HAUSDORFF DIMENSION OF EXTREMELY SLOW MINIMAL DYNAMICAL SYSTEMS AND HÖLDER PRESERVING DIFFERENTIABLE EXTENSIONS

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Abstract. We study continuous functions f from compact perfect subsets $\mathfrak{C}$ of $\mathbb{R}$ *onto* $\mathfrak{C}$ with vanishing derivative everywhere. We show that the domain of such function can have Hausdorff dimension d for any $d \in [0,1)$ and that it can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$ such that F is α-Hölder for every $\alpha \in (0,1)$. This last part is deduced from a novel generalization of Jarník's differentiable extension theorem stating that every differentiable map $f\colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is compact, admits a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ which preserves Hölder continuity of f.

1. Introduction

A *dynamical system* is any continuous function f from a metric (or, more general, topological) space $\langle X, d\rangle$ into itself. It is a *minimal system* when the orbit $O(x) := \{x, f(x), f^2(x), \ldots\}$ of every $x \in X$ is dense in X and it is a *Cantor system* when X is homeomorphic to the Cantor ternary set. We say that an $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is *extremely slow*[1] provided, for every $\lambda \in (0,1)$, f is a pointwise contractive with the constant λ,[2] that is, such that for every $x \in X$ there is an open $U \ni x$ for which $d(f(x), f(y)) \le \lambda d(x,y)$ for every $y \in U$. Notice that if $X \subset \mathbb{R}$ is considered with the standard distance, then f is extremely slow if, and only if, $f'(x) = 0$ for every non-isolated $x \in X$.

The first extremely slow minimal dynamical system f from a compact perfect $\mathfrak{C} \subset \mathbb{R}$ onto $\mathfrak{C}$ was described in a 2016 paper [8] of the first author and Jakub Jasinski. Consecutively, such systems were studied in [1, 2, 7, 12].

It was noticed in [8] that any compact $\mathfrak{C} \subset \mathbb{R}$ that admits extremely slow dynamical system has Lebesgue measure 0, so it is nowhere dense in $\mathbb{R}$. The first goal

Date: 05/19/2023.
2020 *Mathematics Subject Classification.* Primary 37C05, 26A16; Secondary: 28A78, 37E99
Key words: minimal dynamical systems, extremely slow dynamical systems, Hausdorff dimension, Hölder property, differentiable extensions.

[1]In papers [2, 12] the authors refer to extremely slow dynamical systems simply as "slow dynamical systems." However, neither of these papers formally contains a definition of this notion and the commonly used notion of "slow-fast dynamics" (see e.g. [16]) treats slowness as "the magnitude of the derivative being less than 1," rather than being equal to 0. So, adding adjective "extremely" to the term we describe seems appropriate.

[2]In [5] such functions are refered to as *pointwise contractive with the same contraction constant*, λ, and denoted as (uPC). Note that this notion is considerably weaker than that of *being locally contractive with constant* λ, which is commonly defined as "for every $x \in X$ there is an open $U \ni x$ for which $d(f(y), f(z)) \le \lambda d(x,y)$ for every $y, z \in U$." For a more detailed discussion of these two (and other related) notions see [5].

of this paper is to show, in Theorem 2.1, that such sets $\mathfrak{C}$ can still be relatively large: they can have Hausdorff dimension arbitrarily close to 1.

All extremely slow dynamical system described so far in the literature are of the form

$$f = h \circ \sigma \circ h^{-1} \colon \mathfrak{C} \to \mathfrak{C}, \tag{1}$$

where $\mathfrak{C} \subset \mathbb{R}$ is compact, $\sigma \colon X \to X$ is a dynamical system, and $h \colon X \to \mathbb{R}$ is an embedding with $\mathfrak{C} = h[X]$. Following [2] we say that a function f given by (1) constitutes an *embedding of* σ *into* $\mathbb{R}$.

The originally constructed extremely slow f from [8], its considerably simplified form described in [7], as well as all examples constructed in this manuscript are of the form (1), with σ being a minimal dynamical system on 2^ω (where 2^ω denotes the set of all sequences from $\omega := \{0, 1, 2, \dots\}$ into $2 := \{0, 1\}$) known as a *binary odometer* or *add-one-and-carry adding machine* and defined as

$$\sigma(\langle 1, 1, \dots, 1, 0, s_{k+1}, s_{k+2}, \dots \rangle) := \langle 0, 0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots \rangle \tag{2}$$

and $\sigma(\langle 1, 1, 1, \dots \rangle) := \langle 0, 0, 0, \dots \rangle$. However, this choice of sigma is dictated only by its simplicity and other dynamical systems can also be embedded into $\mathbb{R}$ as extremely slow systems. In fact, Boroński, Kupka, and Oprocha proved in [2] that every minimal Cantor dynamical system admits extremely slow embedding into $\mathbb{R}$. In addition, in [12] Gangloff and Oprocha characterized all Cantor dynamical systems that admit extremely slow embeddings into $\mathbb{R}$.

The definition of h in the first paper [8] was quite intricate. However, in [7] its definition is of the following considerably simpler format, very similar to one we use in this article:

$$h(s) := \textstyle\sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s \restriction n)}, \tag{3}$$

where $N(s \restriction n)$ is the natural number for which the following 0-1 sequence $\nu(s, n) = \langle t_n, t_{n-1}, \dots, t_0 \rangle := \langle 1, 1 - s_{n-1}, s_{n-2}, \dots, s_0 \rangle$ is its binary representation, that is,

$$N(s \restriction n) := \sum_{i \le n} t_i 2^i = \sum_{i < n-1} s_i 2^i + (1 - s_{n-1}) 2^{n-1} + 2^n. \tag{4}$$

Notice that

$$2^n \le N(s \restriction n) \le \textstyle\sum_{i \le n} 2^i < 2^{n+1} \text{ for every } s \in 2^\omega. \tag{5}$$

1.1. **Hausdorff dimension.** Recall that the Hausdorff dimension $\dim_H(E)$ of a set $E \subset \mathbb{R}$ is defined as follows, see e.g. [9]. For $\rho \ge 0$ and $\delta > 0$ let

$$\mathcal{H}^\rho_\delta(E) := \inf \left\{ \sum_{U \in \mathcal{U}} |U|^\rho \colon \mathcal{U} \text{ is a } \delta\text{-cover of } E \right\},$$

where $|U|$ denotes the diameter of U and by δ-*cover* of E we mean a cover of E by sets of diameter less than or equal to δ. It is well known and easy to see that the value of $\mathcal{H}^\rho_\delta(E)$ remains unchanged when in its definition we allow only δ-covers by open intervals. The *Hausdorff* ρ-*dimensional measure* of E is defined as

$$\mathcal{H}^\rho(E) := \lim_{\delta \to 0^+} \mathcal{H}^\rho_\delta(E). \tag{6}$$

It is well defined, as the map $\rho \mapsto \mathcal{H}^\rho_\delta(E)$ is monotone. The *Hausdorff dimension* $\dim_H(E)$ is defined as the only number $d \geq 0$ such that

$$\mathcal{H}^\rho(E) = \infty \text{ for every } \rho \in [0, d) \text{ and } \mathcal{H}^\rho(E) = 0 \text{ for every } \rho > d.$$

Remark 1.1. If $\mathfrak{C}_0 := h[2^\omega]$ for h given by (3), then $\dim_H(\mathfrak{C}_0) = 0$.

Proof. For every $n < \omega$ and $t \in 2^n$ let $[t] := \{s \in 2^\omega : t \subset s\}$. Notice that if $s \in [t]$, then

$$\sum_{i<n} 2s_i 3^{-(i+1)N(s\restriction i)} \leq h(s) \leq \sum_{i<n} 2s_i 3^{-(i+1)N(s\restriction i)} + \sum_{i=n}^{\infty} 2 \cdot 3^{-(i+1)N(s\restriction i)}.$$

So, as $N(s \restriction i) \geq N(s \restriction n) \geq 2^n$ for any $i \geq n$,

$$|h([s])| \leq \sum_{i=n}^{\infty} 2 \cdot 3^{-(i+1)N(s\restriction i)} \leq \sum_{i=n}^{\infty} 2 \cdot 3^{-(i+1)2^n} \leq \sum_{i=0}^{\infty} 2 \cdot 3^{-(n+1)2^n - i} = 3 \cdot 3^{-(n+1)2^n}.$$

Therefore, $\mathcal{U} = \{h[t] : t \in 2^n\}$ is a 3^{-n}-cover of $\mathfrak{C}_0$ and, for every $d > 0$,

$$\mathcal{H}^d_{3^{-n}}(\mathfrak{C}_0) \leq \sum_{s \in 2^n} |h[s]|^d \leq 2^n \cdot 3 \cdot 3^{-d(n+1)2^n} \to_{n\to\infty} 0.$$

Hence, indeed $\mathfrak{C}_0$ has Hausdorff dimension 0. □

The remark shows that $\mathfrak{C}_0$ is, in the sense of Hausdorff dimension, as small as it gets. One of the goals of this paper is to show, see Theorem 2.1, that there are compact perfect sets $\mathfrak{C}$ admitting extremely slow minimal dynamical systems that are as big as possible, that is, of Hausdorff dimension arbitrarily close to 1.

1.2. **α-Hölder property.** Let $\alpha \in (0, 1]$ and f be a function from $P \subset \mathbb{R}$ into $\mathbb{R}$. Recall that f *is α-Hölder* provided there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha \text{ for all } x, y \in P.$$

In [2, Question 1.4] the authors ask if every minimal Cantor dynamical system can be embedded into $\mathbb{R}$ as (extremely) slow Cantor dynamical system f such that f can be extended to differentiable F which is also α-Hölder for some $\alpha \in (0, 1)$. The second goal of this paper is to show, see Theorem 3.1, that the answer to this question is positive for the binary odometer σ defined in (2).

In what follows we will use the following two simple and certainly known facts on Hölder continuity. For reader convenience we include here their proofs.

Fact 1.2. *Let $\alpha \in (0, 1]$ and $P \subset \mathbb{R}$ be closed. Assume that there is a $C > 0$ such that $F : \mathbb{R} \to \mathbb{R}$ is α-Hölder with constant C on the closure $\mathrm{cl}(J)$ of every connected component J of $\mathbb{R} \setminus P$, that is,*

$$|F(x) - F(y)| \leq C|x - y|^\alpha \text{ for all } x, y \in \mathrm{cl}(J). \tag{7}$$

If $F \restriction P$ is α-Hölder, then F is also α-Hölder.

Proof. Increasing C, if necessary, we can assume that also

$$|F(x) - F(y)| \leq C|x - y|^\alpha \text{ for all } x, y \in P.$$

Choose any $p < q$. It is enough to show that $|F(p) - F(q)| \leq 3C|p - q|^\alpha$.

If $[p,q]\cap P=\emptyset$, then $[p,q]$ is contained in a single connected component of $\mathbb{R}\setminus P$, so the inequality holds. So, assume that $[p,q]\cap P\neq\emptyset$, let $x=\min[p,q]\cap P$, and $y=\max[p,q]\cap P$. Then,

$$\begin{aligned}|F(p)-F(q)| &\leq |F(p)-F(x)|+|F(x)-F(y)|+|F(y)-F(q)|\\ &\leq C|p-x|^\alpha+C|x-y|^\alpha+C|y-q|^\alpha\leq 3C|p-q|^\alpha\end{aligned}$$

as needed. □

Fact 1.3. *Let $a<b$, $P\subset[a,b]$ be compact, and $F\colon(-\infty,a]\cup P\cup[b,\infty)\to\mathbb{R}$ be such that $F\restriction(-\infty,a]\equiv F(\min P)$ and $F\restriction[b,\infty)\equiv F(\max P)$. Then, for every $\alpha\in(0,1]$, F is α-Hölder if, and only if, $F\restriction P$ is α-Hölder.*

Proof. Forward implication is obvious. To see the other implication, assume that $F\restriction P$ is α-Hölder with constant C, choose $x_1,x_2\in(-\infty,a]\cup P\cup[b,\infty)$ such that $x_1\leq x_2$, and define $r:=\max\{x_1,\min P\}$ and $s:=\min\{x_2,\max P\}$. Then

$$|F(x_1)-F(x_2)|=|F(r)-F(s)|\leq C|r-s|^\alpha\leq C|x_1-x_2|^\alpha,$$

that is, indeed F is α-Hölder. □

It is well known (see e.g. [11]) and easy to see that

(8) if P is compact, f is α-Hölder, and $\beta\in(0,\alpha)$, then f is also β-Hölder.

2. Slow systems of any Hausdorff dimension

The entire content of this section is devoted to prove the following result.

Theorem 2.1. *For every $d\in[0,1)$ there exists an extremely slow minimal dynamical system $f_d\colon\mathfrak{C}_d\to\mathfrak{C}_d$ such that $\mathfrak{C}_d\subset\mathbb{R}$ is compact and has Hausdorff dimension d.*

By Remark 1.1, we can assume that $d>0$. So, fix $d\in(0,1]$ and define $p:=2^{-1/d}$. Notice that $p\in(0,1/2]$ and

$$d=\log_p(1/2).$$

For $n<\omega$ define $\hat{n}:=0$ when $n<2$ and $\hat{n}:=\lfloor\log_2\log_2 n\rfloor$ for $n\geq 2$. Thus, the map $n\mapsto\hat{n}$ is nondecreasing. Also, for $n\geq 2$ we have $\hat{n}\leq\log_2\log_2 n<\hat{n}+1$, so $2^{\hat{n}}\leq\log_2 n<2\cdot 2^{\hat{n}}$ and, by (5), for every $s\in 2^\omega$,

$$\frac{1}{2}\log_2 n\leq 2^{\hat{n}}\leq N(s\restriction\hat{n})<2^{\hat{n}+1}\leq 2\log_2 n, \tag{9}$$

where N is as defined in (4). Function f_d is defined similarly as f given by (1), (3), and (4):

$$f_d:=h_d\circ\sigma\circ h_d^{-1}\colon\mathfrak{C}_d\to\mathfrak{C}_d, \tag{10}$$

where $\sigma\colon 2^\omega\to 2^\omega$ is the add-one-and-carry adding machine defined in (2) while $h_d\colon 2^\omega\to\mathbb{R}$ is defined as

$$h_d(s):=\textstyle\sum_{n=0}^\infty s_n p^{n+\psi(s\restriction n)},$$

where $\psi(s\restriction n):=N(s\restriction\hat{n})^2$. Notice that the sequence $\langle\psi(s\restriction n)\rangle_n$ is nondecreasing. Similarly as earlier, we put $\mathfrak{C}_d:=h_d[2^\omega]$.

In the rest of this section we prove that f_d, for $d\in(0,1)$, is as claimed in Theorem 2.1. We also indicate where the argument does not work for $d=1$.

2.1. **Geometrical description of $\mathfrak{C}_d$.** For $n<\omega$ and $t\in 2^n$ let

$$I_t := [a_t,b_t] \quad \text{where} \quad a_t := \inf h_d([t]) \ \text{ and } \ b_t := \sup h_d([t]).$$

Let $\psi_n := (2\log_2 n)^2$ and notice that, by (9), $\psi_n > \psi(t)$. Next, we will show

$$p^{n+\psi_n} \le |I_t| < \frac{1}{1-p}\, p^{n+\psi(t)} \le p^{n-1+\psi(t)}. \tag{11}$$

To see this, let $s\in 2^\omega$ be an extension of t such that $s_i=1$ for every $i\ge n$ and notice that $|I_t| = b_t - a_t = \sum_{i=n}^\infty p^{i+\psi(s\restriction i)}$. Then, the lower estimate of $|I_t|$ in (11) is justified by $\sum_{i=n}^\infty p^{i+\psi(s\restriction i)} > p^{n+\psi(s\restriction n)} \ge p^{n+\psi_n}$, while the upper estimates by

$$\sum_{i=n}^\infty p^{i+\psi(s\restriction i)} < \sum_{i=n}^\infty p^{i+\psi(t)} = \frac{1}{1-p}\, p^{n+\psi(t)} \le p^{n-1+\psi(t)},$$

where the last inequality holds since $\frac{p}{1-p}\le 1$ for any $p\in[0,1/2]$.

Next, notice that if $t\hat{\ }j\in 2^{n+1}$ is an extension of t such that $(t\hat{\ }j)_n = j$, then (11) and the inequality $\psi(t\hat{\ }0)\le\psi(t\hat{\ }1)$ imply that

$$b_{t\hat{\ }0} = a_t + |I_{t\hat{\ }0}| < a_t + p^{(n+1)-1+\psi(t\hat{\ }0)} \le a_t + p^{n+\psi(t\hat{\ }1)} = a_{t\hat{\ }1}. \tag{12}$$

This means that the intervals in the family $\mathcal{C}_n := \{I_t\colon t\in 2^n\}$ are pairwise disjoint, so that if $s,t\in 2^\omega$ are distinct, then the sets $\{h_d(s)\} = \bigcap_{n<\omega} I_{s\restriction n}$ and $\{h_d(t)\} = \bigcap_{n<\omega} I_{t\restriction n}$ are disjoint. This implies that h_d is indeed an embedding, what was implicitly assumed in our definition (10) of function f_d. Moreover, our set $\mathfrak{C}_d = h_d[2^\omega]$ can be also represented in the standard geometric format of the Cantor ternary set:

$$\mathfrak{C}_d = \bigcap_{n<\omega} \bigcup \mathcal{C}_n.$$

We will use this representation when calculating Hausdorff dimension of $\mathfrak{C}_d$.

2.2. **Hausdorff dimension of $\mathfrak{C}_d$.** So far, we proved that f_d is a well defined minimal Cantor dynamical system. In this subsection we will show that $\dim_H(\mathfrak{C}_d)=d$. This will be deduced from the following lemma.

Lemma 2.2. *Let $p\in(0,1/2]$ and for every $n<\omega$ let $\mathcal{C}_n$ be a family of 2^n pairwise disjoint closed intervals such that each $I\in\mathcal{C}_n$ contains two intervals from $\mathcal{C}_{n+1}$. Assume that there is a sequence $\langle \psi_n\in[0,\infty)\colon n<\omega\rangle$ such that $\lim_{n\to\infty}\frac{\psi_n}{n+1}=0$ and*

- *the length of every $I\in\mathcal{C}_n$ is between $p^{n+\psi_n}$ and p^{n-1}.*

Then $C := \bigcap_{n<\omega}\bigcup\mathcal{C}_n$ has Hausdorff dimension $\rho := \log_p(1/2)$.

Notice that our set $\mathfrak{C}_d$ satisfies the assumptions of this lemma: by (11) the length of every $I_t\in\mathcal{C}_n$ is between $p^{n+\psi_n}$ and $p^{n-1+\psi(t)}\le p^{n-1}$. Also, clearly $\lim_{n\to\infty}\frac{\psi_n}{n+1} = \lim_{n\to\infty}\frac{(2\log_2 n)^2}{n+1} = 0$. Therefore, the lemma implies that $\dim_H(\mathfrak{C}_d) = \log_p(1/2) = d$.

Note that assumptions about $\mathcal{C}_n$'s implies that every element of $\mathcal{C}_{n+1}$ is included in exactly one element of $\mathcal{C}_n$.

The last paragraph of the following proof comes from [10, Mass distribution principle 4.2].

Proof of Lemma 2.2. To see $\dim_H(C) \le \rho$ notice that, by $\bullet$, for every $n < \omega$ all sets in $\mathcal{C}_n$ have diameters less than or equal to p^{n-1}. So, $\mathcal{C}_n$ is a p^{n-1}-cover of C. Moreover, $\sum_{I\in\mathcal{C}_n} |I|^\rho \le 2^n (p^{n-1})^\rho = 2^n \cdot (1/2)^{n-1} = 2$. Therefore, by the property (6), $\mathcal{H}^\rho(C) = \lim_{n\to\infty} \mathcal{H}^\rho_{p^n}(C) \le 2$.

To see that $\dim_H(C) \ge \rho$, fix an $\eta \in (0,\rho)$. It is enough to show that $\mathcal{H}^\eta(C) > 0$. For this, let μ_0 be the standard product measure on 2^ω (i.e., such that $\mu_0([s]) = 2^{-n}$ for every $s \in 2^n$) and define a measure μ on $\mathbb{R}$ (referred sometimes as a mass distribution of C) such that $\mu(U) = \mu_0(\bigcup\{[t]\colon I_t \subset U\})$ for every open $U \subset \mathbb{R}$. In particular, $\mu(I) = 2^{-n}$ for every $I \in \mathcal{C}_n$.

Since $\lim_{n\to\infty} \frac{n+\psi_n}{n+1} = 1 < \frac{\rho}{\eta}$ we can find an $m < \omega$ such that

$$\frac{n+\psi_n}{n+1} \le \frac{\rho}{\eta} \text{ whenever } n \ge m. \tag{13}$$

The number $M := \sup\left\{\frac{\psi_n}{n+1} \colon n < \omega\right\}$ is finite, since $\lim_{n\to\infty} \frac{\psi_n}{n+1} = 0$. Define $\delta := p^{(M+1)(m+1)} > 0$ and notice that

$$|I|^\eta \ge \frac{1}{2}\mu(I) \text{ whenever } |I| \le \delta \text{ and } I \in \mathcal{C}_n \text{ for an } n < \omega. \tag{14}$$

Indeed, our assumption on the lengths of the intervals $I \in \mathcal{C}_n$ implies that $p^{n+\psi_n} \le |I| \le \delta = p^{(M+1)(m+1)}$. Since

$$n + \psi_n = \left(\frac{n+\psi_n}{n+1}\right)(n+1) \le \left(1 + \frac{\psi_n}{n+1}\right)(n+1) \le (M+1)(n+1),$$

we get $p^{(M+1)(n+1)} \le p^{n+\psi_n} \le p^{(M+1)(m+1)}$. Therefore, $n \ge m$ and, by (13), $(n+\psi_n)\eta \le \rho(n+1)$. So, $|I|^\eta \ge p^{(n+\psi_n)\eta} \ge p^{\rho(n+1)} = 2^{-(n+1)} = \frac{1}{2}\mu(I)$.

The key fact for the rest of our argument is that

$$|U|^\eta \ge \frac{1}{8}\mu(U) \quad \text{for every open interval } U \text{ with } |U| \le \delta. \tag{15}$$

To see (15), take an open interval U with $|U| \le \delta$. If $U \cap C = \emptyset$, then $\mu(U) = 0$ and (15) holds. So assume that $U \cap C \ne \emptyset$ and let $n < \omega$ be the smallest such that U contains some $J \in \mathcal{C}_n$. Then $|J| \le |U| \le \delta$. Moreover, by the minimality of n, the family $\mathcal{F}$ of all $I \in \mathcal{C}_n$ intersecting U can have at most 4 elements. In particular, $J \subset U \subset \bigcup\mathcal{F}$ and, by (14),

$$\mu(U) \le \mu(\bigcup\mathcal{F}) = \sum_{I\in\mathcal{F}} \mu(I) \le 4\mu(J) \le 8|J|^\eta \le 8|U|^\eta,$$

implying (15).

Finally notice that if $\mathcal{U}$ is a δ-cover of C by open intervals then, by (15),

$$\sum_{U\in\mathcal{U}} |U|^\eta \ge \sum_{U\in\mathcal{U}} \frac{1}{8}\mu(U) \ge \frac{1}{8}\mu\left(\bigcup\mathcal{U}\right) = \frac{1}{8}\mu(C) = \frac{1}{8}.$$

Thus $\mathcal{H}^\eta(C) = \lim_{\delta\to 0^+} \mathcal{H}^\eta_\delta(C) \ge \frac{1}{8} > 0$, as needed. □

2.3. **The derivative of f_d for $d < 1$.** It remains to show that $f_d'(x) = 0$ for every $x \in \mathfrak{C}_d$. The argument for this is very similar to one used in [7] to show the same result for f defined by (1), (3), and (4).

Notice that $d \in (0,1)$ ensures that $p = 2^{-1/d} < 1/2$ so that $\frac{p}{1-p} < 1$. We start with the following two observations.

(a) For every $s \in 2^\omega$ there is a $k \in \omega$ such that $N(\sigma(s) \restriction \hat{n}) = N(s \restriction \hat{n}) + 1$ for every $n > k$.

(b) If $n = \min\{i \in \omega \colon s_i \neq t_i\}$ for some distinct $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ from 2^ω, then $\left(1 - \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)} \leq |h_d(s) - h_d(t)| \leq \left(1 + \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}$.

To see (a) note that $N(\sigma(s) \restriction \ell) = N(s \restriction \ell) + 1$ whenever $s \restriction \ell \neq \langle 1, \ldots, 1, 0\rangle$. Since $s \restriction \ell = \langle 1, \ldots, 1, 0\rangle$ for at most one $\ell < \omega$, there exists k_0 such that $N(\sigma(s) \restriction \ell) = N(s \restriction \ell) + 1$ provided $\ell > k_0$. Then $k = 2^{2^{k_0+1}}$ is as needed since then $n > k$ implies $\hat{n} > k_0$.

To see (b), take s and t as in its assumption. Notice that $\widehat{n+1} \leq n$, so

$$\psi(s \restriction n+1) = (N(s \restriction \widehat{n+1}))^2 = (N(t \restriction \widehat{n+1}))^2 = \psi(t \restriction n+1). \tag{16}$$

We may assume that $s_n = 1$ and $t_n = 0$. Let $u = t \restriction n = s \restriction n$. Then, using the notation from Subsection 2.1, we have $h_d(t) \in I_{u^\frown 0} = [a_{u^\frown 0}, b_{u^\frown 0}]$ and $h_d(s) \in I_{u^\frown 1} = [a_{u^\frown 1}, b_{u^\frown 1}]$. So, by (12), $a_{u^\frown 0} \leq h_d(t) \leq b_{u^\frown 0} < a_{u^\frown 1} \leq h_d(s) \leq b_{u^\frown 1}$. In particular, by (11) and (16),

$$\begin{aligned}
|h_d(s) - h_d(t)| &\geq a_{u^\frown 1} - b_{u^\frown 0} = \left(a_u + p^{n+\psi(u^\frown 1)}\right) - (a_u + |I_{u^\frown 0}|) \\
&\geq p^{n+\psi(u^\frown 1)} - \frac{1}{1-p}\, p^{n+1+\psi(u^\frown 0)} = \left(1 - \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}
\end{aligned}$$

and

$$\begin{aligned}
|h_d(s) - h_d(t)| &\leq b_{u^\frown 1} - a_{u^\frown 0} = (a_u + p^{n+\psi(u^\frown 1)} + |I_{u^\frown 1}|) - a_u \\
&\leq p^{n+\psi(u^\frown 1)} + \frac{1}{1-p}\, p^{n+1+\psi(u^\frown 0)} = \left(1 + \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)},
\end{aligned}$$

the desired (b).

To see that $f'_d(h_d(s)) = 0$ for an $s \in 2^\omega$, choose a $k \in \omega$ satisfying (a) and let $\delta > 0$ be such that $0 < |h_d(s) - h_d(t)| < \delta$ implies that $n = \min\{i \in \omega \colon s_i \neq t_i\}$ is greater than k. Fix a $t \in 2^\omega$ for which $0 < |h_d(s) - h_d(t)| < \delta$. Then we have $n = \min\{i \in \omega \colon s_i \neq t_i\} = \min\{i \in \omega \colon \sigma(s)_i \neq \sigma(t)_i\}$ and, using (b) for the pairs $\langle s, t\rangle$ and $\langle \sigma(s), \sigma(t)\rangle$, we obtain

$$\tfrac{|f_d(h_d(s)) - f_d(h_d(t))|}{|h_d(s) - h_d(t)|} = \tfrac{|h_d(\sigma(s)) - h_d(\sigma(t))|}{|h_d(s) - h_d(t)|} \leq \tfrac{\left(1+\frac{p}{1-p}\right)p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1-\frac{p}{1-p}\right)p^{n+\psi(s\restriction n+1)}}. \tag{17}$$

Also, using (a), we get

$$\begin{aligned}
\psi(\sigma(s) \restriction n+1) - \psi(s \restriction n+1) &= (N(\sigma(s) \restriction \widehat{n+1}))^2 - (N(s \restriction \widehat{n+1}))^2 \\
&= (N(s \restriction \widehat{n+1}) + 1)^2 - (N(s \restriction \widehat{n+1}))^2 \\
&\geq N(s \restriction \widehat{n+1}).
\end{aligned}$$

From this, (17), and letting $c := \frac{\left(1+\frac{p}{1-p}\right)}{\left(1-\frac{p}{1-p}\right)}$, we get

$$\frac{|f_d(h_d(s)) - f_d(h_d(t))|}{|h_d(s) - h_d(t)|} \leq \frac{\left(1 + \frac{p}{1-p}\right) p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1 - \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}} \leq c \cdot p^{N(s\restriction \widehat{n+1})}. \tag{18}$$

Hence $f'_d(h_d(s)) = 0$, as $p^{N(s\restriction\widehat{n+1})}$ is arbitrarily small for δ small enough.

3. Hölder property of maps f_d

The goal of this section is to prove the following result.

Theorem 3.1. *For every $d \in (0,1)$ the extremely slow minimal dynamical system $f_d\colon \mathfrak{C}_d \to \mathfrak{C}_d$ defined in the previous section is α-Hölder for any $\alpha \in (0,1)$. Moreover, there exists a differentiable extension $F_d\colon \mathbb{R} \to \mathbb{R}$ of f_d such that F_d is α-Hölder for every $\alpha \in (0,1)$.*

The first step in the proof of this theorem is

Lemma 3.2. *Every $f_d\colon \mathfrak{C}_d \to \mathfrak{C}_d$, with $d \in (0,1]$, is α-Hölder for any $\alpha \in (0,1)$.*

Proof. Fix $\alpha \in (0,1)$. Since $\mathfrak{C}_d$ is compact, it is enough to prove that

(19) there is a $k < \omega$ such that f_d is α-Hölder on any set $h([u])$ with $u \in 2^k$.

To see that (19) implies the lemma, first notice that, by the assumptions of (19), there is a $C_1 > 0$ such that f_d is α-Hölder with constant C_1 on any set $h([u])$ with $u \in 2^k$. Moreover, if $E = \bigcup_{u\in 2^k}\{\min h([u]), \max h([u])\}$, then there exists a $C_2 > 0$ such that $|f_d(r) - f_d(s)| \le C_2|r-s|^\alpha$ for all $r, s \in E$. We claim that f_d is α-Hölder with constant $C := 3\max\{C_1, C_2\}$. To see this, choose $x_1, x_2 \in \mathfrak{C}_d$ with $x_1 \le x_2$ and let $u_1, u_2 \in 2^k$ be such that $x_1 \in h([u_1])$ and $x_2 \in h([u_2])$. If $u_1 \ne u_2$, let $r := \max h([u_1])$ and $s := \min h([u_2])$; otherwise put $r = s = x_1$. Then

$$\begin{aligned}
|f_d(x_1) - f_d(x_2)| &= |f_d(x_1) - f_d(r)| + |f_d(r) - f_d(s)| + |f_d(s) - f_d(x_2)| \\
&\le C_1|x_1 - r|^\alpha + C_2|r-s|^\alpha + C_1|s - x_2|^\alpha \\
&\le C_1|x_1 - x_2|^\alpha + C_2|x_1 - x_2|^\alpha + C_1|x_1 - x_2|^\alpha \\
&\le C|x_1 - x_2|^\alpha
\end{aligned}$$

as needed.

To see that (19) is satisfied, fix $k < \omega$ and $u \in 2^k$. Take distinct $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ from $[u]$ with $n = \min\{i \in \omega\colon s_i \ne t_i\}$. So $n > k$. Then, for $c_\alpha := \frac{\left(1+\frac{p}{1-p}\right)}{\left(1-\frac{p}{1-p}\right)^\alpha}$, we get the following simple variation of (18):

$$\frac{|f_d(h(s)) - f_d(h(t))|}{|h(s) - h(t)|^\alpha} \le \frac{\left(1+\frac{p}{1-p}\right) p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1-\frac{p}{1-p}\right)^\alpha p^{\alpha(n+\psi(s\restriction n+1))}} \le \frac{c_\alpha p^n}{p^{\alpha(n+\psi(s\restriction n+1))}}. \tag{20}$$

Also, by (9), we have $\psi(s \restriction n+1) = (N(s\restriction \widehat{n+1}))^2 \le (2\log_2(n+1))^2$, so

$$\frac{c_\alpha p^n}{p^{\alpha(n+\psi(s\restriction n+1))}} \le \frac{c_\alpha p^n}{p^{\alpha(n+\psi(s\restriction n+1))}} \le \frac{c_\alpha p^n}{p^{\alpha(n+(2\log_2(n+1))^2)}}. \tag{21}$$

Since $\frac{\alpha(n+(2\log_2(n+1))^2)}{n} \to_{n\to\infty} \alpha < 1$, there is a $k < \omega$ such that for every $n > k$ we have $\frac{\alpha(n+(2\log_2(n+1))^2)}{n} < 1$, that is, $\alpha(n + (2\log_2(n+1))^2) < n$. Therefore, by (20) and (21), for every $n > k$

$$\frac{|f_d(h(s)) - f_d(h(t))|}{|h(s) - h(t)|^\alpha} \le \frac{c_\alpha p^n}{p^{\alpha(n+(2\log_2(n+1))^2)}} \le \frac{c_\alpha p^n}{p^n} = c_\alpha,$$

that is, f_d is indeed α-Hölder on any set $h([u])$ with $u \in 2^k$. □

Theorem 3.1 follows immediately from this lemma and Theorem 4.1 from the next section.

4. Differential extensions preserving Hölder continuity

Jarník's differentiable extension theorem states that *every real valued differentiable function from a closed subset of $\mathbb{R}$ into $\mathbb{R}$ has a differentiable extension.* For the fascinating history of this theorem and its proof see [3]. (Compare also [6].) For its generalizations, see [13] and [4]. To prove Theorem 3.1, we will need the following generalization of Jarník's differentiable extension theorem, which is of interest by its own right.

Theorem 4.1. *Every differentiable map $f\colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is closed, admits a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ such that if P is compact, then F preserves Hölder continuity of f, that is, if f is α-Hölder for some $\alpha \in (0,1]$, then so is F.*

Proof. We can assume that P is compact and that the set

$$H := \{\alpha \in (0,1]\colon \ f \text{ is } \alpha\text{-Hölder}\}$$

is not empty, since otherwise the result follows immediately from Jarník's differentiable extension theorem. By (8), if $\alpha \in H$ and $\beta \in (0,\alpha)$, then $\beta \in H$.

Let $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ be the linear interpolation[3] of f which is constant on each unbounded connected component of $\mathbb{R} \setminus P$, choose $a < b$ such that $P \subset (a,b)$, and define $\tilde{P} := (-\infty, a] \cup P \cup [b, \infty)$ together with $\tilde{f} := \bar{f} \restriction \tilde{P}$. Then $\tilde{f}$ is still differentiable and, by Fact 1.3 used with $F = \tilde{f}$, $\tilde{f}$ is α–Hölder for every $\alpha \in H$. In addition, $\bar{f}$ is also the linear interpolation of $\tilde{f}$.

Let $\mathcal{J}$ be the family of all connected components of $\mathbb{R} \setminus \tilde{P}$ and Π be the set of all endpoints of the intervals in $\mathcal{J}$. Notice that all $J \in \mathcal{J}$ are bounded.

It is easy to see (compare e.g. [3]) that $\bar{f}$ is differentiable at all points $x \in \mathbb{R} \setminus \Pi$. Also, $\bar{f}$ is differentiable at least from one side at every $x \in \Pi$. Moreover,

$$\bar{f} \text{ is } \alpha\text{-Hölder for every } \alpha \in H. \tag{22}$$

Indeed, this follows from Fact 1.2 used with $F = \bar{f}$ as long as there is $C > 0$ such that

$$|\bar{f}(x) - \bar{f}(y)| \le C|x-y|^\alpha \text{ for every } J \in \mathcal{J} \text{ and all } x, y \in \mathrm{cl}(J). \tag{23}$$

To see (23), fix an $\alpha \in H$ and let C be such that $|\tilde{f}(x) - \tilde{f}(y)| \le C|x-y|^\alpha$ for all $x, y \in \tilde{P}$. Now, if $J = (a,b)$ and $p, q \in \mathrm{cl}(J)$, then $\frac{|\bar{f}(p)-\bar{f}(q)|}{|p-q|} = \frac{|\tilde{f}(a)-\tilde{f}(b)|}{|a-b|}$ and

$$\frac{|\bar{f}(p)-\bar{f}(q)|}{|p-q|^\alpha} = |p-q|^{1-\alpha}\frac{|\bar{f}(p)-\bar{f}(q)|}{|p-q|} \le |a-b|^{1-\alpha}\frac{|\tilde{f}(a)-\tilde{f}(b)|}{|a-b|} = \frac{|\tilde{f}(a)-\tilde{f}(b)|}{|a-b|^\alpha} \le C,$$

justifying (23) and (22).

The proof of Jarník's differentiable extension theorem presented in [3] obtains F by modifying $\bar{f}$ on the family $\mathcal{K} = \{K_n\colon n < \omega\}$ of small pairwise disjoint closed intervals, each contained in the closure of an $J \in \mathcal{J}$ and sharing with J one endpoint. More specifically, for each $n < \omega$ one finds a continuous function $f_n\colon \mathbb{R} \to \mathbb{R}$ with support contained in K_n and defines

$$F := \bar{f} + \sum_{n<\omega} f_n. \tag{24}$$

This modification ensures differentiability at points $x \in \Pi$ from appropriate sides that needed adjustment, while the small size of each K_n ensures preservation of other (unilateral, pointwise) differentiability of $\bar{f}$. In general, it is not clear that

[3] This means that $\bar{f}$ is linear on the closure of every connected component of $\mathbb{R} \setminus P$.

such defined F must preserve Hölder continuity. But we will show that some small modification of the definitions of functions f_n indeed ensures such preservation. Note that without loss of generality we may assume that each $x \in \Pi$ belongs to exactly one K_n, as f_n may be the zero function.

To see this, note that functions f_n are defined in [3] as $f_n(x) := \int_{-\infty}^{x} h_n(t)\, dt$, where h_n is continuous on K_n and zero on its complement, see [3, Figure 3]. Also,

(i) if f is Lipschitz with constant L, then $h_n[\mathbb{R}] \subset [-2L, 2L]$.

In addition, if the lengths of the intervals K_n are further shrinking and new functions h_n are the horizontal proportional shrinking versions of their original selves, then F defined by (24) remains everywhere differentiable. Hence, to finish the proof we just need to show that if numbers $|K_n|$ are small enough, then F is α-Hölder for every $\alpha \in H$.

Towards this goal, let $A = \sup H$ and choose a non-decreasing sequence $\langle \alpha_n \rangle_n$ in H converging to A such that if $A \in H$, then $\alpha_n = A$ for all $n < \omega$. If $1 \in H$, that is, f is Lipschitz, then no change is necessary. Indeed, by (i), F defined by (24) is already Lipschitz,[4] so, every $\alpha \in (0,1)$, the property (8) implies that $F \restriction [a,b]$ is α-Hölder and, by Fact 1.3 used with $P = [a,b]$, so is F.

Thus, we may assume that $1 \notin H$ and so, $\alpha_n < 1$ for all $n < \omega$. For every $n < \omega$ decrease the length of K_n so that the resulting F satisfies

$$|F(x) - F(y)| \le |x-y|^{\alpha_i} \text{ for every } x, y \in K_n \text{ and } i \le n. \tag{25}$$

To see that this can be done, notice that shrinking of K_n does not change Lipschitz constant L_n of F on K_n, which is bounded by the sum of the Lipschitz constant of $\bar{f}$ on K_n and the supremum of $|h_n|[K_n]$. Since, for any $x, y \in K_n$, we have $|F(x) - F(y)| \le L_n|x-y| = L_n|x-y|^{1-\alpha_n}|x-y|^{\alpha_n} \le L_n|K_n|^{1-\alpha_n}|x-y|^{\alpha_i}$, it is enough to shrink K_n so that its new diameter D_n satisfies $L_n D_n^{1-\alpha_n} \le 1$.

To finish the proof, it is enough to show that F defined by (24) and satisfying (25) is as needed. Indeed, clearly F is everywhere differentiable. Next, fix an $\alpha \in H$. To finish the proof it is enough to show that F is α-Hölder. For this, choose an $i < \omega$ such that $\alpha \le \alpha_i$. We claim that

$$F \text{ is } \alpha_i\text{-Hölder.} \tag{26}$$

This will be proved by applying Fact 1.2 to F and the set $\tilde{P}$. Let C be such that $\bar{f}$ is α_i-Hölder with a constant C and let $\mathcal{J}_0$ be the set of all $J \in \mathcal{J}$ such that $K_n \subset \mathrm{cl}(J)$ for some $n < i$. Notice that

$$F \restriction \mathrm{cl}(J) \text{ is } \alpha_i\text{-Hölder with a constant } C+2 \text{ for every } J \in \mathcal{J} \setminus \mathcal{J}_0. \tag{27}$$

Indeed, let $J = (p,q)$. Find a $K \in \mathcal{K}$ containing p and put $K^p := K$. Analogously let K^q be the unique $K \in \mathcal{K}$ satisfying $q \in K$. Notice that, by (25) and the definition of $\mathcal{J}_0$, F on K^p, as well as on K^q, is α_i-Hölder with a constant 1. Finally, to see (27), choose $x_1, x_2 \in [p,q]$ with $x_1 \le x_2$. We need to show that $|F(x_1) - F(x_2)| \le (C+2)|x_1 - x_2|^{\alpha_i}$. If both points x_1 and x_2 are in either K^p or K^q, then this inequality holds. So, assume that this is not the case and let

[4] A Lipschitz differentiable extension version of Jarník's theorem can also be found in [14].

$r := \max\{x_1, \max K^p\}$ and $s := \min\{x_2, \min K^q\}$. Then

$$\begin{aligned} |F(x_1) - F(x_2)| &\le |F(x_1) - F(r)| + |F(r) - F(s)| + |F(s) - F(x_2)| \\ &\le |x_1 - r|^{\alpha_i} + C|r - s|^{\alpha_i} + |s - x_2|^{\alpha_i} \\ &\le (C+2)|x_1 - x_2|^{\alpha_i} \end{aligned}$$

justifying (27). Next notice that, for every $J \in \mathcal{J}_0$, $F \restriction \mathrm{cl}(J)$ is Lipschitz, so, by (8), it is also α_i-Hölder some constant C_J. Combining this with (27), we conclude that the assumption (7) of Fact 1.2 is satisfied with a constant $\max\{C+2, \max_{J\in\mathcal{J}_0} C_J\}$. This completes the proof of (26).

Finally, to see that F is α-Hölder, notice that, by (26), $F \restriction [a, b]$ is α_i-Hölder. So, by (8) and the inequality $\alpha \le \alpha_i$, $F \restriction [a, b]$ is also α-Hölder. So, by Fact 1.3, F is indeed α-Hölder. □

5. Final remarks and open problems

Although for $d = 1$ our minimal dynamical system $f_d : \mathfrak{C}_d \to \mathfrak{C}_d$ is well defined and, by Lemma 2.2, $\mathfrak{C}_d$ has Hausdorff dimension $d = 1$, it is not clear if this f_d is extremely slow. Specifically, the number p associated with $d = 1$ equals to $p = 2^{-1/d} = \frac{1}{2}$, so that the number $1 - \frac{p}{1-p}$ in the estimation (b) from Subsection 2.3 becomes 0. This renders the estimate useless. Thus, the following problems remain open.

Problem 5.1. Does there exist an extremely slow minimal dynamical system $f : C \to C$ such that $C \subset \mathbb{R}$ is compact and of Hausdorff dimension 1?

Notice that for any $n \ge 2$, if a function $F : \mathfrak{C}_d^n \to \mathfrak{C}_d^n$ is defined by a formula $F(\langle x_i\rangle_{i=1}^n) = \langle f_d(x_i)\rangle_{i=1}^n$, then clearly F is an extremely slow dynamical system with $\dim_H(\mathfrak{C}_d^n) \ge n \dim_H(\mathfrak{C}_d) = nd$, see e.g. [15]. Thus, an extremely slow dynamical system on a compact subset of $\mathbb{R}^n$ can have Hausdorff dimension arbitrarily close to n, so greater than 1. However, such defined F is not a minimal dynamical system.

Another interesting question, natural in the context of this work, is

Problem 5.2. Does there exist an extremely slow dynamical system f on a compact $C \subset \mathbb{R}$ such that f is Lipschitz?

References

[1] J. P. Boroński, J. Kupka, and P. Oprocha, *Edrei's conjecture revisited*, Ann. Henri Poincaré **19** (2018), no. 1, 267–281, DOI 10.1007/s00023-017-0623-9.

[2] ———, *All minimal Cantor systems on slow*, Bull. Lond. Math. Soc. **51** (2019), no. 6, 937–944, DOI 10.1112/blms.12275.

[3] M. Ciesielska and K. C. Ciesielski, *Differentiable extension theorem: a lost proof of V. Jarník*, J. Math. Anal. Appl. **454** (2017), no. 2, 883–890, DOI 10.1016/j.jmaa.2017.05.032.

[4] K. C. Ciesielski, *Smooth extension theorems for one variable maps*, J. Math. Anal. Appl. **479** (2019), no. 2, 1893–1905, DOI 10.1016/j.jmaa.2019.07.030. MR3987938

[5] K. C. Ciesielski and J. Jasinski, *Fixed point theorems for maps with local and pointwise contraction properties*, Canad. J. Math. **70** (2018), no. 3, 538–594, DOI 10.4153/CJM-2016-055-2.

[6] K. C. Ciesielski and J. B. Seoane-Sepúlveda, *Differentiability versus continuity: Restriction and extension theorems and monstrous examples*, Bull. Amer. Math. Soc. **56** (2019), no. 2, 211–260, DOI 10.1090/bull/1635.

[7] K. C. Ciesielski, *Monsters in calculus*, Amer. Math. Monthly **125** (2018), no. 8, 739–744, DOI 10.1080/00029890.2018.1502011.

[8] K. C. Ciesielski and J. Jasinski, *An auto-homeomorphism of a Cantor set with derivative zero everywhere*, J. Math. Anal. Appl. **434** (2016), no. 2, 1267–1280, DOI 10.1016/j.jmaa.2015.09.076.
[9] K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986.
[10] K. Falconer, *Fractal geometry*, 3rd ed., John Wiley & Sons, Ltd., Chichester, 2014.
[11] R. Fiorenza, *Hölder and locally Hölder continuous functions, and open sets of class $C^k, C^{k,\lambda}$*, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2016.
[12] S. Gangloff and P. Oprocha, *A Cantor dynamical system is slow if and only if all its finite orbits are attracting*, Discrete Contin. Dyn. Syst. **42** (2022), no. 6, 3039–3064, DOI 10.3934/dcds.2022007.
[13] M. Koc and L. Zajíček, *A joint generalization of Whitney's C^1 extension theorem and Aversa-Laczkovich-Preiss' extension theorem*, J. Math. Anal. Appl. **388** (2012), no. 2, 1027–1037, DOI 10.1016/j.jmaa.2011.10.049.
[14] J. Mařík, *Derivatives and closed sets*, Acta Math. Hungar. **43** (1984), no. 1-2, 25–29, DOI 10.1007/BF01951320.
[15] M. Reyes, *On the dimension print of the Cartesian product of linear sets*, Topology, measures, and fractals (Warnemünde, 1991), Math. Res., vol. 66, Akademie-Verlag, Berlin, 1992, pp. 100–108.
[16] T. Witelski and M. Bowen, *Methods of mathematical modelling*, Springer Undergraduate Mathematics Series, Springer, Cham, 2015.

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