A GENERAL THEORY OF IMAGE SEGMENTATION II: MULTIREGION COMPETITION IN DIFFERENT FRAMEWORKS

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A general theory of image segmentation II: multiregion competition in different frameworks

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***Abstract*— The main subject of this paper is a theoretical study of image segmentation algorithms in the multiregion competition setting. In particular, we investigate such segmentations when the competition is based on a value of the gradient magnitude of the image. The idealized model for such segmentation is placed in our theoretical segmentation framework from [7], [8]. We show that this model is represented by the gradient based relative fuzzy connectedness, RFC, algorithm. We also show that the model is weakly represented by a level set multiregion competition algorithm and that both algorithms are weakly model-equivalent. A particular consequence of this theoretical result is that the difficulties attributed to the level set multiregion competition algorithms could be avoided by using the gradient based RFC algorithm. We also describe a natural model for the fuzzy connectedness algorithm used with the homogeneity based affinity in the RFC setting and also for absolute fuzzy connectedness algorithms.**

I. Introduction

Image segmentation—the process of partitioning the image domain into meaningful object regions—is perhaps the most challenging and critical problem in image processing and analysis. Its central position in image processing comes from the fact that the delineation of objects is usually the first step in other higher level processing tasks, like image interpretation, diagnosis, analysis, visualization, and virtual object manipulation.

The segmentation literature is enormous. General segmentation frameworks may be broadly classified into three groups: boundary-based [6], [11], [12], [13], [15], [16], [19], [20], [21], region-based [1], [2], [3], [4], [25], [26], [27], [28], [29], and hybrid [5], [14]. As the nomenclature indicates, in the first two groups the focus is on recognizing and delineating the boundary or the region occupied by the object in the image. In the third group, the focus is on exploiting the complementary strengths of each of boundary-based and region-based strategies to overcome their individual shortcomings.

Despite this vast literature, our knowledge in this subject has several serious and fundamental gaps: (I) Most of the papers confuse and mix up several disparate aspects of the theory: the description of idealized segmentation models that form the theoretical basis for the algorithms; the description of the segmentation algorithms for digital images; the numerical issues related to the implementation of the algorithms; and filtering issues (denoising, artifact removing, debluring, etc) related to imperfect image acquisition. (II) A lack of methods allowing comparison of different segmentation models at the theoretical level, especially when models were introduced in different mathematical frameworks such as differential equations (often implemented via variational methods), differential geometry, graph theory, etc. (III) A lack of definitions and methods that relate idealized models with the related algorithms. (IV) A lack of rigorous methods allowing a theoretical comparison of segmentation algorithms.

In an attempt to address these issues, we introduced in [7], [8] a general segmentation theory framework and used it to show that the level set based delineation algorithm from [16] is (weakly model-) equivalent to a simple and fast gradient based absolute fuzzy connectedness algorithm of [29]. The present paper is a continuation of the work initiated in [7], [8], with the goal of extending the theory to image segmentations in the multiregion competition setting.

Image segmentation algorithms expressed in a multiregion competition setting have been studied in many different frameworks: fuzzy connectedness [23], [28], [24], [9], level set related [31], [17], [30], [18], and watershed [1], [22]. However, their theories also have the gaps (I)-(IV) described above, which we will address in this paper. Moreover, as noted in a recent paper [18], multiregional segmentation algorithms in a level set setting still face many challenges, like insuring that the segmented regions are disjoint and insuring repeatability of segmentations. In this paper, we present an equivalent alternative which resolves these issues.

II. Preliminaries

We will use the following terminology and notation, which follows closely those from [7], [8]. The discussion and motivation behind these notions can be found in [7], [8].

An *(n-dimensional) idealized image* is any function F from a bounded connected subset Ω of the n-dimensional Euclidean space $\mathbb{R}^n$ into $\mathbb{R}^\ell$. In what follows, we will always assume that Ω is an open, bounded, connected subset of $\mathbb{R}^n$, and often it will be just an n-dimensional cube $\Omega = (a,b)^n$. For the gradient based models we will also assume that F is differentiable.

A *delineation model* $\mathcal{M}$ for a class $\mathcal{F}$ of idealized images is any mapping $\langle F,\vec{p}\rangle \stackrel{\mathcal{M}}{\mapsto} O$, which, for any image $F\colon\Omega\to\mathbb{R}^\ell$ from $\mathcal{F}$ and any additional parameters $\vec{p}$ (like initialization seeds), associates a subset O of Ω interpreted as an *object*

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of the image F indicated by the parameters. We will write $\mathcal{M}(F,\vec{p})$ for the output O of $\mathcal{M}$ applied to $\langle F,\vec{p}\rangle$. A *segmentation model* $\mathcal{M}$ for a class $\mathcal{F}$ of idealized images and k objects, $k\geq 1$, is any mapping $\langle F,\vec{p}\rangle \overset{\mathcal{M}}{\mapsto} \langle O_1,\ldots,O_k\rangle$, which, for any image $F\colon\Omega\to\mathbb{R}^\ell$ from $\mathcal{F}$ and any additional task-speciffic parameters $\vec{p}$, associates a sequence $\langle O_1,\ldots,O_k\rangle$ of (usually pairwise disjoint) subsets O_i of Ω interpreted as the *objects* of the image F indicated by the parameters. We will write $\mathcal{M}(F,\vec{p})$ for the output $\langle O_1,\ldots,O_k\rangle$ of $\mathcal{M}$ applied to $\langle F,\vec{p}\rangle$. Any segmentation model $\mathcal{M}$ for k objects will be identified with k delineation models $\mathcal{M}_1,\ldots,\mathcal{M}_k$, that is, $\mathcal{M}(F,\vec{p})=\langle\mathcal{M}_1(F,\vec{p}),\ldots,\mathcal{M}_k(F,\vec{p})\rangle$.

A *(n-dimensional) digital image* is any function f from a finite subset C of $\mathbb{R}^n$ into $\mathbb{R}^\ell$. A digital image $f\colon C\to\mathbb{R}^\ell$ is a *digitization of an idealized image* $F\colon\Omega\to\mathbb{R}^\ell$ provided f is the restriction $F\restriction C$ of F to C, that is, $C\subset\Omega$ and $f(c)=F(c)$ for every $c\in C$. (This is the simplest scenario. Other possibilities for the definition of this notion are described in [7, Remark 4].)

A *(digital) delineation algorithm* $\mathcal{A}$ is any effectively defined mapping $\langle f,\vec{\theta}\rangle \overset{\mathcal{A}}{\mapsto} P$, which, to any digital image $f\colon C\to\mathbb{R}^\ell$ (possibly restricted to some subclass) and a parameter vector $\vec{\theta}$ (of additional task-specific information, like seeds), associates a subset P of C interpreted as a *segment* of the image f indicated by the parameters. We will write $\mathcal{A}(f,\vec{\theta})$ for the output P of $\mathcal{A}$ applied to $\langle f,\vec{\theta}\rangle$. A *(digital) segmentation algorithm* $\mathcal{A}$ of k objects is identified with a mapping $\mathcal{A}=\langle\mathcal{A}_1,\ldots,\mathcal{A}_k\rangle$, where each $\mathcal{A}_i$ is a delineation algorithm.

The premise for connecting the delineation algorithm $\mathcal{A}$ with a delineation model $\mathcal{M}$ is: *The better the resolution of the digital approximation f of the idealized image F, the closer the algorithm's output $\mathcal{A}(f)$ will be to the model output* $\mathcal{M}(F)$. Mathematically, this can be translated into the following statement: *A delineation algorithm $\mathcal{A}$ represents a delineation model $\mathcal{M}$ for an image F provided, for every sequence $\langle f_i\rangle_{i=1}^\infty$ of digitizations of F, if the resolutions of f_is approach the finest possible, then the segments $\mathcal{A}(f_i,\vec{\theta})$ converge to the object $\mathcal{M}(F,\vec{\theta})$.* To remove ambiguity from this definition schema, we need to introduce the following notions.

Let $\mathbb{Z}$ stand for the set of all integer numbers and, for $h>0$, let $(h\mathbb{Z})^n=\{hk\colon k\in\mathbb{Z}\}^n$ be the rectangular grid of points in $\mathbb{R}^n$ with the basic grid spacing h. For an idealized image $F\colon\Omega\to\mathbb{R}^\ell$, we define $\Omega_h=\Omega\cap(h\mathbb{Z})^n$. In what follows, we will consider only the digitizations of F in the form $f_i=F\restriction\Omega_{h/2^i}$. (More general digitizations are discussed in [7], [8].) We also need the following two limit notions. For $\{A_i(\eta)\colon\eta\in\mathbb{R}\ \&\ i=1,2,3,\ldots\}$, we define $\lim_{i,\eta}{}^* A_i(\eta)$ as $\lim_{\eta\to 0^+}\left(\bigcap_{j=1}^\infty\bigcup_{i\geq j}A_i(\eta)\right)$, and for double indexed family, we put $\lim^\dagger_{i,\eta,\varepsilon}A_i(\eta,\varepsilon)\overset{\text{def}}{=}\lim_{\varepsilon\to 0^+}\left(\lim^*_{i,\eta}A_i(\eta,\varepsilon)\right)$. These two limit notions are different from the standard multivariable limit in that the limiting process for $\lim^*$ and $\lim^\dagger$ is hierarchical in nature: first the limit is taken over the index i (controlling convergence of resolution), second, over the parameter η, and finally (in case of $\lim^\dagger$) over the parameter ε. We need to start with the limit over the resolution of f_is, since it controls the rate in which the difference quotient of F converge to a derivative of F, and, for the representation theorem to hold, this convergence must take precedence over any other part of the limiting process. Now, we can formalize the above intuitions. (A more general definition of representability, which allows digitizations $f=F\restriction C$ of F for all possible finite sets C, can be found in [7], [8].)

Let $\mathcal{A}^{\varepsilon,\eta}(f,\vec{\theta})$ be a delineation algorithm, where $\varepsilon,\eta\in\mathbb{R}$, and assume that $\mathcal{A}^{\varepsilon,0}(f,\vec{\theta})=\lim_{\eta\to 0^+}\mathcal{A}^{\varepsilon,\eta}(f,\vec{\theta})$. If $\mathcal{A}^{\varepsilon}(f,\vec{\theta})=\mathcal{A}^{\varepsilon,0}(f,\vec{\theta})$, then we say that the algorithm $\mathcal{A}^{\varepsilon}(f,\vec{\theta})$ *weakly represents* a delineation model $\mathcal{M}(f,\vec{\theta})$ *for a class* $\mathcal{F}$ of idealized images provided, for every $F\colon\Omega\to\mathbb{R}^\ell$ from $\mathcal{F}$, $h>0$, and a parameter $\vec{\theta}$ appropriate for F, the limit $\lim^\dagger_{i,\eta,\varepsilon}\mathcal{A}^{\varepsilon,\eta}(F\restriction\Omega_{h/2^i},\vec{\theta})$ exists and is a dense subset of $\mathcal{M}(F,\vec{\theta})$. A segmentation algorithm $\mathcal{A}=\langle\mathcal{A}_1,\ldots,\mathcal{A}_k\rangle$ *weakly represents* a segmentation model $\mathcal{M}=\langle\mathcal{M}_1,\ldots,\mathcal{M}_k\rangle$ provided each $\mathcal{A}_i$ weakly represents $\mathcal{M}_i$. Two segmentation (delineation) algorithms $\mathcal{A}$ and $\mathcal{A}'$ are *weakly model-equivalent* in a class $\mathcal{F}$ of idealized images provided there exists a segmentation (delineation) model $\mathcal{M}$ for $\mathcal{F}$ such that both $\mathcal{A}$ and $\mathcal{A}'$ weakly represent $\mathcal{M}$.

III. GRADIENT BASED SEGMENTATION MODEL IN A MULTIREGIONAL COMPETITION SETTING

In [7], [8], we described a gradient based delineation model $\mathcal{M}_\triangledown$ and proved that it is represented by two well known delineation algorithms: the absolute fuzzy connectedness algorithm $\mathcal{A}_\triangledown$ used with gradient based affinity [29], and the level set delineation algorithm $\mathcal{A}_{LS}$ in a version from Malladi, Sethian, and Vemuri paper [16]. In this section we will describe a gradient based segmentation model $\mathcal{M}_{\triangledown RFC}$ in a multiregional competition setting and prove that it is represented by the relative fuzzy connectedness, RFC, segmentation algorithm of [28] used with gradient based affinity and denoted here by $\mathcal{A}_{\triangledown RFC}$. We will also show that there is a natural multiregion competition version of the level set delineation algorithm $\mathcal{A}_{LS}$, which is weakly model-equivalent to the segmentation algorithm $\mathcal{A}_{\triangledown RFC}$.

A. Gradient based RFC segmentation model $\mathcal{M}_{\triangledown RFC}$ and related segmentation algorithm $\mathcal{A}_{\triangledown RFC}$

1) The model: Let $F\colon\Omega\to\mathbb{R}^\ell$ be a differentiable idealized image. The task in a multiregion segmentation problem in this idealized setting is to detect in F the exact spatial extent of $k>1$ objects $O_1,\ldots,O_k$ indicated by the parameters $\vec{p}$, that is, finding $\mathcal{M}(F,\vec{p})=\langle O_1,\ldots,O_k\rangle$.

Assume that for a fixed idealized image $F\colon\Omega\to\mathbb{R}^\ell$ and every path (i.e., continuous mapping) $p\colon[a,b]\to\Omega$, possibly restricted to some class of nice paths, we have associated its strength $\mu(p)$. In a general RFC setting [23], [28], [9], we start with transforming the notion of path strength to the global connectivity measure $\mu(X,S)$, that assigns to non-empty sets $X,S\subset\Omega$ the best strength in which X can be connected to S by a path. In particular, in the case of path strength definition based on gradient:

$$\mu(p)=\sup_{t\in[a,b]}|\triangledown F(p(t))|, \tag{1}$$

where the strength is measured by a *reverse inequality* $\geq$ (see [10] for more on this approach), the global connectivity measure is defined by a formula:

$$\mu(X,S)=\inf\{\mu(p)\colon p \text{ is a path from } X \text{ to } S\}. \qquad (2)$$

If $x\in\Omega$, we will write $\mu(x,S)$ in place of $\mu(\{x\},S)$. Notice that, in (2), we do not require that the infimum is achieved. In fact, it is not difficult to find a simple example in which $\mu(X,S)<\mu(p)$ for every path p from X to S.

Any RFC model takes as an input a pair $\langle F,\vec{S}\rangle$, where F is the image and $\vec{S}=\langle S_1,\ldots,S_k\rangle$ is a sequence of the non-empty sets of seeds, where each S_i indicates the ith object. In the case when a seed set S_i is a singleton $\{s_i\}$, we will often replace S_i with s_i in this representation. When better connectivity strength means smaller number and the connectivity strength is defined via (2), in the model output $\mathcal{M}_{RFC}(F,\vec{S})=\langle\mathcal{M}^1_{RFC}(F,\vec{S}),\ldots,\mathcal{M}^k_{RFC}(F,\vec{S})\rangle$, the ith object $\mathcal{M}^i_{RFC}(F,\vec{S})$ is defined as

$$\{x\in\Omega\colon \mu(x,S_i)<\mu(x,S_j) \text{ for all } j\neq i\}. \qquad (3)$$

In other words, the i-th object $\mathcal{M}^i_{RFC}(F,\vec{S})$ is the set of all $x\in\Omega$ that are better connected to the reference seed set S_i than to any other seed set S_j. This is the competition among objects for their members via FC. Note that definition (3) insures that the objects $O_i=\mathcal{M}^i_{RFC}(F,\vec{S})$ are pairwise disjoint. When path strength is gradient based, we denote $\mathcal{M}_{RFC}$ as $\mathcal{M}_{\bigtriangledown RFC}$.

Definition (3) stresses the competitive nature of membership assignment to $\mathcal{M}^i_{RFC}$. However, for our analysis, it will be more convenient to use an equivalent alternative definition for $\mathcal{M}^i_{\bigtriangledown RFC}$. First we note the following simple fact, where we assume that the path strength is defined by (1).

Lemma 1: For every $x,s\in\Omega$ and non-empty $T\subset\Omega$, we have $\max\{\mu(x,s),\mu(x,T)\}\geq\mu(s,T)$. In particular, if $\mu(x,s)<\mu(x,T)$, then $\mu(x,T)=\mu(s,T)$. Moreover, $\mu(x,s)<\mu(x,T)$ if and only if $\mu(x,s)<\mu(s,T)$.

PROOF. First, by way of contradiction, assume that $\max\{\mu(x,s),\mu(x,T)\}<\mu(s,T)$. Then, there are paths p_1 from s to x and p_2 from x to T such that $\mu(p_i)<\mu(s,T)$ for $i=1,2$. Let p be a path from s to T that first follows path p_1 and then p_2. Then, $\mu(p)<\mu(s,T)$, contradicting definition (2) of $\mu(s,T)$.

To prove the second part, assume that $\mu(x,s)<\mu(x,T)$. By the first part, $\mu(x,T)\geq\mu(s,T)$. In addition, the inequality $\mu(x,T)<\mu(s,T)$ implies $\max\{\mu(s,x),\mu(x,T)\}<\mu(s,T)$, contradicting the first part of the lemma. Therefore, indeed, $\mu(x,T)=\mu(s,T)$.

Finally, by the second part, $\mu(x,s)<\mu(x,T)$ implies that $\mu(x,s)<\mu(x,T)=\mu(s,T)$, while $\mu(x,s)<\mu(s,T)$ implies $\mu(x,s)=\mu(s,x)<\mu(s,T)=\mu(x,T)$. ■

For a fixed image F, a seed sequence $\vec{S}=\langle S_1,\ldots,S_k\rangle$, $i\in\{1,\ldots,k\}$, and $s\in S_i$, let $T_i=\bigcup_{j\neq i}S_j$ and $\theta^s_i=\mu(s,T_i)$. By (3), $\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})=\{x\in\Omega\colon\mu(x,S_i)<\mu(x,T_i)\}=\bigcup_{s\in S_i}\{x\in\Omega\colon\mu(x,s)<\mu(x,T_i)\}$. Since, by Lemma 1, $\{x\in\Omega\colon\mu(x,s)<\mu(x,T_i)\}=\{x\in\Omega\colon\mu(x,s)<\mu(s,T_i)\}$, we conclude that

$$\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})=\bigcup_{s\in S_i}\{x\in\Omega\colon\mu(x,s)<\theta^s_i\}. \qquad (4)$$

In other words, $\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})$ is equal to the union, over all $s\in S_i$, of the gradient based absolute FC model outcomes $\mathcal{M}_{\bigtriangledown}(F,\theta^s_i,s)=\{x\in\Omega\colon\mu(x,s)<\theta^s_i\}$ from [7] defined as (see [7, thm 13])

$$\{x\in\Omega\colon\mu(p)<\theta^s_i \text{ for some path } p \text{ from } x \text{ to } s\}.$$

In what follows we will assume that the ith object is identified only by a single seed, that is, that $S_i=\{s_i\}$ for some $s_i\in C$. In this case we define $\theta_i=\theta^{s_i}_i=\mu(s_i,T_i)$ and equation (4) reduces to

$$\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})=\{x\in\Omega\colon\mu(x,s_i)<\theta_i\}. \qquad (5)$$

Clearly, we would like to ascertain that the seeds S_i indicating the ith object $O_i=\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})$ belong to this object. However, this requires some extra assumption on the choice of sets S_i. For example, since objects are pairwise disjoint, inclusions $S_i\subset O_i$ clearly require that sets S_i are also pairwise disjoint. This, however, is not sufficient. The fully characterizing condition is as follows.

Theorem 2: Let F be a differentiable idealized image and $\vec{S}=\langle S_1,\ldots,S_k\rangle$ be a sequence of seed sets. For every $i\in\{1,\ldots,k\}$ we have $S_i\subset\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})$ if and only if $|\bigtriangledown F(s)|<\mu(s,S_j)$ for every $j\neq i$ and $s\in S_i$.

PROOF. Notice that for every path p from $s\in S_i$ to S_i we have $\mu(s,s)\leq|\bigtriangledown F(s)|\leq\mu(p)\leq\mu(s,S_i)\leq\mu(s,s)$, where the first inequality is justified by a constant path. Therefore, by (3), $s\in\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})$ if and only if $|\bigtriangledown F(s)|=\mu(s,s)=\mu(s,S_i)<\mu(s,S_j)$ for every $j\neq i$. ■

It is also worth noting that if S_i is connected and is a subset of $O_i=\mathcal{M}^i_{\bigtriangledown RFC}(F,\vec{S})$, then O_i is also connected, since, in this case, O_i is a union of the paths intersecting S_i, that is, a union of connected sets, each of which intersects a connected core $S_i\subset O_i$.

2) The algorithm: Having discussed delineation and segmentation modesl based on gradient, we turn now to connecting this model with the delineation and segmentation algorithms. First, our discussion will focus on an RFC algorithm and subsequently a level set algorithm, both for multiregion segmentation. Let $f\colon C\to\mathbb{R}^\ell$ be a digital image, where $C\subset(h\mathbb{Z})^n$ for some $h>0$. We will think of f as a digitization of an idealized image $F\colon\Omega\to\mathbb{R}^\ell$, that is, that $C=\Omega_h$ and $f=F\restriction C$. We will assume that there exists an $\alpha\in[h,n^2h]$ such that two spatial elements (spels) $c,d\in C$ are *adjacent* provided $||c-d||\leq\alpha$, where $||x||$ denotes the Euclidean norm of $x=\langle x_1,\ldots,x_n\rangle\in\mathbb{R}^n$, that is, $||x||=\sqrt{x_1^2+\cdots+x_n^2}$. Recall that a *path p in C* is any sequence $\langle c_1,\ldots,c_k\rangle$ of spels in C, where consecutive c_i and c_{i+1} are adjacent; p is from $x\in C$ to $y\in C$ if $c_1=x$ and $c_k=y$; it is from $S\subset C$ to $T\subset C$ if $c_1\in S$ and $c_k\in T$. A gradient based strength of a path $p=\langle c_1,\ldots,c_k\rangle$ is defined as

$$\mu(p)=\max_{i=1,\ldots,k}|\bigtriangledown f(c_i)|, \qquad (6)$$

where $|\triangledown f(c)| = ||\langle D_1 f(c), \ldots, D_n f(c)\rangle||$. Here an approximate partial derivative $D_i f(c)$ is defined as ∞ if none of the spels $c \pm he_i$ belongs to C and by a formula

$$D_i f(c) = \max\left\{\left|\frac{f(c)-f(d)}{h}\right| : d = c \pm he_i \in C\right\}, \quad (7)$$

where e_i is the unit vector in the direction of the ith variable.

For non-empty sets $X, S \subset C$, the discrete global connectivity measure is defined precisely as in the idealized case, $\mu(X,S) = \inf\{\mu(p)\colon p \text{ is a path from } X \text{ to } S\}$. The k-object gradient based RFC algorithm of [28] has an output $\mathcal{A}_{\triangledown RFC}(f,\vec{S}) = \langle \mathcal{A}^1_{\triangledown RFC}(f,\vec{S}), \ldots, \mathcal{A}^k_{\triangledown RFC}(f,\vec{S})\rangle$, where $\mathcal{A}^i_{\triangledown RFC}(f,\vec{S}) = \{x \in C\colon \mu(x,S_i) < \mu(x,S_j) \text{ for all } j \neq i\}$. Here $\vec{S} = \langle S_1, \ldots, S_k\rangle$ and each set S_i represents seeds indicated the ith object. As above, we assume that $S_i = \{s_i\}$ is a singleton. Since the discrete analog of Lemma 1 can be easily established, the analog of (5) also holds. (This was first proved in [10, thm. 6].)

$$\mathcal{A}^i_{\triangledown RFC}(f,\vec{S}) = \{x \in C\colon \mu(x,s_i) < \hat{\theta}_i\}, \quad (8)$$

where $\hat{\theta}_i = \min_{j\neq i} \mu(s_i,s_j)$. Thus, $\mathcal{A}^i_{\triangledown RFC}(f,\vec{S})$ equals $\mathcal{A}_{\triangledown}(f,\hat{\theta}_i,s_i)$ defined in [7] as

$$\{x \in C\colon \mu(p) < \hat{\theta}_i \text{ for some path } p \text{ from } x \text{ to } s_i\}.$$

Theorem 3: Algorithm $\mathcal{A}_{\triangledown RFC}$ represents model $\mathcal{M}_{\triangledown RFC}$ for the class of $\mathcal{C}^1$ idealized images $F\colon \Omega \to \mathbb{R}^\ell$ for which Ω is convex and $|\triangledown F|$ is uniformly continuous on Ω.

In the statement of Theorem 3, we wrote "represents" instead of "weakly represents" since the theorem is true in this stronger form, with essentially unchanged proof. However, since we did not formally define here this stronger representability notion, we will prove only its weak representability version.

PROOF. Fix a $j = 1, \ldots, k$. We need to prove that the delineation algorithm $\hat{\mathcal{A}} = \mathcal{A}^j_{\triangledown RFC}$ weakly represents delineation model $\hat{\mathcal{M}} = \mathcal{M}^j_{\triangledown RFC}$. Let $\hat{\mathcal{A}}^{\varepsilon,\eta}(f,\vec{S}) = \mathcal{A}_{\triangledown}(f,\hat{\theta}_j - \eta, S_j)$. Then, by (8), $\hat{\mathcal{A}}(f,\vec{S}) = \hat{\mathcal{A}}^{\varepsilon,0}(f,\vec{S}) = \lim_{\eta\to 0^+} \hat{\mathcal{A}}^{\varepsilon,\eta}(f,\vec{S})$. Fix $\vec{S}$ and $F\colon \Omega \to \mathbb{R}^\ell$ as in theorem, and let $h > 0$. We need to show that the limit $L = \lim^{\dagger}_{i,\eta,\varepsilon} \mathcal{A}^{\varepsilon,\eta}(F \restriction \Omega_{h/2^i},\vec{S})$ exists and is a dense subset of $\hat{\mathcal{M}}(F,\vec{S})$. Since, by (8), $\hat{\mathcal{M}}(F,\vec{S}) = \mathcal{M}_{\triangledown}(F,\theta_j,S_j)$ and $L = \lim^*_{i,\eta} \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \hat{\theta}_j - \eta, S_j)$, our task reduces to devising the proof that the limit exists and is dense in $\mathcal{M}_{\triangledown}(F,\theta_j,S_j)$. This seems to follow immediately from [7, thm 16], since it was proved there that the limit $\lim^*_{i,\eta} \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \theta_j - \eta, S_j)$ exists and is dense in $\mathcal{M}_{\triangledown}(F,\theta_j,S_j)$. However, $\hat{\theta}_j$ in the definition of L depends on i (and on $F \restriction \Omega_{h/2^i}$) and, in general, is not equal to θ_j. Thus, our proof requires a bit more delicate argument.

Let $\hat{\theta}_j(i)$ be the value of $\hat{\theta}_j$ for the image $F \restriction \Omega_{h/2^i}$. Notice that

$$\lim_i \hat{\theta}_j(i) = \theta_j.$$

To see that $\lim_i \hat{\theta}_j(i) \le \theta_j$, fix a $\delta > 0$, and note that, by the definition of θ_j, for every $j' \neq j$ there exists a path $\hat{p}$ in Ω from S_j to $S_{j'}$ with $\mu(\hat{p}) < \theta_j + \delta/2$. Then, by [7, lem 15], there is an i_0 such that, for every $i > i_0$, there exists a path p in $\Omega_{h/2^i}$ from S_j to $S_{j'}$ which is inside the closed ball $B[\mathrm{range}(\hat{p}),\varepsilon]$ of radius $\varepsilon = 2n\alpha \le 2n^3h/2^i$. Since $|\triangledown F|$ is continuous, by increasing i_0, if necessary, we may assume that $|\triangledown F|(c) < \theta_j + \delta/2$ for every $c \in B[\mathrm{range}(\hat{p}),\varepsilon]$. In particular, this is true for every c from $\hat{p}$. But $|\triangledown(F \restriction \Omega_{h/2^i})|$ given by (6) uniformly approximates $|\triangledown F|$, as proved in [7, lem 22]. In particular, for i_0 big enough, we have $\big||\triangledown(F \restriction \Omega_{h/2^i})|(c) - |\triangledown F|(c)\big| < \delta/2$. So, $\mu(p) < \theta_j + \delta$, proving $\hat{\theta}_j(i) < \theta_j + \delta$ for all i large enough. Since $\delta > 0$ was arbitrary, we conclude that $\lim_i \hat{\theta}_j(i) \le \theta_j$.

To prove the other inequality, by way of contradiction, assume that $\liminf_i \hat{\theta}_j(i) < \theta_j$ and let $\delta > 0$ be such that $\liminf_i \hat{\theta}_j(i) < \theta_j - \delta$. Then, for every i_0, there is an $i > i_0$ such that for every $j' \neq j$ there exists a path p in $\Omega_{h/2^i}$ from S_j to $S_{j'}$ with $\mu(p) < \theta_j - \delta$. Using again uniform convergence of $|\triangledown(F \restriction \Omega_{h/2^i})|$ to $|\triangledown F|$ we conclude that, for i large enough, $\mu(p) < \theta_j - \delta$ implies that $|\triangledown F|(c) < \theta_j - \delta/2$ for every c from p. Let $\hat{p}$ be a path on Ω obtained from p by connecting consecutive spels in p by straight segments. Then $\hat{p}$ is in Ω, since Ω is convex. Now, from uniform continuity of $|\triangledown F|$ and the fact that the consecutive spels in p are of distance at most $\alpha \le n^2h/2^i$, we conclude that for i large enough $|\triangledown F|(x) < \theta_j - \delta/4$ for every x from $\hat{p}$. Thus, $\mu(\hat{p}) < \theta_j - \delta/4$. Since for this particular i such a path exists for every $j' \neq j$, this contradicts the definition of θ_j.

Now it follows from [7, thm 16] that, for every $\eta > 0$ and every r, there is an $i_0 > r$ such that for every $i > i_0$

$$\begin{aligned}\Omega_{h/2^r} \cap \mathcal{M}_{\triangledown}(F,\theta_j - \eta, S_j) &\subseteq \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \theta_j - \eta, S_j)\\ &\subseteq \mathcal{M}_{\triangledown}(F,\theta_j,S_j).\end{aligned}$$

This implies that for every $\eta > 0$ and every r, there is an $i_0 > r$ such that for every $i > i_0$

$$\begin{aligned}\Omega_{h/2^r} \cap \mathcal{M}_{\triangledown}(F,\theta_j - 2\eta, S_j) &\subseteq \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \hat{\theta}_j - \eta, S_j)\\ &\subseteq \mathcal{M}_{\triangledown}(F,\theta_j,S_j).\end{aligned}$$

Indeed, for large i_0 we have $\theta_j - 2\eta < \hat{\theta}_j - \eta < \theta_j - \eta/2$, so

$$\begin{aligned}\Omega_{h/2^r} \cap \mathcal{M}_{\triangledown}(F,\theta_j - 2\eta, S_j) &\subseteq \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \theta_j - 2\eta, S_j)\\ &\subseteq \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \hat{\theta}_j - \eta, S_j)\\ &\subseteq \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \theta_j - \eta/2, S_j)\\ &\subseteq \mathcal{M}_{\triangledown}(F,\theta_j,S_j).\end{aligned}$$

This implies (for an easy proof see [7, thm 16]) that $\lim^*_{i,\eta} \mathcal{A}_{\triangledown}(F \restriction \Omega_{h/2^i}, \hat{\theta}_j - \eta, S_j) = \mathcal{M}_{\triangledown}(F,\theta_j,S_j) \cap \bigcup_r \Omega_{h/2^r}$, so the limit exists and is dense in $\mathcal{M}_{\triangledown}(F,\theta_j,S_j)$. ∎

B. Multiregion competition level set segmentation algorithm $\mathcal{A}_{LS\,RFC}$ representing model $\mathcal{M}_{\triangledown RFC}$

We turn now to the level set setting for connecting it with the model $\mathcal{M}_{\triangledown RFC}$.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For a digital image $f\colon \Omega_h \to \mathbb{R}^\ell$, a simple closed surface $S \subset \Omega$ and $\varepsilon, \theta > 0$ let $\mathcal{A}^{\varepsilon}_{LS}(f,\theta,S)$ denote a delineation algorithm from [16] propagating curve S with speed $(1 + |\triangledown F|)^{-1} - \varepsilon\kappa$, where κ is curvature of the front. It has been argued in [7] that $\mathcal{A}^{\varepsilon}_{LS}$ weakly represents model $\mathcal{M}_{\triangledown}$.

Formally, in the algorithm $\mathcal{A}^{\varepsilon}_{LS}(f,\theta,S)$, one propagates a digital closed surface $S^h \subset \Omega_h$ approximating S, and not S itself. This will be of some importance for the following definition of an algorithm $\mathcal{A}_{LS\,RFC}$ in the multiregion setting for the level set approach. For a sequence $\vec{S}=\langle S_1,\ldots,S_k\rangle$ of simple closed surfaces in Ω, $\varepsilon,h>0$, and a digital image $f\colon \Omega_h \to \mathbb{R}^\ell$ define

$$\mathcal{A}^{\varepsilon}_{LS\,RFC}(f,\vec{S})=\langle \mathcal{A}^{\varepsilon}_{LS}(f,\bar\theta_1,S_1),\ldots,\mathcal{A}^{\varepsilon}_{LS}(f,\bar\theta_k,S_k)\rangle,$$

where $\bar\theta_i=\min_{j\neq i}\mu(S^h_i,S^h_j)$.

Theorem 4: Algorithm $\mathcal{A}_{LS\,RFC}$ weakly represents model $\mathcal{M}_{\triangledown RFC}$ for the class of $\mathcal{C}^1$ idealized images $F\colon\Omega\to\mathbb{R}^\ell$ for which Ω is convex and $|\triangledown F|$ is uniformly continuous on Ω. In particular, the segmentation algorithms $\mathcal{A}_{LS\,RFC}$ and $\mathcal{A}_{\triangledown RFC}$ are weakly equivalent for this class of images.

SKETCH OF PROOF. We already know that the algorithm $\mathcal{A}^{\varepsilon}_{LS}(F\restriction\Omega_{h/2^i},\theta_j,S_j)$ converges to $\mathcal{M}_{\triangledown}(F,\theta_j,S_j)$ in terms of the limit $\lim^{\dagger}$. We need to show that also $\mathcal{A}^{\varepsilon}_{LS}(F\restriction\Omega_{h/2^i},\hat\theta_j,S_j)$ converges to $\mathcal{M}_{\triangledown}(F,\theta_j,S_j)$. But, similarly as in the proof of Theorem 3 it can be shown that $\lim_i\bar\theta_j(i)=\theta_j$. (Note that, in general, $\bar\theta_j(i)\neq\hat\theta_j(i)$, since $\hat\theta_j(i)$ is defined with the help of sets S_m, while $\bar\theta_j(i)$ is defined from sets $S^{h/2^i}_m$.) These two facts can be used to show that $\mathcal{A}^{\varepsilon}_{LS}(F\restriction\Omega_{h/2^i},\hat\theta_j,S_j)$ converges to $\mathcal{M}_{\triangledown}(F,\theta_j,S_j)$. ■

I will add more details later.

C. Invariance properties

We note that model $\mathcal{M}_{\triangledown RFC}$ and algorithm $\mathcal{A}_{\triangledown RFC}$ have the following robustness property.

Theorem 5: For every idealized image F and a sequence of seed sets $\vec{S}=\langle S_1,\ldots S_k\rangle$, if $\mathcal{M}_{\triangledown RFC}(F,\vec{S})=\langle O_1,\ldots,O_k\rangle$, $T_i\subset O_i$ for $i=1,\ldots k$, and $\vec{T}=\langle T_1,\ldots T_k\rangle$, then $\mathcal{M}_{\triangledown RFC}(F,\vec{S})=\mathcal{M}_{\triangledown RFC}(F,\vec{T})$.

Similarly, for every digital image f and a sequence of seed sets $\vec{S}=\langle S_1,\ldots S_k\rangle$, if $\mathcal{A}_{\triangledown RFC}(f,\vec{S})=\langle O_1,\ldots,O_k\rangle$, $T_i\subset O_i$ for $i=1,\ldots k$, and $\vec{T}=\langle T_1,\ldots T_k\rangle$, then $\mathcal{A}_{\triangledown RFC}(f,\vec{S})=\mathcal{A}_{\triangledown RFC}(f,\vec{T})$.

PROOF. First consider the case of $\mathcal{M}_{\triangledown RFC}$. Fix distinct i and j. Since $T_i\subset O_i$, we have $\mu(S_i,T_i)<\theta_i\le\mu(S_i,S_j)$. So, by Lemma 1, $\mu(T_i,S_j)=\mu(S_i,S_j)$. Similarly, $T_j\subset O_j$ implies $\mu(S_j,T_j)<\theta_j\le\mu(S_i,S_j)=\mu(T_i,S_j)$. So, again by Lemma 1, $\mu(T_i,T_j)=\mu(T_i,S_j)=\mu(S_i,S_j)$. Thus, numbers θ_i obtained from $\vec{S}$ and from $\vec{T}$ are identical. The proof is completed by noticing that $\mathcal{M}_{\triangledown}(F,\theta_i,S_i)=\mathcal{M}_{\triangledown}(F,\theta_i,T_i)$, which follows from [7, thm 14], or another application of Lemma 1.

The proof of the same property for $\mathcal{A}_{\triangledown RFC}$ is essentially the same. ■

We note that neither $\mathcal{A}^{\varepsilon}_{LS}$ nor $\mathcal{A}^{\varepsilon}_{LS\,RFC}$ has the above robustness property, since the speed of the front depends on its curvature (it is reduced by $\varepsilon\kappa$), and the curvature of the initial surface can influence the stopping point of the propagation.

Model $\mathcal{M}_{\triangledown RFC}$ is also invariant under the isometric (i.e., distance preserving) transformation of the scene domain. (Compare [7, thm 15].)

Theorem 6: $\mathcal{M}_{\triangledown RFC}(F\circ\mathcal{I},\mathcal{I}[\vec{S}])=\mathcal{I}\left[\mathcal{M}_{\triangledown RFC}(F,\vec{S})\right]$ for every isometry $\mathcal{I}$ of $\mathbb{R}^n$, differentiable image F, and a seed sequence $\vec{S}$.

PROOF. This follows from the similar property of model $\mathcal{M}_{\triangledown}$ (see [7, thm 15]) or directly from the fact that the gradient magnitude remains unchanged under isometrical transformation of the function domain. ■

We point out that algorithm $\mathcal{A}_{\triangledown RFC}$ does not have this property.

IV. MODELS FOR FUZZY CONNECTEDNESS ALGORITHMS USING HOMOGENEITY BASED AFFINITY

In this section, we will further explore delineation models specifically for FC.

Let $f\colon C\to\mathbb{R}^\ell$ be a digital image, where its support C is of the form Ω_h for some $h>0$ and open convex bounded subset Ω of $\mathbb{R}^n$. We will also assume that the adjacency relation is defined with $\alpha=h$; that is, that spels $c,d\in\Omega_h$ are adjacent if and only if $||c-d||\le h$. Thus, any two adjacent spels lie on a line parallel to an axis of $\mathbb{R}^n$. Now, we will define homogeneity based affinity $\psi\colon C^2\to[0,\infty]$ for the image f as follows: $\psi(c,c)=0$ for every $c\in C$, $\psi(c,d)=\infty$ for nonadjacent $c,d\in C$, and, for distinct adjacent $c,d\in C$, we put

$$\psi(c,d)=\frac{|f(c)-f(d)|}{||c-d||}=\frac{|f(c)-f(d)|}{h}. \qquad (9)$$

Here, the idea is that the larger the value of $\psi(c,d)$, the weaker is the affinity (connectivity) between c and d. Note that usually ([23], [24], [27], [28], [29]) the homogeneity based affinity is defined by a formula $\bar\psi(c,d)=e^{-|f(c)-f(d)|/\sigma^2}$ for some $\sigma>0$. However, it was proved in [10] that the affinity functions $\bar\psi$ and ψ are strongly equivalent, in a sense that the output of any version of the standard fuzzy connectedness algorithm remains unchanged, if one of these affinities is replaced by the other. (Of course, in the thresholding case of absolute FC to provide the final segmentation, the threshold needs to be adjusted, but the adjustment is effective and unique.) The gradient based and homogeneity based affinities are similar, in a sense that they both measure the local strength of spel connectedness via magnitude of image intensity rate of change — an approximation of derivative's magnitude. However, the homogeneity based affinity uses only the directional rate of change in the direction of the path, while the gradient based affinity uses the magnitude of the gradient, which has the maximal magnitude among all possible directional derivatives at the same point for each spel in the path.

As before, a path p in C is any sequence $\langle c_1,\ldots,c_k\rangle$ of spels in C, where consecutive c_i and c_{i+1} are adjacent. A path $p=\langle c\rangle$ will be identified with the path $\langle c,c\rangle$. The homogeneity based strength of $p=\langle c_1,\ldots,c_k\rangle$, where we assume that $k>1$ according to the above identification, is defined as the strength of the weakest link $\langle c_i,c_{i+1}\rangle$ in p, that is, as

$$\mu(p)=\max_{i=1,\ldots,k-1}\psi(c_i,c_{i+1}).$$

The output of the homogeneity based absolute fuzzy connectedness, AFC, algorithm $\mathcal{A}_{\psi}$ applied to an image f to obtain an object that is indicated by a set of seeds $S \subset C$ and a threshold θ is defined as

$$\mathcal{A}_\psi(f,S,\theta)=\{x\in C\colon \mu(S,x)<\theta\},$$

where $\mu(S,X) = \inf\{\mu(p)\colon p \text{ is a path from } S \text{ to } X\}$, or, equivalently, as

$$\{x\in C\colon \mu(p)<\theta \text{ for some path } p \text{ from } S \text{ to } x\}.$$

The k-object homogeneity based RFC algorithm applied to an f and a sequence $\vec{S}=\langle S_1,\ldots,S_k\rangle$ of seeds is defined as $\mathcal{A}_{\psi RFC}(f,\vec{S})=\langle \mathcal{A}^1_{\psi RFC}(f,\vec{S}),\ldots,\mathcal{A}^k_{\psi RFC}(f,\vec{S})\rangle$, where

$$\begin{aligned}\mathcal{A}^i_{\psi RFC}(f,\vec{S}) &= \{x\in C\colon \mu(S_i,x)<\mu(S_j,x) \text{ for all } j\neq i\}\\ &= \{x\in C\colon \mu(S_i,x)<\hat{\theta}_i\}\end{aligned}$$

and $\hat{\theta}_i = \min_{j\neq i}\mu(S_i,S_j)$. The second equation above in the definition of $\mathcal{A}^i_{\psi RFC}$ is justified by a discrete analog of Lemma 1 proved in [10].

A. Idealized models $\mathcal{M}_\psi$ and $\mathcal{M}_{\psi RFC}$ for $\mathcal{A}_\psi$ and $\mathcal{A}_{\psi RFC}$

Fix an open, convex, and bounded subset Ω of $\mathbb{R}^n$. Let $\mathcal{P}_0$ be the family of all straight segment paths in Ω parallel to any of the axes and let $\mathcal{P}$ consist of the paths which are finite unions of the segments from $\mathcal{P}_0$. For a continuously differentiable idealized image $F\colon\Omega\to\mathbb{R}^\ell$, define the strength of a path $p\colon[a,b]\to\mathbb{R}^\ell$ from $\mathcal{P}_0$ as

$$\mu(p)=\sup\{|DF(p(t))|\colon t\in[a,b]\},$$

where DF is the directional (so partial) derivative of F in the direction of p. Also, if $p=p_1\cup\cdots\cup p_s\in\mathcal{P}$ is a union of segments p_i from $\mathcal{P}_0$, then we put $\mu(p)=\sup_{i=1,\ldots,s}\mu(p_i)$. Also, for $X,S\subset\Omega$, put $\mu(S,X)=0$ if S intersects X, while for disjoint S and X define

$$\mu(S,X)=\inf\{\mu(p)\colon p\in\mathcal{P} \text{ is a path from } S \text{ to } X\}.$$

For a non-empty $S\subset\Omega$ and $\theta\geq 0$ we define model $\mathcal{M}_\psi$ as

$$\mathcal{M}_\psi(F,S,\theta)=\{x\in\Omega\colon \mu(S,x)<\theta\}.$$

Also, the k-object homogeneity based RFC model applied to image F and sequence $\vec{S}=\langle S_1,\ldots,S_k\rangle$ of seeds is defined as $\mathcal{M}_{\psi RFC}(F,\vec{S})=\langle\mathcal{M}^1_{\psi RFC}(F,\vec{S}),\ldots,\mathcal{M}^k_{\psi RFC}(F,\vec{S})\rangle$, where

$$\mathcal{M}^i_{\psi RFC}(F,\vec{S})=\{x\in\Omega\colon \mu(S_i,x)<\theta_i\}$$

and $\theta_i=\min_{j\neq i}\mu(S_i,S_j)$.

Theorem 7: *Algorithms $\mathcal{A}_\psi$ and $\mathcal{A}_{\psi RFC}$ weakly represent, respectively, models $\mathcal{M}_\psi$ and $\mathcal{M}_{\psi RFC}$ for the class of $\mathcal{C}^1$ idealized images $F\colon\Omega\to\mathbb{R}^\ell$ for which Ω is convex and $|\nabla F|$ is uniformly continuous on Ω.*

SKETCH OF PROOF. The proof that algorithms $\mathcal{A}_\psi$ weakly represents $\mathcal{M}_\psi$ requires only a small modification of the proof of [7, corollary 17] that the gradient based AFC algorithm $\mathcal{A}_\triangledown$ represents model $\mathcal{M}_\triangledown$. It is also similar to the proof of Theorem 3. Basically, if one starts with $S\subset\Omega_h$ and $\theta>0$, then for every $\eta>0$ and r, there is an $i_0>r$ such that, for every $i>i_0$,

$$\begin{aligned}\Omega_{h/2^r}\cap\mathcal{M}_\psi(F,\theta-\eta,S) &\subseteq \mathcal{A}_\psi(F\restriction\Omega_{h/2^i},\theta-\eta,S)\\ &\subseteq \mathcal{M}_\psi(F,\theta,S).\end{aligned}$$

This is proved precisely as [7, thm 16], the only modification being that every path $\hat{p}\colon[a,b]\to\Omega$ from $\mathcal{P}$ is approximated by a path p in $\Omega_{h/2^i}$ with the same number and direction of "segments" that $\hat{p}$ has. Since the homogeneity based affinity, as defined in (9), approximates directional derivative of F used in the definition of $\mu(\hat{p})$, we have $|\mu(\hat{p})-\mu(p)|<\eta$ provided $h/2^i$ is small enough and p is close enough to $\hat{p}$. The proof that the inclusions imply the desired representability is exactly the same as in [7, thm 16] and is very similar to that presented for Theorem 3.

The proof that $\mathcal{A}_{\psi RFC}$ represents $\mathcal{M}_{\psi RFC}$ is essentially the same as that for Theorem 3. ■

It is worth noting that, unlike for the model $\mathcal{M}_\triangledown$, the object $\mathcal{M}_\psi(F,S,\theta)$ need not be open in Ω. However, it is open in a hyperplane parallel to a vector space spanned by some axis in $\mathbb{R}^n$.

B. Invariance properties

First we note that algorithm $\mathcal{A}_{\psi RFC}$ and model $\mathcal{M}_{\psi RFC}$ have the robustness property, described formally in Theorem 5. For the algorithm, the proof that uses repeatedly an appropriate form of Lemma 1 can be found in [23] and [9, cor 2.7]). The robustness of $\mathcal{M}_{\psi RFC}$, which essentially follows from the robustness property of $\mathcal{A}_{\psi RFC}$ and Theorem 7, can be proved as Theorem 5.

Next, note that $\mathcal{M}_{\psi RFC}$ is also invariant under image translations, since translations preserve the directional derivatives.

Theorem 8: $\mathcal{M}_{\psi RFC}(F\circ t,t[\vec{S}])=t\left[\mathcal{M}_{\psi RFC}(F,\vec{S})\right]$ *for every translation t of $\mathbb{R}^n$, differentiable image F, and a seed sequence $\vec{S}$.*

Similar result is also true for $\mathcal{M}_\psi$. However, neither $\mathcal{M}_{\psi RFC}$ nor $\mathcal{M}_\psi$ is invariant under image rotations. For $\mathcal{M}_{\psi RFC}$ this is best seen in the following example.

Example 9: Let Ω be a circle in $\mathbb{R}^2$ centered at the origin and of radius 2, and let $F\colon\Omega\to\mathbb{R}$ be given by $F(x,y)=x$. Let $\vec{S}=\langle\{(0,0)\},\{(0,1)\}\rangle$. Then $\mathcal{M}_{\psi RFC}(F,\vec{S})=\langle\emptyset,\emptyset\rangle$, since the straight segment path joining the seeds has the best possible strength 0. However, if r is a 30° counter clockwise rotation about the origin, then $\mathcal{M}_{\psi RFC}(F\circ r,r[\vec{S}])=\langle L_0,L_1\rangle$, where L_i is an intersection of Ω with the vertical line containing $r((0,i))$. This is the case since the partial derivatives are constant and equal $\frac{\partial}{\partial y}(F\circ r)=\frac{1}{2}$ and $\frac{\partial}{\partial x}(F\circ r)=\frac{\sqrt{3}}{2}$.

The lack of invariance of $\mathcal{M}_{\psi RFC}$ and $\mathcal{M}_\psi$ under rotation comes from our restriction of the direction of paths, which in turn restricts the directional derivatives used in the definition of path strength to the partial derivatives. One might suggest to relax the path restriction, to allow either all piecewise smooth or piecewise straight paths. Indeed, this seems to solve the problem, since the resulting model, call it $\mathcal{M}^*_\psi$, is indeed rotation invariant. However, there are two problems with this approach. First of all, there are difficulties in defining the algorithm representing the resulting model, since in the scenes with support Ω_h it is difficult to approximate the directional derivatives in all possible directions. (Nevertheless, including all directions can be handled, following the ideas from [2].) However, there is a less obvious, but more serious, problem: for a very large class of $\mathcal{C}^1$ images $F\colon \Omega \to \mathbb{R}$, the delineated object is trivial—it equals the entire scene domain Ω. This is best seen in the following simple example.

Example 10: Let Ω be a circle in $\mathbb{R}^2$ centered at the origin and for every $\sigma > 0$ let $F_\sigma\colon \Omega \to \mathbb{R}$ be the Gaussian $F_\sigma(x,y) = e^{-(x^2+y^2)/\sigma^2}$. Although the magnitude of gradient (and partial derivative along each axis) can be arbitrary large, we have $\mathcal{M}^*_\psi(F_\sigma, \theta, S) = \Omega$ for arbitrary σ, $\theta > 0$, and non-empty $S \subset \Omega$. This is the case, since between any $c, d \in \Omega$ there is a path p from c to d with the magnitude of directional derivative along p being smaller than θ at every point on p. The path is formed as a spiral following closely the circles centered at the origin.

REFERENCES

[1] S. Beucher, “The watershed transformation applied to image segmentation,” in: *10th Pfefferkorn Conf. Signal and Image Processing in Microscopy and Microanalysis* (1992), 299–314.

[2] Y. Boykov and V. Kolmogorov, Computing geodesics and minimal surfaces via graph cuts, *International Conference on Computer Vision* **I** (2003), 26–33.

[3] Y. Boykov, V. Kolmogorov, D. Cremers, and A. Delong, An Integral Solution to Surface Evolution PDEs via Geo-Cuts, *International Conference on Computer Vision*, LNCS 3953, vol.III (2006), 409–422.

[4] Y. Boykov, O. Veksler, and R. Zabih, “Fast approximate energy minimization via graph cuts,” *IEEE Trans. Pattern Anal. Machine Intell.* **23**(11) (2001), 1222–1239.

[5] A. Chakraborty, L. Staib, and J. Duncan, “Deformable boundary finding in medical images by integrating gradient and region information,” *IEEE Trans. Med. Imag.* **15**(6) (1996), 859–870.

[6] T. Chan and J. Shen, *Image processing and analysis: variational, PDE, wavelet, and stochastic methods*, Society of Industrial and Applied Mathematics, 2005.

[7] K.C. Ciesielski and J.K. Udupa, “A general theory of image segmentation: level set segmentation in the fuzzy connectedness framework,” Technical Report MIPG-335, Medical Image Processing Group, Department of Radiology, University of Pennsylvania, Philadelphia, 2006.

[8] K.C. Ciesielski and J.K. Udupa, “A General Theory of Image Segmentation: Level Set Segmentation in the Fuzzy Connectedness Framework,” *Medical Imaging 2007: Image Processing, SPIE Proceedings* **6512** (2007).

[9] K.C. Ciesielski, J.K. Udupa, P.K. Saha, and Y. Zhuge, “Iterative relative fuzzy connectedness for multiple objects, allowing multiple seeds,” *Computer Vision and Image Understanding*, to appear; Technical Report MIPG-327, Medical Image Processing Group, Department of Radiology, University of Pennsylvania, Philadelphia, 2005; for preprint see http://jacobi.math.wvu.edu/˜kcies/publications.html

[10] K.C. Ciesielski, J.K. Udupa, and Y. Zhuge, “Affinity functions in fuzzy connectedness based image segmentation,” Technical Report MIPG-334, Medical Image Processing Group, Department of Radiology, University of Pennsylvania, Philadelphia, 2006.

[11] T. Cootes, C. Taylor, and D. Cooper, “Active shape models-their training and application,” *Computer Vision and Image Understanding* **61** (1995), 38–59.

[12] T. Cootes, G. Edwards, and C. Taylor, “Active appearance models,” *IEEE Trans. Pattern Anal. Machine Intell.* **23**(6) (2001), 681–685.

[13] A.X. Falcão, J.K. Udupa, S. Samarasekera, S. Sharma, B.E. Hirsch, and R. Lotufo, “User-steered image segmentation paradigms: live wire and live lane,” *Graph. Models Image Process* **60**(4) (1998), 233–260.

[14] C. Imielinska, D. Metaxas, J.K. Udupa, Y. Jin, and T. Chen, “Hybrid segmentation of anatomical data,” in: *Proceedings of MICCAI* (2001), 1048–1057.

[15] M. Kass, A. Witkin, and D. Terzopoulos, “Snakes: Active contour models,” *Int. J. Comput. Vision* **1** (1987), 321–331.

[16] R. Malladi, J.A. Sethian, and B.C. Vemuri, “Shape Modeling with Front Propagation: A Level Set Approach,” *IEEE Transactions on Pattern Analysis and Machine Intelligence* **17**(2) (1995), 158–175.

[17] A.R. Mansouri and J. Konrad, “Multiple motion segmentation with level sets,” in: *Proc. IEEE Intern. Conf. Image Processing* **II** (1999), 126–130.

[18] A. Mansouri, A. Mitiche, and C. Vázquez, “Multiregional competition: A level set extension of region competition to multiple region image partitioning,” *Computer Vision and Image Understanding* **101** (2006), 137–150.

[19] T. McInerney and D. Terzopoulos, “Deformable models in medical image analysis: A survey,” *Medical Image Analysis* **1**(2) (1996), 91–108.

[20] D. Morgenthaler and A. Rosenfeld, “Multidimensional edge detection by hypersurface fitting,” *IEEE Transactions on Pattern Analysis and Machine Intelligence* **3** (1981), 482–486.

[21] D. Mumford and J. Shah, “Optimal approximations by piecewise smooth functions and associated variational problems,” *Communications on Pure and Applied Mathematics* **42** (1989), 577–685.

[22] L. Najman, M. Couprie, and G. Bertrand, Watersheds, extension maps, and the emergence paradigm, preprint.

[23] P.K. Saha and J.K. Udupa, “Relative fuzzy connectedness among multiple objects: Theory, algorithms, and applications in image segmentation,” *Computer Vision and Image Understanding* **82**(1) (2001), 42–56.

[24] P. K. Saha and J. K. Udupa, Iterative relative fuzzy connectedness and object definition: theory, algorithms, and applications in image segmentation. In: *Proceedings of IEEE Workshop on Mathematical Methods in Biomedical Image Analysis*, Hilton Head, South Carolina 2002, 28–35.

[25] J.A. Sethian, *Fast Marching Methods and Level Sets Methods. Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science*, Cambridge Univ. Press, 1999.

[26] M. Trivedi and J. Bezdek, “Low-level segmentation of aerial images with fuzzy clustering,” *IEEE Trans. Systems, Man, and Cybernetics* **16**(4) (1986), 589–598.

[27] J.K. Udupa and P.K. Saha, “Fuzzy connectedness in image segmentation,” *Proceedings of the IEEE*, **91**(10) (2003), 1649–1669.

[28] J.K. Udupa, P.K. Saha, and R.A. Lotufo, “Relative fuzzy connectedness and object definition: Theory, algorithms, and applications in image segmentation,” *IEEE Transactions on Pattern Analysis and Machine Intelligence* **24** (2002), 1485–1500.

[29] J.K. Udupa and S. Samarasekera, “Fuzzy connectedness and object definition: theory, algorithms, and applications in image segmentation,” *Graphical Models and Image Processing* **58**(3) (1996), 246–261.

[30] L.A. Vese and T.F. Chan, “A multiphase level set framework for image segmentation using the Mumford and Shah model,” *Int. J. Comput. Vis.* **50**(3) (2002), 271–293.

[31] S. Zhu and A. Yuille, “Region competition: unifying snakes, region growing, and bayes/MDL for multiband image segmentation,” *IEEE Trans. Pattern Anal. Mach. Intell.* **18**(9) (1996), 884–900.