C and of unit length. In particular, the object $P^\psi_{s\theta} = \{c \in C : \theta \ge \mu_\psi(s,c)\}$ represents those spels $c \in C$ which can be reached from s without ever making a step higher than θ. Note that all we measure in this setting is the actual change of the altitude while making the step. Thus, this value can be small, even if the step is made on a very steep slope, as long as the path approximately follows the altitude contour lines — this is why on steep hills the roads zigzag, allowing for a small incline of the motion. On the other hand, the measure of the same step would be large, if measured with some form of gradient induced homogeneity based affinity!

2.2. Object feature based affinity

There are two principal differences between the object feature based and the homogeneity based affinities. (1) The definition of the object feature based affinity requires some prior knowledge on the intensities of the objects we like to uncover, while the definition of the homogeneity based affinity is completely independent of such knowledge. (2) The homogeneity based affinity is represented in terms of (the approximation of) the derivative f' of the intensity function f, while the object feature based affinity is defined directly from f, or it could be some extremely complicated function of f (as in representing texture).

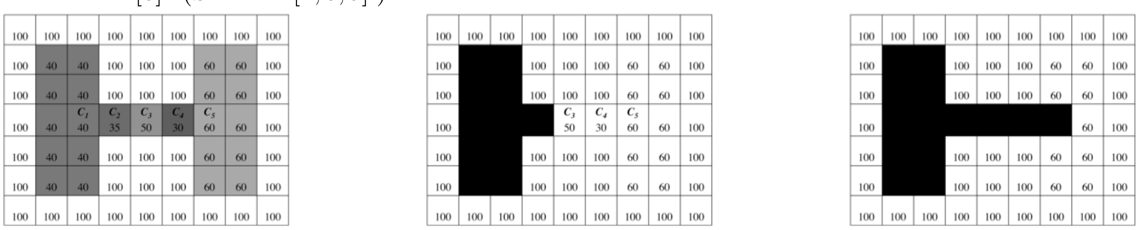
2.2.1. Object feature based affinity: single object case

We will start with the definition of the object feature based affinity, denoted $\phi(c, d)$, in terms of only a single object O. To define ϕ, we need to start with an approximate expected (average) intensity value m for the spels in the object. We will also assume that we have a standard deviation $\sigma > 0$ of the distribution of intensity for this object. Then, the intuition behind ϕ can be expressed with a pseudo-affinity formula $\bar{\varphi}_0(c) = |f(c) - m|$ — the smaller the value of $\bar{\varphi}_0(c)$ is, the closer is c's intensity to the object intensity, and the better c is connected to object O. (Since the range of ϕ is $\langle L, \preceq\rangle = \langle [0, \infty], \geq\rangle$, the notion of "$\preceq$-stronger" translates into "smaller in the $\leq$ sense.") It is also convenient, for facilitating a definition of the object feature based affinity for multiple objects, to rescale this formula to $\bar{\varphi}(c) = |f(c) - m|/\sigma$. This translates to a proper definition of ϕ, as a function on the pairs $\langle c, d\rangle$ of spels, as $\max\{\bar{\varphi}(c), \bar{\varphi}(d)\} = \max\{|f(c) - m|, |f(d) - m|\}/\sigma$ for distinct adjacent c and d, and, in general,

$$\phi(c, d) = \begin{cases} 0 & \text{for } c = d \\ \max\{|f(c) - m|, |f(d) - m|\}/\sigma & \text{for } \|c - d\| = 1 \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

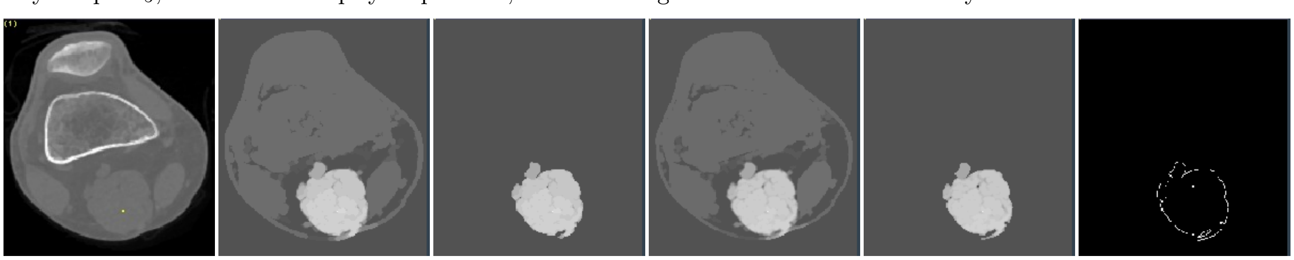
Note that, in reference [6], for distinct adjacent spels c and d, $\phi(c, d)$ is defined as $|(f(c) + f(d))/2 - m|$ in place of $\max\{\bar{\varphi}(c), \bar{\varphi}(d)\}$. Although this carries similar intuitions, the averaging of the values of $f(c)$ and $f(d)$ loses information on how far the intensity of each spel is from m. For example, a difficulty with this definition is shown in Figure 4. Object $P^{\kappa}_{s,6}$ delineated with κ includes spels c_2, c_3, c_4, c_5, but no other spels adjacent to c_5. (The intensity averages of the consecutive spels in the path $\langle c_1, c_2, c_3, c_4, c_5\rangle$ are respectively 37.5, 42.5, 40, 45, that is, closer to $m = 40$ than $\theta = 6$. It does not include any other spel c adjacent to c_5, since for such c the average $\frac{f(c)+f(c_5)}{2} = 60$ is $20 > \theta$ units from m.) Both including the spels c_3, c_4, c_5 in the object as well as after including c_5, excluding other spels adjacent to c_5 defies intuitions behind the object feature based affinity. Notice also that, the object $P^{\phi}_{s,6}$ delineated with ϕ does not include c_3, since $\phi(c_2, c_3) = \max\{|35 - 40|, |50 - 40|\} = 10 > 6 = \theta$.

Once again, we can replace $\phi(c, d)$ with $g_\sigma(\phi(c, d))$ for some Gaussian-like function to get an equivalent affinity in the standard form. In particular, for $g_\sigma(x) = e^{-x^2/\sigma^2}$ this leads to $\bar{\varphi}(c) = \exp(-\frac{(f(c)-m)^2}{\sigma^2})$, one of the formulas used in [6]. (See also [4, 8, 9].)



(a) (b) (c)

Fig. 4. (a) A schematic scene with each rectangular cell representing a single spel. A number in each spel indicates its intensity. We delineate an object indicated by a seed $s = c_1$, assuming that its average intensity is $m = 40$. We also assume $\sigma = 1$. In (b) the shaded area depicts object $P^{\phi}_{s,6}$ (i.e., with $\theta = 6$) delineated with the affinity ϕ defined in (3). The region correctly excludes spel c_3, since the difference between its intensity and m exceeds threshold value $\theta = 6$. The shaded region in (c) represents object $P^{\kappa}_{s,6}$, where $\kappa(c, d) = |(f(c) + f(d))/2 - m|$. Not only it incorrectly leaks all the way to spel c_5, but it also abruptly stops there, after reaching an area of uniform intensity.



(a) (b) (c) (d) (e) (f)

Fig. 5. (a) A 2D scene, same as in Fig. 1(a), with an indicated seed. (b), (c) Connectivity scene and an AFC object corresponding to the indicated seed and affinity $g_\sigma \circ \phi$. (d) and (e): same as in (b) and (c) but for the affinity defined as $g_\sigma\left((f(c) + f(d))/2 - m\right)$. (f) shows the symmetric difference between images (c) and (e).

The difference between ϕ and κ that was illustrated in Figure 4 above is also demonstrated on the CT scene image from Fig. 1 in the following Figure 5. Figs 5 (b) and (c) show, respectively, the connectivity image and AFC object corresponding to the affinity $g_\sigma \circ \phi$ which, for distinct adjacent c and d, is equal to $g_\sigma(\phi(c,d)) = \min\{\exp(-\frac{(f(c)-m)^2}{\sigma^2}), \exp(-\frac{(f(c)-m)^2}{\sigma^2})\}$. Figs 5 (d) and (e) are similar images obtained for the affinity $\hat{\kappa}$ defined, for distinct adjacent c and d, as $g_\sigma\left((f(c)+f(d))/2-m\right)$. The object shown in (f), generated with affinity $\hat{\kappa}$, is slightly bigger than that for $g_\sigma \circ \phi$, shown in (c). Fig. 5(f) shows the symmetric difference between these two segmentation results.

The object feature based connectivity measure of one object has also a nice topographical map interpretation. For understanding this, consider a modified scene $\bar{\mathcal{C}} = \langle C, |f(\cdot)-m|\rangle$ (called *membership scene* in [9]) as a topographical map. Then the number $\mu_\phi(c,d)$ represents the lowest possible elevation (in $\bar{\mathcal{C}}$) which one must reach (a mountain pass) in order to get from c to d, where each step is on a location from C and is of unit length. Notice that $\mu_\phi(c,d)$ is precisely the *degree of connectivity* as defined by Rosenfeld [3]. By the above analysis, we brought Rosenfeld's connectivity also into the affinity framework introduced by [9], particularly as another object feature component of affinity.

2.2.2. Object feature based affinity: case of multiple objects

The single object connectivity measure μ_ϕ can be useful in object definition only if we define it by using absolute connectedness definition, AFC. To find an object via RFC or IRFC methods, we need to have μ_ϕ defined for at least two objects. So, suppose that the scene consists of $n > 1$ objects with expected average intensities $m_1, \ldots, m_n$ and standard deviations $\sigma_1, \ldots, \sigma_n$, respectively. Then we have n different object feature based affinities $\hat{\phi}_i(c,d)$, defined for $c \neq d$ as $\max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\alpha(c,d)$, where $\bar{\varphi}_i(c) = |f(c)-m_i|/\sigma_i$, and their respective connectivity measures $\mu_{\hat{\phi}_i}$. We like to combine affinities $\hat{\phi}_i$ to get the cumulative object feature based affinity ϕ. (Obtaining a single affinity at the end becomes essential in order to fulfill the theoretical requirements of fuzzy connectedness. See [4, 8].) But how to define such a ϕ?

The idea behind the formula for ϕ is to define $\phi(c,d)$ as the best among all numbers $\hat{\phi}_i(c,d)$. One possible choice for $\phi(c,d)$, one used in the literature so far, is $\min_{i=1,\ldots,n} \hat{\phi}_i(c,d)$. The problem with this choice is that we never know which value of $\hat{\phi}_i(c,d)$ was used to determine $\phi(c,d)$. An alternative approach is as follows. Since the values of $\hat{\phi}_i(c,d) = \max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\sigma_i$ are the most valuable when this number is small and because difficulties occur when $\hat{\phi}_i(c,d) = \hat{\phi}_j(c,d)$ for $i \neq j$, we will eliminate the information in $\bar{\varphi}_i(c)$ when this value exceeds $\bar{\varphi}_j(c)$ for some j. This is made formal below.

For distinct $i, j \in \{1, \ldots, n\}$, let $\delta_i^j \geq 0$ be the largest number with the property that $\frac{|x-m_i|}{\sigma_i} < \frac{|x-m_j|}{\sigma_j}$ for every $x \in (m_i - \delta_i^j, m_i + \delta_i^j)$. (If $\sigma_i = \sigma_j$, then δ_i^j is just half of the distance between m_i and m_j.) Thus, if $x_i^j \in \{m_i - \delta_i^j, m_i + \delta_i^j\}$ is between m_i and m_j, then $\bar{\varphi}_i(c) < \frac{|x_i^j - m_i|}{\sigma_i} = \frac{\delta_i^j}{\sigma_i} = \frac{|x_i^j - m_j|}{\sigma_j} < \bar{\varphi}_j(c)$ for each $c \in C$ provided $|f(c)-m_i| < \delta_i^j$. Let $\varepsilon_i = \min_{j \neq i} \delta_i^j$ and $I_i = (m_i - \varepsilon_i, m_i + \varepsilon_i)$. Then intervals I_i, $i \in \{1, \ldots, n\}$, are pairwise disjoint. Function φ_i is defined as a truncation of $\bar{\varphi}_i$ to the interval I_i, that is, by a formula

$$\varphi_i(c) = \varphi_i^{I_i}(c) = \begin{cases} \bar{\varphi}_i(c) & \text{for } f(c) \in I_i \\ \infty & \text{otherwise.} \end{cases}$$

Then $\varphi_i(c) < \infty$ implies $f(c) \in I_i = (m_i - \varepsilon_i, m_i + \varepsilon_i)$. For $c \neq d$ put $\phi_i(c,d) = \max\{\varphi_i(c), \varphi_i(d)\}/\alpha(c,d)$; that is, $\phi_i(c,d) = 0$ when $c = d$, $\phi_i(c,d) = \max\{\varphi_i(c), \varphi_i(d)\}$ for $||c-d|| = 1$, and $\phi_i(c,d) = \infty$ otherwise, and let

$$\phi(c,d) = \min_{i=1,\ldots,n} \phi_i(c,d). \tag{4}$$

Clearly, truncating each $\bar{\varphi}_i$ to $\varphi_i = \varphi_i^{I_i}$ is causing the loss of some information. In fact, the most common definition of ϕ used in the literature till now, see e.g. [4], coincides with ours if one drops the matter of truncation: define $\bar{\phi}(c,d) = \min_{i=1,\ldots,n} \bar{\phi}_i(c,d)$, where $\bar{\phi}_i(c,d) = \max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\alpha(c,d)$ for $c \neq d$. Then $\mu_{\bar{\phi}}$ is defined as usual. Clearly, at the first glance it seems that affinity $\bar{\phi}$ is superior to its truncated version ϕ defined above and that the information truncation makes the ability to distinguish among objects weaker. Although, to some extent, this is a legitimate concern, it should be noted that the objects obtained with the use of $\bar{\phi}$ may be bigger than those obtained with the use of ϕ.

3. HOW TO COMBINE DIFFERENT AFFINITIES?

In this section, we will discuss the issue of how to combine two or more different affinities of the sort described in the previous section into one affinity. We will also examine which parameters in the definitions of the combined affinity are redundant, in the sense that their change leads to an equivalent affinity.

3.1. Affinity combining methods

Assume that for some $k \geq 2$ we have affinity functions $\kappa_i \colon C \times C \to \langle L_i, \preceq_i \rangle$ for $i = 1, \ldots, k$. For example, we can have $k = 2$, $\kappa_1 = \psi$, and $\kappa_2 = \phi$. The most flexible way of combining all these affinities into a single affinity κ is to put $\kappa(c, d) = \langle \kappa_1(c, d), \ldots, \kappa_k(c, d) \rangle$ and define an appropriate linear order $\preceq$ on $L = L_1 \times \cdots \times L_k$. To understand this formalism better, we will start with the following examples, which also constitute our practical approach to the affinity combining problem.

Example 3.1. **(Weighted Averages)** Assume that all linear orderings L_i are equal to the same ordering $\langle L_0, \preceq_0 \rangle$ which is either $\langle [0, \infty], \geq \rangle$ or $\langle [0, 1], \leq \rangle$ and fix a vector $\mathbf{w} = \langle w_1, \ldots, w_k \rangle$ of numbers from $[0, 1]$ (weights) such that $w_1 + \cdots + w_k = 1$; we allow a weight w_i to be equal to 0 (meaning "ignore influence of κ_i") assuming that $0 \cdot \infty = 0$ and $0^0 = \infty^0 = 1$.

Additive Average: Let $h^{add}_{\mathbf{w}}(\mathbf{a}) = w_1 a_1 + \cdots + w_k a_k$ for $\mathbf{a} = \langle a_1, \ldots, a_k \rangle \in (L_0)^k$. If we define order $\leq^{add}_{\mathbf{w}}$ as $\mathbf{a} \leq^{add}_{\mathbf{w}} \mathbf{b} \iff h^{add}_{\mathbf{w}}(\mathbf{a}) \preceq_0 h^{add}_{\mathbf{w}}(\mathbf{b})$, then $\kappa \colon C \times C \to \langle L, \leq^{add}_{\mathbf{w}} \rangle$ is equivalent to $\kappa^a_{\mathbf{w}} \colon C \times C \to \langle L_0, \preceq_0 \rangle$ defined as $\kappa^a_{\mathbf{w}}(c, d) = h^{add}_{\mathbf{w}}(\kappa_1(c, d), \ldots, \kappa_k(c, d))$. For $k = 2$, the affinity $\kappa^a_{\mathbf{w}} = w_1 \kappa_1 + w_2 \kappa_2$ has been considered in [6].

Multiplicative Average: Let $h^{mul}_{\mathbf{w}}(\mathbf{a}) = a_1^{w_1} \cdots a_k^{w_k}$ for $\mathbf{a} = \langle a_1, \ldots, a_k \rangle \in (L_0)^k$. If we define order $\leq^{mul}_{\mathbf{w}}$ as $\mathbf{a} \leq^{mul}_{\mathbf{w}} \mathbf{b} \iff h^{mul}_{\mathbf{w}}(\mathbf{a}) \preceq_0 h^{mul}_{\mathbf{w}}(\mathbf{b})$, then $\kappa \colon C \times C \to \langle L, \leq^{mul}_{\mathbf{w}} \rangle$ is equivalent to $\kappa^m_{\mathbf{w}} \colon C \times C \to \langle L_0, \preceq_0 \rangle$ defined as $\kappa^m_{\mathbf{w}}(c, d) = h^{mul}_{\mathbf{w}}(\kappa_1(c, d), \ldots, \kappa_k(c, d))$. For $k = 2$, the affinity $\kappa^m_{\mathbf{w}} = \kappa_1^{w_1} \kappa_2^{w_2}$ has been already considered in [6].

Recall that the *lexicographical order* $\leq_{lex}$ on $L = L_1 \times \cdots \times L_k$ is defined for distinct $\mathbf{a} = \langle a_1, \ldots, a_k \rangle, \mathbf{b} = \langle b_1, \ldots, b_k \rangle \in L$ as $\mathbf{a} <_{lex} \mathbf{b} \iff a_i \prec_i b_i$, where $i = \min\{j \colon a_j \neq b_j\}$.

Example 3.2. **(Lexicographical Order)** Affinity function $\kappa_{lex} \colon C \times C \to \langle L, \leq_{lex} \rangle$ establishes the strongest possible hierarchy between the coordinate affinities κ_i: in establishing whether $\kappa_{lex}(a, b) \leq_{lex} \kappa_{lex}(c, d)$, the values $\kappa_i(a, b)$ and $\kappa_i(c, d)$ are completely irrelevant, unless $\kappa_j(a, b) = \kappa_j(c, d)$ for all $j < i$, in which case $\kappa_i(a, b) \prec_i \kappa_i(c, d)$ implies $\kappa_{lex}(a, b) <_{lex} \kappa_{lex}(c, d)$.

Notice that κ_{lex} cannot be expressed in the form of $h(\kappa_1, \ldots, \kappa_k)$ for any continuous function on $[0, 1]^k$ or on $[0, \infty]^k$. In what follows, we will restrict our attention to the situation when $k = 2$. In this case the lexicographical order is defined as $\langle a_1, a_2 \rangle <_{lex} \langle b_1, b_2 \rangle$ provided either $a_1 \prec_1 b_1$ or $a_1 = b_1$ and $a_2 \prec_2 b_2$. The lexicographical order approach is quite appealing in case when $\kappa_1 = \psi$ and $\kappa_2 = \phi$ as the decision whether $\mu_\kappa(c, s) \leq_{lex} \mu_\kappa(c, t)$ becomes hierarchical in nature: if $\mu_\psi(c, s) < \mu_\psi(c, t)$, then $\mu_\kappa(c, s) \leq_{lex} \mu_\kappa(c, t)$ independent of the values of $\mu_\phi(c, s)$ and $\mu_\phi(c, t)$; only when the homogeneity based connectivity measure cannot decide the matter, that is, when $\mu_\psi(c, s) = \mu_\psi(c, t)$, we decide on the direction of $\leq_{lex}$ between $\mu_\kappa(c, s)$ and $\mu_\kappa(c, t)$ based on the direction of $\preceq_2$ between $\mu_\phi(c, s)$ and $\mu_\phi(c, t)$. Thus, we treat the homogeneity based connectivity measure as dominant over object feature based connectivity measure. (Note that this will become reversed if $\kappa_1 = \phi$ and $\kappa_2 = \psi$.) However, there is more to it. If $\mu_\psi(c, s) = \mu_\psi(c, t)$, then we decide about $\mu_\kappa(c, s) \leq_{lex} \mu_\kappa(c, t)$ only along the paths $p \in \mathcal{P}_{cs}$ and $q \in \mathcal{P}_{ct}$ with $\mu_\psi(p) = \mu_\psi(q) = \mu_\psi(c, s)$. Only to these paths we apply μ_ϕ measure. Thus, we use the object based feature measure in this schema in a considerably more sophisticated way than what is suggested by the threshold-like interpretation described in Section 2. It should be also clear that, if we agree that we should give priority to homogeneity based connectivity measure in the RFC approach, this is precisely the way we should proceed.

Next, consider the coordinate order preserving property of the combined affinity $\kappa(c, d) = \langle \kappa_0(c, d), \kappa_1(c, d) \rangle$:

(C) for every $i = 0, 1$ and c, d, c', d', if $\kappa_i(c, d) = \kappa_i(c', d')$, then $\kappa(c, d) \prec \kappa(c', d') \Leftrightarrow \kappa_{1-i}(c, d) \prec_{1-i} \kappa_{1-i}(c', d')$.

Property (C) says that if one of the coordinate affinities does not distinguish between two pairs of spels, then the combined affinity decides on this pair according to the other coordinate affinity. This seems to be a very natural and desirable property. It is easy to see that, by design, the κ_{lex} affinity has this property. However, in general, (C) is not satisfied for the multiplicative average $\kappa^m_{\mathbf{w}}$: if $\kappa_i(c, d) = \kappa_i(c', d') = 0$, then $\kappa^m_{\mathbf{w}}(c, d) = \kappa^m_{\mathbf{w}}(c', d') = 0$ independently of the value of κ_{1-i} on these pairs. A similar problem arises for $\kappa_i(c, d) = \kappa_i(c', d') = \infty$, although for $\kappa_i(c, d) = \kappa_i(c', d') \in (0, \infty)$ the equivalence from (C) is satisfied. This creates a problem especially with the truncated version of the object-feature based affinity, since, in this case, affinity is equal to ∞ for many adjacent pairs of spels. Condition (C) also fails for $\kappa_{\mathbf{w}}^{add}$ when $\kappa_{\mathbf{w}}^{add}(c,d) = \kappa_{\mathbf{w}}^{add}(c',d') = \infty$, although for $\kappa_{\mathbf{w}}^{add}(c,d) = \kappa_{\mathbf{w}}^{add}(c',d') < \infty$ the equivalence is satisfied. In particular, (C) holds for $\kappa_{\mathbf{w}}^{add}$ formed with the coordinate affinities with range $\langle [0,1], \leq \rangle$.

Notice that the property (C) fails only if we allow values 0 or ∞ in the range of κ's. Therefore, if we like to insure (C), we can always replace κ_i's with their equivalent forms with the range in $(0,\infty)$ (e.g. by replacing ∞ with some large but finite number), which will insure (C) in the above described combining methods.

3.2. Counting essential parameters

Next, let us turn our attention to the determination of the number of parameters essential in defining the affinities presented in the previous section. We will consider here only the parameters explicitly mentioned there, since any implicit parameters (like the parameters for getting intensity function from the actual acquisition data) could not be handled by the methods we will employ. This exercise is useful in tuning the FC segmentation methods to different applications. It is also useful in comparing these methods with others. Recall that for a $\sigma \in (0,\infty)$ we defined $g_\sigma\colon [0,\infty] \to [0,1]$ by $g_\sigma(x) = e^{-x^2/\sigma^2}$.

Homogeneity based affinity, ψ, is defined as $\psi(c,d) = |f(c) - f(d)|$ for $||c-d|| \leq 1$ and $\psi(c,d) = \infty$ otherwise. As such, there are no parameters in this definition. In its standard form, $g_\sigma \circ \psi$, the parameter σ is redundant, since, by Corollary 1.2, $g_\sigma \circ \psi$ is equivalent to ψ. This beautiful characteristic says that FC partitioning of a scene utilizing homogeneity based affinity is an inherent property of the scene and is independent of any parameters, beside a threshold in case of AFC.

Object feature based affinity for one object, ϕ, is defined by $\phi(c,d) = \max\{|f(c) - m_1|, |f(d) - m_1|\}/\sigma_1$ for $||c-d|| = 1$, $\phi(c,d) = 0$ for $c = d$, and $\phi(c,d) = \infty$ otherwise. From the two parameters, m_1 and σ_1, present in this definition, only m_1 is essential. Parameter σ_1 is redundant, since function $\sigma_1 \cdot \phi$ is independent of its value and $\sigma_1 \cdot \phi$ is equivalent to ϕ, as $\sigma_1 \cdot \phi = h \circ \phi$ for an increasing function $h(x) = \sigma_1 x$. As before, the standard form $g_\sigma \circ \phi$ of ϕ is equivalent to it, so the only essential parameter in the definition of $g_\sigma \circ \phi$ is the number m_1.

Object feature based affinity for multiple objects. Suppose that the affinity is defined for $n > 1$ different objects for which $\bar{m} = \langle m_1, \ldots, m_n \rangle$ and $\bar{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ represent their average intensities and standard deviations, respectively. Let $\phi_{\bar{m},\bar{\sigma}}$ represent the object feature affinity in its main truncated form and let $\bar{\phi}_{\bar{m},\bar{\sigma}}$ stand for its untruncated version. (See Section 2.2.2.) Then $\sigma_1 \cdot \phi_{\bar{m},\bar{\sigma}} = \phi_{\bar{m},\bar{\delta}}$ and $\sigma_1 \cdot \bar{\phi}_{\bar{m},\bar{\sigma}} = \bar{\phi}_{\bar{m},\bar{\delta}}$, where $\bar{\delta} = \langle 1, \delta_2, \ldots, \delta_n \rangle$ and $\delta_i = \sigma_i/\sigma_1$. Since $\sigma_1 \cdot \phi_{\bar{m},\bar{\sigma}}$ is equivalent to $\phi_{\bar{m},\bar{\sigma}}$, affinity $\phi_{\bar{m},\bar{\sigma}}$ depends essentially only on $2n-1$ parameters $m_1, \ldots, m_n, \delta_2, \ldots, \delta_n$. The same is true for its standard form $g_\sigma \circ \phi_{\bar{m},\bar{\sigma}}$ as well as for their untruncated counterparts $\bar{\phi}_{\bar{m},\bar{\sigma}}$ and $g_\sigma \circ \bar{\phi}_{\bar{m},\bar{\sigma}}$.

In what follows, we will assume that $w, \sigma, \tau \in (0,1)$ and that ϕ is equal to either $\phi_{\bar{m},\bar{\delta}}$ or to $\bar{\phi}_{\bar{m},\bar{\delta}}$, so it has $2n-1$ essential parameters. Then we have the following methods of combining, denoted m1–m5, for homogeneity and object feature based affinities.

- m1 The additive average $\kappa = (1-w)\psi + w\phi$ of ψ and ϕ has $2n$ parameters. It is equivalent to $\psi + x\phi$, where $x = \frac{w}{1-w} \in (0,\infty)$. Note that if ϕ is replaced by an equivalent affinity $\sigma_1\phi$, then the resulting average affinity $(1-w)\psi + w\sigma_1\phi$ is also equivalent to $\psi + x\phi$ with $x \in (0,\infty)$. Note also that κ does not satisfy property (C), unless we insure that ψ and ϕ admit no ∞ value.
- m2 The additive average $\kappa = (1-w)g_\sigma \circ \psi + w g_\tau \circ \phi$ of $g_\sigma \circ \psi$ and $g_\tau \circ \phi$ has $2n+2$ essential parameters. Since $\kappa = e^{\ln(1-w) - \psi^2/\sigma^2} + e^{\ln w - \phi^2/\tau^2}$, this operation strangely mixes additive and multiplicative modifications of ψ and ϕ. The additional two parameters, σ and τ, are of importance in this mix. This affinity does satisfy property (C).
- m3 The multiplicative average $\kappa = \psi^{(1-w)}\phi^w$ of ψ and ϕ has $2n$ parameters and it is equivalent to $\psi\phi^x$, where $x = \frac{w}{1-w} \in (0,\infty)$, as $\kappa = (\psi\phi^x)^{1-w}$. If ϕ is replaced by an equivalent affinity $\sigma_1\phi$, then the resulting average $(\psi\sigma_1^x\phi^x)^{1-w}$ is also equivalent to $\psi\phi^x$ with $x \in (0,\infty)$, since function $h(t) = (\sigma_1^x t)^{1-w}$ is increasing as a composition of two increasing functions. This κ does not satisfy property (C), unless we insure that ψ and ϕ admit no 0 and ∞ values.
- m4 The multiplicative average $\kappa = (g_\sigma \circ \psi)^{(1-w)}(g_\tau \circ \phi)^w$ of $g_\sigma \circ \psi$ and $g_\tau \circ \phi$ has $2n+2$ parameters, but only $2n$ of them are essential. This is so since $\kappa = (e^{-\psi^2/\tau^2})^{1-w}(e^{-\phi^2/\sigma^2})^w = (e^{-\psi^2 - x\phi^2})^{(1-w)/\tau^2}$, where $x = \frac{\tau^2}{\sigma^2}\frac{w}{1-w} \in (0,\infty)$, is equivalent to $\psi^2 + x\phi^2$. The same is true if ϕ is replaced by $\sigma_1\phi$. This κ does not satisfy property (C), unless we insure that ψ and ϕ admit no ∞ value.

m5 There are only two essential possibilities for lexicographical order of ψ and ϕ: $\langle \psi, \phi \rangle$ and $\langle \phi, \psi \rangle$, even if we allow replacement of each of the coordinate affinities by any of their equivalent forms, including but not restricted to $g_\sigma \circ \psi$ and $\sigma_1 \phi$, $g_\tau \circ \phi$, or $g_\tau \circ (\sigma_1 \phi)$. This follows from Proposition 1.1, since for any pair $\langle \psi^*, \phi^* \rangle$ such that ψ^* is equivalent to ψ and ϕ^* is equivalent to ϕ, there are strictly monotone functions g and h such that $\psi^* = g \circ \psi$ and $\phi^* = h \circ \phi$, and then $\langle \psi^*, \phi^* \rangle = \langle g, h \rangle \circ \langle \psi, \phi \rangle$, so $\langle g, h \rangle$ establishes the equivalence of $\langle \psi, \phi \rangle$ and $\langle \psi^*, \phi^* \rangle$.

4. CONCLUDING REMARKS

The analysis presented in Section 1 shows that, from the perspective of FC methodology, the only essential attribute of an affinity function is its order. In particular, many transformations (like gaussian) of the natural affinity definitions (like derivative-driven homogeneity based affinity) are of esthetic value only and do not influence the FC segmentation outcomes. Nevertheless, such transformations may play a role in combining different affinities, as can be seen in methods m1 and m2, since only one of them has the property (C).

The analysis from Section 1 forms also the foundation of the investigation presented in Section 3, of which parameters in the definitions of homogeneity and object-feature based affinities, as well as their combinations, are of importance. In particular, we uncovered that many of the parameters in these definitions are of no consequence. Thus, for the tasks of application-driven optimization of the parameters, the number of parameters to be optimized is reduced.

In Section 2, we discussed two commonly used affinities, homogeneity and object-feature based, and interpreted them, respectively, as approximations of the directional derivatives and the distance from the object's average intensities. We also pointed out some theoretical deficiencies with the standard format of the object-feature based affinity in the case of multiple objects and proposed a truncated version of such affinity, which avoids theoretical difficulties, but loses some information along the way. In Section 3, combining the results from the previous sections, we discussed five distinct ways of constructing full affinity functions (m1-m5).

We did not undertake any empirical evaluation studies in this paper. A theoretical study preceding such an evaluation becomes essential to understand what affinity forms are distinct, what are redundant, and what parameters are essential/redundant. This paper constitutes a first such step. Analysis similar to the one conducted in this paper for FC can be carried out for other frameworks, such as level sets, watersheds, and graph cuts.

Also, as mentioned in Section 2.1, in the definition of the homogeneity based affinity it makes sense to use the notion of the gradient as a base for its definition, instead of the notion of the directional derivative. The discussion of the gradient induced homogeneity based affinity is a part of our forthcoming paper.

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