The presented construction is just very close to [Theorem 7.18: W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, 1964].

Weierstrass monster for calculus students

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Abstract

We present a simple example of a Weierstrass Monster—a continuous nowhere differentiable function $f\colon \mathbb{R} \to \mathbb{R}$—that is accessible to anyone familiar with geometric series and epsilon-delta definition of derivative. As such, it can be incorporated into one variable calculus.

The construction of Weierstrass Monster that follows contains elements of those given by van der Waerden [5], McCarthy [1], and Minassian and Gaisser [3]. However, it uses mathematical tools simpler than those and other documented constructions of such a function. (See e.g. [4, 2].)

For every n in $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, let $f_n(x) = \min_{k\in\mathbb{Z}} \left|x - \frac{k}{8^n}\right|$ be the distance from x in $\mathbb{R}$ to the set $\frac{1}{8^n}\mathbb{Z} = \{\frac{k}{8^n} : k \in \mathbb{Z}\}$. Then

$$f(x) = \sum_{n=0}^{\infty} 4^n f_n(x) \textit{ is continuous nowhere differentiable.}$$

Figure 1 shows this Monster function and the first two approximations of it. Note that $4^n f_n(x) \le 4^n \frac{1}{2} 8^{-n} = 2^{-n-1}$ for every $n \in \mathbb{Z}^+$.

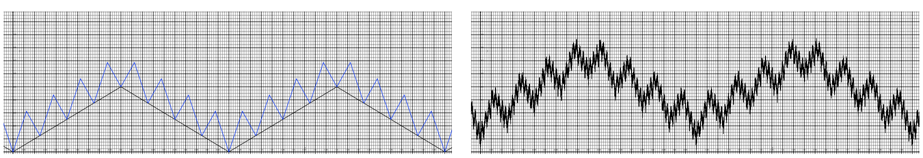

Figure 1: The graphs of: f_0 (lower left), $f_0 + f_1$ (upper left), and f (right)

Continuity of f: Choose $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that $|x_0 - x| < \delta$ implies $|f(x_0) - f(x)| < \varepsilon$. To see this, choose $n \in \mathbb{Z}^+$ such that $\frac{1}{2^{n+1}} < \frac{\varepsilon}{3}$. Since $F_n(x) = \sum_{i=0}^{n} 4^i f_i(x)$ is continuous, there exists a $\delta > 0$ such that $|F_n(x_0) - F_n(x)| < \frac{\varepsilon}{3}$ provided $|x_0 - x| < \delta$. Thus, $|x_0 - x| < \delta$ implies that

$$\begin{aligned} |f(x_0) - f(x)| &= \left|\left(F_n(x_0) + \sum_{i=n+1}^{\infty} 4^i f_i(x_0)\right) - \left(F_n(x) + \sum_{i=n+1}^{\infty} 4^i f_i(x)\right)\right| \\ &\le |F_n(x_0) - F_n(x)| + \left|\sum_{i=n+1}^{\infty} 4^i f_i(x_0)\right| + \left|\sum_{i=n+1}^{\infty} 4^i f_i(x)\right| \\ &\le \frac{\varepsilon}{3} + \left(\sum_{i=n+1}^{\infty} \frac{1}{2^{i+1}}\right) + \left(\sum_{i=n+1}^{\infty} \frac{1}{2^{i+1}}\right) = \frac{\varepsilon}{3} + 2\left(\frac{1}{2^{n+1}}\right) < \varepsilon. \end{aligned}$$

Nowhere differentiability of f: Fix an $n \in \mathbb{Z}^+$. For every $k \in \mathbb{Z}$, let $x_k = \frac{k}{8^n}$. Then, for every $i \geq n$, we have $x_k, x_{k+1} \in \frac{1}{8^i}\mathbb{Z}$ and $f_i(x_k) = f_i(x_{k+1}) = 0$. Also, we have $\frac{f_i(x_k)-f_i(x_{k+1})}{x_k-x_{k+1}} = \pm 1$ for every $i < n$. Thus, using inequalities $|a-b| \geq |a| - |b|$ and $|a_1 + \cdots + a_{n-2}| \leq |a_1| + \cdots + |a_{n-2}|$,

$$\begin{aligned}
\left|\frac{f(x_k)-f(x_{k+1})}{x_k-x_{k+1}}\right| &= \left|\frac{\sum_{i=0}^{n-1} 4^i f_i(x_k) - \sum_{i=0}^{n-1} 4^i f_i(x_{k+1})}{x_k-x_{k+1}}\right| \\
&= \left|\sum_{i=0}^{n-1} \frac{f_i(x_k)-f_i(x_{k+1})}{x_k-x_{k+1}}\, 4^i\right| = \left|\sum_{i=0}^{n-1} \pm 4^i\right| \geq \left|\pm 4^{n-1}\right| - \left|\sum_{i=0}^{n-2} \pm 4^i\right| \\
&\geq 4^{n-1} - \sum_{i=0}^{n-2} \left|\pm 4^i\right| = 4^{n-1} - \sum_{i=0}^{n-2} 4^i = 4^{n-1} - \frac{4^{n-1}-1}{3} > \frac{2}{3}4^{n-1}.
\end{aligned}$$

To proceed further, notice that for every $a < b < c$ and any function f,

$$\max\left\{\frac{|f(c)-f(b)|}{c-b}, \frac{|f(b)-f(a)|}{b-a}\right\} \geq \frac{|f(c)-f(a)|}{c-a}. \tag{1}$$

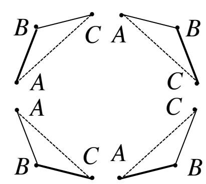

Figure 2: Four slope configurations for (1)

Indeed, let $\overline{AC}$ be the segment joining $A = (a, f(a))$ and $C = (c, f(c))$. If $B = (b, f(b))$ is above $\overline{AC}$, then $\frac{f(c)-f(b)}{c-b} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(a)}{b-a}$; otherwise, $\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(b)}{c-b}$, see Figure 2. This implies (1).[1]

Now, for every $x \in \mathbb{R}$ and every $n \in \mathbb{Z}^+$, there exists a $k \in \mathbb{Z}$ such that $x \in \left[\frac{k}{8^n}, \frac{k+1}{8^n}\right]$. We claim that

($*$) there is $y_n \in \left\{\frac{k}{8^n}, \frac{k+1}{8^n}\right\}$, $y_n \neq x$, such that $\left|\frac{f(x)-f(y_n)}{x-y_n}\right| \geq \left|\frac{f(x_k)-f(x_{k+1})}{x_k-x_{k+1}}\right|$.

Indeed, if x is among the endpoints of $\left[\frac{k}{8^n}, \frac{k+1}{8^n}\right]$, then the other endpoint can serve as y_n. Otherwise, $\frac{k}{8^n} < x < \frac{k+1}{8^n}$ and this follows from (1).

In particular, for every $n \in \mathbb{Z}^+$, there is a y_n such that $0 < |x - y_n| \leq \frac{1}{8^n}$ and $\left|\frac{f(x)-f(y_n)}{x-y_n}\right| \geq \left|\frac{f(x_k)-f(x_{k+1})}{x_k-x_{k+1}}\right| \geq \frac{2}{3}4^{n-1} \to_n \infty$. So, f is not differentiable at x.

References

[1] J. McCarthy, *An everywhere continuous nowhere differentiable function*, Amer. Math. Monthly **60** (1953), 709.

[2] M. Jarnicki and P. Pflug, *Continuous Nowhere Differentiable Functions*, Springer Monographs in Mathematics, New York, 2015.

[3] D.P. Minassian and J.W. Gaisser, *A simple Weierstrass function*, Amer Math. Monthly **91** (1984) 254–256.

[1] Also, the negation of (1) gives a contradiction: $\frac{|f(c)-f(a)|}{c-a} \leq \frac{|f(c)-f(b)|}{c-a} + \frac{|f(b)-f(a)|}{c-a} = \frac{c-b}{c-a}\frac{|f(c)-f(b)|}{c-b} + \frac{b-a}{c-a}\frac{|f(b)-f(a)|}{b-a} \overset{\text{by } \neg(1)}{<} \frac{c-b}{c-a}\frac{|f(c)-f(a)|}{c-a} + \frac{b-a}{c-a}\frac{|f(c)-f(a)|}{c-a} = \frac{|f(c)-f(a)|}{c-a}$.

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